The infinitesimal Hopf algebra and the operads of planar forests

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ABSTRACT. We introduce two operads which own the set of planar forests as a basis. With its usual product and two other products defined by different types of graftings, the algebra of planar rooted trees $\mathcal{H}$ becomes an algebra over these operads. The compatibility with the infinitesimal coproduct of $\mathcal{H}$ and these structures is studied. As an application, an inductive way of computing the dual basis of $\mathcal{H}$ for its infinitesimal pairing is given. Moreover, three Cartier-Quillen-Milnor-Moore theorems are given for the operads of planar forests and a rigidity theorem for one of them.

KEYWORDS. Infinitesimal Hopf algebra, Planar rooted trees, Operads.

AMS CLASSIFICATION. 16W30, 05C05, 18D50.

Contents

Introduction

The Connes-Kreimer Hopf algebra of rooted trees, introduced in [1, 6, 7, 8], is a commutative, non cocommutative Hopf algebra, its coproduct being given by admissible cuts of trees. A non commutative version, the Hopf algebra of planar rooted trees, is introduced in [3, 5]. We furthemore introduce in [4] an infinitesimal version of this object, replacing admissible cuts by left admissible cuts: this last object is here denoted by $\mathcal{H}$. Similarly with the Hopf case, $\mathcal{H}$ is a self-dual object and it owns a non-degenerate, symmetric Hopf pairing, denoted by $\langle -,- \rangle$. This pairing is related to a partial order on the set of planar rooted forests, making it isomorphic to the Tamari poset. As a consequence, $\mathcal{H}$ is given a dual basis denoted by $(f_F)_{F\in\mathbf{F}}$, indexed by the set $\mathbf{F}$ of planar forest. In particular, the sub-family $(f_t)_{t\in\mathbf{T}}$ indexed by the set of planar rooted trees $\mathbf{T}$ is a basis of the space of primitive elements of $\mathcal{H}$.

The aim of this text is to introduce two structures of operad on the space of planar forests. We introduce two (non-symmetric) operads $\mathbb{P}_\searrow$ and $\mathbb{P}_\nearrow$ defined in the following way:

1. $\mathbb{P}_\searrow$ is generated by m and $\searrow\in\mathbb{P}_\searrow(2)$, with relations:
$$\left\{\begin{array}{rcl} m\circ(\searrow,I) &=& \searrow\circ(I,m),\\ m\circ(m,I) &=& m\circ(I,m),\\ \searrow\circ(m,I) &=& \searrow\circ(I,\searrow). \end{array}\right.$$
2. $\mathbb{P}_\nearrow$ is generated by m and $\nearrow\in\mathbb{P}_\nearrow(2)$, with relations:
$$\left\{\begin{array}{rcl} m\circ(\nearrow,I) &=& \nearrow\circ(I,m),\\ m\circ(m,I) &=& m\circ(I,m),\\ \nearrow\circ(\nearrow,I) &=& \nearrow\circ(I,\nearrow). \end{array}\right.$$

We then introduce two products on $\mathcal{H}$ or on its augmentation ideal $\mathcal{M}$, denoted by $\nearrow$ and $\searrow$. The product $F\nearrow G$ consists of grafting F on the left leave of G and the product $F\searrow G$ consists of grafting F on the left root of G. Together with its usual product m, $\mathcal{M}$ becomes both a $\mathbb{P}_\searrow$- and a $\mathbb{P}_\nearrow$-algebra. More precisely, $\mathcal{M}$ is the free $\mathbb{P}_\searrow$- and $\mathbb{P}_\nearrow$-algebra generated by a single element $\bullet$. As a consequence, $\mathbb{P}_\searrow$ and $\mathbb{P}_\nearrow$ inherits a combinatorial representation using planar forests, with composition iteratively described using the products $\searrow$ and $\nearrow$.

We then give several applications of these operadic structures. For example, the antipode of $\mathcal{H}$ is described in term of the operad $\mathbb{P}_\searrow$. We show how to compute elements f_t's, with $t\in\mathbf{T}$, using the action of $\mathbb{P}_\searrow$, and the elements f_F's, $F\in\mathbf{F}$ from the preceding ones using the action of $\mathbb{P}_\nearrow$. Combining all these results, it is possible to compute by induction the basis $(f_F)_{F\in\mathbf{F}}$.

We finally study the compatibilities of products m, $\nearrow$, $\searrow$, the coproduct $\tilde{\Delta}$ and the coproduct $\tilde{\Delta}_\nearrow$ dual of $\nearrow$. This leads to the definition of two types of $\mathbb{P}_\nearrow$-bialgebras, and one type of $\mathbb{P}_\searrow$-bialgebras. Each type then define a suboperad of $\mathbb{P}_\nearrow$ or $\mathbb{P}_\searrow$ corresponding to primitive elements of $\mathcal{M}$, which are explicitively described:

1. The first one is a free operad, generated by the element $\genfrac{}{}{0pt}{}{\bullet}{\bullet}-\bullet\bullet\in\mathbb{P}_\nearrow(2)$. As a consequence, the space of primitive elements of $\mathcal{H}$ admits a basis $(p_t)_{t\in\mathbf{T}_b}$ indexed by the set of planar binary trees. The link with the basis $(f_t)_{t\in\mathbf{T}}$ is given with the help of the Tamari order.
2. The second one admits a combinatorial representation in terms of planar rooted trees. It is generated by the corollas $c_n\in\mathbb{P}_\nearrow(n)$, $n\geq 2$, with the following relations: for all $k,l\geq 2$,
$$c_k\circ(c_l,\underbrace{I,\ldots,I}_{k-1 \text{ times}})=c_l\circ(\underbrace{I,\ldots,I}_{l-1\text{ times}},c_k).$$

3. The third one admits a combinatorial representation in terms of planar rooted trees, and is freely generated by $\mathfrak{l} \in \mathbb{P}_{\searrow}(2)$.

We also give the definition of a double $\mathbb{P}_{\nearrow}$-bialgebra, combining the two types of $\mathbb{P}_{\nearrow}$-bialgebras already introduced. We then prove a rigidity theorem: any double $\mathbb{P}_{\nearrow}$-bialgebra connected as a coalgebra is isomorphic to a decorated version of $\mathcal{M}$.

This text is organised as follows: the first section gives several recalls on the infinitesimal Hopf algebra of planar rooted trees and its pairing. The two products $\searrow$ and $\nearrow$ are introduced in section 2, as well as the combinatorial representation of the two associated operads. The applications to the computation of $(f_F)_{F\in\mathbf{F}}$ is given in section 3. Section 4 is devoted to the study of the suboperads of primitive elements and the last section deals with the rigidity theorem for double $\mathbb{P}_{\nearrow}$-bialgebras.

Notations.

1. We shall denote by K a commutative field, of any characteristic. Every vector space, algebra, coalgebra, etc, will be taken over K.
2. Let (A, Δ, ε) be a counitary coalgebra. Let $1 \in A$, non zero, such that $\Delta(1) = 1 \otimes 1$. We then define the non counitary coproduct:

$$\tilde{\Delta} : \left\{ \begin{array}{rcl} Ker(\varepsilon) & \longrightarrow & Ker(\varepsilon) \otimes Ker(\varepsilon) \\ a & \longrightarrow & \tilde{\Delta}(a) = \Delta(a) - a \otimes 1 - 1 \otimes a. \end{array} \right.$$

We shall use the Sweedler notations $\Delta(a) = a^{(1)} \otimes a^{(2)}$ and $\tilde{\Delta}(a) = a' \otimes a''$.

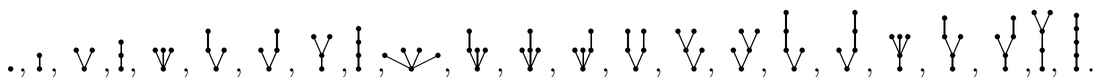
1 Planar rooted forests and their infinitesimal Hopf algebra

We here recall some results and notations of [4].

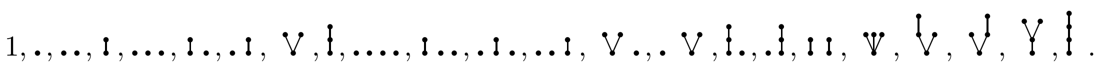
1.1 Planar trees and forests

1. The set of planar trees is denoted by $\mathbf{T}$, and the set of planar forests is denoted by $\mathbf{F}$. The weight of a planar forest is the number of its vertices. For all $n \in \mathbb{N}$, we denote by $\mathbf{F}(n)$ the set of planar forests of weight n.

   Examples. Planar rooted trees of weight ≤ 5:

   Planar rooted forests of weight ≤ 4:

2. The algebra $\mathcal{H}$ is the free associative, unitary algebra generated by $\mathbf{T}$. As a consequence, a linear basis of $\mathcal{H}$ is given by $\mathbf{F}$, and its product is given by the concatenation of planar forests.
3. We shall also need two partial orders and a total order on the set $Vert(F)$ of vertices of $F \in \mathbf{F}$, defined in [3, 4]. We put $F = t_1 \dots t_n$, and let s, s' be two vertices of F.
   (a) We shall say that $s \geq_{high} s'$ if there exists a path from s' to s in F, the edges of F being oriented from the roots to the leaves. Note that $\geq_{high}$ is a partial order, whose Hasse graph is the forest F.

(b) If s and s' are not comparable for $\geq_{high}$, we shall say that $s \geq_{left} s'$ if one of these assertions is satisfied:

i. s is a vertex of t_i and s' is a vertex of t_j, with $i < j$.

ii. s and s' are vertices of the same t_i, and $s \geq_{left} s'$ in the forest obtained from t_i by deleting its root.

This defines the partial order $\geq_{left}$ for all forests F, by induction on the the weight.

(c) We shall say that $s \geq_{h,l} s'$ if $s \geq_{high} s'$ or $s \geq_{left} s'$. This defines a total order on the vertices of F.

1.2 Infinitesimal Hopf algebra of planar forests

1. Let $F \in \mathbf{F}$. An *admissible cut* is a non empty cut of certain edges and trees of F, such that each path in a non-cut tree of F meets at most one cut edge. The set of admissible cuts of F will be denoted by $Adm(F)$. If c is an admissible cut of F, the forest of the vertices which are over the cuts of c will be denoted by $P^c(t)$ (branch of the cut c), and the remaining forest will be denoted by $R^c(t)$ (trunk of the cut). An admissible cut of F will be said to be *left-admissible* if, for all vertices x and y of F, $x \in P^c(F)$ and $x \leq_{left} y$ imply that $y \in P^c(F)$. The set of left-admissible cuts of F will be denoted by $Adm^l(F)$.

2. $\mathcal{H}$ is given a coproduct by the following formula: for all $F \in \mathbf{F}$,

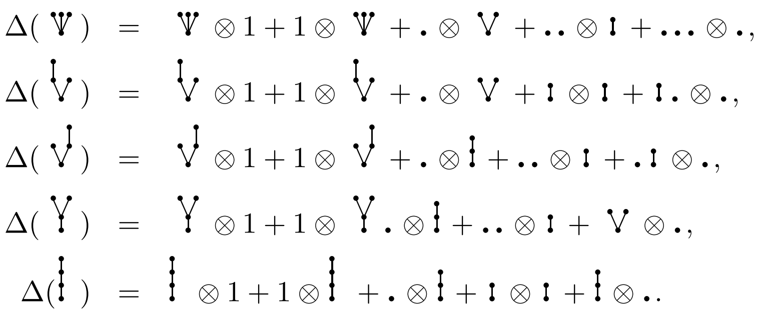
$$\Delta(F) = \sum_{c \in \mathcal{A}dm^l(F)} P^c(F) \otimes R^c(F) + F \otimes 1 + 1 \otimes F.$$

Then $(\mathcal{H}, \Delta)$ is an infinitesimal bialgebra, that is to say: for all $x, y \in \mathcal{H}$,

$$\Delta(xy) = (x \otimes 1)\Delta(y) + \Delta(x)(1 \otimes y) - x \otimes y.$$

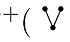
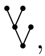
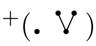
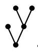
Examples.

$$\begin{array}{rcl}
\Delta(\cdot) &=& \cdot \otimes 1 + 1 \otimes \cdot,\\
\Delta(\cdot\cdot) &=& \cdot\cdot \otimes 1 + 1 \otimes \cdot\cdot + \cdot \otimes \cdot,\\
\Delta(\mathrm{I}) &=& \mathrm{I} \otimes 1 + 1 \otimes \mathrm{I} + \cdot \otimes \cdot,\\
\Delta(\mathrm{I}\cdot) &=& \mathrm{I}\cdot \otimes 1 + 1 \otimes \mathrm{I}\cdot + \cdot \otimes \cdot\cdot + \mathrm{I} \otimes \cdot,\\
\Delta(\mathrm{V}) &=& \mathrm{V} \otimes 1 + 1 \otimes \mathrm{V} + \cdot\cdot \otimes \cdot + \cdot \otimes \mathrm{I},\\
\Delta(\vdots) &=& \vdots \otimes 1 + 1 \otimes \vdots + \mathrm{I} \otimes \cdot + \cdot \otimes \mathrm{I},\\
\Delta(\cdot\cdot\cdot\cdot) &=& \cdot\cdot\cdot\cdot \otimes 1 + 1 \otimes \cdot \otimes \cdot\cdot\cdot + \cdot\cdot \otimes \cdot\cdot + \cdot\cdot\cdot \otimes \cdot,\\
\Delta(\mathrm{I}\cdot\cdot) &=& \mathrm{I}\cdot\cdot \otimes 1 + 1 \otimes \mathrm{I}\cdot\cdot + \cdot \otimes \cdot\cdot\cdot + \mathrm{I} \otimes \cdot\cdot + \mathrm{I}\cdot \otimes \cdot,\\
\Delta(\cdot\mathrm{I}\cdot) &=& \cdot\mathrm{I}\cdot \otimes 1 + 1 \otimes \cdot\mathrm{I}\cdot + \cdot \otimes \mathrm{I}\cdot + \cdot\cdot \otimes \cdot\cdot + \cdot\mathrm{I} \otimes \cdot,\\
\Delta(\cdot\cdot\mathrm{I}) &=& \cdot\cdot\mathrm{I} \otimes 1 + 1 \otimes \cdot\cdot\mathrm{I} + \cdot \otimes \cdot\mathrm{I} + \cdot\cdot \otimes \mathrm{I} + \cdot\cdot\cdot \otimes \cdot,\\
\Delta(\cdot\mathrm{V}) &=& \cdot\mathrm{V} \otimes 1 + 1 \otimes \cdot\mathrm{V} + \cdot \otimes \mathrm{V} + \cdot\cdot \otimes \mathrm{I} + \cdot\cdot\cdot \otimes \cdot,\\
\Delta(\cdot\vdots) &=& \cdot\vdots \otimes 1 + 1 \otimes \cdot\vdots + \cdot \otimes \vdots + \cdot\cdot \otimes \mathrm{I} + \cdot\mathrm{I} \otimes \cdot,\\
\Delta(\mathrm{V}\cdot) &=& \mathrm{V}\cdot \otimes 1 + 1 \otimes \mathrm{V} + \cdot \otimes \mathrm{I}\cdot + \cdot\cdot \otimes \cdot\cdot + \mathrm{V} \otimes \cdot,\\
\Delta(\vdots\cdot) &=& \vdots\cdot \otimes 1 + 1 \otimes \vdots + \cdot \otimes \mathrm{I}\cdot + \mathrm{I} \otimes \cdot\cdot + \vdots \otimes \cdot,\\
\Delta(\mathrm{I}\mathrm{I}) &=& \mathrm{I}\mathrm{I} \otimes 1 + 1 \otimes \mathrm{I}\mathrm{I} + \cdot \otimes \cdot\mathrm{I} + \mathrm{I} \otimes \mathrm{I} + \mathrm{I}\cdot \otimes \cdot,
\end{array}$$



We proved in [4] that $\mathcal{H}$ is an infinitesimal Hopf algebra, that is to say has an antipode S. This antipode satisfies $S(1) = 1$, $S(t) \in Prim(\mathcal{H})$ for all $t \in \mathbf{T}$, and $S(F) = 0$ for all $F \in \mathbf{F} - (\mathbf{T} \cup \{1\})$.

1.3 Pairing on $\mathcal{H}$

1. We define the operator $B^+ : \mathcal{H} \longrightarrow \mathcal{H}$, which associates, to a forest $F \in \mathbf{F}$, the tree obtained by grafting the roots of the trees of F on a common root. For example, $B^+($  $) =$ , and $B^+($  $) =$ .

2. The application γ is defined by:

$$\gamma : \left\{ \begin{array}{rcl} \mathcal{H} & \longrightarrow & \mathcal{H} \\ t_1 \ldots t_n \in \mathbf{F} & \longrightarrow & \delta_{t_1, \bullet} t_2 \ldots t_n. \end{array} \right.$$

3. There exists a unique pairing $\langle -, - \rangle : \mathcal{H} \times \mathcal{H} \longrightarrow K$, satisfying:

   i. $\langle 1, x \rangle = \varepsilon(x)$ for all $x \in \mathcal{H}$.

   ii. $\langle xy, z \rangle = \langle y \otimes x, \Delta(z) \rangle$ for all $x, y, z \in \mathcal{H}$.

   iii. $\langle B^+(x), y \rangle = \langle x, \gamma(y) \rangle$ for all $x, y \in \mathcal{H}$.

   Moreover:

   iv. $\langle -, - \rangle$ is symmetric and non-degenerate.

   v. If x and y are homogeneous of different weights, $\langle x, y \rangle = 0$.

   vi. $\langle S(x), y \rangle = \langle x, S(y) \rangle$ for all $x, y \in \mathcal{H}$.

   This pairing admits a combinatorial interpretation using the partial orders $\geq_{left}$ and $\geq_{high}$ and is related to the Tamari order on planar binary trees, see [4].

4. We denote by $(f_F)_{F \in \mathbf{F}}$ the dual basis of the basis of forests for the pairing $\langle -, - \rangle$. In other terms, for all $F \in \mathbf{F}$, f_F is defined by $\langle f_F, G \rangle = \delta_{F,G}$, for all forest $G \in \mathbf{F}$. The family $(f_t)_{t \in \mathbf{T}}$ is a basis of the space $Prim(\mathcal{H})$ of primitive elements of $\mathcal{H}$.

2 The operads of forests and graftings

2.1 A few recalls on non-Σ-operads

1. We shall work here with non-Σ-operads [11]. Recall that such an object is a family $\mathbb{P} = (\mathbb{P}(n))_{n \in \mathbb{N}}$ of vector spaces, together with a composition for all $n, k_1, \ldots, k_n \in \mathbb{N}$:

$$\left\{ \begin{array}{rcl} \mathbb{P}(n) \otimes \mathbb{P}(k_1) \otimes \ldots \otimes \mathbb{P}(k_n) & \longrightarrow & \mathbb{P}(k_1 + \ldots + k_n) \\ p \otimes p_1 \otimes \ldots \otimes p_n & \longrightarrow & p \circ (p_1, \ldots, p_n). \end{array} \right.$$

The following associativity condition is satisfied: for all $p \in \mathbb{P}(n)$, $p_1 \in \mathbb{P}(k_1)$, ..., $p_n \in \mathbb{P}(k_n)$, $p_{1,1}, \ldots, p_{n,k_n} \in \mathbb{P}$,

$$\begin{array}{rl} & (p \circ (p_1, \ldots, p_n)) \circ (p_{1,1}, \ldots, p_{1,k_1}, \ldots, p_{n,1}, \ldots, p_{n,k_n}) \\ = & p \circ (p_1 \circ (p_{1,1}, \ldots, p_{1,k_1}), \ldots, p_n \circ (p_{n,1}, \ldots, p_{n,k_n})). \end{array}$$

Moreover, there exists a unit element $I \in \mathbb{P}(1)$, satisfying: for all $p \in \mathbb{P}(n)$,

$$\left\{ \begin{array}{rcl} p \circ (I, \ldots, I) & = & p, \\ I \circ p & = & p. \end{array} \right.$$

An operad is a non-Σ-operad $\mathbb{P}$ with a right action of the symmetric group S_n on $\mathbb{P}(n)$ for all n, satisfying a certain compatibility with the composition.

2. Let $\mathbb{P}$ be a non-Σ-operad. A $\mathbb{P}$-algebra is a vector space A, together with an action of $\mathbb{P}$:

$$\left\{ \begin{array}{rcl} \mathbb{P}(n) \otimes A^{\otimes n} & \longrightarrow & A \\ p \otimes a_1 \otimes \ldots \otimes a_n & \longrightarrow & p.(a_1, \ldots, a_n), \end{array} \right.$$

satisfying the following compatibility: for all $p \in \mathbb{P}(n)$, $p_1 \in \mathbb{P}(k_1)$, ..., $p_n \in \mathbb{P}(k_n)$, for all $a_{1,1}, \ldots, a_{n,k_n} \in A$,

$$\begin{array}{rl} & (p \circ (p_1, \ldots, p_n)).(a_{1,1}, \ldots, a_{1,k_1}, \ldots, a_{n,1} \ldots, a_{n,k_n}) \\ = & p.(p_1.(a_{1,1}, \ldots, a_{1,k_1}), \ldots, p_n.(a_{n,1}, \ldots, a_{n,k_n})). \end{array}$$

Moreover, $I.a = a$ for all $a \in A$.

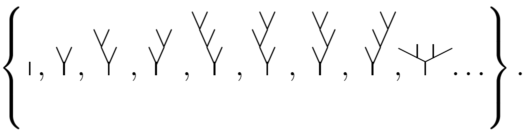
In particular, if V is a vector space, the free $\mathbb{P}$-algebra generated by V is:

$$F_{\mathbb{P}}(V) = \bigoplus_{n \in \mathbb{N}} \mathbb{P}(n) \otimes V^{\otimes n},$$

with the action of $\mathbb{P}$ given by:

$$\begin{array}{rl} & p.\left((p_1 \otimes a_{1,1} \otimes \ldots \otimes a_{1,k_1}), \ldots, (p_n \otimes a_{n,1} \otimes \ldots \otimes a_{n,k_n})\right) \\ = & (p \circ (p_1, \ldots, p_n)) \otimes a_{1,1} \otimes \ldots \otimes a_{1,k_1} \otimes \ldots \otimes a_{n,1} \otimes \ldots \otimes a_{n,k_n}. \end{array}$$

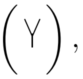
3. Let $\mathbf{T}_b$ be the set of planar binary trees:

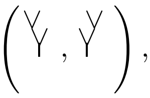
$\mathbf{T}_b =$ 

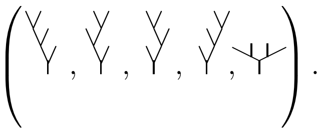
For all $n \in \mathbb{N}$, $\mathbb{T}_b(n)$ is the vector space generated by the elements of $\mathbf{T}_b$ with n leaves:

$\mathbb{T}_b(0) = (0)$,

$\mathbb{T}_b(1) = Vect\,(\mid)$,

$\mathbb{T}_b(2) = Vect$ 

$\mathbb{T}_b(3) = Vect$ 

$\mathbb{T}_b(4) = Vect$ 

The family of vector spaces $\mathbb{T}_b$ is given a structure of non-Σ-operad by graftings on the leaves. More precisely, if $t, t_1, \ldots, t_n \in \mathbf{T}_b$, t with n leaves, then $t \circ (t_1, \ldots, t_n)$ is the binary tree obtained by grafting t_1 on the first leave of t, t_2 on the second leave of t, and so on (note that the leaves of t are ordered from left to right). The unit is $\mid$.

It is known that $\mathbb{T}_b$ is the free non-Σ-operad generated by $\curlyvee \in \mathbb{T}_b(2)$. Similarly, given elements $m_1, \ldots, m_k$ in $\mathbb{P}(2)$, it is possible to describe the free non-Σ-operad $\mathbb{P}$ generated by these elements in terms of planar binary trees whose internal vertices are decorated by $m_1, \ldots, m_k$.

2.2 Presentations of the operads of forests

Definition 1

1. $\mathbb{P}_\searrow$ is the non-Σ-operad generated by m and $\searrow \in \mathbb{P}_\searrow(2)$, with relations:

$$\left\{\begin{array}{rcl} m \circ (\searrow, I) & = & \searrow \circ (I, m), \\ m \circ (m, I) & = & m \circ (I, m), \\ \searrow \circ (m, I) & = & \searrow \circ (I, \searrow). \end{array}\right.$$

2. $\mathbb{P}_\nearrow$ is the non-Σ-operad generated by m and $\nearrow \in \mathbb{P}_\nearrow(2)$, with relations:

$$\left\{\begin{array}{rcl} m \circ (\nearrow, I) & = & \nearrow \circ (I, m), \\ m \circ (m, I) & = & m \circ (I, m), \\ \nearrow \circ (\nearrow, I) & = & \nearrow \circ (I, \nearrow). \end{array}\right.$$

Remark. We shall prove in [2] that these quadratic operads are Koszul.

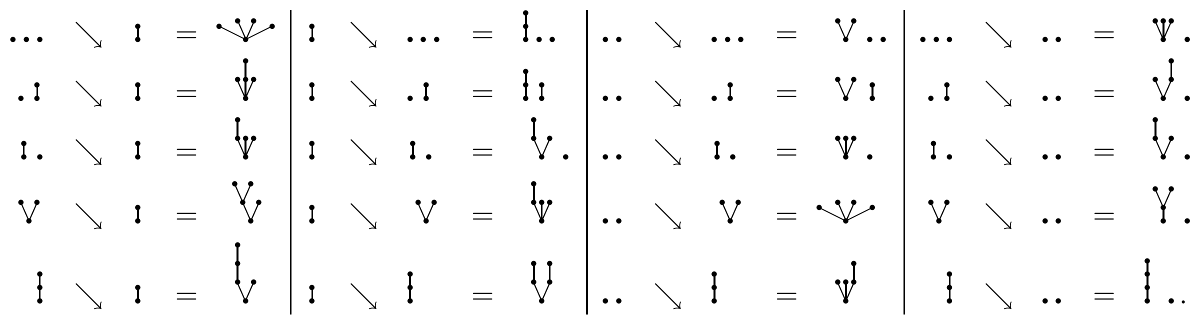
2.3 Grafting on the root

Let $F, G \in \mathbf{F} - \{1\}$. We put $G = t_1 \ldots t_n$ and $t_1 = B^+(G_1)$. We define:

$$F \searrow G = B^+(FG_1) t_2 \ldots t_n.$$

In other terms, F is grafted on the root of the first tree of G, on the left. In particular, $F \searrow \bullet = B^+(F)$.

Examples.



Obviously, $\searrow$ can be inductively defined in the following way: for $F, G, H \in \mathbf{F} - \{1\}$,

$$\left\{\begin{array}{rcl} F \searrow \bullet & = & B^+(F), \\ F \searrow (GH) & = & (F \searrow G) H \\ F \searrow B^+(G) & = & B^+(FG). \end{array}\right.$$

We denote by $\mathcal{M}$ the augmentation ideal of $\mathcal{H}$, that is to say the vector space generated by the elements of $\mathbf{F} - \{1\}$. We extend $\searrow : \mathcal{M} \otimes \mathcal{M} \longrightarrow \mathcal{M}$ by linearity.

Proposition 2 *For all $x, y, z \in \mathcal{M}$:*

$$x \searrow (yz) = (x \searrow y)z, \tag{1}$$
$$x \searrow (y \searrow z) = (xy) \searrow z. \tag{2}$$

Proof. We can restrict ourselves to $x, y, z \in \mathbf{F} - \{1\}$. Then (1) is immediate. In order to prove (2), we put $z = B^+(z_1)z_2$, $z_1, z_2 \in \mathbf{F}$. Then:

$$x \searrow (y \searrow z) = x \searrow (B^+(yz_1)z_2) = B^+(xyz_1)z_2 = (xy) \searrow (B^+(z_1)z_2) = (xy) \searrow z,$$

which proves (2). $\Box$

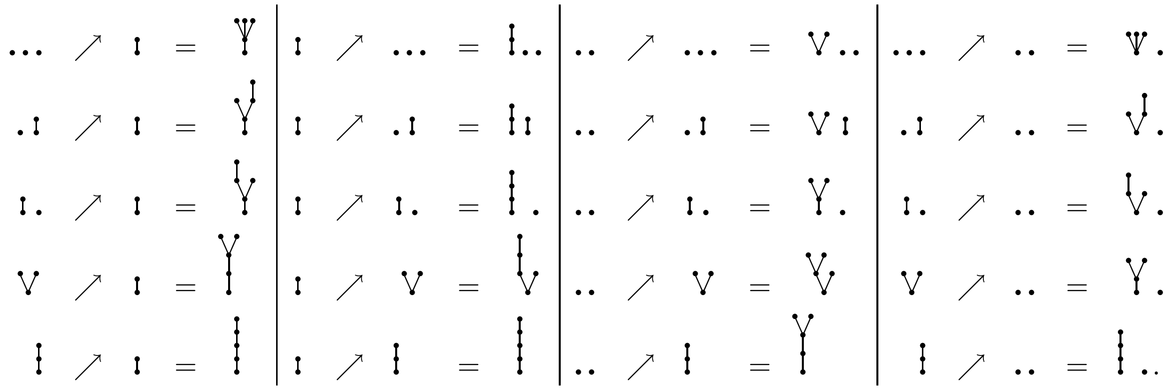
Corollary 3 *$\mathcal{M}$ is given a graded $\mathbb{P}_\searrow$-algebra structure by its products m and by $\searrow$.*

Proof. Immediate, by proposition 2. $\Box$

2.4 Grafting on the left leave

Let $F, G \in \mathbf{F}$. Suppose that $G \neq 1$. Then $F \nearrow G$ is the planar forest obtained by grafting F on the leave of G which is at most on the left. For $G = 1$, we put $F \nearrow 1 = F$. In particular, $F \nearrow \centerdot = B^+(F)$.

Examples.



In an obvious way, $\nearrow$ can be inductively defined in the following way: for $F, G, H \in \mathbf{F}$,

$$\begin{cases} F \nearrow 1 &= F, \\ F \nearrow (GH) &= (F \nearrow G)H \text{ if } G \neq 1, \\ F \nearrow B^+(G) &= B^+(F \nearrow G). \end{cases}$$

We extend $\nearrow : \mathcal{H} \otimes \mathcal{H} \longrightarrow \mathcal{H}$ by linearity.

Proposition 4 *1. For all $x, z \in \mathcal{H}$, $y \in \mathcal{M}$:*

$$x \nearrow (yz) = (x \nearrow y)z. \tag{3}$$

2. For all $x, y, z \in \mathcal{H}$:

$$x \nearrow (y \nearrow z) = (x \nearrow y) \nearrow z.$$

So $(\mathcal{H}, \nearrow)$ is an associative algebra, with unitary element 1.

Proof. Note that (3) is immediate for $x, y, z \in \mathbf{F}$, with $y \neq 1$. This implies the first point. In order to prove the second point, we consider:

$$Z = \{z \in \mathcal{H} \,/\, \forall x, y \in \mathcal{H},\ x \nearrow (y \nearrow z) = (x \nearrow y) \nearrow z\}.$$

Let us first prove that $1 \in Z$: for all $x, y \in \mathcal{H}$,

$$x \nearrow (y \nearrow 1) = x \nearrow y = (x \nearrow y) \nearrow 1.$$

Let $z_1, z_2 \in Z$. Let us show that $z_1 z_2 \in Z$. By linearity, we can separate the proof into two cases:

1. $z_1 = 1$. Then it is obvious.
2. $\varepsilon(z_1) = 0$. Let $x, y \in \mathcal{H}$. By the first point:

$$\begin{array}{rcl} x \nearrow (y \nearrow (z_1 z_2)) &=& x \nearrow ((y \nearrow z_1) z_2)) \\ &=& (x \nearrow (y \nearrow z_1)) z_2 \\ &=& ((x \nearrow y) \nearrow z_1) z_2 \\ &=& (x \nearrow y) \nearrow (z_1 z_2). \end{array}$$

So Z is a subalgebra of $\mathcal{H}$. Let us show that it is stable by B^+. Let $z \in Z$, $x, y \in \mathcal{H}$. Then:

$$\begin{array}{rcl} x \nearrow (y \nearrow B^+(z)) &=& x \nearrow B^+(y \nearrow z) \\ &=& B^+(x \nearrow (y \nearrow z)) \\ &=& B^+((x \nearrow y) \nearrow z) \\ &=& (x \nearrow y) \nearrow B^+(z). \end{array}$$

So Z is a subalgebra of $\mathcal{H}$, stable by B^+. Hence, $Z = \mathcal{H}$. $\square$

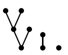
Remarks.

1. (3) is equivalent to: for any $x, y, z \in \mathcal{H}$,

$$x \nearrow (yz) - \varepsilon(y) x \nearrow z = (x \nearrow y) z - \varepsilon(y) xz.$$

2. Let $F \in \mathbf{F} - \{1\}$. There exists a unique family $(\bullet F_1, \ldots, \bullet F_n)$ of elements of $\mathbf{F}$ such that:

$$F = (\bullet F_1) \nearrow \ldots \nearrow (\bullet F_n).$$

For example,  $= (\bullet\bullet) \nearrow (\bullet\bullet) \nearrow (\bullet\, \mathfrak{l}\, \bullet)$. As a consequence, $(\mathcal{H}, \nearrow)$ is freely generated by $\bullet\mathbf{F}$ as an associative algebra.

Corollary 5 *$\mathcal{M}$ is given a graded $\mathbb{P}_{\nearrow}$-algebra structure by its product m and by $\nearrow$.*

Proof. Immediate, by proposition 4. $\square$

2.5 Dimensions of $\mathbb{P}_{\searrow}$ and $\mathbb{P}_{\nearrow}$

We now compute the dimensions of $\mathbb{P}_{\searrow}(n)$ and $\mathbb{P}_{\nearrow}(n)$ for all n and deduce that $\mathcal{M}$ is the free $\mathbb{P}_{\searrow}$- and $\mathbb{P}_{\nearrow}$-algebra generated by $\bullet$.

Notation. We denote by r_n the number of planar rooted forests and we put $R(X) = \sum_{n=1}^{+\infty} r_n X^n$. It is well-known (see [3, 13]) that $R(X) = \dfrac{1 - 2X - \sqrt{1 - 4X}}{2X}$.

Proposition 6 *For $\overset{?}{\to} \in \{\searrow, \nearrow\}$ and all $n \in \mathbb{N}^*$, in the $\mathbb{P}_{\overset{?}{\to}}$-algebra $\mathcal{M}$:*

$$\mathbb{P}_{\overset{?}{\to}}(n).(\bullet, \ldots, \bullet) = Vect(\text{planar forests of weight } n).$$

As a consequence, $\mathcal{M}$ is generated as a $\mathbb{P}_{\overset{?}{\to}}$-algebra by $\bullet$.

Proof. $\subseteq$. Immediate, as $\mathcal{M}$ is a graded $\mathbb{P}_{\overset{?}{\to}}$-algebra.

$\supseteq$. Induction on n. For $n = 1$, $I.(\bullet) = \bullet$. For $n \geq 2$, two cases are possible.

1. $F = F_1F_2$, $weight(F_i) = n_i < n$. By the induction hypothesis, there exists $p_1, p_2 \in \mathbb{P}_{\overset{?}{\to}}$, such that $F_1 = p_1.(\bullet, \ldots, \bullet)$ and $F_2 = p_2.(\bullet, \ldots, \bullet)$. Then $(m \circ (p_1, p_2)).(\bullet, \ldots, \bullet) = m.(F_1, F_2) = F_1F_2$.
2. $F \in \mathbf{T}$. Let us put $F = B^+(G)$. Then there exists $p \in \mathbb{P}_{\overset{?}{\to}}$, such that $p.(\bullet, \ldots, \bullet) = G$. Then:
$$\left\{ \begin{array}{rcccl} (\searrow \circ (p, I)).(\bullet, \ldots, \bullet) & = & G \searrow \bullet & = & F, \\ (\nearrow \circ (p, I)).(\bullet, \ldots, \bullet) & = & G \nearrow \bullet & = & F. \end{array} \right.$$

Hence, in both cases, $F \in \mathbb{P}_{\overset{?}{\to}}(n).(\bullet, \ldots, \bullet)$. $\Box$

Corollary 7 *For all $\overset{?}{\to} \in \{\searrow, \nearrow\}$, $n \in \mathbb{N}^*$, $dim(\mathbb{P}_{\overset{?}{\to}}(n)) \geq r_n$.*

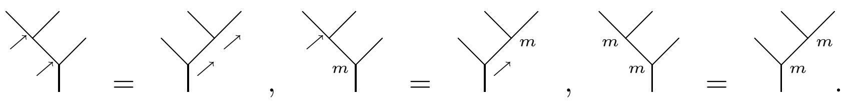
Proof. Because we proved the surjectivity of the following application:

$$ev_{\overset{?}{\to}} : \left\{ \begin{array}{rcl} \mathbb{P}_{\overset{?}{\to}}(n) & \longrightarrow & Vect(\text{planar forests of weight } n) \\ p & \longrightarrow & p.(\bullet, \ldots, \bullet). \end{array} \right.$$

$\Box$

Lemma 8 *For all $\overset{?}{\to} \in \{\searrow, \nearrow\}$, $n \in \mathbb{N}^*$, $dim(\mathbb{P}_{\overset{?}{\to}}(n)) \leq r_n$.*

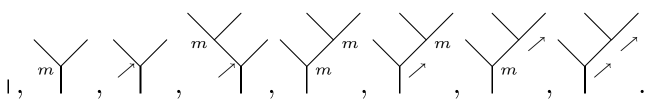
Proof. We prove it for $\overset{?}{\to} = \nearrow$. Let us fix $n \in \mathbb{N}^*$. Then $\mathbb{P}_{\nearrow}(n)$ is linearly generated by planar binary trees whose internal vertices are decorated by m and $\nearrow$. The following relations hold:



In the sequel of the proof, we shall say that such a tree is *admissible* if it satisfies the following conditions:

1. For each internal vertex s decorated by m, the left child of s is a leave.
2. For each internal vertex s decorated by $\nearrow$, the left child of s is a leave or is decorated by m.

For example, here are the admissible trees with 1, 2 or 3 leaves:



The preceding relations imply that $\mathbb{P}_{\nearrow}(n)$ is linearly generated by admissible trees with n leaves. So $dim(\mathbb{P}_{\nearrow}(n))$ is smaller than a_n, the number of admissible trees with n leaves. For $n \geq 2$, we denote by b_n the number of admissible trees with n leaves whose root is decorated by

m, and by c_n the number of admissible trees with n leaves whose root is decorated by $\nearrow$. We also put $b_1 = 1$ and $c_1 = 0$. Finally, we define:

$$A(X) = \sum_{n\geq 1} a_n X^n, \qquad B(X) = \sum_{n\geq 1} b_n X^n, \qquad C(X) = \sum_{n\geq 1} c_n X^n.$$

Immediately, $A(X) = B(X) + C(X)$. Every admissible tree with $n \geq 2$ leaves whose root is decorated by m is of the form $m \circ (I, t)$, where t is an admissible tree with $n-1$ leaves. Hence, $B(X) = XA(X) + X$. Moreover, every admissible tree with $n \geq 2$ leaves whose root is decorated by $\nearrow$ is of the form $\nearrow \circ (t_1, t_2)$, where t_1 is an admissible tree with k leaves whose eventual root is decorated by m and t_2 an admissible tree with $n-k$ leaves $(1 \leq k \leq n-1)$. Hence, for all $n \geq 2$, $c_n = \displaystyle\sum_{k=1}^{n-1} b_k a_{n-k}$, so $C(X) = B(X)A(X)$. As a conclusion:

$$\left\{\begin{array}{rcl} A(X) & = & B(X) + C(X), \\ B(X) & = & XA(X) + X, \\ C(X) & = & B(X)A(X). \end{array}\right.$$

So $A(X) = XA(X) + X + B(X)A(X) = XA(X) + X + XA(X)^2 + XA(X)$, and:

$$XA(X)^2 + (2X-1)A(X) + X = 0.$$

As $a_1 = 1$:

$$A(X) = \frac{1 - 2X - \sqrt{1-4X}}{2X} = R(X).$$

So, for all $n \geq 1$, $dim(\mathbb{P}_{\nearrow}(n)) \leq a_n = r_n$. The proof is similar for $\mathbb{P}_{\searrow}$. $\Box$

As immediate consequences:

Theorem 9 *For* $\stackrel{?}{\rightarrow} \in \{\searrow, \nearrow\}$, $n \in \mathbb{N}^*$, $dim(\mathbb{P}_{\stackrel{?}{\rightarrow}}(n)) = r_n$. *Moreover, the following application is bijective:*

$$ev_{\stackrel{?}{\rightarrow}} : \left\{\begin{array}{rcl} \mathbb{P}_{\stackrel{?}{\rightarrow}}(n) & \longrightarrow & Vect(\textit{planar forests of weight } n) \subseteq \mathcal{M} \\ p & \longrightarrow & p.(\bullet, \ldots, \bullet). \end{array}\right.$$

Corollary 10 1. $(\mathcal{M}, m, \searrow)$ *is the free* $\mathbb{P}_{\searrow}$*-algebra generated by* $\bullet$.

2. $(\mathcal{M}, m, \nearrow)$ *is the free* $\mathbb{P}_{\nearrow}$*-algebra generated by* $\bullet$.

2.6 A combinatorial description of the composition

Let $\stackrel{?}{\rightarrow} \in \{\searrow, \nearrow\}$. We identify $\mathbb{P}_{\stackrel{?}{\rightarrow}}$ and the vector space of non-empty planar forests via theorem 9. In other terms, we identify $F \in \mathbf{F}(n)$ and $ev_{\stackrel{?}{\rightarrow}}^{-1}(F) \in \mathbb{P}_{\stackrel{?}{\rightarrow}}(n)$.

Notations.

1. In order to distinguish the compositions in $\mathbb{P}_{\searrow}$ and $\mathbb{P}_{\nearrow}$, we now denote:

   (a) $F \circ_{\searrow} (F_1, \ldots, F_n)$ the composition of $\mathbb{P}_{\searrow}$,

   (b) $F \circ_{\nearrow} (F_1, \ldots, F_n)$ the composition of $\mathbb{P}_{\nearrow}$.

2. In order to distinguish the action of the operads $\mathbb{P}_{\searrow}$ and $\mathbb{P}_{\nearrow}$ on $\mathcal{M}$, we now denote:

   (a) $F \bullet_{\searrow} (x_1, \ldots, x_n)$ the action of $\mathbb{P}_{\searrow}$ on $\mathcal{M}$,

   (b) $F \bullet_{\nearrow} (x_1, \ldots, x_n)$ the action of $\mathbb{P}_{\nearrow}$ on $\mathcal{M}$.

Our aim in this paragraph is to describe the compositions of $\mathbb{P}_{\searrow}$ and $\mathbb{P}_{\nearrow}$ in term of forests. We shall prove the following result:

Theorem 11 *1. The composition of $\mathbb{P}_{\searrow}$ in the basis of planar forests can be inductively defined in this way:*

$$\left\{\begin{array}{rcl} \bullet \circ_{\searrow} (H) &=& H,\\ B^+(F) \circ_{\searrow} (H_1,\ldots,H_{n+1}) &=& (F \circ_{\searrow} (H_1,\ldots,H_n)) \searrow H_{n+1},\\ FG \circ_{\searrow} (H_1,\ldots,H_{n_1+n_2}) &=& F \circ_{\searrow} (H_1,\ldots,H_{n_1}) G \circ_{\searrow} (H_{n_1+1},\ldots,H_{n_1+n_2}). \end{array}\right.$$

2. The composition of $\mathbb{P}_{\nearrow}$ in the basis of planar forests can be inductively defined in this way:

$$\left\{\begin{array}{rcl} \bullet \circ_{\nearrow} (H) &=& H,\\ B^+(F) \circ_{\nearrow} (H_1,\ldots,H_{n+1}) &=& (F \circ_{\nearrow} (H_1,\ldots,H_n)) \nearrow H_{n+1},\\ FG \circ_{\nearrow} (H_1,\ldots,H_{n_1+n_2}) &=& F \circ_{\nearrow} (H_1,\ldots,H_{n_1}) G \circ_{\nearrow} (H_{n_1+1},\ldots,H_{n_1+n_2}). \end{array}\right.$$

Examples. Let $F_1, F_2, F_3 \in \mathbf{F} - \{1\}$.

$$\begin{array}{rcl|rcl} \bullet\bullet \circ_{\nearrow} (F_1,F_2) &=& F_1F_2, & \bullet\bullet \circ_{\searrow} (F_1,F_2) &=& F_1F_2,\\ \overset{\bullet}{\underset{\bullet}{|}} \circ_{\nearrow} (F_1,F_2) &=& F_1 \nearrow F_2, & \overset{\bullet}{\underset{\bullet}{|}} \circ_{\searrow} (F_1,F_2) &=& F_1 \searrow F_2,\\ \bullet\bullet\bullet \circ_{\nearrow} (F_1,F_2,F_3) &=& F_1F_2F_3, & \bullet\bullet\bullet \circ_{\searrow} (F_1,F_2,F_3) &=& F_1F_2F_3,\\ \bullet\overset{\bullet}{\underset{\bullet}{|}} \circ_{\nearrow} (F_1,F_2,F_3) &=& F_1(F_2 \nearrow F_3), & \bullet\overset{\bullet}{\underset{\bullet}{|}} \circ_{\searrow} (F_1,F_2,F_3) &=& F_1(F_2 \searrow F_3),\\ \overset{\bullet}{\underset{\bullet}{|}}\bullet \circ_{\nearrow} (F_1,F_2,F_3) &=& (F_1 \nearrow F_2)F_3, & \overset{\bullet}{\underset{\bullet}{|}}\bullet \circ_{\searrow} (F_1,F_2,F_3) &=& (F_1 \searrow F_2)F_3,\\ \overset{\bullet\ \bullet}{\underset{\bullet}{\vee}} \circ_{\nearrow} (F_1,F_2,F_3) &=& (F_1F_2) \nearrow F_3, & \overset{\bullet\ \bullet}{\underset{\bullet}{\vee}} \circ_{\searrow} (F_1,F_2,F_3) &=& (F_1F_2) \searrow F_3,\\ \begin{smallmatrix}\bullet\\|\\\bullet\\|\\\bullet\end{smallmatrix} \circ_{\nearrow} (F_1,F_2,F_3) &=& (F_1 \nearrow F_2) \nearrow F_3, & \begin{smallmatrix}\bullet\\|\\\bullet\\|\\\bullet\end{smallmatrix} \circ_{\searrow} (F_1,F_2,F_3) &=& (F_1 \searrow F_2) \searrow F_3. \end{array}$$

Proposition 12 *Let $\overset{?}{\to} \in \{\searrow, \nearrow\}$.*

1. $\bullet$ is the unit element of $\mathbb{P}_{\overset{?}{\to}}$.

2. $\bullet\bullet = m$ in $\mathbb{P}_{\overset{?}{\to}}(2)$. Consequently, in $\mathbb{P}_{\overset{?}{\to}}$, $\bullet\bullet \circ (F,G) = FG$ for all $F, G \in \mathbf{F} - \{1\}$.

3. Let $F, G \in \mathbf{F}$. In $\mathbb{P}_{\overset{?}{\to}}$, $\overset{\bullet}{\underset{\bullet}{|}} = \overset{?}{\to}$. Consequently, $\overset{\bullet}{\underset{\bullet}{|}} \circ_{\overset{?}{\to}} (F,G) = F \overset{?}{\to} G$ for all $F, G \in \mathbf{F} - \{1\}$.

Proof.

1. Indeed, $ev_{\overset{?}{\to}}(\bullet) = \bullet = ev_{\overset{?}{\to}}(I)$. Hence, $\bullet = I$.

2. By definition, $ev_{\overset{?}{\to}}(\bullet\bullet) = \bullet\bullet = ev_{\overset{?}{\to}}(m)$. So $\bullet\bullet = m$ in $\mathbb{P}_{\overset{?}{\to}}(2)$. Moreover, for all $F, G \in \mathbf{F} - \{1\}$:

$$\begin{array}{rcl} ev_{\overset{?}{\to}}(FG) &=& FG\\ &=& m \bullet_{\overset{?}{\to}} (F,G)\\ &=& m \bullet_{\overset{?}{\to}} (F \bullet_{\overset{?}{\to}} (\bullet,\ldots,\bullet), G \bullet_{\overset{?}{\to}} (\bullet,\ldots,\bullet))\\ &=& \left(m \circ_{\overset{?}{\to}} (F,G)\right) \bullet_{\overset{?}{\to}} (\bullet,\ldots,\bullet)\\ &=& ev_{\overset{?}{\to}}(m \circ_{\overset{?}{\to}} (F,G)). \end{array}$$

So $FG = m \circ_{\overset{?}{\to}} (F,G) = \bullet\bullet \circ_{\overset{?}{\to}} (F,G)$.

3. Indeed, $ev_{\overset{?}{\to}}(\mathbf{l}) = \bullet\overset{?}{\to}\bullet = ev_{\overset{?}{\to}}(\overset{?}{\to})$. So $\mathbf{l} = \overset{?}{\to}$ in $\mathbb{P}_{\overset{?}{\to}}(2)$. Moreover:

$$\begin{array}{rcl}
ev_{\overset{?}{\to}}(F\overset{?}{\to}G) &=& F\overset{?}{\to}G\\
&=& \overset{?}{\to}\overset{?}{\bullet\!\!\to}(F,G)\\
&=& \overset{?}{\to}\overset{?}{\bullet\!\!\to}(F\overset{?}{\bullet\!\!\to}(\bullet,\ldots,\bullet),G\overset{?}{\bullet\!\!\to}(\bullet,\ldots,\bullet))\\
&=& (\overset{?}{\to}\overset{?}{\ominus\!\!\to}(F,G)).(\bullet,\ldots,\bullet)\\
&=& ev_{\overset{?}{\to}}(\overset{?}{\to}\overset{?}{\ominus\!\!\to}(F,G)).
\end{array}$$

So, $F\overset{?}{\to}G = \overset{?}{\to}\overset{?}{\ominus\!\!\to}(F,G) = \mathbf{l}\overset{?}{\ominus\!\!\to}(F,G)$.

$\square$

Proposition 13 *1. Let $F,G \in \mathbf{F}$, different from 1, of respective weights n_1 and n_2. Let $H_{1,1},\ldots,H_{1,n_1}$ and $H_{2,1},\ldots,H_{2,n_2} \in \mathbf{F}-\{1\}$. Let $\overset{?}{\to} \in \{\searrow,\nearrow\}$. Then, in $\mathbb{P}_{\overset{?}{\to}}$:*

$$(FG)\overset{?}{\ominus\!\!\to}(H_{1,1},\ldots,H_{1,n_1},H_{2,1},\ldots,H_{2,n_2}) = F\overset{?}{\ominus\!\!\to}(H_{1,1},\ldots,H_{1,n_1})G\overset{?}{\ominus\!\!\to}(H_{2,1},\ldots,H_{2,n_2}).$$

2. Let $F \in \mathbf{F}$, of weight $n \geq 1$. Let $H_1,\ldots,H_{n+1} \in \mathbf{F}$. In $\mathbb{P}_{\overset{?}{\to}}$:

$$B^+(F)\overset{?}{\ominus\!\!\to}(H_1,\ldots,H_{n+1}) = (F\overset{?}{\ominus\!\!\to}(H_1,\ldots,H_n))\overset{?}{\to}H_{n+1}.$$

Proof.

1. Indeed, in $\mathbb{P}_{\overset{?}{\to}}$:

$$\begin{array}{rl}
& (FG)\overset{?}{\ominus\!\!\to}(H_{1,1},\ldots,H_{1,n_1},H_{2,1},\ldots,H_{2,n_2})\\
=& (m\overset{?}{\ominus\!\!\to}(F,G))\overset{?}{\ominus\!\!\to}(H_{1,1},\ldots,H_{1,n_1},H_{2,1},\ldots,H_{2,n_2})\\
=& m\overset{?}{\ominus\!\!\to}(F\overset{?}{\ominus\!\!\to}(H_{1,1},\ldots,H_{1,n_1}),G\overset{?}{\ominus\!\!\to}(H_{2,1},\ldots,H_{2,n_2}))\\
=& F\overset{?}{\ominus\!\!\to}(H_{1,1},\ldots,H_{1,n_1})G\overset{?}{\ominus\!\!\to}(H_{2,1},\ldots,H_{2,n_2})).
\end{array}$$

2. In $\mathbb{P}_{\overset{?}{\to}}$:

$$\begin{array}{rcl}
B^+(F)\overset{?}{\ominus\!\!\to}(H_1,\ldots,H_{n+1}) &=& (F\overset{?}{\to}\bullet)\overset{?}{\ominus\!\!\to}(H_1,\ldots,H_{n+1})\\
&=& (\mathbf{l}\overset{?}{\ominus\!\!\to}(F,\bullet))\overset{?}{\ominus\!\!\to}(H_1,\ldots,H_{n+1})\\
&=& \mathbf{l}\overset{?}{\ominus\!\!\to}(F\overset{?}{\ominus\!\!\to}(H_1,\ldots,H_n),\bullet\overset{?}{\ominus\!\!\to}(H_{n+1}))\\
&=& \mathbf{l}\overset{?}{\ominus\!\!\to}(F\overset{?}{\ominus\!\!\to}(H_1,\ldots,H_n),H_{n+1})\\
&=& (F\overset{?}{\ominus\!\!\to}(H_1,\ldots,H_n))\overset{?}{\to}H_{n+1}.
\end{array}$$

$\square$

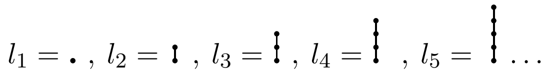
Combining propositions 12 and 13, we obtain theorem 11.

3 Applications to the infinitesimal Hopf algebra $\mathcal{H}$

3.1 Antipode of $\mathcal{H}$

We here give a description of the antipode of $\mathcal{H}$ in terms of the action $\bullet\!\!\searrow$ of the operad $\mathbb{P}_{\searrow}$.

Notations. For all $n \in \mathbb{N}^*$, we denote $l_n = (B^+)^n(1) \in \mathbf{F}(n)$. For example:

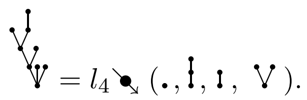


Lemma 14 *Let $t \in \mathbf{T}$. There exists a unique $k \in \mathbb{N}^*$, and a unique family $(t_2 \ldots, t_k) \in \mathbf{T}^{k-1}$ such that:*

$$t = l_k \searrow (\bullet, t_2, \ldots, t_k).$$

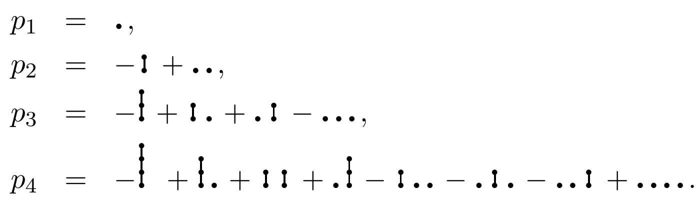
Proof. Induction on the weight n of t. If $n = 1$, then $t = \bullet$, so $k = 1$ and the family is empty. We suppose the result at all rank $< n$. We put $t = B^+(s_1 \ldots s_m)$. Necessarily, $t_k = B^+(s_2 \ldots s_m)$ and $l_{n-1} \searrow (\bullet, t_2, \ldots, t_{k-1}) = s_1$. We conclude with the induction hypothesis on s_1. $\square$

Example.



Definition 15 For all $n \in \mathbb{N}^*$, we put $p_n = \displaystyle\sum_{k=1}^{n} \sum_{\substack{a_1+\ldots+a_k=n \\ \forall i,\, a_i>0}} (-1)^k l_{a_1} \ldots l_{a_k}$.

Examples.



Remark that p_n is in fact the antipode of l_n in $\mathcal{H}$. It is also the antipode of l_n in the non commutative Connes-Kreimer Hopf algebra of planar trees [3].

Corollary 16 *Let $t \in \mathbf{T}$, written under the form $t = l_k \searrow (t_1, \ldots, t_k)$, with $t_1 = \bullet$. Then:*

$$S(t) = p_k \searrow (t_1, \ldots, t_k).$$

Proof. Corollary of proposition 15 of [4], observing that left cuts are cuts on edges from the root of t_i to the root of t_{i+1} in t, for $i = 1, \ldots, n-1$. $\square$

3.2 Inverse of the application γ

Proposition 17 *The restriction $\gamma : Prim(\mathcal{H}) \longrightarrow \mathcal{H}$ is bijective.*

Proof. By proposition 21 of [4]:

$$\gamma_{|Prim(\mathcal{H})} : \left\{ \begin{array}{rcl} Prim(\mathcal{H}) & \longrightarrow & \mathcal{H} \\ f_{B^+(F)}\, (F \in \mathbf{F}) & \longrightarrow & f_F. \end{array} \right.$$

So this restriction is clearly bijective. $\square$

We shall denote $\gamma^{-1}_{|Prim(\mathcal{H})} : \mathcal{H} \longrightarrow Prim(\mathcal{H})$ the inverse of this restriction. Then, for all $F \in \mathbf{F}$, $\gamma^{-1}_{|Prim(\mathcal{H})}(f_F) = f_{B^+(F)}$. Our aim is to express $\gamma^{-1}_{|Prim(\mathcal{H})}$ in the basis of forests.

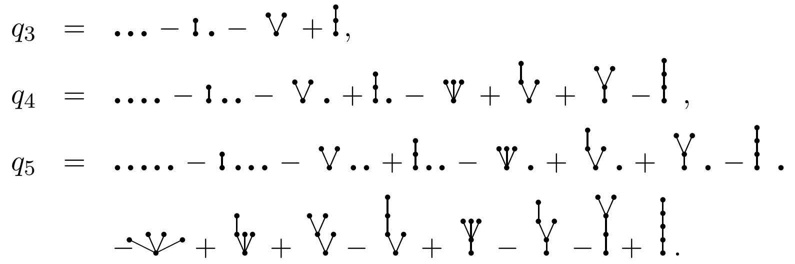
We define inductively a sequence $(q_n)_{n\in\mathbb{N}^*}$ of elements of $\mathbb{P}_\searrow$:

$$\begin{cases} q_1 &= \bullet \in \mathbb{P}_\searrow(1),\\ q_2 &= \bullet\bullet - \mathfrak{l} \in \mathbb{P}_\searrow(2),\\ q_{n+1} &= (\bullet\bullet - \mathfrak{l}) \overset{\circ}{\searrow} (q_n, \bullet) \in \mathbb{P}_\searrow(n+1) \text{ for } n \geq 1. \end{cases}$$

For all $F \in \mathbf{F}$, $\bullet\bullet \overset{\circ}{\searrow} (F, \bullet) = F\bullet$ and $\mathfrak{l} \overset{\circ}{\searrow} (F, \bullet) = B^+(F)$. So, q_n can also be defined in the following way:

$$\begin{cases} q_1 &= \bullet \in \mathbb{P}_\searrow(1),\\ q_{n+1} &= q_n\bullet - B^+(q_n) \in \mathbb{P}_\searrow(n+1) \text{ for } n \geq 1. \end{cases}$$

Examples.



Lemma 18 *Let $F \in \mathbf{F} - \{1\}$, and $t \in \mathbf{T}$. Then, in $\mathcal{H}$:*

$$\Delta(F \searrow t) = (F \searrow t) \otimes 1 + 1 \otimes (F \searrow t) + F' \otimes F'' \searrow t + Ft' \otimes t'' + F \otimes t.$$

Proof. The non-empty and non-total left-admissible cuts of the tree $F \searrow t$ are:

- The cut on the edges relating F to t. For this cut c, $P^c(F \searrow t) = F$ and $R^c(F \searrow t) = t$.
- Cuts acting only on edges of F or on edges relating F to t, at the exception of the preceding case. For such a cut, there exists a unique non-empty, non-total left-admissible cut c' of F, such that $P^c(F \searrow t) = P^{c'}(F)$ and $R^c(F \searrow t) = R^{c'}(F) \searrow t$.
- Cuts acting on edges of t. Then necessarily $F \subseteq P^c(F \searrow t)$. For such a cut, there exists a unique non-empty, non-total left-admissible cut c' of t, such that $P^c(F \searrow t) = FP^{c'}(t)$ and $R^c(F \searrow t) = R^{c'}(t)$.

Summing these cuts, we obtain the announced compatibility. $\square$

Proposition 19 *Let $F = t_1 \dots t_n \in \mathbf{F}$. Then:*

$$\gamma^{-1}_{|Prim(\mathcal{H})}(F) = q_{n+1} \overset{\bullet}{\searrow} (\bullet, t_1, \dots, t_n).$$

Proof. *First step.* Let us show the following property: for all $x \in Prim(\mathcal{H})$, $t \in \mathbf{T}$, $q_2 \overset{\bullet}{\searrow} (x, t)$ is primitive. By lemma 18, using the linearity in F:

$$\begin{aligned} \Delta(x \searrow t) &= (x \searrow t) \otimes 1 + 1 \otimes (x \searrow t) + x \otimes t + xt' \otimes t'',\\ \Delta(xt) &= xt \otimes 1 + 1 \otimes xt + x \otimes t + xt' \otimes t'',\\ \Delta(q_2 \overset{\bullet}{\searrow} (x,t)) &= \Delta(xt - x \searrow t)\\ &= (xt - x \searrow t) \otimes 1 + 1 \otimes (xt - x \searrow t). \end{aligned}$$

Second step. Let us show that for all $x \in Prim(\mathcal{H})$, $t_1, \dots, t_n \in \mathbf{T}$, $q_{n+1} \overset{\bullet}{\searrow} (x, t_1, \dots, t_n) \in Prim(\mathcal{H})$ by induction on n. This is obvious for $n = 0$, as $q_1 \overset{\bullet}{\searrow} (x) = x$. Suppose the result at rank $n-1$. Then:

$$\begin{aligned} q_{n+1} \overset{\bullet}{\searrow} (x, t_1, \dots, t_n) &= (q_2 \overset{\circ}{\searrow} (q_n, I)) \overset{\bullet}{\searrow} (x, t_1, \dots, t_n)\\ &= q_2 \overset{\bullet}{\searrow} (\underbrace{q_n \overset{\bullet}{\searrow} (x, t_1, \dots, t_{n-1})}_{\in Prim(\mathcal{H})}, t_n) \in Prim(\mathcal{H}), \end{aligned}$$

by the first step. As the tree $\bullet$ is primitive, we deduce that, for all forest $F = t_1 \dots t_n \in \mathbf{F}$, $q_{n+1} \bullet\!\!\searrow (\bullet, t_1, \dots, t_n) \in Prim(\mathcal{H})$.

Third step. Let us show that for all $x, y \in \mathcal{M}$, $\gamma(q_2 \bullet\!\!\searrow (x,y)) = \gamma(x)y$. We can limit ourselves to $x, y \in \mathbf{F} - \{1\}$. Then $q_2 \bullet\!\!\searrow (x,y) = xy - x \searrow y$. Moreover, by definition of $\searrow$, $x \searrow y$ is a forest whose first tree is not equal to $\bullet$. Hence, $\gamma(q_2 \bullet\!\!\searrow (x,y)) = \gamma(xy) - 0 = \gamma(x)y$.

Last step. Let us show by induction on n that $\gamma(q_{n+1} \bullet\!\!\searrow (\bullet, t_1, \dots, t_n)) = t_1 \dots t_n$. As $q_1 \bullet\!\!\searrow (\bullet) = \bullet$, this is obvious if $n = 0$. Let us suppose the result at rank $n-1$. By the third step:

$$\begin{aligned}
\gamma(q_{n+1} \bullet\!\!\searrow (\bullet, t_1, \dots, t_n)) &= \gamma(q_2 \bullet\!\!\searrow (q_n \bullet\!\!\searrow (\bullet, t_1, \dots, t_{n-1}), t_n)) \\
&= \gamma(q_n \bullet\!\!\searrow (\bullet, t_1, \dots, t_{n-1}))t_n \\
&= t_1 \dots t_n.
\end{aligned}$$

Consequently, $x = q_{n+1} \bullet\!\!\searrow (\bullet, t_1, \dots, t_n) \in Prim(\mathcal{H})$, and satisfies $\gamma(x) = t_1 \dots t_n$, which proves proposition 19. $\square$

Examples. Let $t_1, t_2, t_3 \in \mathbf{T}$.

$$\begin{aligned}
\gamma^{-1}_{|Prim(\mathcal{H})}(t_1) &= \bullet t_1 - \bullet \searrow t_1, \\
\gamma^{-1}_{|Prim(\mathcal{H})}(t_1 t_2) &= \bullet t_1 t_2 - (\bullet \searrow t_1) t_2 - (\bullet t_1) \searrow t_2 + (\bullet \searrow t_1) \searrow t_2, \\
\gamma^{-1}_{|Prim(\mathcal{H})}(t_1 t_2 t_3) &= \bullet t_1 t_2 t_3 - (\bullet \searrow t_1) t_2 t_3 - (\bullet t_1) \searrow t_2 t_3 + (\bullet \searrow t_1) \searrow t_2 t_3 - (\bullet t_1 t_2) \searrow t_3 \\
&\quad + (\bullet \searrow t_1 t_2) \searrow t_3 + ((\bullet t_1) \searrow t_2) \searrow t_3 - ((\bullet \searrow t_1) \searrow t_2) \searrow t_3.
\end{aligned}$$

3.3 Elements of the dual basis

Lemma 20 *For all $x, y \in \mathcal{H}$, $\Delta(x \nearrow y) = x \nearrow y^{(1)} \otimes y^{(2)} + x^{(1)} \otimes x^{(2)} \nearrow y - x \otimes y$. In other terms, $(\mathcal{H}, \nearrow, \Delta)$ is an infinitesimal Hopf algebra.*

Proof. We restrict to $x = F \in \mathbf{F} - \{1\}$, $y = G \in \mathbf{F} - \{1\}$. The non-empty and non-total left-admissible cuts of the tree $F \nearrow G$ are:

- The cut on the edges relating F to G. For this cut c, $P^c(F \nearrow G) = F$ and $R^c(F \nearrow G) = G$.
- Cuts acting only on edges of F or on edges relating F to G, at the exception of the preceding case. For such a cut, there exists a unique non-empty, non-total left-admissible cut c' of F, such that $P^c(F \nearrow G) = P^{c'}(F)$ and $R^c(F \nearrow G) = R^{c'}(F) \nearrow G$.
- Cuts acting on edges of G. Then necessarily $F \subseteq P^c(F \nearrow G)$. For such a cut, there exists a unique non-empty, non-total left-admissible cut c' of t, such that $P^c(F \nearrow G) = F \nearrow P^{c'}(G)$ and $R^c(F \nearrow G) = R^{c'}(G)$.

Summing these cuts, we obtain, denoting $\Delta(F) = F \otimes 1 + 1 \otimes F + F' \otimes F''$ and $\Delta(G) = G \otimes 1 + 1 \otimes G + G' \otimes G''$:

$$\begin{aligned}
\tilde{\Delta}(F \nearrow G) &= (F \nearrow G) \otimes 1 + 1 \otimes (F \nearrow G) + F \otimes G + F' \otimes F'' \nearrow G + F \nearrow G' \otimes G'' \\
&= (F \otimes 1) \nearrow \Delta(G) + \Delta(F) \nearrow (1 \otimes G) - F \otimes G.
\end{aligned}$$

So $(\mathcal{H}, \nearrow, \Delta)$ is an infinitesimal bialgebra. As it is graded and connected, it has an antipode. $\square$

Proposition 21 *Let $F = t_1 \dots t_n \in \mathbf{F}$. Then $f_F = f_{t_n} \nearrow \dots \nearrow f_{t_1}$.*

Proof. *First step.* We show the following result: for all $F \in \mathbf{F}$, $t \in \mathbf{T}$, $f_F \nearrow f_t = f_{tF}$. We proceed by induction on the weight n of F. If $n = 0$, then $F = 1$ and the result is obvious. We now suppose that the result is true at all rank $< n$. Let be $G \in \mathbf{F}$, and let us prove that $\langle f_F \nearrow f_t, G \rangle = \delta_{tF,G}$. Three cases are possible.

1. $G=1$. Then $\langle f_F \nearrow f_t, G\rangle = \langle f_F \nearrow f_t, 1\rangle = \varepsilon(f_F \nearrow f_t) = 0 = \delta_{tF,G}$.

2. $G=G_1G_2$, $G_i \neq 1$. Then, by lemma 20:

$$\begin{array}{rcl}
\langle f_F \nearrow f_t, G\rangle &=& \langle \Delta(f_F \nearrow f_t), G_2 \otimes G_1\rangle \\
&=& \displaystyle\sum_{F_1F_2=F} \langle f_{F_2} \otimes f_{F_1} \nearrow f_t, G_2 \otimes G_1\rangle \\
&& +\langle f_F \nearrow f_t \otimes 1 + f_F \nearrow 1 \otimes f_t, G_2 \otimes G_1\rangle - \langle f_F \otimes f_t, G_2 \otimes G_1\rangle \\
&=& \displaystyle\sum_{\substack{F_1F_2=F,\\ weight(F_1)<n}} \langle f_{F_2} \otimes f_{F_1} \nearrow f_t, G_2 \otimes G_1\rangle + \langle 1 \otimes f_F \nearrow f_t, G_2 \otimes G_1\rangle \\
&& +\langle f_F \nearrow f_t \otimes 1, G_2 \otimes G_1\rangle + \langle f_F \otimes f_t, G_2 \otimes G_1\rangle - \langle f_F \otimes f_t, G_2 \otimes G_1\rangle \\
&=& \displaystyle\sum_{\substack{F_1F_2=F,\\ weight(F_1)<n}} \langle f_{F_2} \otimes f_{tF_1}, G_2 \otimes G_1\rangle \\
&=& \displaystyle\sum_{\substack{F_1F_2=F,\\ weight(F_1)<n}} \delta_{F_2,G_2}\delta_{tF_1,G_1} \\
&=& \delta_{tF,G}.
\end{array}$$

3. $G=B^+(G_1)$. Note that $f_F \nearrow f_t$ is a linear span of forests $H_1 \nearrow H_2$, with H_1, $H_2 \neq 1$. By definition of $\nearrow$, the first tree of such a forest is not $\centerdot$. Hence, $\gamma(f_F \nearrow f_t)=0$ and:

$$\langle f_F \otimes f_t, G\rangle = \langle \gamma(f_F \otimes f_t), G_1\rangle = 0 = \delta_{tF,G},$$

as $tF \notin \mathbf{T}$ because $F \neq 1$.

Second step. We now prove proposition 21 by induction on n. It is obvious for $n=1$. Suppose the result at rank $n-1$. By the first step:

$$f_{t_1\ldots t_n} = f_{t_2\ldots t_n} \nearrow f_{t_1} = (f_{t_n} \nearrow \ldots \nearrow f_{t_2}) \nearrow f_{t_1} = f_{t_n} \nearrow \ldots \nearrow f_{t_2} \nearrow f_{t_1},$$

using the induction hypothesis for the second equality. $\Box$

Remarks.

1. As an immediate corollary, because $\nearrow$ is associative, for all forests $F_1,\ldots,F_k \in \mathbf{F}$, $f_{F_1\ldots F_k} = f_{F_k} \nearrow \ldots \nearrow f_{F_1}$.

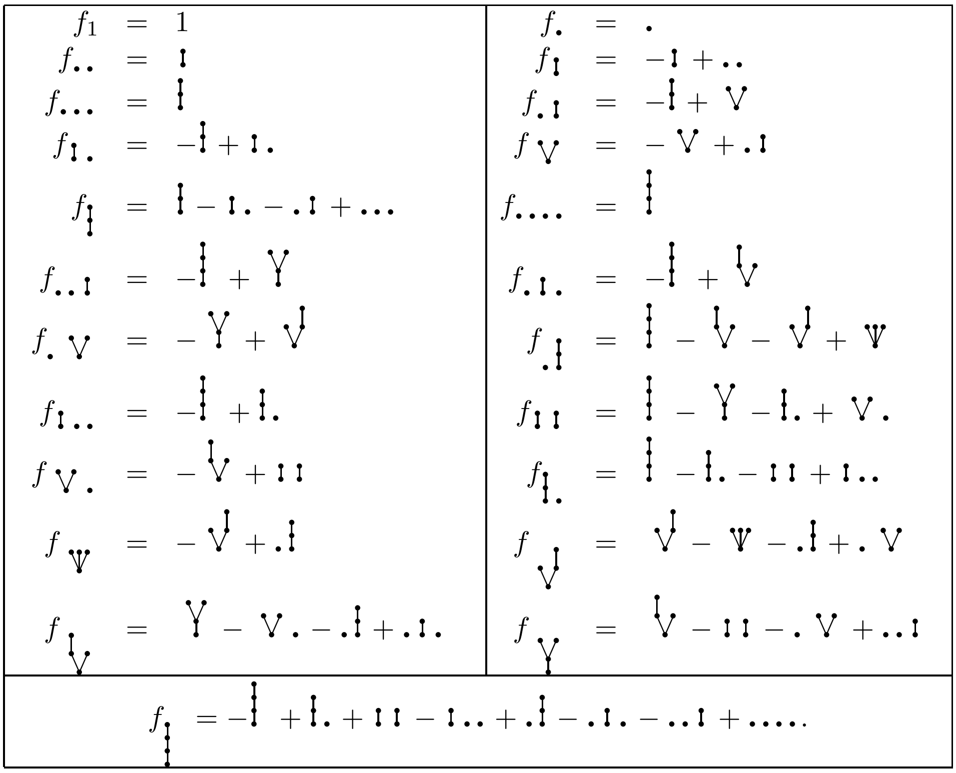
2. In term of operads, proposition 21 can be rewritten in the following way:

Corollary 22 *Let $F_1,\ldots,F_n \in \mathbf{F}$. Then $f_{F_1\ldots F_n} = l_n \mathbin{\nearrow\!\!\!\!\circ} (f_{F_n},\ldots,f_{F_1})$.*

Remark. Hence, the dual basis $(f_F)_{F\in\mathbf{F}}$ can be inductively computed, using proposition 21 of [4], together with propositions 19 and 21 of the present text:

$$\left\{\begin{array}{rcl}
f_1 &=& 1,\\
f_{t_1\ldots t_n} &=& f_{t_n} \nearrow \ldots \nearrow f_{t_1},\\
f_{B^+(t_1\ldots t_n)} &=& \gamma_{|Prim(\mathcal{H})}^{-1}(f_{t_1\ldots t_n}).
\end{array}\right.$$

For example:



4 Primitive suboperads

4.1 Compatibilities between products and coproducts

We define another coproduct $\Delta_{\nearrow}$ on $\mathcal{H}$ in the following way: for all $x, y, z \in \mathcal{H}$,

$$\langle \Delta_{\nearrow}(x), y \otimes z \rangle = \langle x, z \nearrow y \rangle.$$

Lemma 23 *For all forest $F \in \mathbf{F}$, $\Delta_{\nearrow}(F) = \sum\limits_{\substack{F_1, F_2 \in \mathbf{F} \\ F_1 F_2 = F}} F_1 \otimes F_2$.*

Proof. Let $F, G, H \in \mathbf{F}$. Then:

$$\begin{array}{rcl} \langle \Delta_{\nearrow}(F), f_G \otimes f_H \rangle &=& \langle F, f_H \nearrow f_G \rangle \\ &=& \langle F, f_{GH} \rangle \\ &=& \delta_{F,GH} \\ &=& \sum\limits_{\substack{F_1, F_2 \in \mathbf{F} \\ F_1 F_2 = F}} \langle F_1 \otimes F_2, f_G \otimes f_H \rangle. \end{array}$$

As $(f_F)_{F \in \mathbf{F}}$ is a basis of $\mathcal{H}$ and $\langle -, - \rangle$ is non degenerate, this proves the result. $\Box$

Remark. As a consequence, the elements of $\mathbf{T}$ are primitive for this coproduct.

We now have defined three products, namely m, $\nearrow$, and $\searrow$, and two coproducts, namely $\tilde{\Delta}$ and $\tilde{\Delta}_{\nearrow}$, on $\mathcal{M}$, obtained from Δ and $\Delta_{\nearrow}$ by substracting their primitive parts. The following properties sum up the different compatibilities.

Proposition 24 *For all $x, y \in \mathcal{M}$:*

$$\begin{aligned}
\tilde{\Delta}(xy) &= (x \otimes 1)\tilde{\Delta}(y) + \tilde{\Delta}(x)(1 \otimes y) + x \otimes y, &(4)\\
\tilde{\Delta}(x \nearrow y) &= (x \otimes 1) \nearrow \tilde{\Delta}(y) + \tilde{\Delta}(x) \nearrow (1 \otimes y) + x \otimes y, &(5)
\end{aligned}$$

$$\begin{aligned}
\tilde{\Delta}_{\nearrow}(xy) &= (x \otimes 1)\tilde{\Delta}_{\nearrow}(y) + \tilde{\Delta}_{\nearrow}(x)(1 \otimes y) + x \otimes y, &(6)\\
\tilde{\Delta}_{\nearrow}(x \nearrow y) &= (x \otimes 1) \nearrow \tilde{\Delta}_{\nearrow}(y), &(7)\\
\tilde{\Delta}_{\nearrow}(x \searrow y) &= (x \otimes 1) \searrow \tilde{\Delta}_{\nearrow}(y). &(8)
\end{aligned}$$

Proof. It remains to consider the compatibility between $\nearrow$ or $\searrow$ and $\tilde{\Delta}_{\nearrow}$. Let $F, G \in \mathbf{F} - \{1\}$. We put $G = t_1 \dots t_n$, where the t_i's are trees. Then $F \nearrow G = (F \nearrow t_1)t_2 \dots t_n$, and $F \nearrow t_1$ is a tree. Hence:

$$\begin{aligned}
\tilde{\Delta}_{\nearrow}(F \nearrow G) &= \sum_{i=1}^{n-1} (F \nearrow t_1)t_2 \dots t_i \otimes t_{i+1} \dots t_n\\
&= \sum_{i=1}^{n-1} F \nearrow (t_1 t_2 \dots t_i) \otimes t_{i+1} \dots t_n\\
&= (F \otimes 1) \nearrow \tilde{\Delta}_{\nearrow}(G).
\end{aligned}$$

The proof is similar for $F \searrow G$. So all these compatibilities are satisfied. $\Box$

Remark. There is no similar compatibility between $\tilde{\Delta}$ and $\searrow$. In particular, lemma 19 is not available for $t \notin \mathbf{T}$.

This justifies the following definitions:

Definition 25

1. A $\mathbb{P}_{\nearrow}$-bialgebra of type 1 is a family $(A, m, \nearrow, \tilde{\Delta})$, such that:
   (a) $(A, m, \nearrow)$ is a $\mathbb{P}_{\nearrow}$-algebra.
   (b) $(A, \tilde{\Delta})$ is a coassociative, non counitary coalgebra.
   (c) Compatibilities (4) and (5) are satisfied.
2. A $\mathbb{P}_{\nearrow}$-bialgebra of type 2 is a family $(A, m, \nearrow, \tilde{\Delta}_{\nearrow})$, such that:
   (a) $(A, m, \nearrow)$ is a $\mathbb{P}_{\nearrow}$-algebra.
   (b) $(A, \tilde{\Delta}_{\nearrow})$ is a coassociative, non counitary coalgebra.
   (c) Compatibilities (6) and (7) are satisfied.
3. A $\mathbb{P}_{\searrow}$-bialgebra is a family $(A, m, \searrow, \tilde{\Delta}_{\nearrow})$, such that:
   (a) $(A, m, \searrow)$ is a $\mathbb{P}_{\searrow}$-algebra.
   (b) $(A, \tilde{\Delta}_{\nearrow})$ is a coassociative, non counitary coalgebra.
   (c) Compatibilities (6) and (8) are satisfied.

Example. The augmentation ideal $\mathcal{M}$ of the infinitesimal Hopf algebra of trees $\mathcal{H}$ is both a $\mathbb{P}_{\nearrow}$-infinitesimal bialgebra of type 1 and 2, and also a $\mathbb{P}_{\searrow}$-infinitesimal bialgebra.

If A is a bialgebra of such a type, we denote by $Prim(A)$ the kernel of the coproduct. We deduce the definition of the following suboperads:

Definition 26 Let $n \in \mathbb{N}$. We put:

$$\left\{\begin{array}{rcl}
\mathbb{PRIM}^{(1)}_{\nearrow}(n) &=& \left\{p \in \mathbb{P}_{\nearrow}(n) \,/\, \begin{array}{c}\text{For all } A,\ \mathbb{P}_{\nearrow}\text{-infinitesimal bialgebra of type 1,}\\ \text{and for } a_1,\ldots,a_n \in Prim(A),\\ p.(a_1,\ldots,a_n) \in Prim(A).\end{array}\right\},\\[2em]
\mathbb{PRIM}^{(2)}_{\nearrow}(n) &=& \left\{p \in \mathbb{P}_{\nearrow}(n) \,/\, \begin{array}{c}\text{For all } A,\ \mathbb{P}_{\nearrow}\text{-infinitesimal bialgebra of type 2,}\\ \text{and for } a_1,\ldots,a_n \in Prim_{\nearrow}(A),\\ p.(a_1,\ldots,a_n) \in Prim_{\nearrow}(A).\end{array}\right\},\\[2em]
\mathbb{PRIM}_{\searrow}(n) &=& \left\{p \in \mathbb{P}_{\searrow}(n) \,/\, \begin{array}{c}\text{For all } A,\ \mathbb{P}_{\searrow}\text{-infinitesimal bialgebra,}\\ \text{and for } a_1,\ldots,a_n \in Prim_{\nearrow}(A),\\ p.(a_1,\ldots,a_n) \in Prim_{\nearrow}(A).\end{array}\right\}.
\end{array}\right.$$

We identify $\mathbb{P}_{\nearrow}(n)$ and $\mathbb{P}_{\searrow}(n)$ with the homogeneous component of weight n of $\mathcal{M}$. We put $Prim(\mathcal{M}) = Ker(\tilde{\Delta})$ and $Prim_{\nearrow}(\mathcal{M}) = Ker(\tilde{\Delta}_{\nearrow})$. We obtain:

Proposition 27 *1. For all $n \in \mathbb{N}$:*

$$\mathbb{PRIM}^{(1)}_{\nearrow}(n) = \{p \in \mathbb{P}_{\nearrow}(n) \,/\, p \nearrow (\bullet,\ldots,\bullet) \in Prim(\mathcal{M})\} = \mathbb{P}_{\nearrow}(n) \cap Prim(\mathcal{M}).$$

2. For all $n \in \mathbb{N}$:

$$\mathbb{PRIM}^{(2)}_{\nearrow}(n) = \{p \in \mathbb{P}_{\nearrow}(n) \,/\, p \nearrow (\bullet,\ldots,\bullet) \in Prim_{\nearrow}(\mathcal{M})\} = \mathbb{P}_{\nearrow}(n) \cap Prim_{\nearrow}(\mathcal{M}).$$

3. For all $n \in \mathbb{N}$:

$$\mathbb{PRIM}_{\searrow}(n) = \{p \in \mathbb{P}_{\searrow}(n) \,/\, p \searrow (\bullet,\ldots,\bullet) \in Prim_{\nearrow}(\mathcal{M})\} = \mathbb{P}_{\searrow}(n) \cap Prim_{\nearrow}(\mathcal{M}).$$

Proof. As $\mathcal{M}$ is a $\mathbb{P}_{\nearrow}$-infinitesimal bialgebra, by definition:

$$\mathbb{PRIM}^{(1)}_{\nearrow}(n) \subseteq \{p \in \mathbb{P}_{\nearrow}(n) \,/\, p \nearrow (\bullet,\ldots,\bullet) \in Prim(\mathcal{M})\}.$$

Moreover, $\{p \in \mathbb{P}_{\nearrow}(n) \,/\, p \nearrow (\bullet,\ldots,\bullet) \in Prim(\mathcal{M})\} = \mathbb{P}_{\nearrow}(n) \cap Prim(\mathcal{M})$, as, for all $p \in \mathbb{P}_{\nearrow}(n)$, $p \nearrow (\bullet,\ldots,\bullet) = p \in \mathcal{M}$.

We now show that $\{p \in \mathbb{P}_{\nearrow}(n) \,/\, p \nearrow (\bullet,\ldots,\bullet) \in Prim(\mathcal{M})\} \subseteq \mathbb{PRIM}^{(1)}_{\nearrow}(n)$. We take $p \in \mathbb{P}_{\nearrow}(n)$, such that $p \nearrow (\bullet,\ldots,\bullet) \in Prim(\mathcal{M})$. Let $\mathcal{D} = \{1,\ldots,n\}$ and let A be the free $\mathbb{P}_{\nearrow}$-algebra generated by $\mathcal{D}$ (with a unit). It can be described as the associative algebra $\mathcal{H}^{\mathcal{D}}$ generated by the set of planar rooted trees decorated by $\mathcal{D}$, and can be given a structure of $\mathbb{P}_{\nearrow}$-infinitesimal bialgebra. As $\mathcal{M}$ is freely generated by $\bullet$ as a $\mathbb{P}_{\nearrow}$-algebra, there exists a unique morphism of $\mathbb{P}_{\nearrow}$-algebras from $\mathcal{M}$ to $\mathcal{M}^{\mathcal{D}}$, augmentation ideal of $\mathcal{H}^{\mathcal{D}}$:

$$\xi : \left\{\begin{array}{rcl} \mathcal{M} & \longrightarrow & \mathcal{M}^{\mathcal{D}}\\ \bullet & \longrightarrow & \bullet_1 + \ldots + \bullet_n. \end{array}\right.$$

As $\bullet \in Prim(\mathcal{M})$ and $\bullet_1 + \ldots + \bullet_n \in Prim(A)$, ξ is a $\mathbb{P}_{\nearrow}$-infinitesimal bialgebra morphism from $\mathcal{M}$ to $\mathcal{M}^{\mathcal{D}}$. So, $\xi(p \nearrow (\bullet,\ldots,\bullet)) \in Prim(A)$.

Let $F \in A$ be a forest, and $s_1 \geq_{h,l} \ldots \geq_{h,l} s_k$ its vertices. For all $i \in \{1,\ldots,k\}$, we put d_i the decoration of s_i. The *decoration word* associated to F is the word $d_1 \ldots d_n$. It belongs to $M(\mathcal{D})$, the free monoid generated by the elements of $\mathcal{D}$. For all $w \in M(\mathcal{D})$, Let A_w be the subspace of A generated by forests whose decoration word is w. This defines a $M(\mathcal{D})$-gradation of A, as a $\mathbb{P}_{\nearrow}$-infinitesimal bialgebra of type 1.

Consider the projection $\pi_{1,\ldots,n}$ onto $A_{1,\ldots,n}$. We get:

$$\begin{array}{rcl} \pi_{1,\ldots,n}\circ\xi(p\nearrow\!\!\!\!\bullet\,(\bullet,\ldots,\bullet)) & \in & Prim(A),\\ & = & \pi_{1,\ldots,n}(p\nearrow\!\!\!\!\bullet\,(\xi(\bullet),\ldots,\xi(\bullet)))\\ & = & \pi_{1,\ldots,n}(p\nearrow\!\!\!\!\bullet\,(\bullet_1+\ldots+\bullet_n,\ldots,\bullet_1+\ldots+\bullet_n))\\ & = & p\nearrow\!\!\!\!\bullet\,(\bullet_1,\ldots,\bullet_n). \end{array}$$

So $p\nearrow\!\!\!\!\bullet\,(\bullet_1,\ldots,\bullet_n)\in Prim(A)$.

Let B be a $\mathbb{P}_\nearrow$-infinitesimal bialgebra and let $a_1,\ldots,a_n\in Prim(B)$. As $\mathcal{M}^{\mathcal{D}}$ is freely generated by the $\bullet_i$'s, there exists a unique morphism of $\mathbb{P}_\nearrow$-algebras:

$$\chi:\left\{\begin{array}{rcl} A & \longrightarrow & B\\ \bullet_i & \longrightarrow & a_i. \end{array}\right.$$

As the $\bullet_i$ and the a_i's are primitive, χ is a $\mathbb{P}$-infinitesimal bialgebra morphism. So:

$$\xi(p\nearrow\!\!\!\!\bullet\,(\bullet_1,\ldots,\bullet_n))=p.(\xi(\bullet_1),\ldots,\xi(\bullet_n))=p.(a_1,\ldots,a_n)\in\chi(prim(\mathcal{M}^{\mathcal{D}}))\subseteq Prim(A).$$

Hence, $p\in\mathbb{PRIM}^{(1)}_\nearrow(n)$. The proof is similar for $\mathbb{PRIM}^{(2)}_\nearrow$ and $\mathbb{PRIM}_\searrow$. $\square$

4.2 Suboperad $\mathbb{PRIM}^{(1)}_\nearrow$

Lemma 28 *We define inductively the following elements of $\mathbb{P}_\nearrow$:*

$$\left\{\begin{array}{rcl} q_1 & = & \bullet,\\ q_{n+1} & = & (\bullet\bullet-\mathbf{I})\nearrow\!\!\!\!\circ\,(q_n,\bullet)\;=\;q_n\bullet-B^+(q_n),\ \textit{for } n\geq 1. \end{array}\right.$$

Then, for all $n\geq 1$, q_n belongs to $\mathbb{PRIM}^{(1)}_\nearrow$. Moreover, for all $x_1,\ldots,x_n\in Prim(\mathcal{M})$:

$$\gamma(q_n\nearrow\!\!\!\!\bullet\,(x_1,\ldots,x_n))=\gamma(x_1)x_2\ldots x_n.$$

Remark. These q_n's are the same as the q_n's defined in section 3.2.

Proof. Let us remark that $f_{\mathbf{I}}=\bullet\bullet-\mathbf{I}\in Prim(\mathcal{M})$. By proposition 27, $\bullet\bullet-\mathbf{I}\in\mathbb{PRIM}^{(1)}_\nearrow(2)$. As $\mathbb{PRIM}^{(1)}_\nearrow$ is a suboperad of $\mathbb{P}_\nearrow$, it follows that all the q_n's belongs to $\mathbb{PRIM}^{(1)}_\nearrow(n)$.

Let $x_1,\ldots,x_n\in Prim(\mathcal{M})$. Let us show that $\gamma(q_n\nearrow\!\!\!\!\bullet\,(x_1,\ldots,x_n))=\gamma(x_1)x_2\ldots x_n$ by induction on n. If $n=1$, this is immediate. For $n=2$, $q_2\nearrow\!\!\!\!\bullet\,(x_1x_2)=x_1x_2-x_1\nearrow x_2$. Moreover, $x_1\nearrow x_2$ is a linear span of forests whose first tree is not $\bullet$. So $\gamma(q_2\nearrow\!\!\!\!\bullet\,(x_1,x_2))=\gamma(x_1x_2)-0=\gamma(x_1)x_2$.

Suppose now the result true at rank $n-1$. Then:

$$\begin{array}{rcl} q_n\nearrow\!\!\!\!\bullet\,(x_1,\ldots,x_n) & = & q_2\nearrow\!\!\!\!\bullet\,(\underbrace{q_{n-1}\nearrow\!\!\!\!\bullet\,(x_1,\ldots,x_{n-1})}_{\in Prim(\mathcal{M})},x_n),\\ \gamma(q_n\nearrow\!\!\!\!\bullet\,(x_1,\ldots,x_n)) & = & \gamma(q_2\nearrow\!\!\!\!\bullet\,(q_{n-1}\nearrow\!\!\!\!\bullet\,(x_1,\ldots,x_{n-1}),x_n))\\ & = & \gamma(q_{n-1}\nearrow\!\!\!\!\bullet\,(x_1,\ldots,x_{n-1}))x_n\\ & = & \gamma(x_1)x_2\ldots x_n. \end{array}$$

$\square$

Theorem 29 *The non-Σ-operad $\mathbb{PRIM}^{(1)}_\nearrow$ is freely generated by $\mathbf{I}-\bullet\bullet$.*

Proof. Let us first show that the family $(q_n)_{n\geq 1}$ generates $\mathbb{PRIM}^{(1)}_{\nearrow}$. Let $\mathbb{P}$ be the suboperad of $\mathbb{PRIM}^{(1)}_{\nearrow}$ generated by the q_n's. Let us prove by induction on k that $\mathbb{PRIM}^{(1)}_{\nearrow}(k) = \mathbb{P}(k)$. If $k = 1$, $\mathbb{P}(1) = \mathbb{PRIM}^{(1)}_{\nearrow}(1) = K\bullet$. Suppose the result at all ranks $\leq k-1$. By the rigidity theorem for infinitesimal bialgebra of [10], a basis of $\mathcal{H}$ is $(f_{t_1}\ldots f_{t_n})_{t_1\ldots t_n\in\mathbf{F}}$, so a basis of $Prim(\mathcal{M})$ is:

$$\left(\gamma^{-1}_{Prim(\mathcal{H})}(f_{t_1}\ldots f_{t_n})\right)_{t_1\ldots t_n\in\mathbf{F}}.$$

So, a basis of $\mathbb{PRIM}^{(1)}_{\nearrow}(k)$ is $\left(\gamma^{-1}_{Prim(\mathcal{H})}(f_{t_1}\ldots f_{t_n})\right)_{\substack{t_1\ldots t_n\in\mathbf{F}\\ weight(t_1\ldots t_n)=k-1}}$. By lemma 28:

$$\gamma^{-1}_{Prim(\mathcal{H})}(f_{t_1}\ldots f_{t_n}) = q_{n+1}\nearrow(\bullet, f_{t_1},\ldots f_{t_n}).$$

By the induction hypothesis, the f_{t_i}'s belongs to $\mathbb{P}$. So:

$$\gamma^{-1}_{Prim(\mathcal{H})}(f_{t_1}\ldots f_{t_n}) = q_{n+1}\nearrow(\bullet, f_{t_1},\ldots f_{t_n}) \in \mathbb{P}(n).$$

So $\mathbb{PRIM}^{(1)}_{\nearrow} = \mathbb{P}$.

Moreover, if we denote by $\mathbb{P}'$ the suboperad of $\mathbb{PRIM}^{(1)}_{\nearrow}$ generated by q_2, then, immediately, $\mathbb{P}' \subseteq \mathbb{P}$. Finally, by induction on n, $q_n \in \mathbb{P}'(n)$ for all $n \geq 1$ and $\mathbb{P} \subseteq \mathbb{P}'$. So $\mathbb{P}' = \mathbb{P} = \mathbb{PRIM}^{(1)}_{\nearrow}$ is generated by q_2.

Let $\mathbb{P}_{q_2}$ be the non-Σ-operad freely generated by q_2. There is a non-Σ-operad epimorphism:

$$\Psi : \left\{\begin{array}{rcl} \mathbb{P}_{q_2} & \longrightarrow & \mathbb{PRIM}^{(1)}_{\nearrow} \\ q_2 & \longrightarrow & q_2. \end{array}\right.$$

The dimension of $\mathbb{P}_{q_2}(n)$ is the number of planar binary rooted trees with n leaves, that is to say the Catalan number $c_n = \dfrac{(2n-2)!}{(n-1)!n!}$. On the other side, the dimension of $\mathbb{PRIM}^{(1)}_{\nearrow}(n)$ is the number of planar rooted trees with n vertices, that is to say c_n. So Ψ is an isomorphism. $\square$

In other terms, in the language of [9]:

Theorem 30 *The triple of operads $(\mathbb{A}ss, \mathbb{P}^{\Sigma}_{\nearrow}, \mathbb{FREE}_2)$, where $\mathbb{P}^{\Sigma}_{\nearrow}$ is the symmetrisation of $\mathbb{P}_{\nearrow}$ and $\mathbb{FREE}_2$ is the free operad generated by an element in $\mathbb{FREE}_2(2)$, is a good triple of operads.*

Remark. Note that if A is a $\mathbb{P}_{\nearrow}$-bialgebra of type 1, then $(A, m, \tilde{\Delta})$ is a non unitary infinitesimal bialgebra. Hence, if $(K \oplus A, m, \Delta)$ has an antipode S, then $-S$ is an eulerian idempotent for A.

4.3 Another basis of $Prim(\mathcal{H})$

Recall that $\mathbb{T}_b$ is freely generated (as a non-Σ-operad) by $\curlyvee$. In particular, if $t_1, t_2 \in \mathbf{T}_b$, we denote:

$$t_1 \vee t_2 = \curlyvee \circ (t_1, t_2).$$

Every element $t \in \mathbf{T}_b - \{\shortmid\}$ can be uniquely written as $t = t^l \vee t^r$.

There exists a morphism of operads:

$$\Theta : \left\{\begin{array}{rcl} \mathbf{T}_b & \longrightarrow & \mathbb{P}_{\nearrow} \\ \curlyvee & \longrightarrow & \bullet\bullet - \mathfrak{l}. \end{array}\right.$$

By theorem 29, Θ is injective and its image is $\mathbb{PRIM}^{(1)}_{\nearrow}$. So, we obtain a basis of $\mathbb{PRIM}^{(1)}_{\nearrow}$ indexed by $\mathbf{T}_b$, given by $p_t = \Theta(t)$. It is also a basis of $Prim(\mathcal{M})$, which can be inductively computed by:

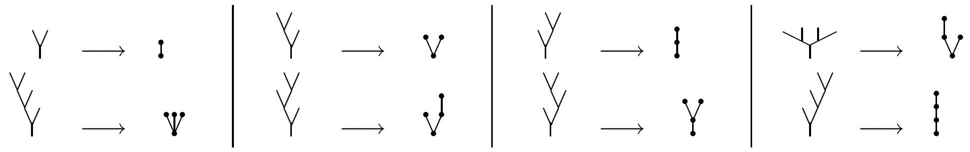
$$\left\{\begin{array}{rcl} p_{|} & = & \bullet, \\ p_{t_1 \vee t_2} & = & (\bullet\bullet - \mathfrak{l}) \not\nearrow (p_{t_1}, p_{t_2}) \;=\; p_{t_1}p_{t_2} - p_{t_1} \nearrow p_{t_2}. \end{array}\right.$$

4.4 From the basis $(f_t)_{t\in\mathbf{T}}$ to the basis $(p_t)_{t\in\mathbf{T}_b}$

We define inductively the application $\kappa : \mathbf{T}_b \longrightarrow \mathbf{T}$ in the following way:

$$\kappa : \left\{\begin{array}{rcl} \mathbf{T}_b & \longrightarrow & \mathbf{T} \\ | & \longrightarrow & \bullet, \\ t_1 \vee t_2 & \longrightarrow & \kappa(t_2) \searrow \kappa(t_1). \end{array}\right.$$

Examples.



It is easy to show that κ is bijective, with inverse given by:

$$\kappa^{-1} : \left\{\begin{array}{rcl} \mathbf{T} & \longrightarrow & \mathbf{T}_b \\ \bullet & \longrightarrow & |, \\ B^+(s_1 \ldots s_m) & \longrightarrow & \kappa^{-1}(B^+(s_2 \ldots s_m)) \vee \kappa^{-1}(s_1). \end{array}\right.$$

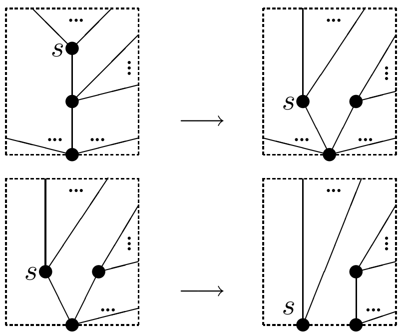
Let us recall the partial order $\leq$, defined in [4], on the set $\mathbf{F}$ of planar forests, making it isomorphic to the Tamari poset.

Definition 31 Let $F \in \mathbf{F}$.

1. An *admissible transformation* on F is a local transformation of F of one of the following types (the part of F which is not in the frame remains unchanged):

First kind:

Second kind:



2. Let F and $G \in \mathbf{F}$. We shall say that $F \leq G$ if there exists a finite sequence $F_0, \ldots, F_k$ of elements of $\mathbf{F}$ such that:

   (a) For all $i \in \{0, \ldots, k-1\}$, F_{i+1} is obtained from F_i by an admissible transformation.

   (b) $F_0 = F$.

   (c) $F_k = G$.

The aim of this section is to prove the following result:

Theorem 32 *Let $t \in \mathbf{T}_b$. Then* $p_t = \sum_{\substack{s \in \mathbf{T} \\ s \leq \kappa(t)}} f_s$.

Proof. By induction on the number n of leaves of t. If $n = 1$, then $t = \mid$ and $p_{\mid} = \bullet = f_{\bullet}$. Suppose the result at all ranks $\leq n-1$. As p_t is primitive, we can put:

$$p_t = \sum_{s \in \mathbf{T}} a_s f_s.$$

Write $t = t_1 \vee t_2$. By the induction hypothesis:

$$p_{t_1} = \sum_{\substack{s_1 \in \mathbf{T} \\ s_1 \leq \kappa(t_1)}} f_{s_1} \text{ and } p_{t_2} = \sum_{\substack{s_2 \in \mathbf{T} \\ s_2 \leq \kappa(t_2)}} f_{s_2}.$$

As $t = t_1 \vee t_2$, $p_t = (\bullet\bullet - \mathfrak{l}) \oslash\!\!\nearrow (p_{t_1}, p_{t_2}) = p_{t_1} p_{t_2} - p_{t_1} \nearrow p_{t_2}$. So, for all $s \in \mathbf{T}$, as s is primitive for $\Delta_{\nearrow}$:

$$\begin{array}{rcl} a_s & = & \langle p_t, s \rangle \\ & = & \langle p_{t_1} p_{t_2} - p_{t_1} \nearrow p_{t_2}, s \rangle \\ & = & \langle p_{t_2} \otimes p_{t_1}, \Delta(s) - \Delta_{\nearrow}(s) \rangle \\ & = & \langle p_{t_2} \otimes p_{t_1}, \Delta(s) \rangle \\ & = & \displaystyle\sum_{\substack{s_1 \in \mathbf{T} \\ s_1 \leq \kappa(t_1)}} \sum_{\substack{s_2 \in \mathbf{T} \\ s_2 \leq \kappa(t_2)}} \langle f_{s_2} \otimes f_{s_1}, \Delta(s) \rangle. \end{array}$$

So a_s is the number of left-admissible cuts c of s, such that $P^c(s) \leq \kappa(t_2)$ and $R^c(s) \leq \kappa(t_1)$.

Suppose that $a_s \neq 0$. Then, there exists a left-admissible cut c of s, such that $P^c(s) \leq \kappa(t_2)$ and $R^c(s) \leq \kappa(t_1)$. As s is a tree, $s \leq \kappa(t_2) \searrow \kappa(t_1) = \kappa(t)$. Moreover, by considering the degree of $P^c(s)$, this cut c is unique, so $a_s = 1$. Reciproquely, if $s \leq \kappa(t)$, if c is the unique left admissible cut such that $weight(P^c(s)) = weight(t_2)$, then $P^c(s) \leq \kappa(t_2)$ and $R^c(s) \leq \kappa(t_1)$. So $a_s \neq 0$. Hence, $(s \leq \kappa(t)) \Longrightarrow (a_s \neq 0) \Longrightarrow (a_s = 1) \Longrightarrow (s \leq \kappa(t))$. This proves theorem 32. $\Box$

Let μ be the Möbius function of the poset $\mathbf{F}$ ([12, 13]). By the Möbius inversion formula:

Corollary 33 *Let $s \in \mathbf{T}$. Then* $f_s = \sum_{t \in T_b,\, \kappa(t) \leq s} \mu(\kappa(t), s) p_t$.

4.5 Suboperad $\mathbb{PRIM}^{(2)}_{\nearrow}$

For all $n \in \mathbb{N}$, we put $c_{n+1} = B^+(\bullet^n)$. In other terms, c_{n+1} is the corolla tree with $n+1$ vertices, or equivalently with n leaves.

Examples. $c_1 = \bullet$, $c_2 = \mathfrak{l}$, $c_3 = \vee$, $c_4 = \Psi$, $c_5 = \ldots$

Lemma 34 *The set $\mathbf{T}$ is a basis of the operad $\mathbb{PRIM}^{(2)}_{\nearrow}$. As an operad, $\mathbb{PRIM}^{(2)}_{\nearrow}$ is generated by the c_n's, $n \geq 2$. Moreover, for all $k, l \geq 2$,*

$$c_k \oslash\!\!\nearrow (c_l, \underbrace{\bullet, \ldots, \bullet}_{k-1 \text{ times}}) = c_l \oslash\!\!\nearrow (\underbrace{\bullet, \ldots, \bullet}_{l-1 \text{ times}}, c_k).$$

Proof. The operad $\mathbb{PRIM}^{(2)}_{\nearrow}$ is identified with $Prim_{\nearrow}(\mathcal{M})$ by proposition 27. So $Prim_{\nearrow}(\mathcal{M})$ is equal to $Vect(\mathbf{T})$. Let $\mathbb{P}$ be the suboperad of $\mathbb{PRIM}^{(2)}_{\nearrow}$ generated by the corollas. Let $t \in \mathbf{T}$, of weight n. Let us prove that $t \in \mathbb{P}$ by induction on n. If $n = 1$, then $t = \bullet \in \mathbb{P}$. If $n \geq 2$, we can suppose that $t = B^+(t_1 \ldots t_k)$, with $t_1, \ldots, t_k \in \mathbb{P}$. Then, by theorem 11:

$$c_{k+1} \not\nearrow (t_1, \ldots, t_k, \bullet) = (\bullet^k \not\nearrow (t_1, \ldots, t_k)) \nearrow \bullet = (t_1 \ldots t_k) \nearrow \bullet = B^+(t_1 \ldots t_k) = t.$$

So $t \in \mathbb{P}$. hence, $\mathbb{P} = \mathbb{PRIM}^{(2)}_{\nearrow}$.

Let $k, l \geq 2$. Then, by theorem 11:

$$\begin{array}{rcl} c_k \not\nearrow (c_l, \bullet, \ldots, \bullet) &=& (\bullet^{k-1} \not\nearrow (c_l, \bullet, \ldots, \bullet)) \nearrow \bullet \\ &=& (c_l \bullet^{k-2}) \nearrow \bullet \\ &=& B^+(c_l \bullet^{k-2}) \\ &=& B^+(B^+(\bullet^{l-1})\bullet^{k-2}). \end{array}$$

On the other hand:

$$\begin{array}{rcl} c_l \not\nearrow (\bullet, \ldots, \bullet, c_k) &=& (\bullet^{l-1} \not\nearrow (\bullet, \ldots, \bullet)) \nearrow c_k \\ &=& (\bullet^{l-1}) \nearrow c_k \\ &=& (\bullet^{l-1}) \nearrow B^+(c^{k-1}) \\ &=& B^+(((\bullet^{l-1}) \nearrow \bullet)\bullet^{k-2}) \\ &=& B^+(B^+(\bullet^{l-1})\bullet^{k-2}). \end{array}$$

So $c_k \not\nearrow (c_l, \bullet, \ldots, \bullet) = c_l \not\nearrow (\bullet, \ldots, \bullet, c_k)$. $\square$

Definition 35 The operad $\mathbb{T}$ is the non-Σ-operad generated by elements $c_n \in \mathbb{T}(n)$, for $n \geq 2$, and the following relations: for all $k, l \geq 2$,

$$c_k \circ (c_l, \underbrace{I, \ldots, I}_{k-1 \text{ times}}) = c_l \circ (\underbrace{I, \ldots, I}_{l-1 \text{ times}}, c_k).$$

In other terms, a $\mathbb{T}$-algebra A has a family of n-multilinear products $[., \ldots, .] : A^{\otimes n} \longrightarrow A$ for all $n \geq 2$, with the associativity condition:

$$[[a_1, \ldots, a_l], a_{l+1}, \ldots, a_{l+k}] = [a_1, \ldots, a_{l-1}, [a_l, \ldots, a_{l+k}]].$$

In particular, $[.,.]$ is associative.

Theorem 36 *The operads $\mathbb{T}$ and $\mathbb{PRIM}^{(2)}_{\nearrow}$ are isomorphic.*

Proof. By lemma 34, there is an epimorphism of operads:

$$\left\{ \begin{array}{rcl} \mathbb{T} & \longrightarrow & \mathbb{PRIM}^{(2)}_{\nearrow} \\ c_n & \longrightarrow & c_n. \end{array} \right.$$

In order to prove this is an isomorphism, it is enough to prove that $dim(\mathbb{T}(n)) \leq dim(\mathbb{PRIM}^{(2)}_{\nearrow}(n))$ for all $n \geq 2$. By lemma 34, $dim(\mathbb{PRIM}^{(2)}_{\nearrow}(n))$ is the n-th Catalan number. Because of the defining relations, $\mathbb{T}(n)$ is generated as a vector space by elements of the form $c_l \circ (I, b_2, \ldots, b_l)$, with $b_i \in \mathbb{T}(n_i)$, such that $n_1 + \ldots + n_l = n - 1$. Hence, we define inductively the following subsets the free non-Σ-operad generated by the c_n's, $n \geq 2$:

$$X(n) = \left\{ \begin{array}{l} \{I\} \text{ if } n = 1, \\ \displaystyle\bigcup_{l=2}^{n} \bigcup_{i_2+\ldots+i_l=n-1} c_l \circ (I, X(i_2), \ldots, X(i_l)) \text{ if } n \geq 2. \end{array} \right.$$

Then the images of the elements of $X(n)$ linearly generate $\mathbb{T}(n)$, so $dim(\mathbb{T}(n)) \leq card(X(n))$ for all n. We now put $a_n = card(X(n))$ and prove that a_n is the n-th Catalan number. We denote by $A(h)$ their generating formal series. Then:

$$\begin{cases} a_1 = 1, \\ a_n = \displaystyle\sum_{l=2}^{n} \sum_{i_2+\ldots+i_l=n-1} a_{i_1} \ldots a_{i_l} \text{ if } n \geq 2. \end{cases}$$

In terms of generating series:

$$A(h) - a_1 h = h\frac{A(x)}{1 - A(x)}.$$

So $A(h)^2 - A(h) + h = 0$. As $A(h) = 1$:

$$A(h) = \frac{1 - \sqrt{1 - 4h}}{2}.$$

So a_n is the n-th Catalan number for all $n \geq 1$. $\Box$

In other terms:

Theorem 37 *The triple of operads $(\mathbb{A}ss, \mathbb{P}^{\Sigma}_{\nearrow}, \mathbb{T}^{\Sigma})$ is a good triple of operads.*

Remark. Note that if A is a $\mathbb{P}_{\nearrow}$-bialgebra of type 2, then $(A, m, \tilde{\Delta}_{\nearrow})$ is a non unitary infinitesimal bialgebra. Hence, if $(K \oplus A, m, \Delta_{\nearrow})$ has an antipode $S_{\nearrow}$, then $-S_{\nearrow}$ is an eulerian idempotent for A.

4.6 Suboperad $\mathbb{PRIM}_{\searrow}$

Lemma 38 *The set $\mathbf{T}$ is a basis of the operad $\mathbb{PRIM}_{\searrow}$. As an operad, $\mathbb{PRIM}_{\searrow}$ is generated by $\mathbf{I}$.*

Proof. Let $\mathbb{P}$ be the suboperad of $\mathbb{PRIM}_{\searrow}$ generated by $\mathbf{I}$. Let $t \in \mathbf{T}$, of weight n. Let us prove that $t \in \mathbb{P}$ by induction on n. If $n = 1$ or 2, this is obvious. If $n \geq 2$, suppose that $t = B^+(t_1 \ldots t_k)$. By the induction hypothesis, t_1 and $B^+(t_2 \ldots t_k)$ belong to $\mathbb{P}$. Then:

$$t = t_1 \searrow B^+(t_2 \ldots t_k) = \mathbf{I} \circ\!\!\!\!\searrow (t_1, B^+(t_2 \ldots t_k)).$$

So $t \in \mathbb{P}$. $\Box$

Theorem 39 *The non-Σ-operad $\mathbb{PRIM}_{\searrow}$ is freely generated by $\mathbf{I}$.*

Proof. Similar as the proof of theorem 29. $\Box$

In other terms:

Theorem 40 *The triple of operads $(\mathbb{A}ss, \mathbb{P}^{\Sigma}_{\searrow}, \mathbb{F}_2)$, where $\mathbb{F}_2$ is the free operad generated by an element in $\mathbb{F}_2(2)$, is a good triple of operads.*

Remark. Note that if A is a $\mathbb{P}_{\searrow}$-bialgebra, then $(A, m, \tilde{\Delta})$ is a non unitary infinitesimal bialgebra. Hence, if $(K \oplus A, m, \Delta)$ has an antipode S, then $-S$ is an eulerian idempotent for A.

5 A rigidity theorem for $\mathbb{P}_\nearrow$-algebras

5.1 Double $\mathbb{P}_\nearrow$-infinitesimal bialgebras

Definition 41 A double $\mathbb{P}_\nearrow$-infinitesimal bialgebra is a family $(A, m, \nearrow, \tilde{\Delta}, \tilde{\Delta}_\nearrow)$ where $m, \nearrow : A \otimes A \longrightarrow A$, $\tilde{\Delta}, \tilde{\Delta}_\nearrow : A \longrightarrow A \otimes A$, with the following compatibilities:

1. $(A, m, \nearrow)$ is a (non unitary) $\mathbb{P}_\nearrow$-algebra.

2. For all $x \in A$:
$$\left\{\begin{array}{rcl}(\tilde{\Delta} \otimes Id) \circ \tilde{\Delta}(x) &=& (Id \otimes \tilde{\Delta}) \circ \tilde{\Delta}(x),\\ (\tilde{\Delta}_\nearrow \otimes Id) \circ \tilde{\Delta}_\nearrow(x) &=& (Id \otimes \tilde{\Delta}_\nearrow) \circ \tilde{\Delta}_\nearrow(x),\\ (\tilde{\Delta} \otimes Id) \circ \tilde{\Delta}_\nearrow(x) &=& (Id \otimes \tilde{\Delta}_\nearrow) \circ \tilde{\Delta}(x).\end{array}\right.$$

   In other terms, $(A, \tilde{\Delta}^{cop}, \tilde{\Delta}_\nearrow^{cop})$ is a $\mathbb{P}_\nearrow$-coalgebra.

3. $(A, m, \nearrow, \tilde{\Delta})$ is a $\mathbb{P}_\nearrow$-bialgebra of type 1.

4. $(A, m, \nearrow, \tilde{\Delta}_\nearrow)$ is a $\mathbb{P}_\nearrow$-bialgebra of type 2.

Remark. If $(A, m, \nearrow, \tilde{\Delta}, \tilde{\Delta}_\nearrow)$ is a graded double $\mathbb{P}_\nearrow$-infinitesimal bialgebra, with finite-dimensional homogeneous components, then its graded dual $(A^*, \tilde{\Delta}^{*,op}, \tilde{\Delta}_\nearrow^{*,op}, m^{*,cop}, \nearrow^{*,cop})$ also is.

Theorem 42 *$(\mathcal{M}, m, \nearrow, \tilde{\Delta}, \tilde{\Delta}_\nearrow)$ is a double $\mathbb{P}$-infinitesimal bialgebra.*

Proof. We already now that $(\mathcal{M}, m, \nearrow)$ is a $\mathbb{P}_\nearrow$-algebra. Moreover, $(\mathcal{M}, \tilde{\Delta}^{cop}, \tilde{\Delta}_\nearrow^{cop})$ is isomorphic to $(\mathcal{M}^*, m^*, \nearrow^*)$ via the pairing $\langle -, - \rangle$, so it is a $\mathbb{P}_\nearrow$-coalgebra. It is already known that $(\mathcal{M}, m, \tilde{\Delta})$ and $(\mathcal{M}, \nearrow, \tilde{\Delta})$ are infinitesimal bialgebras. As $(\mathcal{M}, \nearrow, \tilde{\Delta})$ is isomorphic to $(\mathcal{M}^{op}, m^{op}, \tilde{\Delta}_\nearrow^{cop})$ via the pairing $\langle -, - \rangle$, it is also an infinitesimal bialgebra. So all the needed compatibilities are satisfied. $\square$

Remarks.

1. Via the pairing $\langle -, - \rangle$, $\mathcal{M}$ is isomorphic to its graded dual as an double $\mathbb{P}_\nearrow$-infinitesimal bialgebra. As a consequence, as $\mathcal{M}$ is the free $\mathbb{P}_\nearrow$-algebra generated by $\bullet$, then $\mathcal{M}^{cop}$ is also the cofree $\mathbb{P}_\nearrow$-coalgebra cogenerated by $\bullet$.

2. All these results can be easily extended to infinitesimal Hopf algebras of decorated planar rooted trees, in other terms to every free $\mathbb{P}_\nearrow$-algebras.

Lemma 43 *In the double infinitesimal $\mathbb{P}_\nearrow$-algebra $\mathcal{M}$, $Ker(\tilde{\Delta}) \cap Ker(\tilde{\Delta}_\nearrow) = Vect(\bullet)$.*

Proof. $\supseteq$. Obvious.
$\subseteq$. Let $x \in Ker(\tilde{\Delta}) \cap Ker(\tilde{\Delta}_\nearrow)$. Then $\tilde{\Delta}_\nearrow(x) = 0$, so x is a linear span of trees. We can write:
$$x = \sum_{t \in \mathbf{T}} a_t t.$$
Consider the terms in $\mathcal{M} \otimes \bullet$ of $\tilde{\Delta}(x)$. We get $\displaystyle\sum_{t \in \mathbf{T} - \{\bullet\}} a_t B^-(t) \otimes \bullet = 0$, where $B^-(t)$ is the forest obtained by deleting the root of t. So, if $t \neq \bullet$, then $a_t = 0$. So $x \in vect(\bullet)$. $\square$

Remark. This lemma can be extended to any free $\mathbb{P}_\nearrow$-algebra: if V is a vector space, then the free $\mathbb{P}_\nearrow$-algebra $F_{\mathbb{P}_\nearrow}(V)$ generated by V is given a structure of double $\mathbb{P}_\nearrow$-infinitesimal bialgebra by $\tilde{\Delta}(v) = \tilde{\Delta}_\nearrow(v) = 0$ for all $v \in V$. In this case, $Ker(\tilde{\Delta}) \cap Ker(\tilde{\Delta}_\nearrow) = V$ for $F_{\mathbb{P}_\nearrow}(V)$.

5.2 Connected double $\mathbb{P}_\nearrow$-infinitesimal bialgebras

Notations. Let A be a double $\mathbb{P}_\nearrow$-infinitesimal bialgebra. The iterated coproducts will be denoted in the following way: for all $n \in \mathbb{N}$,

$$\tilde{\Delta}^n : \begin{cases} A & \longrightarrow & A^{\otimes(n+1)} \\ a & \longrightarrow & a^{(1)} \otimes \ldots \otimes a^{(n+1)}, \end{cases}$$

$$\tilde{\Delta}^n_\nearrow : \begin{cases} A & \longrightarrow & A^{\otimes(n+1)} \\ a & \longrightarrow & a^{(1)}_\nearrow \otimes \ldots \otimes a^{(n+1)}_\nearrow. \end{cases}$$

Definition 44 Let A be a double $\mathbb{P}_\nearrow$-infinitesimal bialgebra. It will be said *connected* if, for any $a \in A$, every iterated coproduct $A \longrightarrow A^{\otimes(n+1)}$ vanishes on a for a great enough n.

Theorem 45 *Let A be a connected double $\mathbb{P}_\nearrow$-infinitesimal bialgebra. Then A is isomorphic to the free $\mathbb{P}_\nearrow$-algebra generated by $Prim(A) = Ker(\tilde{\Delta}) \cap Ker(\tilde{\Delta}_\nearrow)$ as a double $\mathbb{P}_\nearrow$-infinitesimal bialgebra.*

Proof. *First step.* We shall use the results on infinitesimal Hopf algebras of [4]. We show that $A = Prim(A) + A.A + A \nearrow A$. As $(A, \nearrow, \tilde{\Delta})$ is a connected non unitary infinitesimal bialgebra, it (or more precisely its unitarisation) has an antipode $S_\nearrow$, defined by:

$$S_\nearrow : \begin{cases} A & \longrightarrow & A \\ a & \longrightarrow & \displaystyle\sum_{i=0}^{\infty} (-1)^{i+1} a^{(1)} \nearrow \ldots \nearrow a^{(i+1)}. \end{cases}$$

As $(A, \tilde{\Delta})$ is connected, this makes sense. Moreover, $-S_\nearrow$ is the projector on $Ker(\tilde{\Delta})$ in the direct sum $A = Ker(\tilde{\Delta}) \oplus A \nearrow A$.

In the same order of idea, as $(A, m, \tilde{\Delta}_\nearrow)$ is a connected infinitesimal bialgebra, we can define its antipode $S^\nearrow$ by:

$$S^\nearrow : \begin{cases} A & \longrightarrow & A \\ a & \longrightarrow & \displaystyle\sum_{i=0}^{\infty} (-1)^{i+1} a^{(1)}_\nearrow \ldots a^{(i+1)}_\nearrow, \end{cases}$$

and $-S^\nearrow$ is the projector on $Ker(\tilde{\Delta}_\nearrow)$ in the direct sum $A = Ker(\tilde{\Delta}_\nearrow) \oplus A.A$.

Let $a \in A$, $b \in Ker(\tilde{\Delta}_\nearrow)$. Then $\tilde{\Delta}_\nearrow(a \nearrow b) = (a \otimes 1)\tilde{\Delta}_\nearrow(b) = 0$. So $A \nearrow Ker(\tilde{\Delta}_\nearrow)$ is a subset of $Ker(\tilde{\Delta}_\nearrow)$. Moreover, if $\tilde{\Delta}_\nearrow(a) = 0$, then $(Id \otimes \tilde{\Delta}_\nearrow) \circ \tilde{\Delta}(a) = (\tilde{\Delta} \otimes Id) \circ \tilde{\Delta}_\nearrow(a) = 0$. So $\tilde{\Delta}(a) \in A \otimes Ker(\tilde{\Delta}_\nearrow)$. As a consequence, if $n \geq 1$:

$$\tilde{\Delta}^n(a) = (\tilde{\Delta}^{n-1} \otimes Id) \circ \tilde{\Delta}(a) \in A^{\otimes n} \otimes Ker(\tilde{\Delta}_\nearrow).$$

Hence, for all $n \in \mathbb{N}$, $\tilde{\Delta}^n(Ker(\tilde{\Delta}_\nearrow)) \in A^{\otimes n} \otimes Ker(\tilde{\Delta}_\nearrow)$. Finally, we deduce that $S_\nearrow(Ker(\tilde{\Delta}_\nearrow)) \subseteq Ker(\tilde{\Delta}_\nearrow)$.

Let $a \in A$. Then $S^\nearrow(a) \in Ker(\tilde{\Delta}_\nearrow)$ and $S_\nearrow \circ S^\nearrow(a) \in Ker(\tilde{\Delta}) \cap Ker(\tilde{\Delta}_\nearrow) = Prim(A)$ by the preceding point. Moreover:

$$\begin{array}{rcl} S^\nearrow(a) & = & -a + A.A, \\ S_\nearrow \circ S^\nearrow(a) & = & -S^\nearrow(a) + A \nearrow A, \\ S_\nearrow \circ S^\nearrow(a) & = & a + A.A + A \nearrow A. \end{array}$$

This proves the first step.

Second step. As A is connected, it classically inherits a filtration of $\mathbb{P}_\nearrow$-algebra given by the kernels of the iterated coproducts. We denote by deg_p the associated degree. In particular, for all $x \in A$, $deg_p(x) \leq 1$ if, and only if, $x \in Prim(A)$. Let B be the $\mathbb{P}_\nearrow$-subalgebra of A generated by $Prim(A)$. Let $a \in A$, let us show that $a \in B$ by induction on $n = deg_p(a)$. If $n \leq 1$, then $a \in Prim(A) \subseteq B$. Suppose the result true at all ranks $\leq n-1$. Then, by the first step, we can write:

$$a = b + \sum_i a_i b_i + \sum_j c_j d_j,$$

with $b \in Prim(A)$, a_i, b_i, c_j, $d_j \in A$. Because of the filtration, we can suppose that $deg_p(a_i)$, $deg_p(b_i)$, $deg_p(c_j)$, $deg_p(d_j) < n$. By the induction hypothesis, they belong to B, so $a \in B$.

Last step. So, there is an epimorphism of $\mathbb{P}_\nearrow$-algebras:

$$\phi : \left\{ \begin{array}{rcl} F_{\mathbb{P}_\nearrow}(Prim(A)) & \longrightarrow & A \\ a \in Prim(A) & \longrightarrow & a, \end{array} \right.$$

where $F_{\mathbb{P}_\nearrow}(Prim(A))$ is the free $\mathbb{P}_\nearrow$-algebra generated by $Prim(A)$. As the elements of $Prim(A)$ are primitive both in A and $F_{\mathbb{P}_\nearrow}(Prim(A))$, this is a morphism of double $\mathbb{P}_\nearrow$-infinitesimal bialgebras. Suppose that it is not monic. Take then $x \in Ker(\phi)$, non-zero, of minimal degree. Then it is primitive, so belongs to $Prim(A)$ (lemma 43). Hence, $\phi(a) = a = 0$: this is a contradiction. So ϕ is a bijection. $\Box$

In other terms:

Corollary 46 *The triple of operads* $\left((\mathbb{P}^\Sigma_\nearrow)^{op}, \mathbb{P}^\Sigma_\nearrow, \mathbb{VECT}\right)$ *is a good triple. Here,* $\mathbb{VECT}$ *denotes the operad of vector spaces:*

$$\mathbb{VECT}(k) = \left\{ \begin{array}{rcl} KI & if & k = 1, \\ 0 & if & k \neq 1, \end{array} \right.$$

where I *is the unit of* $\mathbb{VECT}$.

We also showed that $S_\nearrow \circ S^\nearrow$ is the projection on $Prim(A)$ in the direct sum $A = Prim(A) \oplus (A.A + A \nearrow A)$. So $S_\nearrow \circ S^\nearrow$ is the eulerian idempotent.

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