

Localization on quantum graphs with random edge lengths

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Abstract

The spectral properties of the Laplacian on a class of quantum graphs with random metric structure are studied. Namely, we consider quantum graphs spanned by the simple \mathbb{Z}^d -lattice with δ -type boundary conditions at the vertices, and we assume that the edge lengths are randomly independently identically distributed. Under the assumption that the coupling constant at the vertices does not vanish, we show that the operator exhibits the Anderson localization near the spectral edges situated outside a certain forbidden set.

Keywords: quantum graph, random operator, random metric, Anderson localization

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1 Introduction

In the present paper we are discussing the Anderson localization for a special class of random perturbations of periodic structures. As argued in the original paper by Anderson [4], there is no transport in disordered media under certain conditions. This phenomenon was interpreted initially within the framework of spectral theory: as proved in various settings, a generic random perturbation of periodic operators produces a dense pure point spectrum in certain energy intervals; at the same time one can be interested in the dynamical localization consisting in uniform bounds, in both space and energy, for propagating wave packets, which implies the dense pure point spectrum. We refer to the very recent collection of papers [11] providing the state of art in the theory of random Schrödinger operators, in particular, the paper [23] discussing various mathematical interpretations of the Anderson localization.

Up to now, the most studied models of disordered media are either discrete (tight-binding) Hamiltonians or continuous Schrödinger operators. In the last decade, there is an increasing interest in the analysis of quantum Hamiltonians on so-called quantum graphs, i.e. differential operators on singular one-dimensional spaces, see the collection of papers [6, 13, 15, 27]. A quantum graph is composed of one-dimensional differential operators on the edges and boundary conditions at the vertices describing coupling of edges. Such operators provide an effective model for the study of various phenomena in the condensed matter physics admitting an experimental verification, and there is natural question about the influence of random perturbations in such systems [39]. There are numerous possibilities to introduce randomness: combinatorial structure, coefficients of differential expression, coupling, metric, etc. Being locally of one-dimensional nature and admitting a complex global shape, quantum graphs take an intermediate position between the one-dimensional and higher dimensional Schrödinger operators. It seems that the paper [26] considering the random necklace model was the first one discussing random interactions and the Anderson localization in the quantum graph setting. Later, these results were generalized for radial tree configurations [22], where Anderson localization at all energies was proved. Both papers used a machinery specific for one-dimensional operators. The paper [1] addressed the spectral analysis on quantum tree graphs with random edge lengths; it appears that the Anderson localization does not hold near the bottom of the spectrum at least in the weak disorder limit and one always has some absolutely continuous spectrum. Another important class of quantum graphs is given by \mathbb{Z}^d -lattices. The paper [14] studied the situation where each edge carries a random potential and showed the Anderson localization near the bottom of the spectrum. Some generalizations were then obtained in [20, 21]. The case of random coupling was considered by the present authors in [25]; recently we learned on an earlier paper [10] where some preliminary estimates for the same model were obtained.

The present Letter is devoted to the study of quantum graphs spanned by the \mathbb{Z}^d -lattice where the edge lengths are random independent identically distributed variables. We consider the free Laplacian on each edge with δ -type boundary conditions and show, under certain technical assumptions, that the operator exhibits the Anderson localization at the bottom of the spectrum, i.e. that the bottom of the spectrum is pure point with exponentially decaying eigenfunctions.

There are two basic methods of proving localization for random operators: the multiscale analysis going back to Fröhlich and Spencer [16] and the Aizenman-Molchanov method [3]. The Aizenman-Molchanov method gives explicit and efficient criteria for localization in terms of the Green function but only works under special assumptions on the way the randomness enters the problem (we used this method for the study of the random coupling model in [25]), which does not hold in the situation we are studying. On the other hand, the multiscale analysis is a rather universal tool which can handle very abstract situations [38]. It is a certain iterative procedure which works as far as some input data are available (see the paper [14] discussing the multiscale analysis for quantum graphs). Below we are concentrating on obtaining the the most important necessary ingredients, more precisely, the Wegner estimate and the initial scale estimate. We are mostly interested in the spectral interpretation of the Anderson localization, and our results, being combined with the multiscale analysis, prove the presence of the dense pure point spectrum in respective energy ranges. Nevertheless, they also can be used for the study of the dynamical localization; we refer to [23] for details.

During the revision of the Letter the preprint [30] appeared, which studies a similar model (but with a different problem setting) and contains an alternative proof of the Wegner estimate for the zero coupling constants.

2 Random length model on a quantum graph lattice

We recall here some basic constructions for quantum graphs. For the general theory see e.g. the reviews [18, 28, 29] and the collections of papers [6, 13, 15, 27].

Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a countable directed graph with \mathcal{V} and \mathcal{E} being the sets of vertices and edges, respectively. For an edge $e \in \mathcal{E}$, we denote by ιe its initial vertex and by τe its terminal vertex. For $e \in \mathcal{E}$ and $v \in \mathcal{V}$ we write $v \sim e$ or $e \sim v$ if $v \in \{\iota e, \tau e\}$. The degree of a vertex is the number $\deg v := \#\{e \in \mathcal{E} : e \sim v\}$.

For $0 < l_{\min} < l_{\max} < \infty$ consider a function $l : \mathcal{E} \rightarrow [l_{\min}, l_{\max}]$. Sometimes we will write l_e instead of $l(e)$; this number will be interpreted as the length of the edge e . Replacing each edge e by a copy of the segment $[0, l_e]$ in such a way that 0 corresponds to ιe and l_e to τe , one obtains a so-called metric graph. Our aim is to study a special-type differential operator on such a structure.

In the space $\mathcal{H} := \bigoplus_{e \in \mathcal{E}} \mathcal{H}_e$, $\mathcal{H}_e = L^2[0, l_e]$ consider an operator H acting as $f =: (f_e) \mapsto (-f_e'') =: Hf$ on the domain consisting of the functions $f \in \bigoplus_{e \in \mathcal{E}} H^2[0, l_e]$ satisfying the Kirchhoff boundary conditions, i.e. for any $v \in \mathcal{V}$ one has

$$f_e(l_e) = f_b(0) =: f(v), \quad \tau e = \iota b = v, \quad (1)$$

and

$$f'(v) = \alpha f(v), \quad f'(v) := \sum_{e: \iota e = v} f_e'(0) - \sum_{e: \tau e = v} f_e'(l_e), \quad (2)$$

where α is a real number, the so-called coupling constant (for simplicity, we assume that the coupling constants are the same for all vertices, which is sufficient for our purposes). It is known that the operator thus obtained is self-adjoint [28]. We denote this operator by $H(\Gamma, l, \alpha)$.

We are going to study a special case of underlying combinatorial configuration, namely a periodic lattice with random edge lengths. Let $\mathcal{V} = \mathbb{Z}^d$, $d \geq 1$, and h_j , $j = 1, \dots, d$, be the canonical basis of \mathbb{Z}^d . Set

$$\mathcal{E}_d := \{(m, m + h_j), \quad m \in \mathbb{Z}^d, \quad j = 1, \dots, d\},$$

where for an edge $e = (v, v') \in \mathbb{Z}^d$, $v, v' \in \mathbb{Z}^d$, one has $\iota e := v$, $\tau e := v'$. For this graph $\Gamma^d := (\mathbb{Z}^d, \mathcal{E}_d)$ consider the operator $H(l, \alpha) := H(\Gamma^d, l, \alpha)$. In the present paper, we study some spectral properties of such operators under the assumption that l_e are random independent identically distributed variables.

Namely, on (Ω, \mathbb{P}) , a probability space, let $(l_e^\omega)_{e \in \mathcal{E}}$ be a family of independent identically distributed (i.i.d.) random variables whose common distribution has a Lipschitz continuous density ρ with support $[l_{\min}, l_{\max}]$. By a random Hamiltonian on the quantum graph, we mean the family of operators $H^\omega(\alpha) := H(l^\omega, \alpha)$.

For $n \in \mathbb{Z}^d$ consider the shifts τ_n acting on the set of edges, $\tau_n(m, m') = (m + n, m' + n)$ and the operators $H(l_n^\omega, \alpha)$ with $l_n^\omega(e) := l^\omega(\tau_n e)$. Clearly, $H(l_n^\omega, \alpha)$ is unitarily equivalent to $H(l^\omega, \alpha)$ for any n as $H(l_n^\omega, \alpha)U_n = U_n H(l^\omega, \alpha)$, $U_n(f_e) = (f_{\tau_n e})$. In terms of the theory of random operators, the shifts τ_n form a measure preserving ergodic family on Ω which allows one to use the standard results from the theory of random operators [9, 37]. In particular, one obtains the non-randomness of the spectrum and the spectral components: there exist closed subsets $\Sigma_j = \Sigma_j(\alpha) \subset \mathbb{R}$ and a subset $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') = 1$ such that $\text{spec}_j H^\omega(\alpha) = \Sigma_j$, $j \in \{\text{pp}, \text{ac}, \text{sc}\}$, for any $\omega \in \Omega'$. We recall that the pure point spectrum $\text{spec}_{\text{pp}} H$ is the closure of the set of the eigenvalues of H). Let $\Sigma(\alpha) = \Sigma_{\text{pp}} \cup \Sigma_{\text{ac}} \cup \Sigma_{\text{sc}}$ be the almost sure spectrum of $H^\omega(\alpha)$. The aim of the paper is to show that there are intervals $I \subset \Sigma$ such that

$\Sigma_{ac} \cap I = \Sigma_{sc} \cap I = \emptyset$; this means that the interval I is densely filled with eigenvalues of H^ω and there is no continuous spectrum in I for almost all ω , or, in other words, the spectrum of H^ω in I is *almost surely dense pure point*.

Let us describe first the almost sure spectrum as a set. For $u > 0$ and $\beta \in \mathbb{R}$ denote by $P_{u,\beta}$ the periodic Kronig-Penney operator acting in $L^2(\mathbb{R})$ i.e. the point interaction Hamiltonian,

$$P_{u,\beta} := -\frac{d^2}{dx^2} + \beta \sum_{k \in \mathbb{Z}} \delta(\cdot - ku).$$

The operator can be correctly defined through the associated sesquilinear form

$$\langle f, P_{u,\beta} g \rangle = \langle f', g' \rangle_{L^2} + \beta \sum_{k \in \mathbb{Z}} \overline{f(ku)} g(ku), \quad f, g \in H^1(\mathbb{R}),$$

and is unitarily equivalent to the operator $H(l_u, \beta)$ for $d = 1$, where l_u is the constant function, $l_u(e) \equiv u$ for all $e \in \mathcal{E}$. It is well known [2] that

$$\text{spec } P_{u,\beta} = \{k^2 : \exists k \geq 0, \cos ku + \frac{\beta}{2k} \sin ku \in [-1, 1]\}. \quad (3)$$

In particular, each Dirichlet eigenvalue $(\pi n)^2/u^2$, $n = 1, 2, \dots$, is a spectral edge, and the bands depend continuously on both α and u . As follows from the general theory of random operators [9, 37], one has

$$\Sigma \equiv \Sigma(\alpha) := \bigcup_{u \in [l_{\min}, l_{\max}]} \text{spec } H(l_u, \alpha). \quad (4)$$

On the other hand, as shown in [35], one has the identity $\text{spec } H(l_u, \alpha) := \text{spec } P_{u,\alpha/d}$. Hence the almost sure spectrum $\Sigma(\alpha)$ of $H^\omega(\alpha)$ is a union of bands, and the bottom of the spectrum is given by

$$\inf \Sigma(\alpha) = \begin{cases} k^2 : k \in (0, \pi/l_{\max}) \text{ and } \cos kl_{\max} + \frac{\alpha}{2kd} \sin kl_{\max} = 1, & \alpha > 0, \\ 0, & \alpha = 0, \\ -k^2 : k > 0 \text{ and } \cosh kl_{\min} + \frac{\alpha}{2kd} \sinh kl_{\min} = 1, & \alpha < 0. \end{cases} \quad (5)$$

Define the set

$$\Delta := \bigcup_{n \in \mathbb{Z}} \left[\frac{\pi^2 n^2}{l_{\max}^2}, \frac{\pi^2 n^2}{l_{\min}^2} \right]. \quad (6)$$

The set consists of the spectra of the operator $H(l_u)$ with Dirichlet boundary condition (which formally corresponds to $\alpha = \infty$) at each vertex when u ranges over the support of the random variables $(l_e^\omega)_{e \in \mathcal{E}}$ and the point 0. Our main result is

Theorem 1. (a) *Let $\alpha \neq 0$ and E_0 be an edge of $\Sigma(\alpha) \cap (0, +\infty)$ that is not contained in Δ . Then, there exists $\varepsilon > 0$ such that the spectrum of H^ω in $(E_0 - \varepsilon, E_0 + \varepsilon) \cap \Sigma$ is almost surely dense pure point and the corresponding eigenfunctions are exponentially decaying.*

(b) *There exists $a > 1$ and $\varepsilon > 0$ such that for $\alpha \in (-\infty, -a) \cup (-1/a, 0)$ the spectrum of $H^\omega(\alpha)$ in $[\inf \Sigma, \inf \Sigma + \varepsilon)$ is almost surely dense pure point with exponentially decaying eigenfunctions.*

An immediate corollary of the above theorem and the equalities (4) is

Theorem 2. *There exist $a > 1$ and $\varepsilon > 0$ such that for $\alpha \in (-\infty, -a) \cup (-1/a, 0) \cup (0, +\infty)$ the spectrum of H^ω in $[\inf \Sigma, \inf \Sigma + \varepsilon)$ is almost surely dense pure point with exponentially decaying eigenfunctions.*

Remark 3. An elementary analysis of the condition (4) shows that, if $l_{\max} - l_{\min}$ is sufficiently small and $\alpha \neq 0$, there are band edges of Σ in $(0, +\infty) \setminus \Delta$.

Our results do not establish the localization for the important case $\alpha = 0$. In this case, both the Wegner estimate near $\inf \Sigma = 0$ and the initial scale estimate (see theorems 4 and 5 below) fail to hold. The operator $H(l^\omega, 0)$ is analogous to the acoustic operator (see e.g. [32] and references therein); for this operator, at least in dimension one, it is known that localization does not hold in its strongest form at the bottom of the spectrum (see [19]). In the case $\alpha = 0$, the reduced operator we use to study the random quantum graph is a discrete version of the acoustic operator.

As in [25], the assumptions of theorems 1 and 2 also imply dynamical localization (see remark 7 in [25]).

3 Multiscale analysis and finite-volume operators

Let Λ be a subset of edges from \mathcal{E}_d . Denote $\mathcal{V}_\Lambda := \{\iota e : e \in \Lambda\} \cup \{\tau e : e \in \Lambda\}$ and consider the graph $\Gamma_\Lambda^d := (\mathcal{V}_\Lambda, \Lambda)$. Note that this graph has no isolated vertices. We will call the operator $H_\Lambda(l, \alpha) := H(\Gamma_\Lambda^d, l, \alpha)$ the finite-volume Hamiltonian associated to Λ . For random operators with random length functions l^ω , we write $H_\Lambda^\omega(\alpha) := H_\Lambda(l^\omega, \alpha)$.

In what follows, we consider Hamiltonians associated with finite cubes $\Lambda = \Lambda(n)$ constructed as follows: take $n \in \mathbb{N}$ and denote by $\Lambda(n)$ the set of edges e such that at least one of the vertices $v \in \{\iota e, \tau e\}$ satisfies $|v| \leq n$; for the corresponding set of vertices, we write $\mathcal{V}(n) := \mathcal{V}_{\Lambda(n)}$.

As mentioned previously, the Anderson localization for random operators can be established using a certain iterative procedure called the multiscale analysis. In order to start the multiscale analysis one needs to verify the validity of several conditions for a fixed interval $I \subset \mathbb{R}$, which then imply the localization in some subset of I in various settings, see e.g. [38, Section 3.2].

The first group of conditions uses very few information on the nature of random interactions, i.e. whether one has random edge length, random potential on the edges or the random coupling constants etc. and usually need only some uniform bounds for the random variables. These conditions are as follows:

- (a) the finite-volume Hamiltonians H_Λ^ω and $H_{\Lambda'}^\omega$ corresponding to any two non-overlapping finite sets of edges $\Lambda, \Lambda' \subset \mathcal{E}_d$ are independent,
- (b) the finite-volume operator obeys a Weyl estimate for the eigenvalues in I , i.e. there exists a constant $C > 0$ such that the number of the eigenvalues of H_n^ω in I can be estimated from above by Cn^d for all $n \in \mathbb{Z}$ and almost all $\omega \in \Omega$,
- (c) there exists a geometric resolvent inequality which provides some uniform bounds for the resolvents of finite-volume operators in terms of the operators corresponding to smaller finite volumes,
- (d) a generalized spectral theorem (Schnol-type theorem). This means that the existence of a non-trivial solution f to $H^\omega f = E f$ (i.e. $-f_e'' = E f_e$ on each edge and the boundary conditions at the vertices are satisfied) with a suitable bound at infinity implies $E \in \text{spec } H^\omega$ (for $E \in I$) and, moreover, the spectrum of H^ω in I is the closure of the values $E \in I$ with the above property.

Depending on the concrete problem, these conditions can be weakened, see e.g. [23, Section 4]. Note that the first condition (a) is trivially satisfied in our case. The conditions (b) and (c) were shown in [14] for equilateral lattices, and the proof goes in our case with minor modifications

for any interval I . The generalized spectral theorem, the condition (d), holds in any interval I as well due to the results of [7] (see also [31] for generalizations).

The second group of conditions are very sensible to the way the randomness enters the system. These two conditions are

- (e) The Wegner estimate showing that the probability for H_Λ^ω to have an eigenvalue in the ε -neighborhood of $E \in I$ can be globally bounded by $C\varepsilon^a |\Lambda|^{bd}$ with some $a, b \geq 1$.
- (f) initial scale estimate showing that the probability for H_Λ^ω to have an eigenvalue in the $|\Lambda|^{-a}$ -neighborhood of *some* $E_0 \in I$ can be bounded by $|\Lambda|^{-b}$ with some suitable $a, b \geq 0$ (which depend on the dimension and other parameters).

When all the above conditions are satisfied, the multiscale analysis shows Anderson localization in a certain interval around the energy E_0 taken from the condition (f).

Hence, in the present Letter, we are interested in the Wegner estimate and the initial scale estimates for our model, that is, (e) and (f). They imply theorem 1 by the multiscale analysis.

Theorem 4 (The Wegner estimate). (a) *Let $I \subset (0, \infty)$ be an interval such that $\bar{I} \cap \Delta = \emptyset$, then there exists a constant $C = C(I) > 0$ such that for any interval $J \subset I$, and any cube $\Lambda = \Lambda(n)$ there holds*

$$\mathbb{P}\{\text{spec } H_\Lambda^\omega(\alpha) \cap J \neq \emptyset\} \leq C|\Lambda| |J|. \quad (7)$$

(b) *There exists $a > 1$ such that for $\alpha \in (-\infty, -a) \cup (-1/a, 0)$ the Wegner estimate also holds at negative energies near the bottom of the spectrum, i.e. there exists an interval I with $I \ni \inf \Sigma(\alpha)$ and $C = C(I) > 0$ such that for any interval $J \subset I$, and any cube $\Lambda = \Lambda(n)$ the estimate (7) holds.*

Theorem 5 (The initial scale estimate). *Let E be a spectral edge of $H^\omega(\alpha)$ which is not contained in the set Δ defined in (6), then for each $\xi > 0$ and $\beta \in (0, 1)$ there exists $n^* = n^*(\xi, \beta) > 0$ such that, for $n \geq n^*$,*

$$\mathbb{P}\{\text{dist}(\text{spec } H_{\Lambda(n)}^\omega(\alpha), E) \leq n^{\beta-1}\} \leq n^{-\xi}.$$

Remark 6. By (5), for $\alpha \neq 0$ the initial scale estimate is satisfied near the bottom of the spectrum $\inf \Sigma$, which is outside Δ .

When $\alpha = 0$, independently of the random variables l_e^ω and the set Λ , the constant function $f \equiv 1$ satisfies $H_\Lambda^\omega f = 0$. Hence, both the Wegner estimate and the initial scale estimate fail for the energy $E = 0$, and this is the only spectral edge; in this case, the almost sure spectrum is the positive half-line.

Actually, in dimension $d = 1$, the operator $H^\omega(0)$ is unitary equivalent to the free Laplacian and hence shows no Anderson localization (the spectrum is absolutely continuous). Hence, one has a certain similarity to the Schrödinger operators with random vector potentials, where only localization near internal spectral edges is proved so far [17].

We will prove both estimates, theorems 4 and 5 by exploiting a correspondence between the quantum graphs and discrete operators. A similar approach was used in [25] for quantum graphs with random coupling constants and more details on the reduction can be found there.

Denote by D_e^ω the positive Laplacian with the Dirichlet boundary conditions in $L^2[0, l_e^\omega]$ and set $D_\Lambda^\omega := \bigoplus_{e \in \Lambda} D_e^\omega$. Clearly,

$$\text{spec } D_\Lambda^\omega = \bigcup_{e \in \Lambda} \text{spec } D_e^\omega, \quad \text{spec } D_e^\omega = \left\{ \left(\frac{\pi k}{l_e^\omega} \right)^2 : k = 1, 2, \dots \right\}.$$

For $E \notin \text{spec } D_\Lambda^\omega$ consider the operators $M_\Lambda(l^\omega, E)$ acting on $\ell^2(\mathcal{V}_\Lambda)$,

$$M_\Lambda(l^\omega, E)\varphi(v) = \sqrt{E} \left(\sum_{e \in \Lambda: \iota e = v} \frac{1}{\sin l_e^\omega \sqrt{E}} \varphi(\tau e) + \sum_{e \in \Lambda: \tau e = v} \frac{1}{\sin l_e^\omega \sqrt{E}} \varphi(\iota e) - \sum_{e \in \Lambda: v \sim e} \cot l_e^\omega \sqrt{E} \varphi(v) \right). \quad (8)$$

Here and in what follows, the continuous branch of the square root is fixed by the condition $\Im \sqrt{E} \geq 0$ for $E \in \mathbb{R}$.

The meaning of the operators $M_\Lambda(l^\omega, E)$ is as follows. Consider the equation $H_\Lambda^\omega f = Ef$ for $E \notin \text{spec } D^\omega$. On each edge $e \in \Lambda$, f_e satisfies the Dirichlet problem $-f_e'' = Ef_e$, $f_e(0) = f(\iota e)$, $f_e(l_e^\omega) = f(\tau e)$ (see (1) for the definition of the values $f(v)$). Therefore,

$$f_e(t) = f(\iota e) \frac{\sin \sqrt{E}(l^\omega - t)}{\sin \sqrt{E}l^\omega} + f(\tau e) \frac{\sin \sqrt{E}t}{\sin \sqrt{E}l^\omega}.$$

Substituting this representation into the boundary conditions (2) yields $M_\Lambda(l^\omega, E)f_\Lambda = \alpha f_\Lambda$ where $f_\Lambda = (f(v))_{v \in \mathcal{V}_\Lambda}$. For the complete graph, $\Lambda = \mathcal{E}^d$, we simply write $M(l^\omega, E)$ instead of $M_\Lambda(l^\omega, E)$. The map $E \mapsto M_\Lambda(l^\omega, E)$ is obviously analytic outside $\text{spec } D^\omega$. The following characterization of the spectrum of $H_\Lambda^\omega(\alpha)$ shown in [34] will be the key to our analysis:

- an energy $E \notin \text{spec } D_\Lambda^\omega$ is in the spectrum of $H_\Lambda^\omega(\alpha)$ if and only if $\alpha \in \text{spec } M_\Lambda(l^\omega, E)$
- for each such E , one has $\dim \ker (H_\Lambda^\omega(\alpha) - E) = \dim \ker (M_\Lambda(l^\omega, E) - \alpha)$.

For infinite Λ , one needs to use the self-adjoint extension theory [8]; for finite Λ this relation has been known for a long time, see e.g. [5, 12].

We note that similar relations between quantum graphs and discrete operators exist for more general boundary conditions at the vertices, but the corresponding reduced operators $M(E)$ become much more complicated, see [36].

4 Proof of theorem 4 (Wegner estimate)

As noted previously, one has

$$\mathbb{P}\{\text{spec } H_\Lambda^\omega(\alpha) \cap J \neq \emptyset\} = \mathbb{P}\{\exists E \in J : \alpha \in \text{spec } M_\Lambda(l^\omega, E)\}. \quad (9)$$

Note also that, for any $E_* \in I$, one can write $M_\Lambda(l^\omega, E) = M_\Lambda(l^\omega, E_*) + (E - E_*)_\Lambda B_\Lambda(l^\omega, E, E_*)$. Due to analyticity, one can find a constant $b > 0$ such that $\|B_\Lambda(l^\omega, E, E_*)\| \leq b$ for all $E_*, E \in I$ and $\Lambda \subset \mathcal{E}$ and almost all $\omega \in \Omega$.

On the other hand, the condition $\alpha \in \text{spec } M_\Lambda(l^\omega, E)$ implies the existence of a vector $\varphi \in \ell^2(\mathcal{V}_\Lambda)$, $\|\varphi\| = 1$, such that $(M_\Lambda(l^\omega, E) - \alpha)\varphi = 0$. Let E_J be the center of J . The estimate, for $E \in J$,

$$\|(M_\Lambda(l^\omega, E_J) - \alpha)\varphi\| \leq \|(M_\Lambda(l^\omega, E) - \alpha)\varphi\| + |E - E_J| \cdot \|B_\Lambda(l^\omega, E, E_J)\varphi\| \leq b|J|,$$

yields the inequality

$$\mathbb{P}\{\text{spec } H_\Lambda^\omega(\alpha) \cap J \neq \emptyset\} \leq \mathbb{P}\{\text{dist}(\text{spec } M_\Lambda(l^\omega, E_J), \alpha) \leq b|J|\}. \quad (10)$$

In what follows, we denote E_J simply by E to alleviate the notation.

For $e \in \mathcal{E}$, introduce the operators P_1^e, P_2^e, I_e acting on $\ell^2(\mathcal{V}_\Lambda)$:

$$P_1^e f(v) = \begin{cases} \frac{1}{2} (f(\iota e) + f(\tau e)), & v \in \{\iota e, \tau e\}, \\ 0, & \text{otherwise,} \end{cases} \quad (11)$$

$$P_2^e f(v) = \begin{cases} \frac{1}{2} (f(\iota e) - f(\tau e)), & v = \iota e, \\ \frac{1}{2} (f(\tau e) - f(\iota e)), & v = \tau e, \\ 0, & \text{otherwise,} \end{cases} \quad (12)$$

$$I^e f(v) = \begin{cases} f(v), & v \in \{\iota e, \tau e\}, \\ 0, & \text{otherwise.} \end{cases} \quad (13)$$

In terms of these operators, one has

$$M_\Lambda(l^\omega, E) = \sum_{e \in \Lambda} \left(\frac{\sqrt{E}}{\sin l_e^\omega \sqrt{E}} (P_1^e - P_2^e) - \sqrt{E} \cot l_e^\omega \sqrt{E} I^e \right)$$

and

$$\frac{\partial M_\Lambda(l^\omega, E)}{\partial l_e^\omega} = -\frac{E \cos l_e^\omega \sqrt{E}}{\sin^2 l_e^\omega \sqrt{E}} (P_1^e - P_2^e) + \frac{E}{\sin^2 l_e^\omega \sqrt{E}} I^e.$$

Consider first the part (a) of the theorem, i.e. the case $\bar{I} \subset (0, +\infty)$. As $\|P_1^e - P_2^e\| = 1$ and $P_j^e I^e = P_j^e$ for $j \in \{1, 2\}$, one has

$$-\cos l_e^\omega \sqrt{E} (P_1^e - P_2^e) + I^e \geq (1 - |\cos l_e^\omega \sqrt{E}|) I^e.$$

As I does not meet Δ , there exist constants $c_1, c_2 > 0$ such that

$$1 - |\cos l_e^\omega \sqrt{E}| \geq c_1 \quad \text{and} \quad \frac{E}{\sin^2 l_e^\omega \sqrt{E}} \geq c_2 \quad \text{for all } e \in \mathcal{E} \text{ and } E \in I \text{ and a.e. } \omega \in \Omega.$$

Hence,

$$\frac{\partial M_\Lambda(l^\omega, E)}{\partial l_e^\omega} \geq c_1 c_2 I^e \quad \text{for all } e \in \mathcal{E} \text{ and } E \in I$$

so that

$$\sum_{e \in \mathcal{E}} \frac{\partial M_\Lambda(l^\omega, E)}{\partial l_e^\omega} \geq c_1 c_2 \sum_{e \in \mathcal{E}} I^e \geq \beta \text{id}, \quad \beta = c_1 c_2 > 0$$

or

$$D_\Lambda M_\Lambda(l^\omega, E) \geq \beta \text{id} \quad \text{with} \quad D_\Lambda := \sum_{e \in \Lambda} \frac{\partial}{\partial l_e}.$$

Let $E_\Lambda^\omega(a, b)$ denote the spectral projection of $M_\Lambda(l^\omega, E)$ onto the interval (a, b) . There holds

$$\begin{aligned} & \# [\text{spec } M_\Lambda(l^\omega, E) \cap (\alpha - b|J|, \alpha + b|J|)] \\ &= \text{tr } E_\Lambda^\omega(\alpha - b|J|, \alpha + b|J|) = \text{tr} \left[\int_{\alpha - b|J|}^{\alpha + b|J|} \partial_\lambda \chi_{(-\infty, 0]}(M_\Lambda(l^\omega, E_J) - \lambda) d\lambda \right]. \end{aligned} \quad (14)$$

On the other hand, one has

$$\begin{aligned} & -\operatorname{tr} [D_\Lambda \chi_{(-\infty, 0]}(M_\Lambda(l^\omega, E) - \lambda)] \\ & = \operatorname{tr} [\partial_\lambda \chi_{(-\infty, 0]}(M_\Lambda(l^\omega, E) - \lambda) D_\Lambda M_\Lambda(l^\omega, E)] \geq \beta \operatorname{tr} [\partial_\lambda \chi_{(-\infty, 0]}(M_\Lambda(l^\omega, E) - \lambda)]. \end{aligned}$$

The last estimate is possible as both operators under the trace sign are non-negative. Hence,

$$\operatorname{tr} [\partial_\lambda \chi_{(-\infty, 0]}(M_\Lambda(l^\omega, E) - \lambda)] \leq \beta^{-1} \operatorname{tr} \left[\sum_{e \in \mathcal{E}} -\partial_e \chi_{(-\infty, 0]}(M_\Lambda(l^\omega, E) - \lambda) \right],$$

where we denoted for brevity $\partial_e := \partial / \partial l_e^\omega$, and

$$\operatorname{tr} [E_\Lambda^\omega(\alpha - b|J|, \alpha + b|J|)] \leq \beta^{-1} \int_{\alpha - b|J|}^{\alpha + b|J|} \sum_{e \in \mathcal{E}} \operatorname{tr} [-\partial_e \chi_{(-\infty, 0]}(M_\Lambda(l^\omega, E) - \lambda)] d\lambda.$$

Taking the expectation, one obtains

$$\mathbb{E} \operatorname{tr} E_\Lambda(\alpha - b|J|, \alpha + b|J|) \leq \beta^{-1} \sum_{e \in \mathcal{E}} \int_{l_{\min}}^{l_{\max}} \prod_{e' \neq e} \rho(l_{e'}^\omega) dl_{e'}^\omega \int_{\alpha - b|J|}^{\alpha + b|J|} G_e(E, \lambda, \omega) d\lambda \quad (15)$$

where $M_{\Lambda, e}(l^\omega, l, E)$ is the operator $M_\Lambda(l^\omega, E)$ with l_e^ω replaced by l and

$$G_e(E, \lambda, \omega) = - \int_{l_{\min}}^{l_{\max}} \rho(l) \partial_l \operatorname{tr} [\chi_{(-\infty, 0]}(M_{\Lambda, e}(l^\omega, l, E) - \lambda)] dl.$$

As the density ρ is Lipschitz continuous by assumption, one can integrate by parts and obtain

$$G_e(E, \lambda, \omega) = -\rho(l) F_e(l, E, \lambda, \omega) \Big|_{l=l_{\min}}^{l=l_{\max}} + \int_{l_{\min}}^{l_{\max}} \rho'(l) F_e(l, E, \lambda, \omega) dl,$$

where

$$F_e(l, E, \lambda, \omega) := \operatorname{tr} \left[\chi_{(-\infty, 0]}(M_{\Lambda, e}(l^\omega, l, E) - \lambda) - \chi_{(-\infty, 0]}(M_{\Lambda, e}(l^\omega, l_{\min}, E) - \lambda) \right].$$

As $\partial_e M_\Lambda(l^\omega, E)$ is a rank-two operator, the functions $F_e(l, E, \lambda, \omega)$ are uniformly bounded by 2. Hence, the functions G_e are bounded as well, say $|G_e| \leq G$ for some $G > 0$. Plugging this estimate into (15), one obtains

$$\mathbb{E} \operatorname{tr} E_\Lambda(\alpha - b|J|, \alpha + b|J|) \leq G \beta^{-1} \sum_{e \in \mathcal{E}} \int_{\alpha - b|J|}^{\alpha + b|J|} d\lambda = C |\Lambda| |J|, \quad C := \frac{2bG}{\beta} > 0.$$

It remains to observe that

$$\mathbb{P} \left\{ \operatorname{dist}(\operatorname{spec} M_\Lambda(l^\omega, E), \alpha) \leq b|J| \right\} \leq \mathbb{E} \operatorname{tr} E_\Lambda(\alpha - b|J|, \alpha + b|J|). \quad (16)$$

Now, let us now prove part (b) of theorem 4. Below we assume $\alpha < 0$. Take first an arbitrary interval $I = (-E_+, -E_-)$ with $E_+ > E_- > 0$. Note that for $E < 0$, it is more convenient to rewrite

$$M_\Lambda(l^\omega, E) = \sum_{e \in \Lambda} \left(\frac{\sqrt{-E}}{\sinh l_e^\omega \sqrt{-E}} (P_1^e - P_2^e) - \sqrt{-E} \coth l_e^\omega \sqrt{-E} I^e \right).$$

Then,

$$\partial_e M_\Lambda(l^\omega, E) = \frac{|E|}{\sinh^2 l_e^\omega \sqrt{-E}} \left[I_e - \cosh l_e^\omega \sqrt{-E} (P_1^e - P_2^e) \right],$$

and one has

$$F_\Lambda M_\Lambda(l^\omega, E) = M_\Lambda(l^\omega, E) + K_\Lambda(l^\omega, E)$$

with

$$F_\Lambda := -\frac{1}{\sqrt{-E}} \sum_{e \in \Lambda} \tanh l_e^\omega \sqrt{-E} \frac{\partial}{\partial l_e^\omega}, \quad K_\Lambda(l^\omega, E) := \frac{1}{\sqrt{-E}} \sum_{e \in \Lambda} \tanh l_e^\omega \sqrt{-E} I^e.$$

Denote $\gamma = \frac{\tanh l_{\min} \sqrt{E_-}}{\sqrt{E_+}}$, then one has $K_\Lambda(l^\omega, E) \geq \gamma \text{id}$ for all Λ , $E \in I$, and a.e. ω . As in the case $\alpha > 0$, one computes

$$\begin{aligned} & -\text{tr} \left[F_\Lambda \left(\chi_{(-\infty, 0]}(M_\Lambda(l^\omega, E) - \lambda) \right) \right] \\ & \quad = \text{tr} \left[\partial_\lambda \chi_{(-\infty, 0]}(M_\Lambda(l^\omega, E) - \lambda) F_\Lambda M_\Lambda(l^\omega, E) \right] \\ & \quad = \text{tr} \left[\partial_\lambda \chi_{(-\infty, 0]}(M_\Lambda(l^\omega, E) - \lambda) M_\Lambda(l^\omega, E) \right] + \text{tr} \left[\partial_\lambda \chi_{(-\infty, 0]}(M_\Lambda(l^\omega, E) - \lambda) K_\Lambda(l^\omega, E) \right] \\ & \quad \geq \text{tr} \left[\lambda \partial_\lambda \chi_{(-\infty, 0]}(M_\Lambda(l^\omega, E) - \lambda) \right] + \gamma \text{tr} \left[\partial_\lambda \chi_{(-\infty, 0]}(M_\Lambda(l^\omega, E) - \lambda) \right] \\ & \quad = \text{tr} \left[(\gamma + \lambda) \partial_\lambda \chi_{(-\infty, 0]}(M_\Lambda(l^\omega, E) - \lambda) \right]. \end{aligned}$$

Assume that

$$\lambda + \gamma \geq \beta > 0 \text{ for } \lambda \in (\alpha - b|I|, \alpha + b|I|) \equiv (\alpha - b|E_+ - E_-|, \alpha + b|E_+ - E_-|), \quad (17)$$

then

$$\beta \text{tr} \left[\partial_\lambda \chi_{(-\infty, 0]}(M_\Lambda(l^\omega, E) - \lambda) \right] \leq \text{tr} \left[\sum_{e \in \Lambda} \frac{\tanh l_e^\omega \sqrt{-E}}{\sqrt{-E}} \frac{\partial}{\partial l_e^\omega} \chi_{(-\infty, 0]}(M_\Lambda(l^\omega, E) - \lambda) \right].$$

Using this inequality and (14) one can estimate

$$\begin{aligned} \mathbb{E} \text{tr} \left[E_\Lambda(\alpha - b|J|, \alpha + b|J|) \right] &= \text{tr} \left[\int_{\alpha - b|J|}^{\alpha + b|J|} \partial_\lambda \chi_{(-\infty, 0]}(M_\Lambda(l^\omega, E) - \lambda) d\lambda \right] \\ &\leq \beta^{-1} \int_{\alpha - b|J|}^{\alpha + b|J|} \sum_{e \in \Lambda} \frac{\tanh l_e^\omega \sqrt{-E}}{\sqrt{-E}} \frac{\partial}{\partial l_e^\omega} \text{tr} \chi_{(-\infty, 0]}(M_\Lambda(l^\omega, E) - \lambda) d\lambda \end{aligned}$$

This is then estimated exactly as in the proof of the part (a).

Hence we need to chose the interval I in such a way that (17) is satisfied. Let $E := -\inf \text{spec} \Sigma(\alpha)$. If one has $\frac{\tanh l_{\min} \sqrt{E}}{\sqrt{E}} + \alpha > 0$, then (17) holds for $E_- < E < E_+$ with $|E_+ - E_-|$ sufficiently small. On the other hand, by (5),

$$\alpha = -2d\sqrt{E} \tanh \frac{l_{\min} \sqrt{E}}{2}. \quad (18)$$

Therefore, it is sufficient to find values of α for which

$$f(E) := \frac{\tanh l_{\min} \sqrt{E}}{\sqrt{E}} - 2d\sqrt{E} \tanh \frac{l_{\min} \sqrt{E}}{2} > 0.$$

A short computation shows that the sign of $f(E)$ coincides with the sign of $\cosh^2 \frac{l_{\min} \sqrt{E}}{2} - Ed$ which is positive for E sufficiently close to 0 as well as for E sufficiently large, and it remains to note that E is a monotonous function of α due to (18). This completes the proof of theorem 4.

5 Proof of theorem 5 (initial scale estimate)

As in the proof of theorem 4, one can show that, for some $b > 0$, there holds

$$\mathbb{P}\{\text{dist}(\text{spec } H_{\Lambda(n)}(l_\omega, \alpha), E) \leq n^{\beta-1}\} \leq \mathbb{P}\{\text{dist}(\text{spec } M_{\Lambda(n)}(l_\omega, E), \alpha) \leq bn^{\beta-1}\}. \quad (19)$$

Hence theorem 5 is a consequence of

$$\mathbb{P}\{\text{dist}(\text{spec } M_{\Lambda(n)}(l_\omega, E), \alpha) \leq bn^{\beta-1}\} \leq n^{-\xi}. \quad (20)$$

It is well known, see e.g. Section 2.2 in [38] that, in order to prove the estimates (20), it is sufficient to show the Lifshitz tail behavior for the integrated density of states for the operator $M(l_\omega, E)$. Note that, if E is an edge of the almost sure spectrum of $H^\omega(\alpha)$, then α is an edge of the almost sure spectrum of $M(l^\omega, E)$ (see e.g. [25]). Hence, it is sufficient to study the behavior of the integrated density of states of $M(l^\omega, E)$ at the spectral edges.

The operator $M(l^\omega, E)$ is closely related to the random hopping model considered in [24]; below, we use the very constructions of [24] and [33] to obtain the Lifshitz tails. The integrated density of states in our case is defined by

$$k(t) := \lim_{n \rightarrow \infty} \frac{\#\{\lambda \in \text{spec } M_{\Lambda(n)}(l_\omega, E) : \lambda < t\}}{|\mathcal{V}(n)|}.$$

This limit exists almost surely and $t \mapsto k(t)$ is non decreasing. Let $[\mu_{\min}, \mu_{\max}]$ be the almost sure spectrum of $M(l^\omega, E)$. Then, $k(t) = 0$ for $t \leq \mu_{\min}$ and $k(t) = 1$ for $t \geq \mu_{\max}$. Denote also

$$b := \sup_{l \in [l_{\min}, l_{\max}]} \left| \frac{\sqrt{E}}{\sin l \sqrt{E}} \right|.$$

By well-known arguments, see e.g. [38, Section 2.2], in order to prove (20) it is sufficient to show

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\log |\log[1 - k(\mu_{\max} - \varepsilon)]|}{\log \varepsilon} \leq -\frac{d}{2} \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0^+} \frac{\log |\log k(\mu_{\min} + \varepsilon)|}{\log \varepsilon} \leq -\frac{d}{2}. \quad (21)$$

For $n \in \mathbb{N}$ define $M_n^\omega(E) := M(l_n^\omega, E)$ where $l_n^\omega(e) = l_n^\omega(e + \gamma)$ for $\gamma \in (2n+1)\mathbb{Z}^d$. By the Floquet-Bloch theory, the operator $M_n^\omega(E)$ admits a density of states, k_n^ω , satisfying

$$k_n(E) = \frac{1}{(2\pi)^d} \int_{[-\frac{\pi}{2n+1}, \frac{\pi}{2n+1}]} \#\text{spec } M_n^\omega(E, \theta) \cap (-\infty, E) d\theta$$

where $M_n^\omega(E, \theta)$ differs from $M_n^\omega(E)$ only by an operator of rank at most Cn^{d-1} with $C > 0$ independent of n . As suggested in [24], in order to obtain (21), it is sufficient to show the analogous estimates with $k(E)$ replaced by $\mathbb{E}(k_n^\omega(E))$ uniformly in n for sufficiently large n . Then, as noted in [33] and applied in [24], the latter asymptotics can be obtained directly from the following local energy estimate which has been proved in [24, Lemma 2.1]. Let $a \in (0, b)$. For $\varphi \in l^2(\mathbb{Z}^d)$ one has

$$\langle \varphi, M_n^\omega \varphi \rangle \geq \langle \varphi, W_n^\omega \varphi \rangle + a \langle |\varphi|, H_0 |\varphi| \rangle$$

where H_0 is the free Laplace operator in (\mathbb{Z}^d) ,

$$H_0 u(n) = \sum_{n:|m-n|=1} (u(n) - u(m)),$$

and the potential W_n^ω is given by

$$W_n^\omega(v) = \sum_{e:v \sim v} \beta\left(\frac{\sqrt{E}}{\sin l_e^\omega \sqrt{E}}\right) + \sum_{e:v \sim e} \sqrt{E} \cot l_e^\omega \sqrt{E}$$

with

$$\beta(t) := \begin{cases} -|t|, & |t| \geq a, \\ -a, & \text{otherwise.} \end{cases}$$

Then, as in [24], following the computations done in [33], one proves Lifshitz tails for $M(l^\omega, E)$ near μ_{\max} or μ_{\min} . This completes the proof of theorem 5. \square

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