On gaps in Rényi β-expansions of unity for $\beta > 1$ an algebraic number.

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Abstract. Let $\beta > 1$ be an algebraic number. We study the strings of zeros ("gaps") in the Rényi β-expansion $d_\beta(1)$ of unity which controls the set $\mathbb{Z}_\beta$ of β-integers. Using a version of Liouville's inequality which extends Mahler's and Güting's approximation theorems, the strings of zeros in $d_\beta(1)$ are shown to exhibit a "gappiness" asymptotically bounded above by $\log(\mathrm{M}(\beta))/\log(\beta)$, where $\mathrm{M}(\beta)$ is the Mahler measure of β. The proof of this result provides in a natural way a new classification of algebraic numbers > 1 with classes called $\mathrm{Q}_i^{(j)}$ which we compare to Bertrand-Mathis's classification with classes C_1 to C_5 (reported in an article by Blanchard). This new classification relies on the maximal asymptotic "quotient of the gap" value of the "gappy" power series associated with $d_\beta(1)$. As a corollary, all Salem numbers are in the class $\mathrm{C}_1 \cup \mathrm{Q}_0^{(1)} \cup \mathrm{Q}_0^{(2)} \cup \mathrm{Q}_0^{(3)}$; this result is also directly proved using a recent generalization of the Thue-Siegel-Roth Theorem given by Corvaja.

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1 Introduction

The exploration of the links between symbolic dynamics and number theory of β-expansions, when $\beta > 1$ is an algebraic number or more generally a real number, started with Bertrand-Mathis [Be1] [Be2]. Bertrand-Mathis, in Blanchard [Bl], reported a classification of real numbers according to their β-shift, using the properties of the Rényi β-expansion $d_\beta(1)$ of 1. A lot of questions remain open concerning the distribution of the algebraic numbers $\beta > 1$ in this classification. The Rényi β-expansion of 1 is important since it controls the β-shift [Pa] and the discrete and locally finite set $\mathbb{Z}_\beta \subset \mathbb{R}$ of β-integers [B-K] [E-VG] [Ga] [G1]. The aim of this note is to give a new Theorem (Theorem 1.1) on the gaps (strings of $0'$s) in $d_\beta(1)$ for algebraic numbers $\beta > 1$, and investigate how it provides (partial) answers to some questions of [Bl], in particular for Salem numbers (Corollary 1.2).

Theorem 1.1 provides an upper bound on the asymptotic quotient of the gap of $d_\beta(1)$ and is obtained by a version of Liouville's inequality extending Mahler's and Güting's approximation theorems. The proof of Theorem 1.1 turns out to be extremely instructive in itself since it leads to a new classification of the algebraic numbers β as a function of the asymptotics of the gaps in $d_\beta(1)$ and "intrinsic features", namely the Mahler measure $\mathrm{M}(\beta)$, of β (the definition of $\mathrm{M}(\beta)$ is recalled in Section 3). The existence of this double parametrization, symbolic and algebraic, was guessed in [Bl] p 137. This new classification complements Bertrand-Mathis's (Blanchard [Bl] pp 137–139) and both are recalled below for comparison's sake. The question whether an algebraic number $\beta > 1$ is contained in one class or another has already been discussed by many authors [Be1] [Be2] [Be3] [Bl] [Bo] [Bo1] [Bo2] [Bo3] [D-S] [FS] [Li1] [Li2] [Pa] [PF] [Sc] [Sk] and depends at least upon the distribution of the conjugates of β in the complex plane. Only the conjugates of β of modulus strictly greater than unity intervene in Theorem 1.1 via the Mahler measure of β. Corollary 1.2 is readily deduced from this remark. We deduce that Salem numbers belong to $\mathrm{C}_1 \cup \mathrm{C}_2 \cup \mathrm{Q}_0$, whereas the Pisot numbers are in $\mathrm{C}_1 \cup \mathrm{C}_2$ [Th].

Another proof of Corollary 1.2 consists of controlling the gaps of $d_\beta(1)$ by stronger Theorems of Diophantine Geometry which allow suitable collections of places of the number field $\mathbb{Q}(\beta)$ associated with the conjugates of β and the properties of $d_\beta(1)$ to be taken into account simultaneously. This alternative proof of Corollary 1.2, just sketched in Section 4, is obtained using the Theorem of Thue-Siegel-Roth given by Corvaja [A] [C].

Theorem 1.1. *Let $\beta > 1$ be an algebraic number and* $\mathrm{M}(\beta)$ *be its Mahler measure. Denote by $d_\beta(1) := 0.t_1t_2t_3\ldots$, with $t_i \in A_\beta := \{0, 1, 2, \ldots, \lceil \beta - 1 \rceil\}$, the Rényi β-expansion of 1. Assume that $d_\beta(1)$ is infinite and gappy in* *the following sense: there exist two sequences* $\{m_n\}_{n\geq 1}$, $\{s_n\}_{n\geq 0}$ *such that*

$$1 = s_0 \leq m_1 < s_1 \leq m_2 < s_2 \leq \ldots \leq m_n < s_n \leq m_{n+1} < s_{n+1} \leq \ldots$$

with $(s_n - m_n) \geq 2$, $t_{m_n} \neq 0, t_{s_n} \neq 0$ *and* $t_i = 0$ *if* $m_n < i < s_n$ *for all* $n \geq 1$. *Then*

$$\limsup_{n\to+\infty} \frac{s_n}{m_n} \leq \frac{\log(\mathrm{M}(\beta))}{\log(\beta)}. \tag{1.1}$$

Moreover, if $\liminf_{n\to+\infty}(m_{n+1} - m_n) = +\infty$, *then*

$$\limsup_{n\to+\infty} \frac{s_{n+1} - s_n}{m_{n+1} - m_n} \leq \frac{\log(\mathrm{M}(\beta))}{\log(\beta)}. \tag{1.2}$$

As in Ostrowski [Os] the quotient $s_n/m_n \geq 1$ is called *the quotient of the gap*, relative to the nth-gap (assuming $t_j \neq 0$ for all $s_n \leq j \leq m_{n+1}$ to characterize uniquely the gaps). Note that the term "*lacunary*" has often other meanings in literature and is not used here to describe "gappiness". Denote by $\mathcal{L}(S_\beta)$ the language of the β-shift [Bl] [Fr1] [Fr2] [Lo]. Bertrand-Mathis's classification ([Bl] pp 137–139) is as follows:

C_1 : $d_\beta(1)$ is finite.
C_2 : $d_\beta(1)$ is ultimately periodic but not finite.
C_3 : $d_\beta(1)$ contains bounded strings of $0'$s, but is not ultimately periodic.
C_4 : $d_\beta(1)$ does not contain some words of $\mathcal{L}(S_\beta)$, but contains strings of $0'$s with unbounded length.
C_5 : $d_\beta(1)$ contains all words of $\mathcal{L}(S_\beta)$.

Present classes of algebraic numbers, with the notations of Theorem 1.1:

$\mathrm{Q}_0^{(1)}$: $\quad 1 = \lim_{n\to+\infty} \frac{s_n}{m_n}$ with $(m_{n+1} - m_n)$ bounded.

$\mathrm{Q}_0^{(2)}$: $\quad 1 = \lim_{n\to+\infty} \frac{s_n}{m_n}$ with $(s_n - m_n)$ bounded and $\lim_{n\to+\infty}(m_{n+1} - m_n) = +\infty$.

$\mathrm{Q}_0^{(3)}$: $\quad 1 = \lim_{n\to+\infty} \frac{s_n}{m_n}$ with $\limsup_{n\to+\infty}(s_n - m_n) = +\infty$.

Q_1 : $\quad 1 < \limsup_{n\to+\infty} \frac{s_n}{m_n} < \frac{\log(\mathrm{M}(\beta))}{\log(\beta)}$.

Q_2 : $\quad \limsup_{n\to+\infty} \frac{s_n}{m_n} = \frac{\log(\mathrm{M}(\beta))}{\log(\beta)}$.

What are the relative proportions of each class in the whole set $\overline{\mathbb{Q}}_{>1}$ of algebraic numbers $\beta > 1$? Comparing C_2, C_3 and $\mathrm{Q}_0^{(1)}$, what are the relative proportions in $\mathrm{Q}_0^{(1)}$ of those β which give ultimate periodicity in $d_\beta(1)$ and those for which $d_\beta(1)$ is not ultimately periodic ? Schmeling ([Sc] Theorem A) has shown that the class C_3 (of real numbers $\beta > 1$) has Hausdorff dimension one. We have:

- $\overline{\mathbb{Q}}_{>1} \cap C_2 \quad \subset \quad Q_0^{(1)}$,
- $\overline{\mathbb{Q}}_{>1} \cap C_3 \quad \subset \quad Q_0^{(1)} \cup Q_0^{(2)}$, with $C_3 \cap Q_0^{(3)} = \emptyset$,
- $\overline{\mathbb{Q}}_{>1} \cap C_4 \quad \subset \quad Q_0^{(3)} \cup Q_1 \cup Q_2$.

The Pisot numbers β are contained in $C_1 \cup Q_0^{(1)}$ since they are such that $d_\beta(1)$ is finite or ultimately periodic (Parry [Pa], Bertrand-Mathis [Be3]). Recall that a Perron number is an algebraic integer $\beta > 1$ such that all the conjugates $\beta^{(i)}$ of β satisfy $|\beta^{(i)}| < \beta$. Conversely, as shown in Lind [Li1], Denker, Grillenberger, Sigmund [D-S] and Bertrand-Mathis [Be2], if $\beta > 1$ is such that $d_\beta(1)$ is ultimately periodic (finite or not), then β is a Perron number. Not all Perron numbers are attained in this way: a Perron number which possesses a real conjugate greater than 1 cannot be such that $d_\beta(1)$ is ultimately periodic ([Bl] p 138). Parry numbers belong to $C_1 \cup C_2$. Let $Q_0 = Q_0^{(1)} \cup Q_0^{(2)} \cup Q_0^{(3)}$.

Corollary 1.2. *Let $\beta > 1$ be a Salem number which does not belong to C_1. Then β belongs to the class Q_0.*

The attribution of Salem numbers to C_1, $Q_0^{(1)}$, $Q_0^{(2)}$ and $Q_0^{(3)}$ is an open problem in general, except in low degree. Boyd [Bo] [Bo3] has shown that Salem numbers of degree 4 belong to C_2, hence to $Q_0^{(1)}$. It is also the case of some Salem numbers of degree 6 and ≥ 8 in the framework of a probabilistic model [Bo2] [Bo3]. In Section 5 we ask the question whether Corollary 1.2 could still be true for Perron numbers.

The definition of the class Q_0 does not make any allusion to β, i.e. to $M(\beta)$, to the conjugates of β, to the minimal polynomial of β or to its length, etc, but takes only into account the quotients of the gaps in $d_\beta(1)$. Hence this class Q_0 can be applied to real numbers $\beta > 1$ in full generality instead of only to algebraic numbers > 1. The question whether there exist transcendental numbers $\beta > 1$ which belong to the class Q_0 was asked in [Bl]; what proportion appears in each subclass ? Examples of transcendental numbers (Komornik-Loreti constant [AC] [KL], Sturmian numbers [CK]) in Q_0 are given in Section 5.

In the present note, we deal with the algebraicity of values of "gapppy" series, deduced from $d_\beta(1)$, at the algebraic point β^{-1}. In a related context, more related to transcendency, Nishioka [N] and Corvaja Zannier [CZ] have followed different paths and applied the Subspace Theorem [Sw] to deduce different results.

2 Definitions

For $x \in \mathbb{R}$ the integer part of x is $\lfloor x \rfloor$ and its fractional part $\{x\} = x - \lfloor x \rfloor$. The smallest integer larger than or equal to x is denoted by $\lceil x \rceil$. For $\beta > 1$ a real number and $z \in [0,1]$ we denote by $T_\beta(z) = \beta z \pmod 1$ the β-transform on $[0,1]$ associated with β [Pa] [Re], and iteratively, for all integers $j \geq 0$, $T_\beta^{j+1}(z) := T_\beta(T_\beta^j(z))$, where by convention $T_\beta^0 = Id$.

Let $\beta > 1$ be a real number. A beta-representation (or β-representation, or representation in base β) of a real number $x \geq 0$ is given by an infinite sequence $(x_i)_{i \geq 0}$ and an integer $k \in \mathbb{Z}$ such that $x = \sum_{i=0}^{+\infty} x_i \beta^{-i+k}$, where the digits x_i belong to a given alphabet ($\subset \mathbb{N}$) [Fr1] [Fr2] [Lo]. Among all the beta-representations of a real number $x \geq 0, x \neq 1$, there exists a particular one called Rényi β-expansion, which is obtained via the greedy algorithm: in this case, k satisfies $\beta^k \leq x < \beta^{k+1}$ and the digits

$$x_i := \lfloor \beta T_\beta^i (\frac{x}{\beta^{k+1}}) \rfloor \qquad\qquad i = 0, 1, 2, \ldots, \tag{2.1}$$

belong to the finite canonical alphabet $\mathbb{A}_\beta := \{0, 1, 2, \ldots, \lceil \beta - 1 \rceil\}$. If β is an integer, then $\mathbb{A}_\beta := \{0, 1, 2, \ldots, \beta - 1\}$; if β is not an integer, then $\mathbb{A}_\beta := \{0, 1, 2, \ldots, \lfloor \beta \rfloor\}$. We denote by

$$\langle x \rangle_\beta := x_0 x_1 x_2 \ldots x_k \,.\, x_{k+1} x_{k+2} \ldots \tag{2.2}$$

the couple formed by the string of digits $x_0 x_1 x_2 \ldots x_k x_{k+1} x_{k+2} \ldots$ and the position of the dot, which is at the k-th position (between x_k and x_{k+1}). By definition the integer part (in base β) of x is $\sum_{i=0}^{k} x_i \beta^{-i+k}$ and its fractional part (in base β) is $\sum_{i=k+1}^{+\infty} x_i \beta^{-i+k}$. If a Rényi β-expansion ends in infinitely many zeros, it is said to be finite and the ending zeros are omitted. If it is periodic after a certain rank, it is said to be eventually periodic (the period is the smallest finite string of digits possible, assumed not to be a string of zeros); for the substitutive approach see [F] [PF].

The Rényi β-expansion which plays an important role in the theory is the Rényi β-expansion of 1, denoted by $d_\beta(1)$ and defined as follows: since $\beta^0 \leq 1 < \beta$, the value $T_\beta(1/\beta)$ is set to 1 by convention. Then using the formulae (2.1)

$$t_1 = \lfloor \beta \rfloor, t_2 = \lfloor \beta\{\beta\} \rfloor, t_3 = \lfloor \beta\{\beta\{\beta\}\} \rfloor, \ldots \tag{2.3}$$

The writing

$$d_\beta(1) = 0.t_1 t_2 t_3 \ldots$$

corresponds to

$$1 = \sum_{i=1}^{+\infty} t_i \beta^{-i}.$$

Links between the set $\mathbb{Z}_\beta$ of beta-integers and $d_\beta(1)$ are evoked in [E-VG] [F-K] [G1] [G2] [V1] [V2]. A real number $\beta > 1$ such that $d_\beta(1)$ is finite or eventually periodic is called a *beta-number* or more recently *a Parry number* (this recent terminology appears in [E-VG]). In particular, it is called a *simple beta-number* or *a simple Parry number* when $d_\beta(1)$ is finite. Beta-numbers (Parry numbers) are algebraic integers [Pa] and all their conjugates lie within a compact subset which looks like a fractal in the complex plane [So]. The conjugates of beta-numbers are all bounded above in modulus by the golden mean $\frac{1}{2}(1+\sqrt{5})$ [So] [F-P].

3 Proof of Theorem 1.1

Since algebraic numbers $\beta > 1$ for which the Rényi β-expansion $d_\beta(1)$ of 1 is finite are excluded, we may consider that β does not belong to $\mathbb{N}$. Indeed, if $\beta = h \in \mathbb{N}$, then $d_h(1) = 0.h$ is finite (Lothaire [Lo], Chap. 7). If $\beta \notin \mathbb{N}$, then $\lceil \beta - 1 \rceil = \lfloor \beta \rfloor$ and the alphabet A_β equals $\{0, 1, 2, \ldots, \lfloor \beta \rfloor\}$, where $\lfloor \beta \rfloor$ denotes the greatest integer smaller than or equal to β.

Let $f(z) := \sum_{i=1}^{+\infty} t_i z^i$ be the "gappy" power series deduced from the representation $d_\beta(1) = 0.t_1 t_2 t_3 \ldots$ associated with the β-shift (*gappy* in the sense of Theorem 1.1). Since $d_\beta(1)$ is assumed to be infinite, its radius of convergence is 1. By definition, it satisfies

$$f(\beta^{-1}) = 1, \tag{3.1}$$

which means that the function value $f(\beta^{-1})$ is algebraic, equal to 1, at the real algebraic number β^{-1} in the open disk of convergence $D(0,1)$ of $f(z)$ in the complex plane. This fact is a general intrinsic feature of the Rényi expansion process which leads to the following important consequence by the theory of admissible power series of Mahler [Ma].

Proposition 3.1.

$$\limsup_{n \to +\infty} \frac{s_n}{m_n} < +\infty. \tag{3.2}$$

Proof. This is a consequence of Theorem 1 in [Ma]. Indeed, if we assume that there exists a sequence of integers (n_i) which tends to infinity such that $\lim_{i \to +\infty} s_{n_i}/m_{n_i} = +\infty$, then $f(z)$ would be *admissible* in the sense of [Ma]. Since $f(z)$ is a power series with nonnegative coefficients, which is not a polynomial, the function value $f(\beta^{-1})$ should not be algebraic. But it equals 1. Contradiction. $\square$

Let us improve Proposition 3.1. Assume that

$$\limsup \frac{s_n}{m_n} > \frac{\log(\mathrm{M}(\beta))}{\log(\beta)} \tag{3.3}$$

and show the contradiction with (1.1) and (1.2). Recall that, if

$$P_\beta(X) = \sum_{i=0}^{d} \alpha_i X^i \ = \ \alpha_d \prod_{i=0}^{d-1} (X - \beta^{(i)})$$

with $d \geq 1$, $\alpha_0 \alpha_d \neq 0$, denotes the minimal polynomial of $\beta = \beta^{(0)} > 1$, having $\beta^{(1)}, \beta^{(2)}, \ldots, \beta^{(d-1)}$ as conjugates, the Mahler measure of β is by definition

$$\mathrm{M}(\beta) := |\alpha_d| \prod_{i=0}^{d-1} \max\{1, |\beta^{(i)}|\}.$$

Güting [Gü] has shown that the approximation of algebraic numbers by algebraic numbers is fairly difficult to realize by polynomials. In the present proof, we use a version of Liouville's inequality which generalizes approximation theorems obtained by Güting [Gü], and apply it to the values of the "polynomial tails" of the power series $f(z)$ at the algebraic number β^{-1}, to obtain the contradiction. Let us write

$$f(z) = \sum_{n=0}^{+\infty} Q_n(z) \tag{3.4}$$

with

$$Q_n(z) := \sum_{i=s_n}^{m_{n+1}} t_i z^i, \qquad n = 0, 1, 2, \ldots. \tag{3.5}$$

By construction the polynomials $Q_n(z)$, of degree m_{n+1}, are not identically zero and $Q_n(1) > 0$ is an integer for all $n \geq 0$.

Denote by $S_n(z) = -1 + \sum_{i=1}^{m_n} t_i z^i$ the m_nth-section polynomial of the power series $f(z) - 1$ for all $n \geq 1$. Recall that, for $R(X) = \sum_{i=0}^{v} \alpha_i X^i \in \mathbb{Z}[X]$, $L(R) := \sum_{i=0}^{v} |\alpha_i|$ denotes the length of the polynomial $R(X)$. We have: $L(S_n) = 1 + \sum_{i=1}^{m_n} t_i = 1 + \sum_{j=0}^{n-1} Q_j(1)$. From Theorem 5 in [Gü] we deduce that only one of the following cases (G-i) or (G-ii) holds, for all $n \geq 1$:

$$(G-i) \qquad\qquad S_n(\beta^{-1}) = 0, \tag{3.6}$$

$$(G-ii) \qquad\qquad \left|S_n(\beta^{-1})\right| \geq \frac{1}{\left(1 + \sum_{j=0}^{n-1} Q_j(1)\right)^{d-1} \left(L(P_\beta^*)\right)^{m_n}}, \tag{3.7}$$

where $P_\beta^*(X) = X^d P_\beta(1/X)$ is the reciprocal polynomial of the minimal polynomial of β, for which $L(P_\beta) = L(P_\beta^*) \in \mathbb{N} \setminus \{0,1\}$.

Case (G-i) is impossible for any n. Indeed, if there exists an integer $n_0 \geq 1$ such that (G-i) holds, then, since all the digits t_i are positive and that $\beta^{-1} > 0$, we would have $t_i = 0$ for all $i \geq s_{n_0}$. This would mean that the Rényi expansion of 1 in base β is finite, which is excluded by assumption. Contradiction. Therefore, the only possibility is (G-ii), which holds for all integers $n \geq 1$. From Lemma 3.10 and Liouville's inequality (Proposition 3.14) in Waldschmidt [W] the inequality (G-ii) can be improved to

$$(L-ii) \qquad \left|S_n(\beta^{-1})\right| \geq \frac{1}{\left(1+\sum_{j=0}^{n-1} Q_j(1)\right)^{d-1} (\mathrm{M}(\beta))^{m_n}}. \tag{3.8}$$

This improvement may be important; recall the well-known inequalities:

$$\mathrm{M}(\beta) \leq L(P_\beta) \leq 2^{\deg(\beta)}\mathrm{M}(\beta)$$

and see [W] p113 for comparison with different heights. On the other hand, since $\left|S_n(\beta^{-1})\right| = \sum_{i=s_n}^{+\infty} t_i \beta^{-i}$ for all integers $n \geq 1$, we deduce

$$\left|S_n(\beta^{-1})\right| \leq \frac{\lfloor\beta\rfloor}{1-\beta^{-1}}\beta^{-s_n} \qquad n = 1, 2, \ldots. \tag{3.9}$$

Putting together (3.8) and (3.9), we deduce that

$$\frac{\beta^{s_n}}{\left(1+\sum_{j=0}^{n-1} Q_j(1)\right)^{d-1} \mathrm{M}(\beta)^{m_n}} \leq \frac{\lfloor\beta\rfloor}{1-\beta^{-1}} \tag{3.10}$$

should be satisfied for $n = 1, 2, 3, \ldots$. Denote

$$u_n := \frac{\beta^{s_n}}{\left(1+\sum_{j=0}^{n-1} Q_j(1)\right)^{d-1} \mathrm{M}(\beta)^{m_n}} \qquad \text{for all } n \geq 1.$$

Proof of (1.1)*:* from (3.3) assumed to be true there exists a sequence of integers (n_i) which tends to infinity and an integer i_0 such that

$$\frac{s_{n_i}}{m_{n_i}} > \frac{\log(\mathrm{M}(\beta))}{\log(\beta)} \qquad \text{for all } i \geq i_0.$$

Now,

$$\left(\frac{1}{1+\lfloor\beta\rfloor m_{n_i}}\right)^{d-1} \left(\frac{\beta^{\frac{s_{n_i}}{m_{n_i}}}}{\mathrm{M}(\beta)}\right)^{m_{n_i}} \leq \frac{1}{\left(1+\sum_{j=0}^{n_i-1} Q_j(1)\right)^{d-1}} \left(\frac{\beta^{\frac{s_{n_i}}{m_{n_i}}}}{\mathrm{M}(\beta)}\right)^{m_{n_i}} \leq u_{n_i}. \tag{3.11}$$

For $i \geq i_0$ the inequality

$$1 = \frac{\beta^{\frac{\log(\mathrm{M}(\beta))}{\log(\beta)}}}{\mathrm{M}(\beta)} < \frac{\beta^{\frac{s_{n_i}}{m_{n_i}}}}{\mathrm{M}(\beta)} \tag{3.12}$$

holds. This implies that the left-hand side member of (3.11) tends exponentially to infinity when i tends to infinity. By (3.11) this forces u_{n_i} to tend to infinity. The contradiction now comes from (3.10) since the sequence (u_n) should be uniformly bounded.

Proof of (1.2)*:* for $n = 1, 2, \ldots$, let us rewrite the n-th quotient

$$\frac{u_{n+1}}{u_n} = \frac{\beta^{s_{n+1}-s_n}}{\mathrm{M}(\beta)^{m_{n+1}-m_n}} \frac{\Big(1 + \sum_{j=0}^{n-1} Q_j(1)\Big)^{d-1}}{\Big(1 + \sum_{j=0}^{n} Q_j(1)\Big)^{d-1}} \tag{3.13}$$

as

$$\frac{u_{n+1}}{u_n} = \frac{\big(\frac{\beta^{\frac{s_{n+1}-s_n}{m_{n+1}-m_n}}}{\mathrm{M}(\beta)}\big)^{m_{n+1}-m_n}}{(m_{n+1} - m_n + 1)^{(d-1)}} \left[(m_{n+1} - m_n + 1)^{(d-1)} \frac{\Big(1 + \sum_{j=0}^{n-1} Q_j(1)\Big)^{d-1}}{\Big(1 + \sum_{j=0}^{n} Q_j(1)\Big)^{d-1}} \right] \tag{3.14}$$

and denote

$$U_n := \frac{1}{(m_{n+1} - m_n + 1)^{(d-1)}} \left(\frac{\beta^{\frac{s_{n+1}-s_n}{m_{n+1}-m_n}}}{\mathrm{M}(\beta)} \right)^{m_{n+1}-m_n} \tag{3.15}$$

and

$$W_n := (m_{n+1} - m_n + 1)^{(d-1)} \left(\frac{1 + \sum_{j=0}^{n-1} Q_j(1)}{1 + \sum_{j=0}^{n} Q_j(1)} \right)^{d-1} \tag{3.16}$$

so that $u_{n+1}/u_n = U_n W_n$.

Lemma 3.2.

$$0 < \liminf_{n \to +\infty} \quad W_n \tag{3.17}$$

Proof. Assume the contrary. Then there exists a subsequence (n_i) of integers which tends to infinity such that $\lim_{i \to +\infty} W_{n_i} = 0$. In other terms, for all $\epsilon > 0$, there exists i_1 such that $i \geq i_1$ implies $W_{n_i} \leq \epsilon$, equivalently

$$(m_{n_i+1} - m_{n_i} + 1)\big(1 + \sum_{j=0}^{n_i-1} Q_j(1)\big) \leq \epsilon^{\frac{1}{d-1}} \times \big(1 + \sum_{j=0}^{n_i} Q_j(1)\big). \tag{3.18}$$

Since, by hypothesis, $t_{s_n} \geq 1$ and $t_{m_n+1} \geq 1$ for all $n \geq 1$, we have: $n_i \leq 1 + \sum_{j=0}^{n_i-1} Q_j(1)$. On the other hand, $Q_{n_i}(1) \leq \lfloor\beta\rfloor(m_{n_i+1} - m_{n_i} + 1)$. Then, from (3.18) with ϵ taken equal to 1, we would have

$$n_i \leq 1 + \sum_{j=0}^{n_i-1} Q_j(1) \leq \frac{Q_{n_i}(1)}{(m_{n_i+1} - m_{n_i} + 1) - 1} \leq \lfloor\beta\rfloor \times \frac{m_{n_i+1} - m_{n_i} + 1}{m_{n_i+1} - m_{n_i}} \leq \frac{3}{2}\lfloor\beta\rfloor. \tag{3.19}$$

But the left-hand side member of (3.19) tends to infinity which is impossible. Contradiction. □

Let us assume that (1.2) does not hold and show the contradiction ; that is, assume that $\liminf_{n\to+\infty}(m_{n+1} - m_n) = +\infty$ and $\limsup_{n\to+\infty}(s_{n+1} - s_n)/(m_{n+1} - m_n) > \log(\mathrm{M}(\beta))/\log(\beta)$ hold. Then

$$1 = \frac{\beta^{\frac{\log(\mathrm{M}(\beta))}{\log(\beta)}}}{\mathrm{M}(\beta)} < \frac{\beta^{\frac{s_{n_i+1}-s_{n_i}}{m_{n_i+1}-m_{n_i}}}}{\mathrm{M}(\beta)} \tag{3.20}$$

for some sequence of integers (n_i) which tends to infinity. This proves that $\limsup_{n\to+\infty} U_n = +\infty$ since $\lim_{i\to+\infty} U_{n_i} = +\infty$ exponentially, by (3.15) and (3.20).

By Lemma 3.2 there exists $r > 0$ such that $W_n \geq r$ for all n large enough. Therefore, $u_{n+1}/u_n = U_n W_n \geq r U_n$ for all n large enough. Since $\limsup_{n\to+\infty} U_n = +\infty$ we conclude that $\limsup u_{n+1}/u_n = +\infty$, hence that $\limsup u_n = +\infty$. This contradicts (3.10) and proves (1.2).

4 A direct proof of Corollary 1.2

Let $\beta > 1$ be a Salem number such that $\beta \notin C_1$. Using the notations of Theorem 1.1 we show that the assumption

$$\limsup_{n\to+\infty} \frac{s_n}{m_n} > 1 \tag{4.1}$$

leads to a contradiction.

Denote by $\mathbb{K}$ the algebraic number field $\mathbb{Q}(\beta)$, considered as a multivalued field with the product formula [C] [Sw] (see also [Lg]).

The present proof is merely an adaptation of that of Theorem 1 in [A], though the aims are different, and therefore does not merit publication. We simply point out a few hints for the interested reader.

The main result which is used is Corollary 1 of the Main Theorem in [C], as in [A]. This is a version of the Thue-Siegel-Roth Theorem given by Corvaja which is stronger than Roth Theorem for number fields [Le] [Sw] to the extent it allows us to introduce a *missing proportion of places* of $\mathbb{K}$ by considering the projective approximation of the point at infinity in $\mathbb{P}^1(\mathbb{K})$. Since β is a Salem number, it is a unit [B-S]. Hence, this missing proportion has just to be chosen among the pairwise distinct Archimedean places of $\mathbb{K}$.

5 On the class Q_0

5.1 Perron numbers

Let us give, after Solomyak ([So], p 483), the example of a Perron number which is not a beta-number and therefore which is not in the class C_2, without knowing whether it is in the class Q_0. This example allows us to estimate the sharpness of the upper bound $\log(\mathrm{M}(\beta))/\log(\beta)$ in (1.1). Recall that a real number $\beta > 1$ is a beta-number if the orbit of $x = 1$ under the transformation $T_\beta : x \to \beta x \pmod 1$ is finite [Lo] [PF]. The set of all conjugates of all beta-numbers is the union of the closed unit disc in the complex plane and the set of reciprocals of zeros of the function class $\{f(z) = 1 + \sum a_j z^j \mid 0 \le a_j \le 1\}$. The closure of this domain, say Φ, is compact and was studied by Flatto, Lagarias and Poonen [F-P] and Solomyak [So]. After [So], the Perron number $\beta = \frac{1}{2}(1+\sqrt{13})$, dominant root of $P_\beta(X) = X^2 - X - 3$, is not a beta-number, though its only conjugate $\beta' = \frac{1}{2}(1 - \sqrt{13})$ lies in the interior $\mathrm{int}(\Phi)$. We have $\mathrm{M}(\beta) = 3$. By Theorem 1.1 the "quotients of the gaps" are asymptotically bounded above by $\log(3)/\log(\beta) = 1.3171\ldots$, a much better bound than the degree $d = 2$ of β (see Lemma 5.1). This does not suffice to conclude that $\frac{1}{2}(1 + \sqrt{13})$ belongs to Q_0.

Do all Perron numbers belong to Q_0 ? Let $\beta > 1$ be a Perron number of degree $d \ge 2$ and denote by $\beta^{(1)}, \beta^{(2)}, \ldots, \beta^{(d-1)}$ the conjugates of $\beta = \beta^{(0)}$, roots of the minimal polynomial $P_\beta(X)$ of β. Let $K_\beta := \max\{|\beta^{(i)}| \mid i = 1, 2, \ldots, d-1\}$.

Lemma 5.1. *Let $n = n_\beta$ (with $2 \le n_\beta \le d$) be the number of conjugates of β of modulus strictly greater than unity (including β). Then*

$$\frac{\log(\mathrm{M}(\beta))}{\log(\beta)} \le n - \frac{n-1}{(d\beta)^{6d^3}\log\beta}. \tag{5.1}$$

Proof. Obvious since (Lemma 2 in [Li2]): $K_\beta < \beta\big(1 - \dfrac{1}{(d\beta)^{6d^3}}\big)$. □

The upper bound (5.1) does not allow us to give a positive answer to the question and has probably to be improved.

5.2 Transcendental numbers

Let us show that the Komornik-Loreti constant [KL] [AC] belongs to $\mathrm{Q}_0^{(1)}$.

Theorem 5.2. *There exists a smallest* $q \in (1,2)$ *for which there exists a unique expansion of* 1 *as* $1 = \sum_{n=1}^{\infty} \delta_n q^{-n}$ *, with* $\delta_n \in \{0,1\}$*. Furthermore, for this smallest* q*, the coefficient* δ_n *is equal to* 0 *(respectively,* 1*) if the sum of the binary digits of* n *is even (respectively, odd). This number* q *can then be obtained as the unique positive solution of* $1 = \sum_{n=1}^{\infty} \delta_n q^{-n}$*. It is equal to* $1.787231650\ldots$*.*

This constant q is named Komornik-Loreti constant. Allouche and Cosnard [AC] have shown the following result.

Theorem 5.3. *The constant* q *is a transcendental number, where the sequence of coefficients* $(\delta_n)_{n\geq 1}$ *is the Prouhet-Thue-Morse sequence on the alphabet* $\{0,1\}$*.*

The uniqueness of the development of 1 in base q given by Theorem 5.2 allows us to write

$$d_q(1) = 0.\delta_1\delta_2\delta_3\ldots,$$

the coefficients δ_n being the digits of the Rényi q-expansion of 1. Since the strings of zeros and 1's in the Prouhet-Thue-Morse sequence are known (Thue, 1906/1912; [AS]) and uniformly bounded, the constant q belongs to the class $\mathrm{Q}_0^{(1)}$.

As second example, let us show that Sturmian numbers in the interval $(1,2)$ (in the sense of [CK]) belong to $\mathrm{Q}_0^{(1)}$.

A real number $\beta > 1$ is called a Sturmian number if $d_\beta(1)$ is a Sturmian word over a binary alphabet $\{a,b\}$, with $0 \leq a < b = \lfloor \beta \rfloor$. Chi and Kwon [CK] have shown the following theorem.

Theorem 5.4. *Every Sturmian number is transcendental.*

Let us consider all the Sturmian numbers $\beta \in (1,2)$ for which the two-letter alphabet is $\{0,1\}$. For such numbers gappiness appears in $d_\beta(1)$ (in the sense of Theorem 1.1). By Theorem 3.3 in [CK] strings of zeros, resp. of 1's, cannot be arbitrarily long. This gives the claim.

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