

VALUATION DOMAINS WITH A MAXIMAL IMMEDIATE EXTENSION OF FINITE RANK.

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ABSTRACT. If R is a valuation domain of maximal ideal P with a maximal immediate extension of finite rank it is proven that there exists a finite sequence of prime ideals $P = L_0 \supset L_1 \supset \cdots \supset L_m \supseteq 0$ such that R_{L_j}/L_{j+1} is almost maximal for each j , $0 \leq j \leq m-1$ and R_{L_m} is maximal if $L_m \neq 0$. Then we suppose that there is an integer $n \geq 1$ such that each torsion-free R -module of finite rank is a direct sum of modules of rank at most n . By adapting Lady's methods, it is shown that $n \leq 3$ if R is almost maximal, and the converse holds if R has a maximal immediate extension of rank ≤ 2 .

Let R be a valuation domain of maximal ideal P , \widehat{R} a maximal immediate extension of R , \widetilde{R} the completion of R in the R -topology, and $Q, \widehat{Q}, \widetilde{Q}$ their respective fields of quotients. If L is a prime ideal of R , as in [5], we define the **total defect** at L , $d_R(L)$, the **completion defect** at L , $c_R(L)$, as the rank of the torsion-free R/L -module $\widehat{(R/L)}$ and the rank of the torsion-free R/L -module $\widetilde{(R/L)}$, respectively. Recall that a local ring R is **Henselian** if each indecomposable module-finite R -algebra is local and a valuation domain is **strongly discrete** if it has no non-zero idempotent prime ideal. The aim of this paper is to study valuation domains R for which $d_R(0) < \infty$. The first example of a such valuation domain was given by Nagata [11]; it is a Henselian rank-one discrete valuation domain of characteristic $p > 0$ for which $d_R(0) = p$. By using a generalization of Nagata's idea, Facchini and Zanardo gave other examples of characteristic $p > 0$, which are Henselian and strongly discrete. More precisely:

Example 0.1. [5, Example 6] *For each prime integer p and for each finite sequence of integers $\ell(0) = 1, \ell(1), \dots, \ell(m)$ there exists a strongly discrete valuation domain R with prime ideals $P = L_0 \supset L_1 \supset \cdots \supset L_m = 0$ such that $c_R(L_i) = p^{\ell(i)}$, $\forall i, 1 \leq i \leq m$.*

So, $d_R(0) = p^{(\sum_{i=1}^m \ell(i))}$ by [5, Corollary 4].

Theorem 0.2. [5, Theorem 8] *Let α be an ordinal number, $\ell : \alpha + 1 \rightarrow \mathbb{N} \cup \{\infty\}$ a mapping with $\ell(0) = 1$ and p a prime integer. Then there exists a strongly discrete valuation domain R and an antiisomorphism $\alpha + 1 \rightarrow \text{Spec}(R)$, $\lambda \mapsto L_\lambda$, such that $c_R(L_\lambda) = p^{\ell(\lambda)}$, $\forall \lambda \leq \alpha$.*

So, if $\ell(\lambda) = 0$, $\forall \lambda \leq \alpha$, except for a finite subset, then $d_R(0) < \infty$ by [5, Corollary 4].

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In 1990, Vámos gave a complete characterization of non-Henselian valuation domains with a finite total defect, and examples of such rings. His results are summarized in the following:

Theorem 0.3. [12, Theorem 5] *Let R be a non-Henselian valuation domain and assume that $d_R(0) < \infty$. Then one of the following holds:*

- (rc) $d_R(0) = 2$, R has characteristic zero, \widehat{Q} is algebraically closed and its cardinality $|\widehat{Q}|^{\aleph_0} = |\widehat{Q}|$. Further, Q is real-closed, the valuation on Q has exactly two extensions to \widehat{Q} and R is almost maximal.
- (y) There is a non-zero prime ideal L of R such that R_L is a maximal valuation ring, and R/P and its field of quotients satisfy (rc).

As Vámos, if R is a domain, we say that $fr(R) \leq n$ (respectively $fr^o(R) \leq n$) if every torsion-free module (respectively every submodule of a free module) of finite rank is a direct sum of modules of rank at most n . By [12, Theorem 3] $fr(R) \geq d_R(0)$ if R is a valuation domain. So, the study of valuation domains R for which $d_R(0) < \infty$ is motivated by the problem of the characterization of valuation domains R for which $fr(R) < \infty$.

When R is a valuation domain which is a \mathbb{Q} -algebra or not Henselian, then $fr(R) < \infty$ if and only if $fr(R) = d_R(0) \leq 2$ by [12, Theorem 10]. Moreover, if $fr(R) = 2$, either R is of type (rc) and $fr^o(R) = 1$ or R is of type (y) and $fr^o(R) = 2$. When R is a rank-one discrete valuation domain, then $fr(R) < \infty$ if and only if $fr(R) = d_R(0) \leq 3$ by [13, Theorem 8] and [1, Theorem 2.6].

In this paper we complete Vámos's results. In Section 1, a description of valuation domains with a finite total defect is given by Theorem 1.7 and Proposition 1.8. In Section 2 we give some precisions on the structure of torsion-free R -modules of finite rank when R satisfies a condition weaker than $d_R(0) < \infty$. In Section 3 we extend to every almost maximal valuation domain the methods used by Lady in [8] to study torsion-free modules over rank-one discrete valuation domains. If R is an almost maximal valuation domain, we prove that $d_R(0) \leq 3$ if $fr(R) < \infty$ and that $fr(R) = d_R(0)$ if $d_R(0) \leq 2$.

For definitions and general facts about valuation rings and their modules we refer to the books by Fuchs and Salce [6] and [7].

1. MAXIMAL IMMEDIATE EXTENSION OF FINITE RANK

We recall some preliminary results needed to prove Theorem 1.7 which gives a description of valuation domains with a finite total defect.

Let M be a non-zero module over a valuation domain R . As in [7, p.338] we set $M^\# = \{s \in R \mid sM \subset M\}$. Then $M^\#$ is a prime ideal of R and is called the **top prime ideal** associated with M .

Proposition 1.1. *Let A be a proper ideal of R and let L be a prime ideal such that $A^\# \subseteq L$ and A is not isomorphic to L . Then R/A is complete in its ideal topology if and only if R_L/A is also complete in its ideal topology.*

Proof. Let $(a_i + A_i)_{i \in I}$ be a family of cosets of R_L such that $a_i \in a_j + A_j$ if $A_i \subset A_j$ and such that $A = \bigcap_{i \in I} A_i$. We may assume that $A_i \subseteq L$, $\forall i \in I$. So, $a_i + L = a_j + L$, $\forall i, j \in I$. Let $b \in a_i + L$, $\forall i \in I$. It follows that $a_i - b \in L$, $\forall i \in I$. If R/A is complete in the R/A -topology, $\exists c \in R$ such that $c + b - a_i \in A_i$, $\forall i \in I$. Hence R_L/A is complete in the R_L/A -topology too.

Conversely let $(a_i + A_i)_{i \in I}$ be a family of cosets of R such that $a_i \in a_j + A_j$ if $A_i \subset A_j$ and such that $A = \bigcap_{i \in I} A_i$. We may assume that $A \subset A_i \subseteq L$, $\forall i \in I$. We put $A'_i = (A_i)_L$, $\forall i \in I$. We know that $A = \bigcap_{a \notin A} La$. Consequently, if $a \notin A$, there exists $i \in I$ such that $A_i \subseteq La$, whence $A'_i \subseteq La$. It follows that $A = \bigcap_{i \in I} A'_i$. Clearly, $a_i \in a_j + A'_j$ if $A'_i \subset A'_j$. Then there exists $c \in R_L$ such that $c \in a_i + A'_i$, $\forall i \in I$. Since $A'_i \subset R$, $\forall i \in I$, $c \in R$. From $A = \bigcap_{j \in I} A'_j$ and $A \subset A_i$, $\forall i \in I$ we deduce that $\forall i \in I$, $\exists j \in I$ such that $A'_j \subset A_i$. We get that $c \in a_i + A_i$ because $c - a_j \in A'_j \subseteq A_i$ and $a_j - a_i \in A_i$. So, R/A is complete in the R/A -topology. \square

Proposition 1.2. [7, Exercise II.6.4] *Let R be a valuation ring and let L be a non-zero prime ideal. Then R is (almost) maximal if and only if R/L is maximal and R_L is (almost) maximal.*

Proof. If R is (almost) maximal, it is obvious that R/L is maximal and by Proposition 1.1 R_L is (almost) maximal. Conversely let A be a non-zero ideal and $J = A^\#$. Suppose that either $J \subset L$ or $J = L$ and A is not isomorphic to L . Since R_L is (almost) maximal it follows that R_L/A is complete in its ideal topology. From Proposition 1.1 we deduce that R/A is complete in its ideal topology. Now, suppose that $L \subset J$. If $A \subset L$ let $t \in J \setminus L$. Thus $A \subset t^{-1}A$. Let $s \in t^{-1}A \setminus A$. Therefore $L \subset tR \subseteq s^{-1}A$. So, $R/s^{-1}A$ is complete in its ideal topology because R/L is maximal, whence R/A is complete too. Finally if $A \cong L$ the result is obvious. \square

Proposition 1.3. *Let $(L_\lambda)_{\lambda \in \Lambda}$ be a non-empty family of prime ideals of R and let $L = \bigcup_{\lambda \in \Lambda} L_\lambda$. Then L is prime, $R_L = \bigcap_{\lambda \in \Lambda} R_{L_\lambda}$ and R_L is maximal if and only if R_{L_λ} is maximal $\forall \lambda \in \Lambda$.*

Proof. It is obvious that L is prime. Let Q be the field of fractions of R . If $x \in Q \setminus R_L$ then $x = \frac{1}{s}$ where $s \in L$. Since $L = \bigcup_{\lambda \in \Lambda} L_\lambda$, $\exists \mu \in \Lambda$ such that $s \in L_\mu$. We deduce that $x \notin R_{L_\mu}$ and $R_L = \bigcap_{\lambda \in \Lambda} R_{L_\lambda}$.

If R_L is maximal, we deduce that R_{L_λ} is maximal $\forall \lambda \in \Lambda$ by Proposition 1.1. Conversely, by [14, proposition 4] R_L is linearly compact in the inverse limit topology. Since R_L is Hausdorff in this linear topology then every nonzero ideal is open and also closed. Hence R_L is linearly compact in the discrete topology. \square

Recall that a valuation domain R is **Archimedean** if its maximal P is the only non-zero prime ideal and an ideal A is **Archimedean** if $A^\# = P$.

Proposition 1.4. [3, Corollary 9] *Let R be an Archimedean valuation domain. If $d_R(0) < \infty$, then R is almost maximal.*

From Propositions 1.1, 1.2, 1.3 and 1.4 we deduce the following:

Proposition 1.5. *Let R be a valuation domain such that $d_R(0) < \infty$ and R/A is Hausdorff and complete in its ideal topology for each non-zero non-Archimedean ideal A . Then R is almost maximal.*

Proof. Let L, L' be prime ideals such that $L' \subset L$. Since $\widehat{(R_L)}$ is a summand of $\widehat{(R)}_L$ we have $d_{R_L}(0) \leq d_R(0)$. On the other hand, by tensoring a pure-composition series of $\widehat{(R_L)}$ with R_L/L' we get a pure-composition series of $\widehat{(R_L/L')}$. So, $d_{R_L}(L') \leq d_R(0)$.

If R is Archimedean the result follows from Proposition 1.4. Suppose that R is not Archimedean, let J be a non-zero ideal and let $(L_\lambda)_{\lambda \in \Lambda}$ be the family of prime ideals properly containing J and properly contained in P . If $\Lambda = \emptyset$ we get that R is almost maximal by applying Propositions 1.4 and 1.2. Else, let $L' = \cup_{\lambda \in \Lambda} L_\lambda$. By Proposition 1.3, $R_{L'}/J$ is maximal for each non-zero prime J . If $L' \neq P$ then R/L' is maximal by Proposition 1.4 and it follows that R/J is maximal by Proposition 1.2. If the intersection K of all non-zero primes is zero then R is almost maximal. If $K \neq 0$ then R_K is Archimedean. We conclude by using Propositions 1.4 and 1.2. \square

Given a ring R , an R -module M and $x \in M$, the **content ideal** $c(x)$ of x in M , is the intersection of all ideals A for which $x \in AM$. We say that M is a **content module** if $x \in c(x)M$, $\forall x \in M$.

Lemma 1.6. *Let U be a torsion-free module such that $U = PU$. Then:*

- (1) $\forall x \in U$, $x \neq 0$, $x \notin c(x)U$;
- (2) let $0 \neq x, y \in U$ and $t \in R$ such that $x = ty$. Then $c(y) = t^{-1}c(x)$;
- (3) if U is uniserial then, for each $x \in U$, $x \neq 0$, $c(x)^\# = U^\#$.

Proof. (1). If $x \in c(x)U$, there exist $a \in R$ and $z \in U$ such that $x = az$ and $c(x) = Ra$. But, since $z \in PU$, we get a contradiction.

(2). Let $0 \neq x, y \in U$ such that $x = ty$. If $s \notin c(y)$ then $x = tsz$ for some $z \in U$ and $st \notin c(x)$. So, $s \notin t^{-1}c(x)$. Conversely, if $s \notin t^{-1}c(x)$ then $st \notin c(x)$. We have $x = stz$ for some $z \in U$. We get that $y = sz$. So, $s \notin c(y)$.

(3). We put $A = c(x)$ and $L = A^\#$. Let $s \notin L$ and $y \in U$ such that $x = ty$ for some $t \in R$. Then $c(y) = t^{-1}A$ and $t \notin A$. So, $t^{-1}A \subseteq L$. Consequently $y \in sU$. Let $s \in L$. If $s \in A$ then $x \notin sU$. If $s \in L \setminus A$ let $t \in s^{-1}A \setminus A$. There exists $y \in U$ such that $x = ty$. Since $c(y) = t^{-1}A$ and $s \in t^{-1}A$ we deduce that $y \notin sU$. \square

This lemma and the previous proposition allow us to show the following theorem.

Theorem 1.7. *Let R be a valuation domain such that $d_R(0) < \infty$. Then there exists a finite family of prime ideals $P = L_0 \supset L_1 \supset \dots \supset L_{m-1} \supset L_m \supseteq 0$ such that R_{L_k}/L_{k+1} is almost maximal, $\forall k$, $0 \leq k \leq m-1$ and R_{L_m} is maximal if $L_m \neq 0$ (or equivalently, for each proper ideal $A \not\subseteq L_k$, $\forall k$, $0 \leq k \leq m$, R/A is Hausdorff and complete in its ideal topology). Moreover, $d_R(0) = \prod_{k=1}^{k=m} c_R(L_k)$.*

Proof. Let $n = d_R(0)$. Then \widehat{R} has a pure-composition series

$$0 = G_0 \subset R = G_1 \subset \dots \subset G_{n-1} \subset G_n = \widehat{R}$$

such that, $\forall k$, $1 \leq k \leq n$, $U_k = G_k/G_{k-1}$ is a uniserial torsion-free module. The family (L_0, \dots, L_m) is defined in the following way: $\forall j$, $0 \leq j \leq m$, there exists k , $1 \leq k \leq n$ such that $L_j = U_k^\#$.

Now, let A be a proper ideal such that R/A is Hausdorff and non-complete in its ideal topology. By [7, Lemma V.6.1] there exists $x \in \widehat{R} \setminus R$ such that $A = c(x+R)$ (Clearly $c(x+R) = B(x)$, the breadth ideal of x). Let U be a pure uniserial submodule of \widehat{R}/R containing $x+R$ and let M be the inverse image of U by the natural map $\widehat{R} \rightarrow \widehat{R}/R$. From the pure-composition series of M with factors R and U , and a pure-composition series of \widehat{R}/M we get a pure-composition series

of \widehat{R} . Since each pure composition series has isomorphic uniserial factors by [7, Theorem XV.1.7], it follows that $U \cong U_k$ for some k , $2 \leq k \leq n$. So, by Lemma 1.6 $A^\# = U^\# = U_k^\#$.

We apply Proposition 1.5 and deduce that R_{L_k}/L_{k+1} is almost maximal $\forall k$, $0 \leq k \leq m-1$ and R_{L_m} is maximal if $L_m \neq 0$.

To prove the last assertion we apply [5, Lemma 2] (The conclusion of this lemma holds if R_L/L' is almost maximal, where L and L' are prime ideals, $L' \subset L$). \square

The following completes the previous theorem.

Proposition 1.8. *Let R be a valuation domain such that $d_R(0) < \infty$, let $(U_k)_{1 \leq k \leq n}$ be the family of uniserial factors of all pure-composition series of \widehat{R} and let $(L_j)_{0 \leq j \leq m}$ be the family of prime ideals defined in Theorem 1.7. Then:*

- (1) $\forall k, 1 \leq k \leq n, U_k \cong R_{U_k^\#}$;
- (2) \widehat{R} has a pure-composition series

$$0 = F_0 \subset R = F_1 \subset \cdots \subset F_{m-1} \subset F_m = \widehat{R}$$

where F_{j+1}/F_j is a free R_{L_j} -module of finite rank, $\forall j, 0 \leq j \leq m-1$.

Proof. (1). Let A be an ideal such that $\exists j, 0 \leq j \leq m, A^\# = L_j$ and $A \not\cong L_j$. In the sequel we put $L_{m+1} = 0$ if $L_m \neq 0$.

First, for each uniserial torsion-free module U , we will show that each family $(x_r + rU)_{r \in R \setminus A}$ has a non-empty intersection if $x_r \in x_t + tU, \forall r, t \in R \setminus A, r \in tR$. As in the proof of Proposition 1.2 we may assume that $L_{j+1} \subset A$. Since R_{L_j}/L_{j+1} is almost maximal and A is an ideal of R_{L_j} the family $(x_r + rU_{L_j})_{r \in R \setminus A}$ has a non-empty intersection. If $r \in L_j \setminus A$, we have $r^{-1}A \subset L_j$. So, if $t \in L_j \setminus r^{-1}A$ then $rt \notin A$ and $rtU_{L_j} \subseteq rU$. It follows that we can do as in the proof of Proposition 1.1 to show that the family $(x_r + rU)_{r \in R \setminus A}$ has a non-empty intersection.

Let $0 = G_0 \subset R = G_1 \subset \cdots \subset G_{n-1} \subset G_n = \widehat{R}$ be a pure-composition series of \widehat{R} whose factors are the $U_k, 1 \leq k \leq n$. By induction on k and by using the pure-exact sequence $0 \rightarrow G_{k-1} \rightarrow G_k \rightarrow U_k \rightarrow 0$, we get that each family $(x_r + rG_k)_{r \in R \setminus A}$ for which $x_r \in x_t + tG_k, \forall r, t \in R \setminus A, r \in tR$, has a non-empty intersection.

Let $k, 2 \leq k \leq n$, be an integer, let $0 \neq x \in G_k \setminus G_{k-1}$ and let $A = c(x + G_{k-1})$. Then $A^\# = U_k^\# = L_j$ for some $j, 1 \leq j \leq m$. We shall prove that $A \cong L_j$. For each $r \in R \setminus A, x = g_r + ry_r$ for some $g_r \in G_{k-1}$ and $y_r \in G_k$. Let $r, t \in R \setminus A$ such that $r \in tR$. Then we get that $g_r \in g_t + tG_k \cap G_{k-1} = g_t + tG_{k-1}$ since G_{k-1} is a pure submodule. If $A \not\cong L_j$ the family $(g_r + rG_{k-1})_{r \in R \setminus A}$ has a non-empty intersection. Let $g \in g_r + rG_{k-1} \forall r \in R \setminus A$. Then $(x - g) \in rG_k, \forall r \in R \setminus A$. Since G_k is a pure-essential extension of a free module, G_k is a content module by [4, Proposition 23]. It follows that $(x - g) \in AG_k$. So $x + G_{k-1} \in AU_k$. But, since $k \geq 2, U_k = PU_k$ because $\widehat{R}/P\widehat{R} \cong R/PR$. So, $x + G_{k-1} \notin AU_k$. From this contradiction we get that $A = sL_j$ for some $0 \neq s \in R$. If $sL_j \neq L_j$ then $x + G_{k-1} = sy + G_{k-1}$ for some $y \in G_k$ because $s \notin A$. It follows that $c(y + G_{k-1}) = L_j$. We put $y' = y + G_{k-1}$. Then, for each $z \in U_k \setminus Ry'$ there exists $t \in R \setminus L_j$ such that $y' = tz$. We get that $U_k = R_{L_j}y'$.

(2). Let $M = \widehat{R}/R$. Then $L_1 = M^\#$. From above we get that $M/L_1M \neq 0$. By [4, Proposition 21] M contains a pure free R_{L_1} -submodule N such that $N/L_1N \cong$

M/L_1M . It follows that $(M/N)^\# = L_2$. We set F_2 the inverse image of N by the natural map $\widehat{R} \rightarrow M$. We complete the proof by induction on j . \square

2. TORSION-FREE MODULES OF FINITE RANK.

In this section we give some precisions on the structure of torsion-free R -modules of finite rank when R satisfies a condition weaker than $d_R(0) < \infty$. The following lemmas are needed.

Lemma 2.1. *Let R be a valuation ring (possibly with zerodivisors), let U be a uniserial module and let L be a prime ideal such that $L \subset U^\#$. Then U_L is a cyclic R_L -module.*

Proof. Let $s \in U^\# \setminus L$ and let $x \in U \setminus sU$. Let $y \in U \setminus Rx$. There exists $t \in P$ such that $x = ty$. Then $t \notin Rs$, whence $t \notin L$. It follows that $U_L = R_Lx$. \square

Lemma 2.2. *Let R be a valuation ring (possibly with zerodivisors), let U and V be uniserial modules such that $V^\# \subset U^\#$. Assume that U_L is a faithful R_L -module, where $L = V^\#$. Then $\text{Ext}_R^1(U, V) = 0$.*

Proof. Let M be an extension of V by U . By lemma 2.1 U_L is a free cyclic R_L -module. Since V is a module over R_L , it follows that V is a summand of M_L . We deduce that V is a summand of M too. \square

Lemma 2.3. *Let R be a valuation domain for which there exists a prime ideal $L \neq P$ such that R/L is almost maximal. Then $\text{Ext}_R^1(U, V) = 0$ for each pair of ideals U and V such that $L \subset U^\# \cap V^\#$.*

Proof. Let M be an extension of V by U . It is easy to check that U/LU and V/LV are non-zero and non divisible R/L -modules. Since R/L is almost maximal $M/LM \cong U/LU \oplus V/LV$ by [6, Proposition VI.5.4]. If $L \neq 0$, it follows that there exist two submodules H_1 and H_2 of M , containing LM , such that $H_1/LM \cong U/LU$ and $H_2/LM \cong V/LV$. For $i = 1, 2$ let $x_i \in H_i \setminus LM$ and let A_i be the submodule of H_i such that A_i/Rx_i is the torsion submodule of H_i/Rx_i . Then $A_i + LM/LM$ is a non-zero pure submodule of H_i/LM which is of rank one over R/L . It follows that $H_i = A_i + LM$. By Lemma 2.1 $M_L \cong V_L \oplus U_L$. We deduce that $LM \cong LM_L$ is a direct sum of uniserial modules. Since $A_i \cap LM$ is a non-zero pure submodule of LM there exists a submodule C_i of LM such that $LM = (A_i \cap LM) \oplus C_i$ by [7, Theorem XII.2.2]. It is easy to check that $H_i = A_i \oplus C_i$. From $M = H_1 + H_2$ and $LM = H_1 \cap H_2$ we deduce that the following sequence is pure exact:

$$0 \rightarrow LM \rightarrow H_1 \oplus H_2 \rightarrow M \rightarrow 0,$$

where the homomorphism from LM is given by $x \mapsto (x, -x)$, $x \in LM$, and the one onto M by $(x, y) \mapsto x + y$, $x \in H_1$, $y \in H_2$. Since $H_1 \oplus H_2$ is a direct sum of uniserial modules, so is M by [7, Theorem XII.2.2]. Consequently $M \cong V \oplus U$. \square

Proposition 2.4. *Let R be a valuation domain. Let G be a torsion-free R -module of finite rank. Then G has a pure-composition series with uniserial factors $(U_k)_{1 \leq k \leq n}$ such that $U_k^\# \supseteq U_{k+1}^\#$, $\forall k$, $1 \leq k \leq n - 1$.*

Proof. G has a pure-composition series

$$0 = H_0 \subset H_1 \subset \cdots \subset H_{n-1} \subset H_n = G$$

such that, $\forall k, 1 \leq k \leq n, V_k = H_k/H_{k-1}$ is a uniserial torsion-free module. Suppose there exists $k, 1 \leq k \leq (n-1)$ such that $V_k^\# \subset V_{k+1}^\#$. By Lemma 2.2, $H_{k+1}/H_{k-1} \cong V_k \oplus V_{k+1}$. So, if H'_k is the inverse image of V_{k+1} by the surjection $H_{k+1} \rightarrow H_{k+1}/H_{k-1}$, if $U_k = H'_k/H_{k-1}$ and $U_{k+1} = H_{k+1}/H'_k$ then $U_k^\# \supset U_{k+1}^\#$. So, in a finite number of similar steps, we get a pure composition series with the required property. \square

Proposition 2.5. *Let R be a valuation domain. Assume that there exists a finite family of prime ideals $P = L_0 \supset L_1 \supset \dots \supset L_{m-1} \supset L_m = 0$ such that R_{L_k}/L_{k+1} is almost maximal $\forall k, 0 \leq k \leq m-1$. Let G be a torsion-free R -module of finite rank. Then G has a pure-composition series*

$$0 = G_0 \subseteq G_1 \subseteq \dots \subseteq G_{m-1} \subseteq G_m \subseteq G_{m+1} = G$$

where G_{j+1}/G_j is a finite direct sum of ideals of $R_{L_j}, \forall j, 0 \leq j \leq m$.

Proof. By Proposition 2.4 G has a pure-composition series

$$0 = H_0 \subset H_1 \subset \dots \subset H_{n-1} \subset H_n = G$$

such that, $\forall k, 1 \leq k \leq n, U_k = H_k/H_{k-1}$ is a uniserial torsion-free module and $U_k^\# \supseteq U_{k+1}^\#, \forall k, 1 \leq k \leq n-1$. Now, for each $j, 1 \leq j \leq m$, let k_j be the greatest index such that $L_j \subset U_{k_j}^\#$. We put $G_j = H_{k_j}$. Then G_{j+1}/G_j is an R_{L_j} -module which is a direct sum of ideals by Lemma 2.3. \square

3. VALUATION DOMAINS R WITH $fr(R) < \infty$.

First we extend to every almost maximal valuation domain the methods used by Lady in [8] to study torsion-free modules over rank-one discrete valuation domains. So, except in Theorem 3.6, we assume that R is an almost maximal valuation ring. We put $K = Q/R$. For each R -module $M, d(M)$ is the divisible submodule of M which is the union of all divisible submodules and M is said to be **reduced** if $d(M) = 0$. We denote by \widehat{M} the **pure-injective hull** of M (see [7, chapter XIII]). If U is a uniserial module then $\widehat{U} \cong \widehat{R} \otimes_R U$ because R is almost maximal. Let G be a torsion-free module of finite rank r . By Proposition 2.5 G contains a submodule B which is a direct sum of ideals and such that G/B is a Q -vector space. We put **corank** $G = \text{rank } G/B$. Now, it is easy to prove the following.

Proposition 3.1. *Let G be a torsion-free R -module of rank r and corank c . Then:*

- (1) G contains a pure direct sum B of ideals, of rank $r - c$, such that G/B is a Q -vector space of dimension c .
- (2) G contains a pure direct sum B' of ideals, of rank r such that G/B' is isomorphic to a quotient of K^c .

An element of $Q \otimes_R \text{Hom}_R(G, H)$ is called a **quasi-homomorphism** from G to H , where G and H are R -modules. Let \mathcal{C}_{ab} be the category having **weakly polyserial** R -modules (i.e modules with composition series whose factors are uniserial) as objects and quasi-homomorphisms as morphisms and let \mathcal{C} be full subcategory of \mathcal{C}_{ab} having torsion-free R -modules of finite rank as objects. Then \mathcal{C}_{ab} is abelian by [7, Lemma XII.1.1]. If G and H are torsion-free of finite rank, then the quasi-homomorphisms from G to H can be identified with the Q -linear maps $\phi : Q \otimes_R G \rightarrow Q \otimes_R H$ such that $r\phi(G) \subseteq H$ for some $0 \neq r \in R$. We say that G and

H are **quasi-isomorphic** if they are isomorphic objects of \mathcal{C} . A torsion-free module of finite rank is said to be **strongly indecomposable** if it is an indecomposable object of \mathcal{C} .

Lemma 3.2. *Let $\phi : Q \otimes_R G \rightarrow Q \otimes_R H$ be a Q -linear map. Then ϕ is a quasi-homomorphism if and only if $(\widehat{R} \otimes_R \phi)(d(\widehat{R} \otimes_R G)) \subset d(\widehat{R} \otimes_R H)$.*

Proof. Assume that ϕ is a quasi-homomorphism. There exists $0 \neq r \in R$ such that $r\phi(G) \subseteq H$. It successively follows that $r(\widehat{R} \otimes_R \phi)(\widehat{R} \otimes_R G) \subseteq \widehat{R} \otimes_R H$ and $(\widehat{R} \otimes_R \phi)(d(\widehat{R} \otimes_R G)) \subseteq d(\widehat{R} \otimes_R H)$.

Conversely, let B be a finite direct sum of ideals which satisfies that G/B is a Q -vector space. There exists a free submodule F of $Q \otimes_R G$ such that $B \subseteq F$. So, $\exists 0 \neq r \in R$ such that $r\phi(B) \subseteq r\phi(F) \subseteq H$. Since $\widehat{R} \otimes_R G = (\widehat{R} \otimes_R B) \oplus (d(\widehat{R} \otimes_R G))$, it follows that $r(\widehat{R} \otimes_R \phi)(\widehat{R} \otimes_R G) \subseteq (\widehat{R} \otimes_R H)$. We deduce that $r\phi(G) \subseteq (\widehat{R} \otimes_R H) \cap (Q \otimes_R H) = H$. \square

Proposition 3.3. *Let G be a torsion-free R -module of rank r and corank c .*

- (1) *If G has no summand isomorphic to an ideal, then $\text{End}(G)$ can be embedded in the ring of $c \times c$ matrices over \widehat{Q} . In particular if $c = 1$, $\text{End}(G)$ is a commutative integral domain.*
- (2) *If G is reduced, then $\text{End}(G)$ can be embedded in the ring of $(r-c) \times (r-c)$ matrices over \widehat{R} . In particular if $c = r-1$, $\text{End}(G)$ is a commutative integral domain.*

Proof. See the proof of [8, Theorem 3.1]. \square

In the sequel we assume that $n = c_R(0) < \infty$. So, there are $n-1$ units π_2, \dots, π_n in $\widehat{R} \setminus R$ such that $1, \pi_2, \dots, \pi_n$ is a basis of \widehat{Q} over Q . By [7, Theorem XV.6.3] there exists an indecomposable torsion-free R -module E with rank n and corank 1. We can define E in the following way: if $(e_k)_{2 \leq k \leq n}$ is the canonical basis of \widehat{R}^{n-1} , if $e_1 = \sum_{k=2}^{k=n} \pi_k e_k$ and V is the Q -vector subspace of \widehat{Q}^{n-1} generated by $(e_k)_{1 \leq k \leq n}$, then $E = V \cap \widehat{R}^{n-1}$. Then a basis element for $d(\widehat{R} \otimes E)$ can be written $u_1 + \pi_2 u_2 + \dots + \pi_n u_n$, where $u_1, \dots, u_n \in E$. Since E is indecomposable it follows that u_1, \dots, u_n is a basis for $Q \otimes E \cong V$.

Theorem 3.4. *Let G be a torsion-free R -module of rank r and corank c . Then the following assertions hold:*

- (1) *The reduced quotient of G is isomorphic to a pure submodule of \widehat{B} where B is a direct sum of $(r-c)$ ideals.*
- (2) *G is the direct sum of ideals of R with a quasi-homomorphic image of E^c .*

Proof. (1) can be shown as the implication (1) \Rightarrow (2) of [8, Theorem 4.1] and (2) as the implication (1) \Rightarrow (3) of [8, Theorem 4.1]. \square

Corollary 3.5. *Let G be a torsion-free R -module of rank r and corank c . Then:*

- (1) *If G has no summand isomorphic to an ideal, then $r \leq nc$.*
- (2) *If G is reduced, then $nc \leq (n-1)r$.*

Proof. This corollary is a consequence of Theorem 3.4 and can be shown as [8, Corollary 4.2]. \square

Theorem 3.6. *Let R be a valuation domain such that $d_R(0) = 2$. Then $fr(R) = 2$. Moreover $fr^o(R) = 1$ if $c_R(0) = 2$ and $fr^o(R) = 2$ if $c_R(0) = 1$.*

Proof. First suppose that $c_R(0) = 2$. So, R is almost maximal and $fr^o(R) = 1$. Let G be an indecomposable torsion-free module with rank r and corank c which is not isomorphic to Q and to an ideal. Then G is reduced and has no summand isomorphic to an ideal of R . From Corollary 3.5 we deduce that $r = 2c$. By Theorem 3.4 G is isomorphic to a pure submodule of \widehat{B} where B is a direct sum of c ideals. Since rank $\widehat{B} = 2c$ it follows that $G \cong \widehat{B}$. So, $c = 1$ and $G \cong \widehat{A}$ for a non-zero ideal A .

If $c_R(0) = 1$ let L be the non-zero prime ideal such that $c_R(L) = d_{R/L}(0) = 2$. Then $fr(R/L) = 2$. Since R_L is maximal it follows that $fr(R) = fr^o(R) = 2$ by [12, Lemma 9 and Lemma 4]. \square

Lemma 3.7. *Every proper subobject of E in \mathcal{C} is a direct sum of ideals.*

Proof. Let G be a proper object of E in \mathcal{C} and let H be the pure submodule of E such that H/G is the torsion submodule of E/G . Since E is indecomposable, E has no summand isomorphic to a direct sum of ideals. So, corank $E/H = 1$ and corank $H = 0$. As corank $H \geq$ corank G we get that G is a direct sum of ideals. \square

Proposition 3.8. *E is an indecomposable projective object of \mathcal{C} .*

Proof. Let $\phi : H \rightarrow E$ be a quasi-epimorphism where H is a torsion-free module of finite rank. Suppose that $H = F \oplus G$ where F is a direct sum of ideals. By Lemma 3.7, $\phi(G)$ is quasi-isomorphic to E . So, we may assume that H has no summand isomorphic to an ideal. By Theorem 3.4 there is a quasi-epimorphism $\psi : E^c \rightarrow H$ where $c =$ corank H . It is sufficient to see that $\phi \circ \psi$ is a split epimorphism in \mathcal{C} . But by Proposition 3.3(1), $Q \otimes \text{End}(E)$ is a subfield of \widehat{Q} , so every quasi-homomorphism $E \rightarrow E$ is either a quasi-isomorphism or trivial and the splitting follows immediately. \square

In the sequel, $Q \otimes_R \text{Hom}_R(R \oplus E, M)$ is denoted by \ddot{M} for each R -module M and the ring $Q \otimes_R \text{End}_R(R \oplus E)$ by Λ .

Theorem 3.9. *The foncteur $Q \otimes_R \text{Hom}_R(R \oplus E, \)$ is an exact fully faithful functor from \mathcal{C} into $\text{mod-}\Lambda$, the category of finitely generated right Λ -modules.*

Proof. By Theorem 3.4 and Proposition 3.8, $R \oplus E$ is a progenerator of \mathcal{C} . For each finite rank torsion-free R -module H , the natural map $Q \otimes_R \text{Hom}_R(R \oplus E, H) \rightarrow \text{Hom}_\Lambda(\ddot{R} \oplus \ddot{E}, \ddot{H})$ is an isomorphism because $\Lambda = \ddot{R} \oplus \ddot{E}$. Thus $Q \otimes_R \text{Hom}_R(F, H) \rightarrow \text{Hom}_\Lambda(\ddot{F}, \ddot{H})$ is an isomorphism if F is a summand of a finite direct sum of modules isomorphic to $R \oplus E$. Let G be a finite rank torsion-free R -module. We may assume that G has no summand isomorphic to an ideal of R . By Proposition 3.8 and Lemma 3.7, there is an exact sequence $0 \rightarrow R^{nc-r} \rightarrow E^c \rightarrow G \rightarrow 0$ in \mathcal{C} . Since both functors are left exact, we get that $Q \otimes_R \text{Hom}_R(G, H) \cong \text{Hom}_\Lambda(\ddot{G}, \ddot{H})$. \square

Lemma 3.10. *If M is a right Λ -module and $M \subseteq \ddot{G}$ for some finite rank torsion-free R -module G , then $M \cong \ddot{H}$ for some torsion-free R -module H .*

Proof. See the proof of [8, Lemma 5.2]. \square

Proposition 3.11. *The ring Λ is a hereditary Artinian Q -algebra such that $(\text{rad } \Lambda)^2 = 0$. There are two simple right Λ -modules, \tilde{R} which is projective and \tilde{K} which is injective.*

Proof. See the proof of [8, Proposition 5.3]. □

Proposition 3.12. *\tilde{Q} is an injective hull for \tilde{R} and \tilde{E} is a projective cover for \tilde{K} .*

Proof. See the proof of [8, Proposition 5.4]. □

Theorem 3.13. *The image of \mathcal{C} under the functor $Q \otimes_R \text{Hom}_R(R \oplus E, _)$ is the full subcategory of $\text{mod-}\Lambda$ consisting of modules with no summand isomorphic to \tilde{K} .*

Proof. See the proof of [8, Theorem 5.5]. □

Let M be a finitely generated (i.e., finite length) right Λ -module. We define **rank** M to be the number of factors in a composition series for M isomorphic to \tilde{R} and **corank** M to be the number of composition factors isomorphic to \tilde{K} .

Proposition 3.14. *The foncteur $Q \otimes_R \text{Hom}_R(R \oplus E, _)$ preserves rank and corank.*

Proof. See the proof of [8, Proposition 5.6]. □

We now consider the functors $\mathbf{D} = \text{Hom}_R(_, Q)$ and $\mathbf{Tr} = \text{Ext}_\Lambda(_, \Lambda)$ which take right Λ -modules to left Λ -modules and conversely. It is well known that \mathbf{D} is an exact contravariant length preserving functor taking projectives to injectives and conversely, and that \mathbf{D}^2 is the identity for finitely generated Λ -modules. Since Λ is hereditary, \mathbf{Tr} is right exact and $\mathbf{Tr}^2 M \cong M$ if M has no projective summand, $\mathbf{Tr} M = 0$ if M is projective. We consider the Coxeter functors $C^+ = \mathbf{DTr}$ and $C^- = \mathbf{TrD}$. Thus $C^+ : \text{mod-}\Lambda \rightarrow \text{mod-}\Lambda$ is left exact and $C^- : \text{mod-}\Lambda \rightarrow \text{mod-}\Lambda$ is right exact. If M has no projective (respectively injective) summand, it is easy to check that M is indecomposable if and only if $C^+ M$ (respectively $C^- M$) is indecomposable.

Proposition 3.15. *Let M be a right Λ -module with rank r and corank c .*

- (1) *If M has no projective summand, then $\text{corank } C^+ M = (n - 1)c - r$ and $\text{rank } C^+ M = nc - r$.*
- (2) *If M has no injective summand, then $\text{rank } C^- M = (n - 1)r - nc$ and $\text{corank } C^- M = r - c$.*

Proof. See the proof of [8, Proposition 5.7]. □

Theorem 3.16. *The following assertions hold:*

- (1) *If $n = 3$, then, up to quasi-homomorphism, the strongly indecomposable torsion-free R -modules are R , Q , E , \tilde{R} and an R -module with rank 2 and corank 1 (corresponding to $C^+ \tilde{Q} = C^- \tilde{R}$).*
- (2) *If $n \geq 4$, there are strongly indecomposable torsion-free R -modules with arbitrarily large rank.*

Proof. We show (1) and (2) as Lady in the proof of [8, Theorem 5.11] by using Proposition 3.15 and [10, Theorem 2]. □

4. SOME OTHER RESULTS AND OPEN QUESTIONS.

Let G be a finite rank torsion-free module over an almost maximal valuation domain R . A **splitting field** for G is a subfield Q' of \widehat{Q} containing Q such that $(Q' \cap \widehat{R}) \otimes_R G$ is a completely decomposable $(Q' \cap \widehat{R})$ -module (i.e. a direct sum of rank one modules). If Q' is a splitting field for G , G is called **Q' -decomposable**. By [9, Theorem 7], each finite rank torsion-free module G has a unique minimal splitting field Q' and $[Q' : Q] < \infty$. So, Lady's results on splitting fields of torsion-free modules of finite rank over rank one discrete valuation domains can be extended to almost maximal valuation domains by replacing \widehat{Q} by Q' in the previous section and by taking \mathcal{C} to be the category whose objects are the finite rank torsion-free Q' -decomposable modules.

Now, R is a valuation domain which is not necessarily almost maximal. We say that an R -module G is **strongly flat** if it is an extension of a free module by a divisible torsion-free module (see [2]). By [6, Lemma V.1.1 and Proposition V.1.2] \widetilde{Q} is a splitting field for each finite rank strongly flat module. So, each finite rank strongly flat module G has a unique minimal splitting field $Q' \subseteq \widetilde{Q}$ and $[Q' : Q] < \infty$. We also can extended Lady's results. In particular:

Theorem 4.1. *Let R be a valuation domain. We consider the following conditions:*

- (1) $\exists l \geq 1$ such that each finite rank strongly flat module is a direct sum of modules of rank at most l ;
- (2) $c_R(0) \leq 3$.

Then (1) \Rightarrow (2). If $c_R(0) \leq 2$ then (1) holds and $l = c_R(0)$.

If $c_R(0) = 3$, it is possible that the proof of [1, Theorem 2.6] should be generalized.

Proof. (1) \Rightarrow (2). We show that $c_R(0) \leq \infty$ in the same way that the condition (a) of [12, Theorem 3] is proven. Then, as above, we use Lady's methods to get $c_R(0) \leq 3$.

If $c_R(0) = 2$ we do as in the proof of Theorem 3.6. □

Some open questions:

- (1) Does henselian valuation domains with finite total defect, which are not strongly discrete, exist?
- (2) For a valuation domain R , does the condition $d_R(0) = 3$ imply $fr(R) = 3$ or $< \infty$? It is possible that the proof of [1, Theorem 2.6] should be generalized.
- (3) For a valuation domain R with finite total defect, does the condition $c_R(L_i) = p$, $\forall i$, $1 \leq i \leq m$, where $p = 2$ or 3 and $(L_i)_{1 \leq i \leq m}$ is the family of prime ideals defined in Theorem 1.7, imply $fr(R) < \infty$ for some $m > 1$? (If $m > 1$, by [12, Theorem 10], R is Henselian and p is the characteristic of its residue field.)

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