Soliton solutions to systems of coupled Schrödinger equations of Hamiltonian type

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1 Introduction

A major role in quantum physics is played by the nonlinear Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta_x\psi + V(x)\psi - \bar{f}(x,\psi), \tag{1.1}$$

where m and $\hbar$ are positive constants, the wave $\psi : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{C}$, $N \geq 3$, V is a potential which is bounded below, and $\bar{f} = f(x, |\psi|)\psi$ is a nonlinear function, for instance in the classical cubic approximation $\bar{f} = |\psi|^2\psi$. One of the questions to which huge attention has been given during the last twenty years is the existence of stationary states (see (1.2) below) for small values of $\hbar$, which appear due to the geometry of the potential.

This paper is devoted to the corresponding question of existence of solutions of some *systems* of Schrödinger equations. Systems of nonlinear Schrödinger type have been widely used in the applied sciences but mathematical study of standing wave solutions was undertaken only very recently, prompted in particular by the discovery of the importance of these systems as models in nonlinear optics (see for instance [4], [9], [26]) and in the study of Bose-Einstein condensates (see [26], [39]). As in the large majority of other papers on the subject we consider here systems of two equations.

So we suppose ψ is a vector function, $\psi = (\psi_1, \psi_2)$, and satisfies a system of equations like (1.1), with $\bar{f} = (\bar{f}_1, \bar{f}_2)$ and $\bar{f}_k = \sum_j f_{kj}(x, |\psi_1|, |\psi_2|)\psi_j$. We will be interested in soliton (standing wave) solutions of these systems, that is, solutions in the form

$$\psi_j(t,x) = e^{i(E/\hbar)t}u_j(x). \tag{1.2}$$

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Substituting (1.2) into (1.1) and setting $b(x) = V(x) - E$ leads to the system of real elliptic partial differential equations (we write $u = u_1, v = u_2$)

$$(S_\hbar) \qquad \begin{cases} -\hbar^2 \Delta u + b(x)u = f_1(x, u, v) & \text{in } \mathbb{R}^N, \\ -\hbar^2 \Delta v + b(x)v = f_2(x, u, v) & \text{in } \mathbb{R}^N. \end{cases}$$

We suppose this system is in variational form, that is, it is the Euler-Lagrange system of some energy functional. This happens when f_1, f_2 are the derivatives of some function $H(x, u, v)$. There are two types of such systems, Lagrangian - when $f_1 = H_u,\ f_2 = H_v$ (the above mentioned examples are of this type), and Hamiltonian - when $f_1 = H_v,\ f_2 = H_u$. The simplest example of a Hamiltonian system is the widely studied Lane-Emden system - when $f_1 = v^p,\ f_2 = u^q$ in $(S_\hbar)$. Even for this system important open questions subsist (see [15]). Hamiltonian systems are very usual in biology, more specifically in models in population dynamics (see [27]) whose stationary states verify systems of type $(S_\hbar)$, for instance with $f_1 = v(u^2 + g_1(v))$, $f_2 = u(v^2 + g_2(u))$. An important difficulty in the study of Hamiltonian systems (as opposed to Lagrangian) is the fact that the energy functional is strongly indefinite, that is, its leading part is respectively coercive and anti-coercive on infinitely dimensional subspaces of the energy space - we refer to [3] for a general discussion. The present article is devoted to this case. Our goal is to get a general existence result for small $\hbar$ in the case of a superlinear and subcritical Hamiltonian system with a well potential.

As in many applications, we consider trapping (or "well"-type) potentials, the standard example being $b(x) \sim |x - x_0|^2$ in a neighbourhood of some $x_0 \in \mathbb{R}^N$. A particular case of our result will be the existence of soliton waves thanks to a global well structure of b, that is,

$$0 = \inf_{x \in \mathbb{R}^N} b(x) < \liminf_{|x| \to \infty} b(x). \tag{1.3}$$

Notice that $\inf_{x \in \mathbb{R}^N} b(x) = 0$ can always be achieved through the choice of E in (1.2).

Unfortunately, as of today PDE theory lacks the means to tackle the existence question under hypothesis (1.3) only, even in the scalar case. However, it turns that we can show that $(S_\hbar)$ has a solution provided the constant $\hbar$ is *sufficiently small*. Note that in practice $\hbar$, the Planck constant, is a very small quantity, so it makes sense to study problem $(S_\hbar)$ at the limit $\hbar \to 0$.

Here are the precise statements. We assume $H(x, u, v)$ is differentiable and strictly convex in $(u, v) \in \mathbb{R}^2$ for all $x \in \mathbb{R}^N$, $H(x, 0, 0) = 0$ and

(H1) there exist constants $p, q, \alpha_k, \beta_k > 1$, such that

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N}, \qquad \frac{\alpha_k}{p+1} + \frac{\beta_k}{q+1} = 1, \tag{1.4}$$

and for some $c_0, d_0 > 0$, $C_k \geq 0$, $D_k \geq 0$ we have for $x \in \mathbb{R}^N$,$(u,v) \in \mathbb{R}^2$

$$c_0|u|^q \leq |H_u(x,u,v)| \leq C_0|u|^q + \sum_{k=1}^m C_k|u|^{\alpha_k-1}|v|^{\beta_k},$$
$$d_0|v|^p \leq |H_v(x,u,v)| \leq D_0|v|^p + \sum_{k=1}^m D_k|u|^{\alpha_k}|v|^{\beta_k-1}.$$

(H2) There exists $\alpha > 2$ such that for all $x \in \mathbb{R}^N$ and $(u,v) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$$uH_u(x,u,v) + vH_v(x,u,v) \geq \alpha H(x,u,v) > 0.$$

A typical example of a function satisfying these hypotheses is $H(x,u,v) = a_0(x)|u|^{p+1} + \sum_1^n a_i(x)|u|^{\alpha_i}|v|^{\beta_i} + a_{n+1}(x)|v|^{q+1}$, under (1.4).

We suppose that the continuous potential $b(x)$ satisfies $b \geq 0$ in $\mathbb{R}^N$ and

(b1) there exists $x_0 \in \mathbb{R}^N$ (say $x_0 = 0$) such that $b(x_0) = 0$;

(b2) there exists $A > 0$ such that the level set $G_A = \{x \in \mathbb{R}^N : b(x) < A\}$ has finite Lebesgue measure.

Note that the conditions (b1)-(b2) include (1.3) as a particular case. We shall also suppose that $b(x)$ is bounded. This condition is made for simplicity, since it is irrelevant to the goal of our paper, which is to use the well geometry of the potential. Actually it is even easier to consider potentials which are large at infinity (then there is no restriction on $\hbar$), since the energy space embeds compactly into Lebesgue spaces, see for instance Theorem 4 in [37].

Note also that (H1) means the problem is superlinear and subcritical, in other words, the couple (p,q) is under the critical hyperbola (given by the inequality (1.4)). In particular, one of the nonlinearities in $(S_\hbar)$ can have growth larger than the exponent $(N+2)/(N-2)$, provided the growth of the other is smaller enough to compensate (note that when $p = q$ (1.4) reduces to $p < (N+2)/(N-2)$). In this case the functional associated to $(S_\hbar)$ is not defined for $u, v \in H^1(\mathbb{R}^N)$. It is nowadays well-known that (1.4) is the right notion of subcriticality for a Hamiltonian system with power-growth nonlinearity, see [7], [21], [35], [36].

The following theorem contains our main result.

Theorem 1 *If $f_1 = H_v$, $f_2 = H_u$, and (H1)-(H2), (b1)-(b2) are satisfied then $(S_\hbar)$ has a nontrivial solution for small $\hbar$.*

We now quote previous works related to this result. There is a huge literature for the scalar case – we refer to [2], [5], [10], [14], [17], [19], [20], [23], [28], [29], [32], [38], [41] and to the references in these papers. Some types of Lagrangian systems with well potentials were studied in [1], [26], [30]. Existence results (for any $\hbar$) for radially invariant Hamiltonian systems in $\mathbb{R}^N$ were established in [16] and [37]. A result similar to Theorem 1 can be found in [33] (see also [34]) in the particular case when $H = F(u) + G(v)$, that is, the right-hand side of the system is independent of x and has no cross-terms in u, v. This restrictive hypothesis is due to the method used in these papers, which extends to systems the arguments in [14]. Finally, in the recent paper [11] a fairly general result was proved on system $(S_\hbar)$, but under the hypothesis that *both* p, q are smaller than (or in some cases equal to) the scalar exponent $(N + 2)/(N - 2)$. The method in [11] is based on an application of a linking theorem to the energy functional associated to $(S_\hbar)$.

The starting point for our work is [38], where the scalar version of Theorem 1 was proved. The method in [38] extends readily to Lagrangian systems, since then the energy functional has the same geometry as the scalar one, but the situation appears to be considerably more involved for Hamiltonian systems. We have used a dual variational structure, relying on the Legendre–Fenchel transformation, which allows us to transform the problem into a new one, to which the Mountain Pass Theorem (without the Palais-Smale condition) applies. However, then one of the key observations – that the generalized mountain pass value tends to zero as $\hbar \to 0$ – turns out to be rather delicate to prove, and the method of proof in [38] fails. We have found a way to deal with this problem by Fourier analysis, a tool that is seldom encountered in this branch of the calculus of variations. Our method will hopefully be useful in other situations as well.

So the main interest of Theorem 1 is twofold – first, it extends and joins together previous existence results of this type, giving an optimal range for the growth of the nonlinearities involved ; and second, its proof is based on a new idea, namely the use of Fourier transforms in the study of the behaviour of generalized critical values.

We finally remark that in the scalar case it has recently been established that standing wave solutions of (1.1) can be shown to exist for nonlinearities which grow supercritically - see [6], [12], [13]. In the light of these results, we expect that our hypotheses on the growth of f_1, f_2 at infinity can be relaxed, at least for some type of nonlinearities.

The paper is organized as follows. The next section is preliminary - we describe the variational setting we use. The main frame of the proof of Theorem 1 is to be found in Section 3. Finally, the core result – the fact that the mountain pass values (and hence the norms and the energy) of the solutions we find tend to zero as $\hbar \to 0$ – is proved in Section 4.

2 The dual variational formulation

We start by recalling some facts which permit us to set up the variational framework for solving system $(S_\hbar)$.

Lemma 2.1 *Let V be bounded and nonnegative function satisfying (b1) and (b2). Then, for every $g \in L^s(\mathbb{R}^N)$, $1 < s < \infty$, and $\hbar > 0$, the problem*

$$-\Delta u + V(\hbar x)u = g \quad in \ \ \mathbb{R}^N$$

possesses a unique solution $u \in W^{2,s}(\mathbb{R}^N)$. In addition, there exits a constant $K > 0$ (which may depend of $\hbar$) such that

$$\|u\|_{W^{2,s}(\mathbb{R}^N)} \leq K\|g\|_{L^s(\mathbb{R}^N)}$$

Proof: Denote $V_\hbar(x) = V(\hbar x)$. For $s \in (1, \infty)$, consider the operator $R_s : W^{2,s}(\mathbb{R}^N) \to L^s(\mathbb{R})$ defined by

$$R_s u = (-\Delta + V_\hbar I)u \quad \text{ for } \ u \in W^{2,s}(\mathbb{R}^N).$$

It follows for instance from Theorem 1 of [31] that

(i) $\mathrm{Ker}\,(R_s - \lambda I) = \mathrm{Ker}\,(R_2 - \lambda I)$, for every $s \in (1, \infty)$.

(ii) $L^s(\mathbb{R}^N) = \mathrm{Ker}\,(R_s - \lambda I) \oplus \mathrm{Im}\,(R_s - \lambda I)$.

Since $V_\hbar \in L^\infty(\mathbb{R}^N)$, it is known (see for example Lemma 3.10 in [40]) that the spectrum $\sigma(R_2) \subset [\Lambda, \infty)$ and $\Lambda_\hbar \in \sigma(R_2)$, where

$$\Lambda_\hbar = \inf\left\{ \int (|\nabla u|^2 + V_\hbar(x)u^2) \mid u \in H^1(\mathbb{R}^N), \int u^2 = 1 \right\}.$$

It follows from Lemma 1 in [38] that $\Lambda_\hbar > 0$. Therefore $0 \notin \sigma(R_2)$. Consequently $\mathrm{Ker}\,(R_s) = \mathrm{Ker}\,(R_2) = \{0\}$ and

$$L^s(\mathbb{R}^N) = \mathrm{Ker}\,(R_s) + \mathrm{Im}(R_s) = \mathrm{Im}(R_s).$$

Thus, $R_s : W^{2,s}(\mathbb{R}^N) \subset L^s(\mathbb{R}^N) \to L^s(\mathbb{R}^N)$ is a isomorphism. Note that R_s is continuous thanks to the immersion $W^{2,s}(\mathbb{R}^N) \subset L^s(\mathbb{R}^N)$. So, there exists a positive constant C such that for all $u \in L^s(\mathbb{R}^N)$

$$\|R_s^{-1} u\|_{W^{2,s}(\mathbb{R}^N)} \leq C\|u\|_{L^s(\mathbb{R}^N)}. \qquad \square$$

Given $p,\ q > 1$ such that $\dfrac{1}{p+1} + \dfrac{1}{q+1} > \dfrac{N-2}{N}$, we define the operators

$$\widetilde{R}_\hbar : L^{\frac{p+1}{p}}(\mathbb{R}^N) \to W^{2,\frac{p+1}{p}}(\mathbb{R}^N), \qquad \widetilde{S}_\hbar : L^{\frac{q+1}{q}}(\mathbb{R}^N) \to W^{2,\frac{q+1}{q}}(\mathbb{R}^N),$$

by

$$\widetilde{R}_\hbar = \widetilde{S}_\hbar = (-\Delta + b_\hbar I)^{-1},$$

where $b_\hbar(x) = b(\hbar x)$. It follows from Lemma 2.1 that the operators $\widetilde{R}_\hbar$ and $\widetilde{S}_\hbar$ are well defined and continuous. Since $1/(q+1) > p/(p+1) - 2/N$ holds, we have the continuous Sobolev embeddings

$$i_1 : W^{2,\frac{p+1}{p}}(\mathbb{R}^N) \to L^{q+1}(\mathbb{R}^N), \qquad i_2 : W^{2,\frac{q+1}{q}}(\mathbb{R}^N) \to L^{p+1}(\mathbb{R}^N),$$

consequently $R_\hbar \doteq i_1 \circ \widetilde{R}_\hbar$, $S_\hbar \doteq i_2 \circ \widetilde{S}_\hbar$ are linear continuous operators.

So we can define the linear operator

$$T_\hbar : L^{\frac{q+1}{q}}(\mathbb{R}^N) \times L^{\frac{p+1}{p}}(\mathbb{R}^N) \to L^{q+1}(\mathbb{R}^N) \times L^{p+1}(\mathbb{R}^N), \quad T_\hbar := \begin{pmatrix} 0 & R_\hbar \\ S_\hbar & 0 \end{pmatrix},$$

that is, for all $f, \phi \in L^{\frac{q+1}{q}}(\mathbb{R}^N)$, $g, \varphi \in L^{\frac{p+1}{p}}(\mathbb{R}^N)$,

$$\langle T_\hbar w, \eta \rangle = \phi R_\hbar g + \varphi S_\hbar f, \quad \forall\, \eta = (\phi, \varphi),\ \forall\, w = (f, g).$$

Let $X = L^{\frac{q+1}{q}}(\mathbb{R}^N) \times L^{\frac{p+1}{p}}(\mathbb{R}^N)$ be the Banach space endowed with the norm

$$\|w\| = \sqrt{\|f\|^2_{\frac{q+1}{q}} + \|g\|^2_{\frac{p+1}{p}}}; \quad w = (f, g) \in X,$$

from now on $\|\cdot\|_s$ and $\int h dx$ will denote the L^s-norm in $\mathbb{R}^N$ and $\int_{\mathbb{R}^N} h(x)dx$, respectively.

The dual functional $\Psi^\hbar : X \to I\!R$ is defined by

$$\Psi^\hbar(w) = \int H^*(x, w)\, dx - \frac{1}{2}\int \langle T_\hbar w, w \rangle\, dx, \qquad w \in X,$$

where H^* is the Legendre-Fenchel transform of H, that is, for all $x \in \mathbb{R}$ and $w = (w_1, w_2) \in \mathbb{R}^2$,

$$H^*(x, w) = \sup_{t \in \mathbb{R}^2} \{w_1 t_1 + w_2 t_2 - H(x, t)\}.$$

Lemma 2.2 *The functional $\Psi^\hbar$ is well defined and C^1 on X^*. Its Fréchet derivative is given by*

$$(\Psi^\hbar)'(w)\eta = \int H^*_w(x, w)\eta\, dx - \int \langle T_\hbar w, \eta \rangle\, dx, \quad \forall\, \eta \in X.$$

If $w=(f,g)$ is a critical point of $\Psi^{\hbar}$, then $(u,v)=T_h w$ is a solution of the system (obtained by $(S_\hbar)$ through the change $x\to\hbar x$)

$$\begin{cases} -\Delta u + b(hx)u = H_v(hx,u,v) & \text{in } \mathbb{R}^N, \\ -\Delta v + b(hx)v = H_u(hx,u,v) & \text{in } \mathbb{R}^N. \end{cases} \qquad (S'_\hbar)$$

Proof: The proof of this lemma is known, for instance we can employ the arguments given in [8] (see Lemma 4.3 there, and also [22]). Let us sketch it for completeness.

The derivative of the second term in $\Psi_\hbar$ is simple to get, by the relation

$$\int \langle \eta, T_\hbar w\rangle dx = \int \langle w, T_\hbar \eta\rangle\, dx, \quad \forall\, \eta, w \in X.$$

Consider the functional

$$\mathcal{H}(z) = \int H(x,z)\, dx, \quad \mathcal{H}: X^* = L^{q+1}(\mathbb{R}^N)\times L^{p+1}(\mathbb{R}^N) \to \mathbb{R},$$

where $z=(u,v)$. From the hypotheses on H it follows that $\mathcal{H}$ is well-defined on X^* and is a C^1-functional. The Legendre-Fenchel transform of $\mathcal{H}$ is given by

$$\mathcal{H}^*(w) = \int H^*(x,w)\, dx, \quad \mathcal{H}^*: X \to \mathbb{R}.$$

Since H is strictly convex the gradient $H_z:\mathbb{R}^2\to\mathbb{R}^2$ is a homeomorphism. Thus, $\mathcal{H}'$ is a bijection from X^* to X, which is continuous and bounded. Furthermore, $\mathcal{H}^*$ is Gâteaux differentiable, $(\mathcal{H}^*)'(w)=(\mathcal{H}')^{-1}(w)$ for every $w\in X$ (this is a characterization of the Legendre-Fenchel transform), and

$$(\mathcal{H}^*)'(w)\eta = \int H^*_w(x,w)\eta\, dx, \quad \forall\, \eta, w\in X.$$

Thus, $(\mathcal{H}^*)': X\to X^*$ is continuous and bounded, which implies that $\mathcal{H}^*$ is Fréchet differentiable. Now, if w is a critical point of $\Psi^\hbar$, it follows that $z=(u,v)=T_h w$ is a solution of $(S'_\hbar)$. In fact, we have

$$(\mathcal{H}^*)'(w) - T_h w = 0 \quad \text{in } X^*,$$

that is

$$(\mathcal{H}')^{-1}(w) - z = 0 \quad \text{in } X^*.$$

As a result,

$$T_h^{-1} z - (\mathcal{H}')(z) = 0 \quad \text{in } W^{2,\frac{p+1}{p}} \times W^{2,\frac{q+1}{q}},$$

because $T_\hbar^{-1}$ is an isomorphism between $W^{2,\frac{p+1}{p}} \times W^{2,\frac{q+1}{q}}$ and $L^{\frac{p+1}{p}} \times L^{\frac{q+1}{q}}$. Thus, $(u,v) = z = T_\hbar w$ is a solution of system $(S'_\hbar)$. $\square$

We say that $w = (f,g)$ is the dual solution associated to (u,v). By making the change of variable $x \mapsto \hbar^{-1}x$ in $\mathbb{R}^N$, system $(S'_\hbar)$ becomes

$$\begin{cases} -\hbar^2\Delta u + b(x)u = H_v(x,u,v) & \text{in } \mathbb{R}^N, \\ -\hbar^2\Delta v + b(x)v = H_u(x,u,v) & \text{in } \mathbb{R}^N. \end{cases} \tag{$S_\hbar$}$$

3 Proof of Theorem 1

We start with the following simple fact.

Lemma 3.1 *The functional $\Psi^\hbar$ has a "mountain pass geometry" on the space X, in the sense that there exist $\rho, \alpha > 0$ and $w \in X$ such that $\Psi^\hbar|_{\partial B_\rho} \geq \alpha$, $\Psi^\hbar(w) < 0$ and $\|w\| > \rho$.*

Proof: It is easy to see that (H1) and (H2) imply that there exist positive constants $c_1 - c_4$ such that

$$c_1|f|^{q+1} + c_2|g|^{p+1} \leq H(x,w) \leq c_3|f|^{q+1} + c_4|g|^{p+1}, \quad w = (f,g).$$

From properties of Legendre-Fenchel transformations, we have

$$d_1|f|^{\frac{q+1}{q}} + d_2|g|^{\frac{p+1}{q}} \leq H^*(x,w) \leq d_3|f|^{\frac{q+1}{q}} + d_4|g|^{\frac{p+1}{p}}, \tag{3.5}$$

for some positive constants $d_1 - d_4$.

By using the Hölder inequality and the boundedness of $R_\hbar$ and $S_\hbar$, for all $w = (f,g) \in X$ we easily get

$$\begin{aligned} \int \langle w, T_\hbar w\rangle &\leq C(\|f\|_{\frac{q+1}{q}}\|g\|_{\frac{p+1}{p}} + \|g\|_{\frac{p+1}{p}}\|f\|_{\frac{q+1}{q}}) \\ &\leq C(\|g\|^2_{\frac{p+1}{p}} + \|f\|^2_{\frac{q+1}{q}}) = C\|w\|^2_X, \end{aligned} \tag{3.6}$$

Then, from (3.5) and (3.6) we get

$$\Psi^\hbar(w) \geq C(\|f\|^{\frac{q+1}{q}}_{\frac{q+1}{q}} + \|g\|^{\frac{p+1}{p}}_{\frac{p+1}{p}}) - C(\|f\|^2_{\frac{q+1}{q}} + \|g\|^2_{\frac{p+1}{p}}).$$

Thus, since $(p+1)/p < 2$ and $(q+1)/q < 2$, for each $\hbar > 0$ there exist constants $\rho,\ \alpha > 0$ such that $\Psi^\hbar|_{\partial B_\rho} \geq \alpha$.

Now, we claim we can find $w \in X$ such that $\Psi^\hbar(w) < 0$ and $\|w\| > \rho$. In fact, there exists $w^+ = (f^+, g^+) \in X$ such that $\int\langle T_\hbar w^+, w^+\rangle > 0$ (indeed, it is sufficient to take $f^+ = g^+ \in C_c^\infty(\mathbb{R}^N)$). By using (3.5) we obtain, for all $t > 0$,

$$\Psi^\hbar(tw^+) \leq Ct^{\frac{q+1}{q}} \int |f|^{\frac{q+1}{q}} + Ct^{\frac{p+1}{p}} \int |g|^{\frac{p+1}{p}} - \frac{t^2}{2} \int \langle T_h w^+, w^+ \rangle,$$

for some positive constant C. Since $\frac{p+1}{p}$, $\frac{q+1}{q} < 2$, the claim follows for $t > 0$ sufficiently large. □

Set

$$\Gamma_\hbar \doteq \{\gamma \in C([0,1], X) \;:\; \gamma(0) = 0,\; \Psi^\hbar(\gamma(1)) < 0\}$$

and

$$c_\hbar = \inf_{\gamma \in \Gamma_\hbar} \max_{t \in [0,1]} \Psi^\hbar(\gamma(t)).$$

Standard critical point theory implies that for each $\hbar > 0$ we can find a sequence $\{w_n^\hbar\}_{n=1}^\infty \subset X$ such that

$$\Psi^\hbar(w_n^\hbar) \to c_\hbar \quad \text{and} \quad (\Psi^\hbar)'(w_n^\hbar) \to 0 \quad \text{as } n \to \infty. \tag{3.7}$$

Our goal will be to show that for sufficiently small values of $\hbar$ each of these sequences possesses an accumulation point, which is nontrivial solution of $(S_\hbar)$.

Lemma 3.2 *For $\hbar > 0$ fixed, the sequence $w_n^\hbar = (f_n^\hbar, g_n^\hbar)$ is bounded in X.*

Proof: From properties of the Legendre-Fenchel transform and (H2) we have

$$H^*(x, w_n^\hbar) \geq (1 - \frac{1}{\alpha}) H_f^*(x, w_n^\hbar) f_n^\hbar + (1 - \frac{1}{\alpha}) H_g^*(x, w_n^\hbar) g_n^\hbar. \tag{3.8}$$

Now

$$\begin{aligned} \int H^*(x, w_n^\hbar) &= \frac{1}{2} \int \langle T_h w_n^\hbar, w_n^\hbar \rangle + \Psi^\hbar(w_n^\hbar) \\ &= \Psi^\hbar(w_n^\hbar) - \frac{1}{2} \langle (\Psi^\hbar)'(w_n^\hbar), w_n^\hbar \rangle + \frac{1}{2} \int H_w^*(x, w_n^\hbar) w_n^\hbar. \end{aligned}$$

Setting $\lambda = \frac{\alpha}{2(\alpha - 1)} < 1$, from (3.7) and (3.8) we obtain

$$(1 - \lambda) \int H^*(x, w_n^\hbar) \leq c_\hbar + o_n(1) \|w_n^\hbar\|_X, \tag{3.9}$$

where $o_n(1)$ is a quantity which tends to zero as $n \to \infty$. By combining (3.5) and (3.9) we get for some $k, K > 0$

$$k \|w_n^\hbar\|_X^\gamma \leq \|f_n^\hbar\|_{L^{\frac{q+1}{q}}}^{\frac{q+1}{q}} + \|g_n^\hbar\|_{L^{\frac{p+1}{p}}}^{\frac{p+1}{p}} \leq K c_\hbar + o_n(1) \|w_n^\hbar\|_X, \tag{3.10}$$

with $\gamma = \min\{1 + 1/p, 1 + 1/q\} > 1$. This trivially implies that $\{w_n^\hbar\}$ is bounded in X, for $\hbar > 0$ fixed. □

With the help of Lemma 3.2 for each $\hbar > 0$ we can extract a subsequence of $\{w_n^\hbar\}$ which converges weakly in X to a function $w^\hbar = (f^\hbar, g^\hbar)$. We affirm that w is a critical point of $\Psi^\hbar$. First, for each $\hbar > 0$ the sequence $z_n = T_h w_n^\hbar$ is clearly bounded in X^*, since $T_\hbar$ is bounded. Another way of writing (3.7) is

$$T_h^{-1} z_n - (\mathcal{H}')(z_n) = o_n(1)$$

(see the proof of Lemma 2.2). Since up to a subsequence we have $z_n \rightharpoonup z$ in $W^{2,\frac{p+1}{p}} \times W^{2,\frac{q+1}{q}}$ we see that the limit function z is a weak solution of $(S_\hbar)$. This implies that $T_h z \in X$ and $w = T_h z$ is a critical point of $\Psi^\hbar$.

It remains to show that $w^\hbar$ is not identically zero. We claim that for small $\hbar$ this is the case. The proof of this claim will be carried out through several steps. First, let $u_n^\hbar$ and $v_n^\hbar$ be the functions given by

$$u_n^\hbar = R_\hbar g_n^\hbar \in W^{2,\frac{p+1}{p}}(\mathbb{R}^N) \quad \text{and} \quad v_n^\hbar = S_\hbar f_n^\hbar \in W^{2,\frac{q+1}{q}}(\mathbb{R}^N), \tag{3.11}$$

that is,

$$-\Delta u_n^\hbar + b(\hbar x) u_n^\hbar = g_n^\hbar \quad \text{and} \quad -\Delta v_n^\hbar + b(\hbar x) v_n^\hbar = f_n^\hbar, \quad x \in \mathbb{R}^N. \tag{3.12}$$

Next, we note that (1.4) permits to us to choose s, t such that $0 < s, t < 2$, $s + t = 2$ and

$$t < \frac{N}{2}, \quad 2 - t < \frac{N}{2}, \quad \frac{N(p-1)}{2(p+1)} < t < \frac{4(q+1) - N(q-1)}{2(q+1)}. \tag{3.13}$$

Then $p + 1 < \frac{2N}{N-2t}$ and $q + 1 < \frac{2N}{N-2s}$, which implies

$$W^{2,\frac{p+1}{p}} \hookrightarrow H^s \hookrightarrow L^{q+1} \quad \text{and} \quad W^{2,\frac{q+1}{q}} \hookrightarrow H^t \hookrightarrow L^{p+1},$$

where H^s, H^t are the usual fractional Sobolev spaces over $\mathbb{R}^N$.

Lemma 3.3 *There exists a constant $\beta > 0$ (independent of $\hbar$) such that for each $\hbar > 0$ we can find $R = R(\hbar) > 0$, for which*

$$\begin{aligned} \|u_n^\hbar\|_{H^s(\mathbb{R}^N)}^{q+1} &\leq \beta c_\hbar^{\frac{(q+1)p}{p+1}} + \beta \|u_n^\hbar\|_{H^s(B_R)}^{q+1} + o_n(1), \\ \|v_n^\hbar\|_{H^t(\mathbb{R}^N)}^{p+1} &\leq \beta c_\hbar^{\frac{(p+1)q}{q+1}} + \beta \|v_n^\hbar\|_{H^t(B_R)}^{p+1} + o_n(1). \end{aligned}$$

Proof: We shall need some functional analysis. For $s \in (0,1)$ let $H^s_{b(\hbar x)}$ be the space of the functions u such that

$$b^{\frac{1}{2}}(\hbar x)u \in L^2(\mathbb{R}^N) \qquad \text{and} \qquad \frac{|u(x)-u(y)|}{|x-y|^{s+\frac{N}{2}}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N).$$

One can also define $H^s_{b(\hbar x)}$ by interpolation between the spaces

$$L^2_{b(\hbar x)} = \{u \,:\, \int b(\hbar x)u^2 < \infty\} \text{ and } H^1_{b(\hbar x)} = \{u \in L^2_{b(\hbar x)} \,:\, \int |\nabla u|^2 < \infty\}.$$

Since $b \in L^\infty(\mathbb{R}^N)$, the inclusion $H^s \subset H^s_{b(\hbar x)}$ holds. On the other hand it is standard to check that $H^s_{b(\hbar x)}(\mathbb{R}^N)$ is embedded into $H^s(B_R)$, for any $s > 0$ and any ball B_R. Once more through Lemma 1 in [38] (see also the argument used in the proof of this lemma) we can prove that $H^s(\mathbb{R}^N) = H^s_{b(\hbar x)}(\mathbb{R}^N)$ under hypotheses (b1) and (b2).

Define $L = -\Delta + b(\hbar x) : H^2 \subset L^2 \to L^2$ (L is a positive operator) and $A^s := (\sqrt{L})^s$, so that $A^s : H^s_{b(\hbar x)} \to L^2$ is a isomorphism between $H^s_{b(\hbar x)}$ and L^2. This is standard functional analysis, for details and references see [16], pages 224-226, where the case $b \equiv 1$ was considered. We observe that $\|u\|_{H^s_{b(\hbar x)}} = \|A^s u\|_{L^2}$. Then the weak formulation of the first equation in (3.12) is

$$\int A^s u_n^\hbar A^t \varphi = \int g_n^\hbar \varphi, \quad \forall\, \varphi \in H^t. \tag{3.14}$$

Putting $\varphi = A^{-t}A^s u_n^\hbar$ into (3.14), we obtain

$$\|u\|^2_{H^s_{b(\hbar x)}} = \int |A^s u_n^\hbar|^2 = \int g_n^\hbar A^{-t} A^s u_n^\hbar.$$

So there exists a positive constant independent of $\hbar$ such that for all $R > 0$

$$\begin{aligned} \|u_n^\hbar\|^2_{H^s_{b(\hbar x)}(\mathbb{R}^N)} &\le \|g_n^\hbar\|_{L^{\frac{p+1}{p}}} \|A^{-t}A^s u_n^\hbar\|_{L^{p+1}} \\ &\le C\|g_n^\hbar\|_{L^{\frac{p+1}{p}}} \|u_n^\hbar\|_{H^s} \\ &= C\|g_n^\hbar\|_{L^{\frac{p+1}{p}}} \left[\|u_n^\hbar\|_{H^s(B_R)} + \|u_n^\hbar\|_{H^s(\mathbb{R}^N \setminus B_R)}\right]. \end{aligned} \tag{3.15}$$

On the other hand, hypotheses (b1) and (b2) imply that we can find $c > 0$ such that for any $\hbar > 0$ there exists $R = R(\hbar)$ for which

$$\|w\|_{H^s_{b(\hbar x)}(\mathbb{R}^N)} \ge c\|w\|_{H^s(\mathbb{R}^N \setminus B_R)}, \qquad \forall\, w \in H^s_{b(\hbar x)}(\mathbb{R}^N) = H^s(\mathbb{R}^N).$$

This inequality (particularly easy to check under (1.3)) follows from Lemma 3 in [38] where the case $s = 1$ was studied, and from an interpolation argument.

Since $x^2 \leq a + bx, x \geq 0$ implies $x \leq C(b + \sqrt{a})$ we get from (3.15)

$$\|u_n^\hbar\|_{H^s(\mathbb{R}^N \setminus B_R)} \leq C\|g_n^\hbar\|_{L^{\frac{p+1}{p}}(\mathbb{R}^N)} + C\|u_n^\hbar\|_{H^s(B_R)}, \tag{3.16}$$

for some positive constant C independent of $\hbar$. Similarly,

$$\|v_n^\hbar\|_{H^t(\mathbb{R}^N \setminus B_R)} \leq C\|f_n^\hbar\|_{L^{\frac{q+1}{q}}(\mathbb{R}^N)} + C\|v_n^\hbar\|_{H^t(B_R)}. \tag{3.17}$$

Recall we already proved (Lemma 3.2 and (3.10)) that there exists a positive constant C independent of $\hbar$ for which

$$\|g_n^\hbar\|_{L^{\frac{p+1}{p}}} \leq C c_\hbar^{\frac{p}{p+1}} + o_n(1) \qquad \text{and} \qquad \|f_n^\hbar\|_{L^{\frac{q+1}{q}}} \leq C c_\hbar^{\frac{q}{q+1}} + o_n(1). \tag{3.18}$$

By combining these with (3.16) and (3.17) we get Lemma 3.3. □

The final and basic ingredient of the proof of Theorem 1 is the following

Lemma 3.4 *We have*

$$\lim_{\hbar \to 0} c_\hbar = 0. \tag{3.19}$$

The proof of this lemma will be given in the next section. We shall now proceed to the proof of Theorem 1.

Proof of Theorem 1. Since $H^*(w)$ is a convex function on $\mathbb{R}^2$ we have $\langle \nabla H^*(w), w\rangle \geq H^*(w)$ for all $w \in \mathbb{R}^2$. Hence

$$\begin{aligned} c_\hbar &= \lim_{n\to\infty} \left[\Psi^\hbar(w_n^\hbar) - \langle (\Psi^\hbar)'(w_n^\hbar), w_n^\hbar\rangle\right] \\ &\leq \frac{1}{2}\lim_{n\to\infty} \int \langle T w_n^\hbar, w_n^\hbar\rangle = \frac{1}{2}\lim_{n\to\infty}\int f_n^\hbar R_\hbar g_n^\hbar + g_n^\hbar S_\hbar f_n^\hbar \\ &\leq \limsup_{n\to\infty}\left(\|f_n^\hbar\|_{L^{\frac{q+1}{q}}}\|R_\hbar g_n^\hbar\|_{L^{q+1}} + \|g_n^\hbar\|_{L^{\frac{p+1}{p}}}\|S_\hbar f_n^\hbar\|_{L^{p+1}}\right). \end{aligned}$$

By the Hölder inequality for each $\varepsilon > 0$ there exists $C = C(\varepsilon) > 0$ such that

$$c_\hbar \leq \varepsilon \limsup_{n\to\infty}(\|f_n^\hbar\|_{L^{\frac{q+1}{q}}}^{\frac{q+1}{q}} + \|g_n^\hbar\|_{L^{\frac{p+1}{p}}}^{\frac{p+1}{p}}) + C\limsup_{n\to\infty}(\|R_\hbar g_n^\hbar\|_{L^{q+1}}^{q+1} + \|S_\hbar f_n^\hbar\|_{L^{p+1}}^{p+1}),$$

so by using (3.18) and by choosing ε sufficiently small we get by the Sobolev embedding and the boundedness of $R_\hbar, S_\hbar$ that

$$\begin{aligned} c_\hbar &\leq C\limsup_{n\to\infty}\left(\|R_\hbar g_n^\hbar\|_{L^{q+1}}^{q+1} + \|S_\hbar f_n^\hbar\|_{L^{p+1}}^{p+1}\right) \\ &\leq C\limsup_{n\to\infty}\|u_n^\hbar\|_{H^s(\mathbb{R}^N)}^{q+1} + C\limsup_{n\to\infty}\|v_n^\hbar\|_{H^t(\mathbb{R}^N)}^{p+1}. \end{aligned}$$

Therefore, by the previous lemma,

$$c_\hbar \le \beta \left(c_\hbar^{\frac{(q+1)p}{p+1}} + c_\hbar^{\frac{(p+1)q}{q+1}} \right) + C \limsup_{n\to\infty} \|u_n^\hbar\|^{q+1}_{H^s(B_R)} + C \limsup_{n\to\infty} \|v_n^\hbar\|^{p+1}_{H^t(B_R)}.$$

Note the embeddings $W^{2,\frac{p+1}{p}} \hookrightarrow H^s$, $W^{2,\frac{q+1}{q}} \hookrightarrow H^t$ are compact on bounded domains, so $\{u_n^\hbar\}, \{v_n^\hbar\}$ converge strongly on B_R as $n \to \infty$. Hence for the limit functions $u^\hbar$, $v^\hbar$ we get

$$\|u^\hbar\|^{q+1}_{H^s(B_R)} + \|v^\hbar\|^{p+1}_{H^t(B_R)} \ge \left[1 - C^{-1}\beta \left(c_\hbar^{\frac{pq-1}{p+1}} + c_\hbar^{\frac{pq-1}{q+1}} \right) \right].$$

However the last quantity is strictly positive for small $\hbar$ (since $c_\hbar \to 0$), which means that the limit functions are not identically zero. Note that of course $(f^\hbar, g^\hbar) = (0,0)$ if and only if $(u^\hbar, v^\hbar) = (0,0)$. □

4 Proof of Lemma 3.4.

We start by observing that

$$c_\hbar \le \inf_{w\in X\setminus\{0\}} \sup_{t\ge 0} \Psi^\hbar(tw) = \inf_{w\in E} \max_{t\ge 0} \Psi^\hbar(tw),$$

where we have set $E = \{w \in X \mid \int \langle Tw, w\rangle > 0\}$. An explicit computation (see Appendix I) shows that for any $w \in E$ we have

$$\max_{t\ge 0} \Psi^\hbar(tw) \le \text{const.} \left(\frac{\|w\|^2}{\int \langle T_\hbar w, w\rangle} \right)^\gamma, \tag{4.20}$$

where

$$\gamma = \max \left\{ \frac{\frac{p+1}{p}}{2 - \frac{p+1}{p}}, \frac{\frac{q+1}{q}}{2 - \frac{q+1}{q}} \right\}.$$

So to prove Lemma 3.4 it will be enough to establish the following claim:

$$\inf_{w\in E} \frac{\|w\|^2}{\int \langle T_\hbar w, w\rangle} \to 0, \quad \text{as } \hbar \to 0. \tag{4.21}$$

In order to verify this, we observe that

$$\inf_{w\in E} \frac{\|w\|^2}{\int \langle T_\hbar w, w\rangle} = \inf_{w\in E\,:\,\|w\|=1} \frac{1}{\int \langle T_\hbar w, w\rangle}.$$

Thus, (4.21) is equivalent to the following result.

Lemma 4.1 *We have*

$$\sup_{w\in E\,:\,\|w\|=1} \int \langle T_\hbar w, w\rangle \, dx \to +\infty \qquad as \;\; \hbar \to 0. \tag{4.22}$$

To facilitate the task of the reader, we first describe the idea behind the proof of (4.22). The point is that if p, q are under the critical hyperbola and s, t are chosen as in (3.13), then it is possible to find (explicitly) a function $g \in L^{\frac{p+1}{p}}(\mathbb{R}^N)$ such that if u satisfies

$$-\Delta u(x) = g(x), \quad x \in \mathbb{R}^N,$$

then u does not belong to the fractional Sobolev space $H^s(\mathbb{R}^N)$, and respectively a function $f \in L^{\frac{q+1}{q}}(\mathbb{R}^N)$ such that the solution of $-\Delta v = f$ is not in H^t. We recall that a function w is in $H^s(\mathbb{R}^N)$ if and only if $w \in L^2(\mathbb{R}^N)$ and its Fourier transform $\widehat{w}(\xi)$ is such that $|\xi|^s\widehat{w}(\xi) \in L^2(\mathbb{R}^N)$. Then, assuming (4.22) does not hold we show we can perturb and cut off the functions f, g, to construct a sequence $w_\hbar = (f_h, g_h)$ such that $\|w_\hbar\| = 1$ and we can control the corresponding $R_h g_h$, $S_\hbar f_\hbar$ in a way which yields a contradiction for small $\hbar$.

Proof of Lemma 4.1. Let us suppose (4.22) does not hold, that is, there exists $C_0 > 0$ such that

$$\int \langle T_\hbar w, w\rangle \, dx \le C_0, \qquad \text{for each } \; w \in E \; \text{ with } \; \|w\| = 1.$$

We start by giving some results from the theory of Fourier transforms, which we shall use. The next theorem is a standard fact from the theory of Fourier transforms of distributions.

Theorem 2 *Suppose the function u_0 has slow growth, that is, there exists $m \in \mathbb{N}$ such that*

$$\int_{\mathbb{R}^N} \frac{|u_0(x)|dx}{(1+|x|)^m} < \infty. \tag{4.23}$$

Then the Fourier transform $\widehat{u}$ exists and belongs to the class of tempered distributions $\mathcal{S}'$. In addition if $\phi \in C_c^\infty(\mathbb{R}^N)$ is such that $\phi \equiv 1$ in B_1, $\phi \equiv 0$ in $\mathbb{R}^N \setminus B_2$ and we set $\phi_n(x) = \phi(x/n)$ then $\widehat{\phi_n u} \to \widehat{u}$ in $\mathcal{S}'$.

We shall use the Fourier transform of the function

$$w_0(x) = \frac{1}{1+|x|^2},$$

and its powers. It is a well-known fact from Fourier analysis that for any $\alpha > 0$ we have

$$\widehat{w_0^\alpha}(\xi) = C(N,\alpha)|\xi|^{\alpha-\frac{N}{2}} K_{\frac{N}{2}-\alpha}(|\xi|), \tag{4.24}$$

(this is for instance formula (3.11) in [24]) ; here $K_\nu(z)$ is the modified Bessel function of the second kind, given by

$$K_\nu(z) = \frac{\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}}|z|^\nu \int_0^\infty \frac{\cos(t)}{(t^2+z^2)^{\nu+\frac{1}{2}}}\,dt = \frac{\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}|z|^\nu} \int_0^\infty \frac{\cos(sz)}{(1+s^2)^{\nu+\frac{1}{2}}}\,ds.$$

Standard analysis shows that $K_\nu(z) > 0$, $K_\nu(z) \in C^\infty(\mathbb{R}\setminus\{0\})$, K decays exponentially as $|z| \to \infty$, and, most importantly, $K_\nu(z) \sim \text{const.}|z|^{-\nu}$ as $z \to 0$. Hence

$$\widehat{w_0^\alpha}(\xi) \sim C(N,\alpha)|\xi|^{2\alpha-N} \qquad \text{as } |\xi| \to 0. \tag{4.25}$$

We now fix $p' > p$ and $q' > q$ such that p', q' are still under the critical hyperbola, $\dfrac{1}{p'+1} + \dfrac{1}{q'+1} > 1 - \dfrac{2}{N}$. We set

$$\alpha = \frac{Np'}{2(p'+1)}, \qquad \beta = \frac{Nq'}{2(q'+1)},$$

so that in particular

$$w_0^\alpha \in L^{\frac{p+1}{p}}(\mathbb{R}^N), \qquad w_0^\beta \in L^{\frac{q+1}{q}}(\mathbb{R}^N).$$

Let u_0, v_0 be the solutions of

$$-\Delta u_0 = k_1 w_0^\alpha, \qquad -\Delta v_0 = k_2 w_0^\beta \qquad \text{in } \mathbb{R}^N, \tag{4.26}$$

where $k_1 = k_1(p,q,N) := \|w_0^\alpha\|^{-1}_{\frac{p+1}{p}}$, $k_2 = k_2(p,q,N) := \|w_0^\beta\|^{-1}_{\frac{q+1}{q}}$. By standard PDE theory u_0 and v_0 are functions which belong to some Lebesgue spaces over $\mathbb{R}^N$ (see for instance Theorem 10.2 (i) in [25]), which in particular implies that they have slow growth, as in (4.23) (by the Hölder inequality). Hence Theorem 2 applies, and, by taking the Fourier transform on both sides of the equations in (4.26) we get

$$\widehat{u_0}(\xi) = k_1|\xi|^{-2}\widehat{w_0^\alpha}(\xi), \qquad \widehat{v_0}(\xi) = k_2|\xi|^{-2}\widehat{w_0^\beta}(\xi). \tag{4.27}$$

Note that $\widehat{u_0}, \widehat{v_0}$ are positive.

Lemma 4.2 *We have*

$$\int_{\mathbb{R}^N} |\xi|^2 \widehat{u_0}(\xi)\widehat{v_0}(\xi)\,d\xi = \infty.$$

Proof. By (4.24) and (4.27) we have

$$|\xi|^2\widehat{u_0}(\xi)\widehat{v_0}(\xi) \sim C(N,\alpha,\beta)|\xi|^{2(\alpha+\beta-1-N)} \qquad \text{as } |\xi| \to 0.$$

However, by the choice of α and β that we made

$$\begin{aligned}\alpha+\beta-1-N &= \frac{N}{2}\left(\frac{p'}{p'+1}+\frac{q'}{q'+1}\right)-1-N\\ &= \frac{N}{2}\left(2-\frac{1}{p'+1}-\frac{1}{q'+1}\right)-1-N\\ &< \frac{N}{2}\left(1+\frac{2}{N}\right)-1-N=-\frac{N}{2},\end{aligned}$$

and the lemma follows. □

We set $u_n = \phi_n u_0 \in C_c^\infty(\mathbb{R}^N), v_n = \phi_n v_0 \in C_c^\infty(\mathbb{R}^N)$, where ϕ_n is a function as in Theorem 2, and

$$\begin{aligned}\widetilde{g}_{n,\hbar} &:= -\Delta u_n + b(\hbar x)u_n,\\ \widetilde{f}_{n,\hbar} &:= -\Delta v_n + b(\hbar x)v_n.\end{aligned}$$

Since u_n, v_n have compact support and $b(0)=0$ for each fixed n we have

$$\widetilde{g}_{n,\hbar} \to -\Delta u_n \quad \text{in } L^{\frac{p+1}{p}}, \qquad \widetilde{f}_{n,\hbar} \to -\Delta v_n \quad \text{in } L^{\frac{q+1}{q}} \qquad \text{as } \hbar \to 0.$$

Clearly

$$-\Delta u_n \to -\Delta u_0 \quad \text{in } L^{\frac{p+1}{p}}, \qquad -\Delta v_n \to -\Delta v_0 \quad \text{in } L^{\frac{p+1}{p}} \qquad \text{as } n \to \infty,$$

and, recalling that we have taken u_0, v_0 so that $\|\Delta u_0\|_{\frac{p+1}{p}} = \|\Delta u_0\|_{\frac{q+1}{q}} = 1$, we see that we can find n_0 such that for each $n \geq n_0$ there exists $\hbar_n$ for which

$$2 \geq \|\widetilde{g}_{n,\hbar}\|_{\frac{p+1}{p}} \geq \frac{1}{2}, \qquad 2 \geq \|\widetilde{f}_{n,\hbar}\|_{\frac{q+1}{q}} \geq \frac{1}{2}, \qquad \text{if } \hbar < \hbar_n.$$

Now set

$$g_{n,\hbar} = \frac{\widetilde{g}_{n,\hbar}}{\sqrt{2}\|\widetilde{g}_{n,\hbar}\|_{\frac{p+1}{p}}}, f_{n,\hbar} = \frac{\widetilde{f}_{n,\hbar}}{\sqrt{2}\|\widetilde{f}_{n,\hbar}\|_{\frac{q+1}{q}}},$$

and $w_{n,\hbar} = (f_{n,\hbar}, g_{n,\hbar})$. So $w_{n,\hbar} \in E$ and $\|w_{n,\hbar}\|_X = 1$. By the hypothesis we made

$$\int \langle T_\hbar w_{n,\hbar}, w_{n,\hbar}\rangle \leq C_0,$$

for all $n \geq n_0$ and all $\hbar < \hbar_n$.

On the other hand, setting $k_{n,\hbar}^{-1} = 2\|\widetilde{g}_{n,\hbar}\|_{L^{\frac{p+1}{p}}}\|\widetilde{g}_{n,\hbar}\|_{L^{\frac{p+1}{p}}}$ (by the above $k_{n,\hbar} \in (1/8, 2)$), we have

$$\begin{aligned}
\int \langle T_\hbar w_{n,\hbar}, w_{n,\hbar}\rangle &= k_{n,\hbar}\int (\widetilde{f}_{\hbar,n} R_\hbar \widetilde{g}_{n,\hbar} + \widetilde{g}_{n,\hbar} S_\hbar \widetilde{f}_{n,\hbar}) \\
&= k_{n,\hbar}\int u_n(-\Delta v_n) + v_n(-\Delta u_n) + 2b(\hbar x)u_n v_n \\
&\geq k_{n,\hbar}\int \widehat{u_n}(\widehat{-\Delta v_n}) + \widehat{v_n}(\widehat{-\Delta u_n}) \\
&= 2k_{n,\hbar}\int |\xi|^2 \widehat{\phi_n u_0}(\xi)\widehat{\phi_n v_0}(\xi)\, d\xi,
\end{aligned}$$

where we used Parseval's identity and the positivity of b, u_n, v_n. Hence

$$\int |\xi|^2 \widehat{\phi_n u_0}(\xi)\widehat{\phi_n v_0}(\xi)\, d\xi \leq 4C_0. \tag{4.28}$$

Note that the definition of the Fourier transform implies $\widehat{\phi_n u_0}(\xi) \to \widehat{u_0}(\xi)$ for each $\xi \neq 0$. Actually (see for instance Theorems 2.16, 5.3, 5.8 in [24]) $\widehat{\phi_n u_0} = \widehat{\phi}_n * \widehat{u}_0 \to \widehat{u}_0$ in any Lebesgue space to which belongs $\widehat{u}_0$, and similarly for $\widehat{v}_0$. Recall we have explicit expressions for $\widehat{u}_0$, $\widehat{v}_0$ and know that they are strictly positive, behave like $|\xi|$ to a negative power as $\xi \to 0$ and decay exponentially as $\xi \to \infty$. It is then simple to check that the negative part of $|\xi|^2\widehat{\phi_n u_0}(\xi)\widehat{\phi_n v_0}(\xi)$ is bounded by an integrable function independently of n, so Fatou's lemma applies to (4.28) and gives a contradiction with Lemma 4.2. Alternatively, one can prove that Fatou's lemma applies to (4.2) by noticing that the integrand in this inequality is $\nabla(\phi_n u_0).\nabla(\phi_n v_0)$ and this scalar product is positive, since ϕ_n, u_0 and v_0 are positive, radial, and decreasing functions.

This completes the proof of Theorem 1. □

5 Appendix

In this appendix we verify estimate (4.20), which we used in Lemma 3.4. First, we note that for $w = (f, g)$

$$\Psi^\hbar(tw) = \frac{A}{\alpha}t^\alpha + \frac{B}{\beta}t^\beta - \frac{C}{2}t^2, \tag{5.29}$$

where

$$\alpha = (p+1)/p, \qquad \beta = (q+1)/q,$$

$$A=\int|g|^{\frac{p+1}{p}},\quad B=\int|f|^{\frac{q+1}{q}},\quad C=\int\langle T_\hbar w,w\rangle.$$

Denoting the right hand side of (5.29) by $h(t)$, it is easy to check that

$$\max\{\Psi^\hbar(tw),t\geq 0\}=h(\bar t),$$

for some $\bar t>0$ if and only if $h'(\bar t)=0$, that is

$$\bar t^2=\frac{A}{C}\bar t^\alpha+\frac{B}{C}\bar t^\beta.$$

This implies that there exists a positive constant K such that

$$\bar t\leq K\left[\left(\frac{A}{C}\right)^{\frac{1}{2-\alpha}}+\left(\frac{B}{C}\right)^{\frac{1}{2-\beta}}\right].$$

Then, for some constant $K'>0$,

$$\begin{aligned}
h(\bar t) &= A(\frac{1}{\alpha}-\frac{1}{2})\bar t^\alpha+B(\frac{1}{\beta}-\frac{1}{2})\bar t^\beta\\
&\leq K'\left[\frac{A^{\frac{2}{2-\alpha}}}{C^{\frac{\alpha}{2-\alpha}}}+\frac{AB^{\frac{\alpha}{2-\beta}}}{C^{\frac{\alpha}{2-\beta}}}+\frac{BA^{\frac{\beta}{2-\alpha}}}{C^{\frac{\beta}{2-\alpha}}}+\frac{B^{\frac{2}{2-\beta}}}{C^{\frac{\beta}{2-\beta}}}\right]\\
&= K'\left[\frac{A^{\frac{2}{2-\alpha}}}{C^{\frac{\alpha}{2-\alpha}}}+\frac{A}{C^{\frac{\alpha}{2}}}\frac{B^{\frac{\alpha}{2-\beta}}}{C^{\frac{\alpha\beta}{2(2-\beta)}}}+\frac{B}{C^{\frac{\beta}{2}}}\frac{A^{\frac{\beta}{2-\alpha}}}{C^{\frac{\alpha\beta}{2(2-\alpha)}}}+\frac{B^{\frac{2}{2-\beta}}}{C^{\frac{\beta}{2-\beta}}}\right].
\end{aligned}$$

By using the Young inequality, we obtain

$$\begin{aligned}
\frac{A}{C^{\frac{\alpha}{2}}}\frac{B^{\frac{\alpha}{2-\beta}}}{C^{\frac{\alpha\beta}{2(2-\beta)}}}+\frac{B}{C^{\frac{\beta}{2}}}\frac{A^{\frac{\beta}{2-\alpha}}}{C^{\frac{\alpha\beta}{2(2-\alpha)}}} &\leq \left(\frac{A}{C^{\frac{\alpha}{2}}}\right)^{\frac{2}{2-\alpha}}+\left(\frac{B^{\frac{\alpha}{2-\beta}}}{C^{\frac{\alpha\beta}{2(2-\beta)}}}\right)^{\frac{2}{\alpha}}\\
&\quad+\left(\frac{B}{C^{\frac{\beta}{2}}}\right)^{\frac{2}{2-\beta}}+\left(\frac{A^{\frac{\beta}{2-\alpha}}}{C^{\frac{\alpha\beta}{2(2-\alpha)}}}\right)^{\frac{2}{\beta}}.
\end{aligned}$$

Thus,

$$h(\bar t)\leq 2K'\left[\left(\frac{A^{\frac{2}{\alpha}}}{C}\right)^{\frac{\alpha}{2-\alpha}}+\left(\frac{B^{\frac{2}{\beta}}}{C}\right)^{\frac{\beta}{2-\beta}}\right]\leq 2K'\left[\frac{A^{\frac{2}{\alpha}}+B^{\frac{2}{\beta}}}{C}\right]^\gamma,$$

where $\gamma=\max\left\{\frac{\alpha}{2-\alpha},\frac{\beta}{2-\beta}\right\}$.

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