ARMA manuscript No.
(will be inserted by the editor)

Solvability of uniformly elliptic fully nonlinear PDE

Boyan SIRAKOV

Abstract

We get existence, uniqueness and non-uniqueness of viscosity solutions of uniformly elliptic fully nonlinear equations of Hamilton-Jacobi-Bellman-Isaacs type, with unbounded ingredients and quadratic growth in the gradient, without hypotheses of convexity or properness. Some of our results are new even for equations in divergence form.

1. Introduction and Main Results

This paper is devoted to the Dirichlet problem

$$\begin{cases} F(D^2u, Du, u, x) = f(x) \ \text{in} \quad \Omega \\ \qquad\qquad\qquad\quad u = \psi(x) \ \text{on} \ \partial\Omega. \end{cases} \tag{1}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, which satisfies an uniform exterior cone condition. We study uniformly elliptic fully nonlinear operators F which can be non-convex, non-proper, with nonlinear growth in the gradient of u, and with unbounded dependence in x. Our results apply to general Hamilton-Jacobi-Bellman-Isaacs equations (basic in many applications to geometry, stochastic control theory, large deviations, game theory, see [Li2], [Ni], the surveys [K2], [Ca], and the references there)

$$\sup_{\alpha\in\mathcal{A}} \inf_{\beta\in\mathcal{B}} F^{\alpha,\beta}(u,x) = 0, \tag{2}$$

for arbitrary index sets $\mathcal{A}$, $\mathcal{B}$. Here $F^{\alpha,\beta}(u,x)$ can for instance be the operator

$$\mathrm{tr}\left(A^{\alpha,\beta}(x)D^2u\right) + <Q^{\alpha,\beta}(x)Du, Du> + <b^{\alpha,\beta}(x), Du> + c^{\alpha,\beta}(x)u - f^{\alpha,\beta}(x),$$

where $A^{\alpha,\beta}, Q^{\alpha,\beta}$ are bounded matrices, $A^{\alpha,\beta}(x) \geq \lambda I$ for some $\lambda > 0$, and $b^{\alpha,\beta} \in L^p(\Omega)$, for some $p > N$, $c^{\alpha,\beta}, f^{\alpha,\beta} \in L^N(\Omega)$. Our main theorem is a statement on the solvability of (1) in terms of Lebesgue norms of these coefficients. The results below are new in much more particular cases than (2), and some of them are new even for equations in divergence form.

We often use as a model the following equation (for which all results below are new as well)

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) + \mu(x)|Du|^2 + b(x)|Du| + c(x)u = f(x), \tag{3}$$

$$\text{with} \quad \mu \in L^\infty(\Omega), \quad b \in L^p(\Omega),\ p > N, \quad c, f \in L^N(\Omega), \tag{4}$$

where $\mathcal{M}^+_{\lambda,\Lambda}$ is the Pucci extremal operator. As is well-known, even if $\mathcal{M}^+_{\lambda,\Lambda}(D^2\cdot) + \mu|D\cdot|^2$ is replaced by a linear operator, one cannot take larger Lebesgue spaces in (4) if a general solvability theory for (3) is to be built.

Our hypothesis on the structure of (1) is : $F(0,0,0,x) \equiv 0$, and for some null set $\mathcal{N} \subset \Omega$ and all $M, N \in \mathcal{S}_N(\mathbb{R}), p, q \in \mathbb{R}^N, u, v \in \mathbb{R}, x \in \Omega \setminus \mathcal{N}$,

$$(S)\begin{cases} F(M,p,u,x) - F(N,q,v,x) \leq \mathcal{M}^+_{\lambda,\Lambda}(M-N) + \mu(|p|+|q|)|p-q| \\ \qquad\qquad\qquad\qquad\qquad\qquad +b(x)|p-q| + d(x)\overline{h}(u,v) \\ \\ F(M,p,u,x) - F(N,q,v,x) \geq \mathcal{M}^-_{\lambda,\Lambda}(M-N) - \mu(|p|+|q|)|p-q| \\ \qquad\qquad\qquad\qquad\qquad\qquad -b(x)|p-q| - d(x)\underline{h}(u,v), \end{cases}$$

where $0 < \lambda \leq \Lambda$, $\mu \in \mathbb{R}^+$, $b \in L^p(\Omega)$ for some $p > N$, $d \in L^N(\Omega)$, $\mu, b, d \geq 0$, and $\overline{h}, \underline{h} \in C(\mathbb{R}^2)$. We recall that Pucci's operators are defined by $\mathcal{M}^+_{\lambda,\Lambda}(M) = \sup_{A\in\mathcal{A}} \mathrm{tr}(AM)$, $\mathcal{M}^-_{\lambda,\Lambda}(M) = \inf_{A\in\mathcal{A}} \mathrm{tr}(AM)$, where $\mathcal{A} \subset \mathcal{S}_N$ denotes the set of matrices whose eigenvalues lie in the interval $[\lambda, \Lambda]$.

When we speak of solutions of (1) we mean L^N-viscosity solutions – we refer to [CCKS] for a general review of this notion (we recall some definitions and results in the next section). Note that viscosity solutions are continuous and that any function in $W^{2,N}_{\mathrm{loc}}(\Omega)$ satisfies (1) almost everywhere – such a solution is called strong – if and only if it is a L^N-viscosity solution.

The next theorem is our main result. As usual, we set $a^\pm = \max\{\pm a, 0\}$.

Theorem 1 *Suppose* (S) *holds with* $\underline{h} = h((u-v)^+)$, $\overline{h} = h((v-u)^+)$, *for some continuous function* h, *such that* $h(0) = 0$. *Let* $c \in L^N(\Omega)$. *Then*

(i) if $c(x) \leq -\bar{c}$ *a.e. in* Ω, *for some constant* $\bar{c} > 0$, *then for any data* $f \in L^N(\Omega)$ *and* $\psi \in C(\partial\Omega)$ *there exists a solution* $u \in C(\overline{\Omega})$ *of*

$$\begin{cases} F(D^2u, Du, u, x) + c(x)u = f(x) & \text{in} \quad \Omega \\ u = \psi(x) & \text{on } \partial\Omega. \end{cases} \tag{5}$$

(ii) there exists a positive constant δ_0, *depending on* $\lambda, \Lambda, N, \|b\|_{L^N(\Omega)}$, *and* $\mathrm{diam}(\Omega)$, *such that if* $f \in L^N(\Omega)$ *and* $\psi \in C(\partial\Omega)$ *satisfy*

$$\|\mu|f| + \mu M c^+ + c^+\|_{L^N(\Omega)} < \delta_0, \tag{6}$$

then there exists a solution $u \in C(\overline{\Omega})$ *of (5) (here* $M = \max_{\partial\Omega}|\psi|$*).*

(iii) If problem (5) with $c^+ \equiv 0$ *or with* $\mu = 0$ *and* $\|c^+\|_{L^N(\Omega)} < \delta_0$ *has a strong solution then this solution is the unique viscosity solution of (5).*

(iv) The strong solutions of (5) are not unique if $\mu > 0$ *and* $c(x) \equiv c > 0$ *(arbitrarily small), even for* $\lambda = \Lambda$, $b = d = f = \psi \equiv 0$.

Remark 1. The choice of $\underline{h}, \overline{h}$ in Theorem 1 means F is nonincreasing in u, that is, F is *proper*. Of course the term $c(x)u$ in (5) can be incorporated into F, with corresponding generalizations of the statements. We have preferred not to do so, for clarity.

Remark 2. An explicit expression for the constant δ_0 in statement (ii) can be deduced from its proof – see (18) in Section 3.

We immediately note that the existence hypothesis (6) and the multiplicity statement (iv) are new, even for equations with bounded continuous coefficients or in the divergence framework, where such equations have been studied extensively (see below). All previous works on problems of this type concerned cases when either $c^+ \equiv 0$ or $\mu = 0$. Allowing nonlinear growth in the gradient for nonproper operators yields qualitatively new phenomena, as the facts that uniqueness breaks down and that the solvability depends on the size of the boundary data attest.

Further, the result in (ii) is optimal, in the sense that, even for the simplest equations satisfying our hypotheses, when $\mu = 0$ and c is sufficiently large, or when μf is large and $c \equiv 0$, or when μ and c are small and M is large there may not exist solutions of (5) (see the end of Section 3). Note also that in the framework of non-divergence form operators with measurable ingredients uniqueness of (viscosity) solutions does not hold in general, even for linear equations – see [Na], [Sa]. So, to have a uniqueness result like in Theorem 1 (iii) some additional hypothesis is needed. Generally, the existence of a strong solution is such an assumption, verified for example by operators which are convex or concave in M and $F(M, 0, 0, x)$ is continuous.

Next, we give references and situate Theorem 1 with respect to previous works. The theory of strong solutions of quasilinear uniformly elliptic equations was developed in the classical works of Ladizhenskaya-Uraltseva and Krylov-Safonov, see [LU1], [LU2], [KSa], [K1]. The well-known papers [CIL], [IL] describe the theory of viscosity solutions of proper equations with continuous ingredients ($c^+ = 0$; F, c, f continuous in $x \in \overline{\Omega}$) – the so-called C-viscosity solutions. For proper fully nonlinear equations with linear growth and bounded measurable coefficients (that is, for $\mu = 0, c^+ = 0$, $b, c \in L^\infty(\Omega)$), the solvability of (5) in the L^N-viscosity sense was proved in [CCKS] and [CKLS]. We are going to use all those important papers. More recently, an existence theorem for equations where F is convex in D^2u, $c \equiv 0$, and $b \in L^{2N}$ close to the boundary was stated in the thesis [F1]. The case $\mu > 0$, $b \in L^\infty(\Omega)$ and $c = 0$ is studied in [KS1], where the authors use [F1].

Theorem 1 (i) is the first statement of this type in the noncontinuous setting. It is inspired by works on equations in divergence form (see below), and has as a starting point the main existence result in [CIL] (see pages 25-26 in that paper), which, applied to (5) with $c(x) \leq -\bar{c}$, would require that F, c and f be continuous in $x \in \overline{\Omega}$, and that the growth in the gradient be strictly smaller than 2. We remove these hypotheses here. The case $c^+ \equiv 0$ in statement (ii) unifies and extends (with a different proof) the results in [CCKS], [CKLS], [F1], [KS1] to the optimal Lebesgue range $b \in L^p, p > N, c, d \in L^N$ and general F. Further, and not less importantly, we get existence for *non-proper* equations, that is, without a hypothesis of monotonicity in u of the operator ($c^+ \not\equiv 0$). As we noted, the uniqueness is then lost. In the fully nonlinear setting there are rather few works on non-proper equations - a related result can be found in [QS1], in the particular case of convex positively homogeneous operators with bounded coefficients (like (3) with $\mu \equiv 0$, $b, c \in L^\infty(\Omega)$), when the solutions are still unique. Some nonproper equations with power growth in u were considered in [QS2].

It is important to note that the solvability of elliptic equations with natural (quadratic) growth in the gradient has been studied very extensively in the divergence framework, where weak solutions can be searched for in Sobolev spaces. A typical example is $\mathrm{div}(A(x)Du) + \mu|Du|^2 + b(x).Du + c_0 u = f(x)$, for $b \in L^N(\Omega), f \in L^q(\Omega)$. Note that in this situation $q = N/2$ is the dividing number for the solutions to be bounded and continuous. We refer to [BMP1], [BMP2], [AGP] for the case $c_0 < 0$, and to [MPS], [FM], [GMP] for the case $c_0 = 0$ (see also the references in these works). Uniqueness in the natural Sobolev spaces was proved in [BM], [BBGK], [BP].

Parts (i), (ii) with $c^+ = 0$, and (iii) of Theorem 1 can be seen as counterparts of these results for (fully nonlinear) equations in non-divergence form, for which the natural weak notion of a solution is the viscosity one. Of course the methods in the two frameworks are different. We stress once more that the statements in Theorem 1 (ii) and (iv) are new even for divergence-form equations which satisfy (S). The (actually not difficult) observation in (iv) opens interesting lines of research on multiplicity of solutions - see the remarks after the proof of (iv) in Section 3.

Let us now make a brief account of the main points in the proof of Theorem 1, which we give in Section 3. The proof of (i) relies on the method of sub- and supersolutions, together with smoothing and approximation techniques. In particular, we extend a fundamental approximation theorem (Theorem 3.8 in [CCKS]) to equations with unbounded ingredients and quadratic growth in the gradient – see Theorem 4 in Section 3. For (ii) we use in addition some recent results on existence and properties of eigenvalues of convex positively homogeneous operators, obtained in [QS1]. More specifically, we take advantage of the fact that the positivity of the eigenvalues guarantees the validity of the comparison principle and the solvability of the associated Dirichlet problem. The uniqueness in (iii) is shown to be a consequence of the ABP inequality (given in the next section). Finally, the proof of (iv) is based on a fixed point theorem and Leray-Schauder degree theory.

A pivot role in the proofs of (i), (ii) and (iv) play statements on availability of *a priori estimates* in the uniform norm for solutions of related equations - see Lemma 33, Proposition 34 and the proof of (iv). These results take different form in each of the three cases, and require different proofs.

Another important point is the global Hölder continuity of the solutions of (1). The following theorem, which is of clear independent interest, is used to approximate some of the equations we consider by equations with regular ingredients (as it implies their set of solutions is precompact in $C(\overline{\Omega})$, whenever a priori estimates are available).

Theorem 2 *Suppose* (S) *holds for* $N = 0, q = 0, v = 0$, *and* $u \in C(\Omega)$ *is a solution of (1). Then there exists* $\alpha \in (0,1)$ *depending only on* N, λ, Λ, p, $\|b\|_{L^p(\Omega)}$, *such that* $u \in C^\alpha_{\mathrm{loc}}(\Omega)$, *and for any subdomain* $\Omega' \subset\subset \Omega$ *we have* $\|u\|_{C^\alpha(\Omega')} \le K$, *where* K *depends on* $N, \lambda, \Lambda, \mu, p, \|b\|_{L^p(\Omega)}, \|c\|_{L^N(\Omega)}, \|f\|_{L^N(\Omega)}$, $\mathrm{dist}(\Omega', \partial\Omega)$, $\sup_{\Omega'}|u|$.

If, in addition, $u \in C(\overline{\Omega})$ *and* Ω *satisfies an uniform exterior cone condition (with size L), then there exist some* $\alpha_0, \rho_0 > 0$, *depending only on* $N, \lambda, \Lambda, L, p$, $\|b\|_{L^p(\Omega)}$, *such that for each ball* B_ρ *with radius* $\rho \le \rho_0$ *and center in* $\overline{\Omega}$

$$\operatorname*{osc}_{\Omega\cap B_\rho} u \le K(\rho^{\alpha_0} + \operatorname*{osc}_{\partial\Omega\cap B_{\sqrt{\rho}}} u),$$

K depends on $N, \lambda, \Lambda, \mu, p, \|b\|_{L^p(\Omega)}, \|c\|_{L^N(\Omega)}, \|f\|_{L^N(\Omega)}, L, \sup_\Omega|u|, \mathrm{diam}(\Omega)$.

Hence if $u|_{\partial\Omega} \in C^\beta(\partial\Omega)$ *then* $u \in C^\alpha(\overline{\Omega})$, *with* $\alpha = \min\{\alpha_0, \beta/2\}$.

Hölder estimates for L^N-viscosity solutions of Pucci equations were obtained by Caffarelli in his seminal work [C], see also [CC]. These estimates were extended to operators with bounded coefficients ($\mu, b, c, f \in L^\infty(\Omega)$) in [W]. An alternative approach to the results in [W] can be found in [F1], see Theorem 1.3 there. In these papers the Hölder estimate is obtained as a consequence of a Harnack inequality for the corresponding equation. Here we will make use of another idea, whose essence is that one does not need to prove a Harnack inequality if only Hölder estimates are aimed at; actually, it is enough to have some comparison between the measures of level sets of the solution, which can be achieved by use of simple barriers and the ABP inequality. This method is originally due to Krylov and Safonov. We will develop it in our setting in Section 4 (giving all the proofs in extenso, as we find it important to provide a full quotable source for the Hölder estimates in this generality). Another adaptation to viscosity solutions of the methods of Krylov and Safonov can be found in the proof of the Harnack inequality for Pucci equations in [CC]. The same approach as in [CC] was used in the work [KS3], which we recently received, where the authors prove the weak Harnack inequality (which also implies the Hölder continuity of the solutions, see the remark after Proposition 42) for operators with linear growth ($\mu = 0$) and $f \in L^{N-\varepsilon_0}$, for some $\varepsilon = \varepsilon_0(N, \lambda, \Lambda) > 0$.

Acknowledgement. The author is indebted to A. Swiech and to an anonymous referee, for some very useful comments.

2. Preliminaries

We start by recalling the definition of a viscosity solution of (1).

Definition 21 *We say that $u \in C(\Omega)$ is a L^p-viscosity subsolution (supersolution) of (1), provided for any $\varepsilon > 0$, any open subset $\mathcal{O} \subset \Omega$, and any $\varphi \in W^{2,p}(\mathcal{O})$ – we call φ a test function – such that*

$$\begin{aligned} &F(D^2\varphi(x), D\varphi(x), u(x), x) \le f(x) - \varepsilon \\ &(F(D^2\varphi(x), D\varphi(x), u(x), x) \ge f(x) + \varepsilon) \qquad \text{a.e. in } \mathcal{O}, \end{aligned}$$

the function $u - \varphi$ cannot achieve a local maximum (minimum) in $\mathcal{O}$. In this case we say that u satisfies $F(D^2u, Du, u, x) \ge (\le) f$ in the L^p-viscosity sense in Ω. We say that u is a solution of (1) if u is at the same time a subsolution and a supersolution of (1).

If both F and f are continuous in x and the above definition holds for all $\varphi \in C^2(\mathcal{O})$, then we speak of C-viscosity subsolution (supersolution). We recall that strong and C-viscosity solutions are L^p-viscosity solutions, see [CCKS], and that whenever a function in $W^{2,N}_{\mathrm{loc}}(\Omega)$ satisfies an inequality $F(D^2u, Du, u, x) \ge (\le) f$ a.e. in Ω then it is a viscosity solution.

We shall make essential use of the Generalized Maximum Principle for elliptic equations, commonly known as the Alexandrov-Bakelman-Pucci (ABP) inequality. It states that for any measurable matrix $A(x)$, $\lambda I \le A(x) \le \Lambda I$, $x \in \Omega$, any $b \in L^N(\Omega)^N$, $f \in L^N(\Omega)$, and any $u \in W^{2,N}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ such that

$$\mathrm{tr}(A(x)D^2u) + b(x).Du \ge f(x),$$

we have

$$\sup_{\Omega} u \le \sup_{\partial\Omega} u^+ + C\|f\|_{L^N(\Gamma)},$$

where C depends on $N, \lambda, \Lambda, \|b\|_{L^N(\Omega)}$, $\mathrm{diam}(\Omega)$, and Γ is the upper contact set of u (see Chapter 9 of [GT] for a proof and references).

A breakthrough in the theory of viscosity solutions of uniformly elliptic equations was the extension of this inequality to viscosity solutions of $\mathcal{M}^+_{\lambda,\Lambda}(D^2u) \ge f$, obtained by Caffarelli in [C]. The result was subsequently shown to hold for equations with bounded measurable coefficients in [W], [CCKS], and with unbounded coefficients in [F2]. A simple self-contained proof of the inequality from [F2] can be found in [KS2]. We give it next.

Theorem 3 ([F2], [KS2]) *Suppose $u \in C(\overline{\Omega})$ is a L^N-viscosity solution of*

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) + b(x)|Du| \ge f(x) \qquad (\text{resp. } \mathcal{M}^-_{\lambda,\Lambda}(D^2u) - b(x)|Du| \le f(x)),$$

in Ω^+ (resp. Ω^-), where $b \in L^p(\Omega)$ for some $p > N$, $f \in L^N(\Omega)$, and $\Omega^\pm = \{x \in \Omega : \pm u(x) > 0\}$. Then

$$\sup_{\Omega} u \le \sup_{\partial\Omega} u^+ + \mathrm{diam}(\Omega).C_1\|f^-\|_{L^N(\Omega^+)} \tag{7}$$

(resp. $\sup_{\Omega} u^- \le \sup_{\partial\Omega} u^- + \mathrm{diam}(\Omega).C_1 \|f^+\|_{L^N(\Omega^-)}$*), where* C_1 *is a constant which depends on* N, λ, Λ*,* $\|b\|_{L^N(\Omega)}$*, diam*(Ω)*, and* C_1 *remains bounded when these quantities are bounded.*

In the sequel we denote $C_A := \mathrm{diam}(\Omega).C_1$*.*

Remark 1. See [A], [GT], [KS2] for an explicit expression of C_A.
Remark 2. Theorem 3 is a scaled version (with respect to $\mathrm{diam}(\Omega)$) of either Theorem 1.2 of [F2] or Proposition 2.8 in [KS2]. Actually, the result is based on an upgrade to unbounded coefficients of the basic Lemma 3.1 in [CCKS], where the correct scaling is given.

We recall some easy properties of Pucci operators (see for instance [CC]).

Lemma 21 *Let* $M, N \in \mathcal{S}_N$*,* $\phi(x) \in C(\overline{\Omega})$ *be such that* $0 < a \le \phi(x) \le A$*. Then*

$$\mathcal{M}^-_{\lambda,\Lambda}(M) = -\mathcal{M}^+_{\lambda,\Lambda}(-M),$$

$$\mathcal{M}^-_{\lambda,\Lambda}(M) = \lambda \sum_{\{\nu_i>0\}} \nu_i + \Lambda \sum_{\{\nu_i<0\}} \nu_i, \qquad \text{where } \{\nu_1, \dots, \nu_N\} = \mathrm{spec}(M),$$

$$\mathcal{M}^-_{\lambda,\Lambda}(M) + \mathcal{M}^-_{\lambda,\Lambda}(N) \le \mathcal{M}^-_{\lambda,\Lambda}(M+N) \le \mathcal{M}^-_{\lambda,\Lambda}(M) + \mathcal{M}^+_{\lambda,\Lambda}(N),$$

$$\mathcal{M}^-_{\lambda,\Lambda}(M) + \mathcal{M}^+_{\lambda,\Lambda}(N) \le \mathcal{M}^+_{\lambda,\Lambda}(M+N) \le \mathcal{M}^+_{\lambda,\Lambda}(M) + \mathcal{M}^+_{\lambda,\Lambda}(N),$$

$$\mathcal{M}^-_{\lambda a,\Lambda A}(M) \le \mathcal{M}^-_{\lambda,\Lambda}(\phi M) \le \mathcal{M}^-_{\lambda A,\Lambda a}(M),$$

We shall also use the following simple fact.

Lemma 22 *Suppose* $u \in C^2(B)$ *is a radial function, say* $u(x) = g(|x|)$*, defined on a ball* $B \subset \mathbb{R}^N$*. Then* $g''(|x|)$ *is an eigenvalue of the matrix* $D^2u(x)$*, and* $|x|^{-1}g'(|x|)$ *is an eigenvalue of multiplicity* $N-1$*.*

The following lemma will help us to deal with the quadratic dependence in the gradient.

Lemma 23 *Let* $u \in W^{2,N}_{\mathrm{loc}}(\Omega)$*. For any* $m > 0$ *set*

$$v = \frac{e^{mu} - 1}{m}, \qquad w = \frac{1 - e^{-mu}}{m}.$$

Then a.e. in Ω *we have* $Dv = (1+mv)Du,\ Dw = (1-mw)Du,$

$$m\lambda|Du|^2 + \mathcal{M}^\pm_{\lambda,\Lambda}(D^2u) \ \le\ \frac{\mathcal{M}^\pm_{\lambda,\Lambda}(D^2v)}{1+mv} \ \le\ m\Lambda|Du|^2 + \mathcal{M}^\pm_{\lambda,\Lambda}(D^2u),$$

$$-m\Lambda|Du|^2 + \mathcal{M}^\pm_{\lambda,\Lambda}(D^2u) \ \le\ \frac{\mathcal{M}^\pm_{\lambda,\Lambda}(D^2w)}{1-mw} \ \le\ -m\lambda|Du|^2 + \mathcal{M}^\pm_{\lambda,\Lambda}(D^2u).$$

and, clearly, $u = 0$ *(resp.* $u > 0$*) is equivalent to* $v = 0$ *(resp.* $v > 0$*) and to* $w = 0$ *(resp.* $w > 0$*).*

The same inequalities hold in the L^N-viscosity sense, that is, if, for example, $u \in C(\Omega)$ is a viscosity solution of

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) + \mu|Du|^2 + b(x)|Du| \geq f(x) \tag{8}$$

then $v = \left(e^{(\mu/\lambda)u} - 1\right)/(\mu/\lambda)$ is a viscosity solution of

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2v) + b(x)|Dv| - (\mu/\lambda)f(x)v \geq f(x), \quad \textit{etc.} \tag{9}$$

Proof. This is a matter of an easy computation and use of Lemma 21 and Definition 21. Suppose first that $u \in W^{2,N}_{\text{loc}}(\Omega)$. Then a.e. in Ω

$$Dv = e^{mu}Du, \qquad D^2v = e^{mu}D^2u + mDu \otimes Du,$$

$$Dw = e^{-mu}Du, \qquad D^2w = e^{-mu}D^2u - mDu \otimes Du,$$

and Lemma 23 follows from Lemma 21, since

$$\text{spec}\,(Du \otimes Du) = \{0, \ldots, 0, |Du|^2\}.$$

If u is only continuous and we suppose v does not satisfy (9), then by Definition 21 there exists a function $\psi \in W^{2,N}_{\text{loc}}(\Omega)$ and $\varepsilon > 0$ such that $v - \psi$ attains a maximum in some open set $\mathcal{O} \subset \Omega$, while for $m = \mu/\lambda$

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2\psi) + b(x)|D\psi| \leq f(x)e^{mu} - \varepsilon \qquad \text{in } \mathcal{O}.$$

By setting $\phi = (1/m)\log(1 + m\psi)$ in this inequality we get a contradiction with the fact that u satisfies (8) in the sense of Definition 21, since $u - \phi$ attains a maximum in $\mathcal{O}$. $\square$

3. Proof of Theorem 1

For any measurable set $A \subset \mathbb{R}^N$ we denote the Lebesgue measure of A by $|A|$ or meas(A). The first lemma concerns the existence of subsolutions and supersolutions in a simple continuous setting.

Lemma 31 *Suppose $\partial\Omega$ is of class C^2. For any positive constants μ, b, c, k, there exist C-viscosity solutions u_1, u_2, such that $u_1 \leq 0 \leq u_2$ in Ω, $u_i = 0$ on $\partial\Omega$, of*

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u_2) + \mu|Du_2|^2 + b|Du_2| - cu_2 \leq -k,$$
$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u_1) - \mu|Du_1|^2 - b|Du_1| - cu_1 \geq k.$$

We can take $u_1 = -u_2$ and $\|u_2\|_{L^\infty(\Omega)} \leq (\lambda/\mu)\left(e^{(\mu k)/(\lambda c)} - 1\right)$.

Proof. The proof of this lemma is based on techniques described in [CIL]. Let us prove that the first inequality has a solution (note the second inequality is obtained from the first by the change $u \to -u$). In view of Lemma 23, it is enough to construct a nonnegative solution of

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2v) + b|Dv| - (c/m)(1+mv)\log(1+mv) \leq -k(1+mv), \quad (10)$$

such that $v = 0$ on $\partial\Omega$, with $m = \mu/\lambda, v$ defined in Lemma 23.

To avoid writing constants, suppose for simplicity $c = m = 1$ (the important point is that $c > 0$). Then $v_1 \equiv e^k - 1 =: B$ is a solution of (10) in Ω. We want to find a neighbourhood of $\partial\Omega$, denoted by $\Omega_\alpha = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < \frac{1}{\alpha}\}$, and a function v_2 which satisfies (10) in Ω_α, such that $v_2 = v_1 + 1$ on $\partial\Omega_\alpha$ and $v_2 = 0$ on $\partial\Omega$. Then the function

$$v = \begin{cases} v_1 & \text{in } \Omega \setminus \Omega_\alpha \\ \min\{v_1, v_2\} & \text{in } \quad \Omega_\alpha \end{cases}$$

is a solution of (10), since the minimum of two viscosity supersolutions is a viscosity supersolution.

So we set $v_2 = (B+1)(1-e^{-1})^{-1}\left(1 - e^{-\alpha d(x)}\right)$ in Ω_α, where $d(x)$ is the distance function to the boundary and α is chosen sufficiently large so that d is C^2 in Ω_α.

Let

$$K = \max_{t \in [0, A+1]} (1+t)(k - \log(1+t)) = \max\{k, e^{k-1}\}.$$

Then, by computing Dv_2 and D^2v_2, and by using the facts that $|Dd| = 1$ and that D^2d is bounded in Ω_α (see for instance Chapter 14.16 in [GT]) we get, with the help of Lemma 21,

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2v_2) + b|Dv_2| \leq -C_1\alpha^2 + C_2\alpha < -K,$$

if α is large enough ; here C_1, C_2 depend on b, B, λ, Λ, and $\partial\Omega$. □

Next we recall the following *comparison* result for C-viscosity solutions, obtained in [IL].

Proposition 31 *Under the conditions of Theorem 1 (i), suppose in addition that $c, f \in C(\overline{\Omega})$, F is continuously differentiable in $x \in \overline{\Omega}$, and for each $R > 0$ there exists $C_R > 0$ such that*

$$\frac{\partial F}{\partial x}(M, p, u, x) \leq C_R(1 + |p|^2 + \|M\|),$$

for all $x \in \overline{\Omega}, u \in [-R, R], p \in \mathbb{R}^N, M \in \mathcal{S}_N$. Then the comparison principle holds for C-viscosity solutions of (1), that is, if $u, v \in C(\overline{\Omega})$ satisfy

$$F(D^2u, Du, u, x) + c(x)u \geq f(x) \quad \textit{and} \quad F(D^2v, Dv, v, x) + c(x)v \leq f(x)$$

in Ω, and $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.

Proof. This is a particular case of Theorem III.1 (1) of [IL], by taking $\omega_R(s) = \mu_R(s) = C_R s$ in hypotheses (3.2) and (3.3) of that paper[1]. □

Corollary 31 *Under the conditions of Proposition 31, problem (5) has a unique C-viscosity solution, provided* $\psi \in C^2(\overline{\Omega})$.

Proof. Suppose first $\partial\Omega$ is smooth. From (S) and Lemma 31 (we replace u by $u - \psi$, that is, k by $k(1 + \|\psi\|_{C_2(\overline{\Omega})} + \|\psi\|^2_{C_1(\overline{\Omega})})$ in that lemma) we infer that (5) has a subsolution and a supersolution which are ordered. Then the existence of a solution of (5) is an immediate consequence of Proposition 31 and Perron's method - Proposition II.1 in [IL].

To extend the result to a domain Ω which satisfies only the uniform exterior cone condition with size L, we approximate Ω by smooth domains Ω_n, which admit exterior cones with size at least $L/2$, such that $\Omega \subset \Omega_n$, and take solutions u_n of (5) in Ω_n. By Lemma 31 $\{u_n\}$ can be taken to be uniformly bounded in $L^\infty(\Omega)$. Then, by our Theorem 2, $\{u_n\}$ is uniformly bounded in $C^\alpha(\Omega_n)$, so, by the compact embedding $C^\alpha \hookrightarrow C^0$, a subsequence of u_n converges uniformly in $\overline{\Omega}$ to a function u, which is then a solution of (5) in Ω, by the convergence results in Chapter 6 of [CIL]. □

The next proposition asserts the existence of *strong* subsolutions and supersolutions of extremal equations of our type.

Proposition 32 *For any* $\mu \geq 0$, $c > 0$, $b \in L^p(\Omega)$ $(p > N)$, $f \in L^N(\Omega)$, $b, f \geq 0$, *there exist* strong *solutions* u_1, u_2 *of*

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u_2) + \mu|Du_2|^2 + b(x)|Du_2| - cu_2 \leq -f(x),$$
$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u_1) - \mu|Du_1|^2 - b(x)|Du_1| - cu_1 \geq f(x),$$

such that $u_1 \leq 0 \leq u_2$ *in* Ω, $u_i = 0$ *on* $\partial\Omega$.

We shall use the following lemmas.

Lemma 32 *Let* $c, m > 0$ *and* $b, f \in C(\overline{\Omega})$. *Then there exists a strong solution of the equation*

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2v) + b(x)|Dv| = (f(x) + (c/m)\log(1 + mv))(1 + mv) \qquad (11)$$

in Ω, $v = 0$ *on* $\partial\Omega$.

Proof. We take the solutions of the problems

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u_2) + m\Lambda|Du_2|^2 + \|b\|_{L^\infty(\Omega)}|Du_2| - cu_2 \leq -\|f\|_{L^\infty(\Omega)},$$
$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u_1) + m\lambda|Du_1|^2 - \|b\|_{L^\infty(\Omega)}|Du_1| - cu_1 \geq \|f\|_{L^\infty(\Omega)},$$

[1] For the reader's convenience we note that there is a misprint in [IL] - the assumption on ω_R in (3.3) there should read $\omega_R/(1+r)$ is bounded, as an inspection of the proof of Theorem III.1 shows ((3.3) is used in (3.14) and the argument after Lemma III.1 in [IL]).

given by Lemma 31. Then by Lemma 23 the functions $v_i = (1/m)(e^{mu_i} - 1)$ are respectively a subsolution and a supersolution of (11). Hence, by the standard sub- and supersolution method, this problem has a solution $v \in C(\overline{\Omega})$ (note the right-hand side of (11) is locally Lipschitz in v). By the well-known regularity results for the equation $\mathcal{M}^+_{\lambda,\Lambda}(D^2v)+b(x)|Dv| = g(x)$ (see [S]) this solution is strong, $v \in W^{2,p}_{\mathrm{loc}} \cap C(\overline{\Omega})$, $p < \infty$. □

Lemma 33 *Let $c, m > 0$ and $\{b_n\} \subset L^p(\Omega)$ $(p > N)$, $\{f_n\} \subset L^N(\Omega)$ be sequences such that $\{b_n\}$ is bounded in $L^N(\Omega)$ and $\{f_n\}$ converges strongly in $L^N(\Omega)$. Then each suite $\{v_n\} \subset C(\overline{\Omega})$ (resp. $\{w_n\} \subset C(\overline{\Omega})$) of solutions of the inequation*

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2v_n) + b_n(x)|Dv_n| \geq (f_n(x) + (c/m)\log(1 + mv_n))(1 + mv_n)$$
$$\mathcal{M}^-_{\lambda,\Lambda}(D^2w_n) - b_n(x)|Dw_n| \leq (f_n(x) + (c/m)\log(1 + mw_n))(1 + mw_n)$$

in Ω is such that $1+mv_n \leq \overline{a}$ (resp. $1+mw_n \geq \underline{a}$) on $\partial\Omega$ for some $\overline{a}(\underline{a}) > 0$ implies

$$1 + mv_n \leq C_0 \qquad (\text{ resp. } 1 + mw_n \geq c_0) \quad \text{ in } \Omega,$$

where c_0, C_0 are positive constants independent of n.

Proof. Take for simplicity $m = c = 1$ and set $z_n = 1 + v_n$, $\widetilde{z}_n = 1 + w_n$. Note $z_n, \widetilde{z}_n > 0$ in Ω and $z_n \leq \overline{a}$, $\widetilde{z}_n \geq \underline{a} > 0$ on $\partial\Omega$. For each $a > 0$ let $\Omega^n_a = \{x \in \Omega \,:\, z_n > a\}$ and $\omega^n_a = \{x \in \Omega \,:\, \widetilde{z}_n < a\}$. We need to show that $\Omega^n_a = \emptyset$ for all n if a is sufficiently large, and that $\omega^n_a = \emptyset$ for all n if a is close to zero.

Since $\{f_n\}$ converges strongly in L^N for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for each $G \subset \Omega$, $|G| < \delta$ implies $\|f_n\|_{L^N(G)} < \varepsilon$ for all n. Set $\varepsilon_0 = 1/(2\overline{C}_A)$ and $\delta_0 = \delta(\varepsilon_0)$, where $\overline{C}_A$ is an upper bound for the constants $C_A(n, \widetilde{\Omega})$ which appear in the ABP inequality for the operator $\mathcal{M}^+_{\lambda,\Lambda}(D^2\cdot)+b_n|D\cdot|$ in any domain $\widetilde{\Omega} \subset \Omega$ – see Theorem 3. By this theorem $\overline{C}_A$ can be taken to depend only on N, λ, Λ, $\mathrm{diam}(\Omega)$, and $\sup_n \|b_n\|_{L^N(\Omega)}$.

Next, since $\{f_n\}$ converges strongly in L^N we can find $a > \overline{a}$ sufficiently large so that $|\{f_n < -\log a\}| < \delta_0$ for all n. If $\Omega^n_a = \emptyset$ for all n we are done. If on the other hand $\Omega^n_a \neq \emptyset$ for some n we apply the ABP inequality in Ω^n_a to

$$\begin{aligned}\mathcal{M}^+_{\lambda,\Lambda}(D^2z_n) + b_n(x)|Dz_n| &= (f_n + \log z_n)z_n \\ &\geq -(f_n + \log a)^- z_n,\end{aligned}$$

getting

$$\begin{aligned}\sup_{\Omega^n_a} z_n &\leq a + \overline{C}_A\|(f_n + \log a)^- z_n\|_{L^N(\Omega^n_a)} \\ &\leq a + \overline{C}_A\|(f_n + \log a)^-\|_{L^N(\Omega)} \sup_{\Omega^n_a} z_n \\ &\leq a + \overline{C}_A\|f_n^-\|_{L^N(\{f_n < -\log a\})} \sup_{\Omega^n_a} z_n \leq a + 1/2 \sup_{\Omega^n_a} z_n\end{aligned}$$

so Ω^n_{2a} is empty for all n.

In order to prove the existence of c_0 we take some $a \in (0, \underline{a})$ sufficiently small to have $|\{f_n > -\log a\}| < \delta_0$ for all n, we apply the ABP inequality to $\mathcal{M}^+_{\lambda,\Lambda}(D^2\widetilde{z}_n) + b_n(x)|D\widetilde{z}_n| \le (f_n + \log a)^+\widetilde{z}_n$ in $\omega^n_a \neq \emptyset$, and get

$$\inf_{\omega^n_a} \widetilde{z}_n \ge a - \overline{C}_A\|(f_n + \log a)^+\|_{L^N(\Omega)} \sup_{\omega^n_a} \widetilde{z}_n \ge a/2,$$

so $\omega^n_{a/2}$ is empty for all n. □

Corollary 32 *Suppose $v_n = w_n$ satisfy both inequalities in Lemma 33 and $v_n = w_n = 0$ on $\partial\Omega$. Set $v_n = (1/m)(e^{mu_n} - 1)$. Then the sequence $\{u_n\}$ is bounded in $L^\infty(\Omega)$.*

Proof. This trivially follows from Lemma 23 and Lemma 33. □

Corollary 33 *Let $\{\mu_n\}, \{b_n\}, \{c_n\}, \{f_n\}$ be sequences of measurable functions, such that $\{\mu_n\}$ is bounded in $L^\infty(\Omega)$, $\{b_n\} \subset L^p(\Omega)$ $(p > N)$ is bounded in $L^N(\Omega)$, $\{f_n\}$ converges strongly in $L^N(\Omega)$, and $c_n(x) \le -\overline{c}$, for some constant $\overline{c} > 0$. Then each sequence $\{u_n\} \subset C(\overline{\Omega})$ of solutions of*

$$\begin{cases} \mathcal{M}^+_{\lambda,\Lambda}(D^2u_n) + \mu_n(x)|Du_n|^2 + b_n(x)|Du_n| + c_n(x)u_n \ge -f_n^-(x) \\ \mathcal{M}^-_{\lambda,\Lambda}(D^2u_n) - \mu_n(x)|Du_n|^2 - b_n(x)|Du_n| + c_n(x)u_n \le f_n^+(x) \end{cases}$$

is bounded in $L^\infty(\Omega)$, provided it is bounded in $L^\infty(\partial\Omega)$.

Proof. The maximum of subsolutions is a viscosity subsolution, so $u_n^+ = \max\{u_n, 0\}$ is a solution of

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u_n^+) + \mu_n(x)|Du_n^+|^2 + b_n(x)|Du_n^+| - \overline{c}u_n^+ \ge -f_n^-(x),$$

(recall $c_n(x) \le -\overline{c} < 0$). We then set $v_n = (1/m)(e^{mu_n^+} - 1)$, with $m = (\sup_n \|\mu_n\|/\lambda)$, and get the first inequality of Lemma 33. Similarly, since the minimum of supersolutions is a supersolution we get

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2(-u_n^-)) - \mu_n(x)|D(-u_n^-)|^2 - b_n(x)|D(-u_n^-)| + c_n(x)(-u_n^-) \le f_n^+(x),$$

so

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u_n^-) + \mu_n(x)|Du_n^-|^2 + b_n(x)|Du_n^-| - \overline{c}u_n^- \ge -f_n^+(x),$$

and we conclude as for u_n^+. □

Proof of Proposition 32. We take sequences of continuous functions b_n, f_n, which approximate b, f in L^p, L^N respectively. By Lemma 32 we know that the problem

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2v_n) + b_n(x)|Dv_n| = ((\Lambda/\lambda)f_n(x) + cu_n)(1 + mv_n), \tag{12}$$

has a strong solution v_n, with $v_n = 0$ on $\partial\Omega$ (we have set $m = \mu/\Lambda$ and $v_n = (1/m)(e^{mu_n} - 1)$). Since the right-hand side of (12) is bounded in $L^N(\Omega)$ – note Corollary 32 shows that $\{u_n\}$ is bounded in $L^\infty(\Omega)$ – by applying the uniform C^α-estimate (Theorem 2), we see that u_n, v_n are bounded in

$C^\alpha(\Omega)$, so a subsequence of u_n converges uniformly on $\overline{\Omega}$ to a function u, and $v_n = (1/m)(e^{mu_n} - 1) \rightrightarrows v = (1/m)(e^{mu} - 1)$.

The right-hand side of (12) converges in $L^N(\Omega)$, while for each $\mathcal{O}$ in Ω and each $\phi \in W^{2,N}(\mathcal{O})$

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2\phi) + b_n(x)|D\phi| \longrightarrow \mathcal{M}^+_{\lambda,\Lambda}(D^2\phi) + b(x)|D\phi| \quad \text{in} \;\; L^N(\mathcal{O}).$$

Note $W^{2,N}$ is embedded in $W^{1,q}$ for all $q < \infty$, so the convergence of the term $b_n(x)|D\phi|$ in L^N is a simple consequence of $p > N$ and the Hölder inequality.

Hence we are in position to apply Theorem 3.8 in [CCKS], which shows that we can pass to the limit in (12) and

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2v) + b(x)|Dv| = ((\Lambda/\lambda)f(x) + cu)(1 + mv).$$

So, by Lemma 23,

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) + \mu(\lambda/\Lambda)|Du|^2 + b(x)|Du| - cu \leq (\Lambda/\lambda)f(x),$$

and we conclude by replacing u by $(\Lambda/\lambda)u$. Note that we can show the uniform boundedness of v_n (or u_n) in $W^{2,N}_{\rm loc}(\Omega)$ by applying to (12) the same cut-off argument as the one used in the proof of Lemma 3.1 in [CCKS]. A precise upgrade of this result to coefficients $b \in L^p(\Omega), p > N$, is given in Proposition 2.6 in [KS2]; actually, (12) can be treated exactly like equation (2.8) in [KS2]. So a subsequence of u_n converges weakly to u also in $W^{2,N}_{\rm loc}(\Omega)$, and u is a strong solution.

The second inequality in Proposition 32 is obtained from the first by the change $u \to -u$. □

Remark. Strictly speaking, the operator $\mathcal{M}^+_{\lambda,\Lambda}(D^2\cdot) + b(x)|D\cdot|$ does not satisfy the hypothesis of Theorem 3.8 in [CCKS], since $b \notin L^\infty(\Omega)$. However the proof of this theorem can be repeated without modifications for this operator, only at its end we have to note that solutions of

$$\begin{cases} \mathcal{M}^+_{\lambda,\Lambda}(D^2\phi_n) + b_n(x)|D\phi_n| = f_n \;\text{in}\;\; \Omega \\ \phi_n = 0 \quad \text{on}\; \partial\Omega \end{cases}$$

where $b_n \in L^\infty(\Omega)$ and $\{b_n\}$ is bounded in $L^N(\Omega)$, (these ϕ_n exist, by the results in [CCKS], [CKLS]) are such that $\phi_n \rightrightarrows 0$ in $\overline{\Omega}$ if $f_n \to 0$ in $L^N(\Omega)$, by the ABP inequality (Theorem 3). □

One of the consequences of this result is a general approximation theorem for operators of our type. It extends Theorem 3.8 in [CCKS] to operators with unbounded coefficients and natural growth in the gradient.

Theorem 4 *Suppose F_n, F are operators which satisfy (S) with $\underline{h}, \overline{h}$ as in Theorem 1. Suppose $f_n, f \in L^N(\Omega)$ and $u_n, u \in C(\Omega)$ are such that u_n is a supersolution (subsolution) of*

$$F_n(D^2u_n, Du_n, u_n, x) = f_n \qquad \text{in} \;\; \Omega, \quad \text{for each} \;\; n,$$

and u_n converges to u locally uniformly in Ω. If for any ball $B \subset \Omega$ and any $\phi \in W^{2,N}(B)$, setting

$$g_n = F_n(D^2\phi, D\phi, u_n, x) - f_n, \quad g = F(D^2\phi, D\phi, u, x) - f(x),$$

we have

$$\|(g - g_n)^+\|_{L^N(B)} \longrightarrow 0 \qquad \left(\|(g - g_n)^-\|_{L^N(B)} \longrightarrow 0\right),$$

then u is a supersolution (subsolution) of $F(D^2u, Du, u, x) = f(x)$ in Ω.

Proof. The proof is identical to the proof of Theorem 3.8 in [CCKS], using (S) and Proposition 32 at the end (we write $F_n - u_n = f_n - u_n = \widetilde{f_n}$ so that Proposition 32 applies to $F_n - u_n$). □

Proof of Theorem 1 (i). With the previous results at hand, the proof is carried out with the help of a standard smoothing argument. For instance, if we have to solve the Dirichlet problem for the model equation (3), we take smooth functions $\mu_n, b_n, c_n, f_n, \psi_n$ which converge to μ, b, c, f, ψ respectively in $L^q(\Omega)$ for all $q < \infty$, $L^p(\Omega)$, $L^N(\Omega)$, $C(\partial\Omega)$, and $\|\mu_n\|_{L^\infty} \leq \|\mu\|_{L^\infty} + 1$, $c_n \leq -\overline{c}/2$. Then by Corollary 31 the approximating problems

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) + \mu_n(x)|Du|^2 + b_n(x)|Du| + c_n(x)u = f_n(x),$$

have solutions u_n, with $u_n = \psi_n$ on $\partial\Omega$. By Corollary 33 $\{u_n\}$ is bounded in $L^\infty(\Omega)$. By Theorem 2 $\{u_n\}$ satisfies the conditions of the Arzela-Ascoli theorem, and hence converges (up to a subsequence) uniformly in $\overline{\Omega}$. Then the solvability of (3) follows from Theorem 4, noting again that $b_n|D\phi| \to b|D\phi|$ and $\mu_n|D\phi|^2 \to \mu|D\phi|^2$ for any $\phi \in W^{2,N} \hookrightarrow W^{1,q}, q < \infty$.

For general F we can use the same regularization argument as in the proof of Theorem 4.1 in [CKLS]. Let f_n, c_n be sequences of continuous functions which converge to f, c in $L^N(\Omega)$, $c_n \leq -\overline{c}/2$, and set

$$F_n(M, p, u, x) = n^N \int_{\mathbb{R}^N} \eta\left(n(x - y)\right) F(M, p, u, y)\, dy, \tag{13}$$

where $\eta \geq 0, \eta \in C^\infty$, is an usual mollifier with compact support and mass 1. Now, for fixed n, the operator $F_n + c_n$ satisfies the conditions of Corollary 31, so the problem $F_n + c_n = f_n$ has a solution u_n. Then we conclude again with the help of Corollary 33, Theorem 2, and the approximation Theorem 4, noting that

$$F_n(D^2\phi, D\phi, u_n, x) \to F(D^2\phi, D\phi, u, x)$$

is a consequence of the Lebesgue dominated convergence theorem. Part (i) of Theorem 1 is proved. □

Now we turn to the proof of Theorem 1 (ii). We shall use the notion of first eigenvalues for convex fully nonlinear elliptic operators with bounded coefficients, recently developed in [QS1] (previous works include [Be], [Li1], [FQ], [BEQ]). We need the following particular case of the results in [QS1].

Theorem 5 ([QS1]) *Given $\lambda,\Lambda>0$, and $b,c\in L^\infty(\Omega)$, $b\ge 0$, there exist numbers $\lambda_1^+\le\lambda_1^-$, and functions $\varphi_1^+,\varphi_1^-\in W^{2,p}_{\rm loc}(\Omega)\cap C(\overline{\Omega})$ for each $p<\infty$, such that*

$$\begin{cases} \mathcal{M}^+_{\lambda,\Lambda}(D^2\varphi_1^\pm)+b(x)|D\varphi_1^\pm|+c(x)\varphi_1^\pm=-\lambda_1^\pm\varphi_1^\pm \ \ in \ \Omega\\ \qquad\qquad \pm\varphi_1^\pm>0 \ \ in \ \ \Omega, \qquad \varphi_1^\pm=0 \qquad\qquad on\ \partial\Omega,\end{cases}$$

In addition, $\lambda_1^+>0$ is a sufficient condition for the Dirichlet problem

$$\begin{cases} \mathcal{M}^+_{\lambda,\Lambda}(D^2u)+b(x)|Du|+c(x)u=f \ \ in \ \Omega\\ \qquad\qquad\qquad\qquad\qquad u=0 \ \ on\ \partial\Omega\end{cases}$$

to have a unique viscosity solution $u\in W^{2,N}_{\rm loc}(\Omega)\cap C(\overline{\Omega})$, for any $f\in L^N(\Omega)$.

The following proposition gives a bound on the eigenvalues in terms of Lebesgue norms of the coefficients.

Proposition 33 *Given $\lambda,\Lambda>0$, and $b,c\in L^\infty(\Omega)$, $b\ge0$, the number λ_1^+ defined in Theorem 5 satisfies*

$$\lambda_1^+\ge|\Omega|^{-1/N}\left(C_A^{-1}-\|c^+\|_{L^N(\Omega)}\right),$$

where C_A is the ABP constant from Theorem 3. Recall C_A depends only on $\lambda,\Lambda,N,\|b\|_{L^N(\Omega)}$, diam$(\Omega)$.

Proof. We apply the ABP inequality (Theorem 3) to

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2\varphi_1^+)+b(x)|D\varphi_1^+|=-(\lambda_1^++c(x))\varphi_1^+\ge-(\lambda_1^++c^+(x))\varphi_1^+,$$

which yields

$$\sup_\Omega\varphi_1^+\le C_A\|\lambda_1^++c^+(x)\|_{L^N(\Omega)}\sup_\Omega\varphi_1^+,$$

so

$$\lambda_1^+|\Omega|^{1/N}+\|c^+\|_{L^N(\Omega)}\ge\|\lambda_1^++c^+(x)\|_{L^N(\Omega)}\ge 1/C_A,$$

and the result follows. □

We can now deduce a first result on solvability for non-proper equations with unbounded coefficients.

Proposition 34 *Given $\lambda,\Lambda>0$, and functions $b\in L^p(\Omega)$ $(p>N)$, $b\ge0$, $c\in L^N(\Omega)$,*

$$\|c^+\|_{L^N(\Omega)}<C_A^{-1}$$

is a sufficient condition for the problem

$$\begin{cases} \mathcal{M}^+_{\lambda,\Lambda}(D^2u)+b(x)|Du|+c(x)u=f \ \ in \ \Omega\\ \qquad\qquad\qquad\qquad\qquad u=0 \ \ on\ \partial\Omega\end{cases}\tag{14}$$

to have a unique viscosity solution $u\in W^{2,N}_{\rm loc}(\Omega)\cap C(\overline{\Omega})$, for any $f\in L^N(\Omega)$. In addition, we have the estimate

$$\|u\|_{L^\infty(\Omega)}\le\frac{C_A}{1-C_A\|c^+\|_{L^N(\Omega)}}\|f\|_{L^N(\Omega)},$$

and $f\le(\ge)0$ in Ω implies $u\ge(\le)0$ in Ω.

Proof. We approximate b, c in L^p, L^N by sequences of continuous functions b_n, c_n, such that $\|c_n^+\|_{L^N(\Omega)} < C_A^{-1}$, so the first eigenvalues of the operators $\mathcal{M}^+_{\lambda,\Lambda}(D^2\cdot)+b_n|D\cdot|+c_n\cdot$ are positive for all n – by the previous proposition. Hence, by Theorem 5, there exist strong solutions of

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u_n) + b_n|Du_n| + c_nu_n = f$$

with $u_n = 0$ on $\partial\Omega$. Note that $f \le (\ge)0$ implies $u_n \ge (\le)0$ in Ω by the maximum principle, which was shown in [QS1] to hold for operators with positive eigenvalues.

Next, we apply the ABP inequality (Theorem 3) to

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u_n) + b_n|Du_n| = -c_nu_n + f \ge -c_n^+u_n + f \qquad \text{on } \{u_n > 0\}$$
$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u_n) + b_n|Du_n| = -c_nu_n + f \le -c_n^+u_n + f \qquad \text{on } \{u_n < 0\}$$

to get

$$\sup_\Omega |u_n| \le C_A^{(n)}\left(\|c_n^+\|_{L^N(\Omega)}\sup_\Omega|u_n| + \|f\|_{L^N(\Omega)}\right),$$

where $C_A^{(n)}$ is the ABP constant for the operator $\mathcal{M}^+_{\lambda,\Lambda}(D^2\cdot) + b_n|D\cdot|$ in Ω. Clearly $C_A^{(n)} \to C_A$, so for n sufficiently large we have

$$\sup_\Omega |u_n| \le \frac{C_A^{(n)}}{1 - C_A^{(n)}\|c_n^+\|_{L^N(\Omega)}}\|f\|_{L^N(\Omega)},$$

that is, u_n is bounded in $L^\infty(\Omega)$. Then, by Theorem 2, u_n is bounded in $C^\alpha(\Omega)$ so a subsequence of u_n converges uniformly in $\overline{\Omega}$ to a solution u of problem (14).

Further, $\mathcal{M}^+_{\lambda,\Lambda}(D^2u_n) + b_n|Du_n| = -c_nu_n + f$ in Ω implies that u_n is bounded in $W^{2,N}_{\text{loc}}(\Omega)$, hence converges weakly in that space, so u is a strong solution. The boundedness in $W^{2,N}_{\text{loc}}$ is proved again as in the proofs of Lemma 3.1 in [CCKS] and Proposition 2.6 in [KS2] .

Finally, if (14) has another solution v, by Lemma 21 the function $w = u - v$ satisfies $w = 0$ on $\partial\Omega$ and

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2w) + b|Dw| \ge -cw \ge -c^+w \qquad \text{on } \{w > 0\}.$$

Applying the ABP inequality in $\{w > 0\}$ and $\|c^+\|_{L^N(\Omega)} < C_A^{-1}$ imply this set is empty, that is, $w \le 0$ in Ω. The same holds for $-w$, so $w \equiv 0$ in Ω. □

Next, we prove a key result on existence of strong subsolutions and supersolutions of (5).

Proposition 35 *Given $\lambda, \Lambda, \mu > 0$, $M \ge 0$, $b \in L^p(\Omega)(p > N)$, $b \ge 0$, $c \in L^N(\Omega)$, and an operator F satisfying the hypotheses of Theorem 1 (ii),* *there exists a constant* $\overline{\delta} > 0$ *depending only on* $\lambda, \Lambda, N, \|b\|_{L^N(\Omega)}$, *diam*$(\Omega)$, *such that*

$$\|\mu|f| + \mu M c^+ + c^+\|_{L^N(\Omega)} \leq \overline{\delta}$$

is a sufficient condition for the existence of functions $\underline{u}, \overline{u} \in W^{2,N}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$, *such that* $\underline{u} \leq 0 \leq \overline{u}$ *in* Ω,

$$F(D^2\underline{u}, D\underline{u}, \underline{u}, x) + c\underline{u} \geq f, \qquad F(D^2\overline{u}, D\overline{u}, \overline{u}, x) + c\overline{u} \leq f \quad \text{in} \quad \Omega,$$

and $\underline{u} = -M$, $\overline{u} = M$ *on* $\partial\Omega$. *In addition, the* L^∞*-norms of* $\underline{u}, \overline{u}$ *are bounded by a constant depending only on* $\lambda, \Lambda, \mu, M, N, \|b\|_{L^N(\Omega)}$, *and diam*$(\Omega)$.

Proof. Let us prove the existence of a positive supersolution. In view of (S) it is enough to find u such that $u \geq 0$ in Ω,

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) + \mu|Du|^2 + b(x)|Du| + c(x)u \leq -f^-(x) - Mc^+(x) \qquad \text{in} \quad \Omega,$$

and $u = 0$ on $\partial\Omega$ (then we take $\overline{u} = u + M$). If $\mu = 0$ the existence of u follows from Proposition 34. So let $\mu > 0$. We set $v = (1/m)(e^{mu} - 1)$ like in Lemma 23, $m = \mu/\lambda$, and $g(x) := f^-(x) + Mc^+(x) \geq 0$. By Lemma 23 we see that it is enough to find a solution $v \geq 0$ of

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2v) + b(x)|Dv| + mg(x)v \leq -g(x) - \frac{c^+(x)}{m}(1+mv)\,\log(1+mv) \quad (15)$$

in Ω, $v = 0$ on $\partial\Omega$.

Let $v \in W^{2,N}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ be the solution of the problem

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2v) + b(x)|Dv| + mg(x)v = -g(x) - \frac{1}{m}\,c^+(x)(1 + mv),$$

that is, of

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2v) + b(x)|Dv| + (mg(x) + c^+(x))v = -\frac{1}{m}(mg(x) + c^+(x)),$$

in Ω, with $v = 0$ on $\partial\Omega$. This problem is solvable and its solution is positive, by Proposition 34, provided

$$\|mg(x) + c^+(x)\|_{L^N(\Omega)} \leq \overline{\delta}, \quad (16)$$

for any $\overline{\delta} \in (0, C_A^{-1})$. Then, by the same proposition,

$$\|v\|_{L^\infty(\Omega)} \leq \frac{C_A\overline{\delta}}{m(1 - C_A\overline{\delta})}\,. \quad (17)$$

Note the function v is a solution of inequality (15) if $\log(1 + mv(x)) \leq 1$ for all $x \in \Omega$, which by (17) follows from

$$\overline{\delta} = (1 - 1/e)\,C_A^{-1}.$$

The existence of a subsolution is proved analogously. □

Remark. The number $\overline{\delta}$ which we found in the last proof gives δ_0 which appears in Theorem 1 (ii). We are going to see that problem (5) has a solution provided

$$\|\mu|f| + \mu M c^+ + \lambda c^+\|_{L^N(\Omega)} < \delta_0 = \lambda\overline{\delta} = \lambda\left(1 - \frac{1}{e}\right) C_A^{-1} \qquad (18)$$

(this implies (16)), where C_A is the ABP constant from Theorem 3.

This value of δ_0 should be improvable. In particular, a solution v of (15) can be found provided the first eigenvalues of the "linearized at zero" operator $\mathcal{M}^+_{\lambda,\Lambda}(D^2\cdot) + b(x)|D\cdot| + (mg(x) + c^+(x))\cdot$ are positive (note these eigenvalues are still to be defined since this operator has unbounded coefficients) and the function $g(x)$ (which appears in the right hand side of (15) as well) is not too large, in an appropriate sense. These questions will be taken up elsewhere.

Proposition 36 *Under the conditions of Theorem 1 (ii), suppose in addition that $c, f \in C(\overline{\Omega})$, F is continuously differentiable in $x \in \overline{\Omega}$, and for each $R > 0$ there exists $C_R > 0$ such that*

$$\frac{\partial F}{\partial x}(M, p, u, x) \le C_R(1 + |p|^2 + \|M\|),$$

for all $x \in \overline{\Omega}, u \in [-R, R], p \in \mathbb{R}^N, M \in \mathcal{S}_N$.

Then there exists a solution $u \in C(\overline{\Omega})$ of (5).

Proof. Suppose (18) holds and let $\underline{u}, \overline{u}$ be the subsolution and supersolution obtained in Proposition 35. We solve the hierarchy of problems in Ω

$$\begin{cases} F(D^2u_n, Du_n, u_n, x) - (c^-(x) + 1)u_n = f(x) - (c^+(x) + 1)u_{n-1} \\ \qquad u_0 = \underline{u}, \qquad u_n = \psi(x) \quad \text{on} \quad \partial\Omega, \quad n \ge 1. \end{cases}$$

Each of these problems is solvable, by Corollary 31 (or by Theorem 1 (i), which we already proved). It is easily seen, by induction in n and by Proposition 31, that

$$\underline{u} \le u_n \le u_{n+1} \le \overline{u} \qquad \text{for all } n \ge 1. \qquad (19)$$

These inequalities imply $\{u_n\}$ is bounded in $L^\infty(\Omega)$, so it has an uniformly convergent subsequence, by Theorem 2 and the Arzela-Ascoli theorem. By the monotonicity in n given by (19) the whole sequence $\{u_n\}$ converges uniformly. Hence its limit u is a solution to (5), by the approximation Theorem 4.

Proof of Theorem 1 (ii). We use the same approximating sequences F_n (given by (13)), f_n, c_n, ψ_n as in the proof of (i). Suppose n is large enough so that (18) holds for F_n, f_n, c_n, and let $\underline{u}_n, \overline{u}_n$ be the subsolution and supersolution obtained in Proposition 35, for the problem $F_n + c_n = f_n$. By Proposition 36 this problem has a solution which is between $\underline{u}_n$ and $\overline{u}_n$.

However (17) implies $\underline{u}_n, \overline{u}_n$ are bounded in $L^\infty(\Omega)$, independently of n (since the constant in (17) depends only on the Lebesgue norms of the coefficients). Hence $\{u_n\}$ is bounded in $L^\infty(\Omega)$, so it has an uniformly convergent subsequence, by Theorem 2 and the Arzela-Ascoli theorem. Again the limit u of this subsequence is a solution to (5), by the approximation Theorem 4. Theorem 1 (ii) is proved. □

Proof of Theorem 1 (iii). Suppose u_1 and u_2 are solutions of (5) and $u_2 \in W^{2,N}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$. Set $u = u_1 - u_2$.

By using (S) we see that u satisfies

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) - (\mu|Du| + 2\mu|Du_2| + b)|Du| - d(x)h(u^+) + c(x)u \le 0.$$

Note that, since u_2 is a strong solution, we can use (S) as if both u_1 and u_2 were strong - this is trivially seen with the help of test functions.

First, if $c \le 0$ in Ω and $\mu > 0$ we use Lemma 23 which implies

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2\widetilde{u}) - \widetilde{b}(x)|D\widetilde{u}| \le 0 \qquad \text{in } \{u<0\} = \{\widetilde{u} < 0\},$$

where $\widetilde{u} = (1/m)(1-e^{-mu})$, $m = \mu/\Lambda$, $\widetilde{b} = 2\mu|Du_2|+b$. Applying Theorem 3 in the set $\{\widetilde{u} < -\varepsilon\}$, we see that it is empty for each $\varepsilon > 0$, so $u \ge 0$ in Ω.

Second, if $\mu = 0$ then we apply Theorem 3 directly to

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) - b|Du| \le -c^+(x)u \qquad \text{in } \{u<0\},$$

and conclude $u^- \equiv 0$ for $\|c^+\| < C_A^{-1}$, like in the proof of Proposition 33.

The fact that $u^+ \equiv 0$ is proved analogously. □

Proof of Theorem 1 (iv). We consider the semilinear problem

$$\begin{aligned} \Delta u + \mu|Du|^2 + c_0 u &= 0 \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{20}$$

$\mu, c_0 > 0$. This clearly is the simplest problem which satisfies the hypotheses of Theorem 1 with quadratic dependence in the gradient and non-trivial positive part of the zero order coefficient.

We want to show that, for all c_0 small and positive, this problem has a solution different from the trivial one $u \equiv 0$. Setting, as before, $v = (1/\mu)\,(e^{\mu u} - 1)$, we see that (20) transforms into

$$-\Delta v = (c_0/\mu)(1+\mu v)\log(1+\mu v) =: f(v) \qquad \text{in } \Omega, \tag{21}$$

and $v = 0$ on $\partial\Omega$. Since $f(0) = 0$, $f(v) > 0$ for $v > 0$,

$$f'(0) = c_0, \qquad \lim_{t\to\infty} \frac{f(t)}{t} = \infty,$$

following the established terminology in the theory of semilinear elliptic equations, the nonlinearity $f(u)$ is *superlinear* provided

$$0 < c_0 < \lambda_1, \tag{22}$$

where $\lambda_1 > 0$ is the first eigenvalue of the Laplacian in Ω.

It is a classical application of the theory of Leray-Schauder degree (see for instance Theorem 3.6.10 in [Ck]) that a superlinear problem of this type possesses a positive classical solution provided it admits *a priori estimates*, that is, if we able to show that all (eventual) positive classical solutions of (21) are uniformly bounded in the L^∞-norm by a constant which, in this case, depends only on Ω and μ.

So let us show (21) admits a priori bounds. We use the well-known "blow-up" method of Gidas and Spruck [GS], adapting it to the logarithmic nonlinearity in (21) . Suppose for contradiction that there exists a sequence v_n of solutions of (21) such that $\|v_n\|_{L^\infty(\Omega)} \to \infty$. Set

$$s_n = \log \|v_n\|_{L^\infty(\Omega)} \,,$$

and make the change of unknowns

$$v_n(x) = e^{s_n} w_n(y), \qquad y = \sqrt{s_n}(x - x_n),$$

where $x_n \in \Omega$ is a point where v_n attains its maximum. Then $0 \le w_n \le 1$, $w_n(0) = 1$, and

$$-\Delta w_n(y) = s_n^{-1} e^{-s_n} f(e^{s_n} w_n(y)) \qquad \text{for } \; y \in \Omega_n,$$

where $\Omega_n := \sqrt{s_n}(\Omega - x_n)$ is a domain which converges either to $\mathbb{R}^N$ or to a half-space in $\mathbb{R}^N$.

It is trivial to see that the right-hand side of the last equation remains bounded (recall $0 \le w_n \le 1$), hence by elliptic estimates w_n converges in $W^{2,p}_{\rm loc}(\mathbb{R}^N)$ to a function w, such that $0 \le w \le 1$ and $w(0) = 1$. Further, we have pointwise, and hence in $L^p_{\rm loc}(\mathbb{R}^N)$ (by Lebesgue dominated convergence)

$$s_n^{-1} e^{-s_n} f(e^{s_n} w_n) \to c_0 w$$

so w is a bounded positive (by the strong maximum principle) solution of

$$-\Delta w = c_0 w$$

in $\mathbb{R}^N$ or in a half-space (with $w = 0$ on the boundary of the half space). This implies that the first eigenvalue of the Laplacian in any ball is larger than c_0, which is a contradiction with $c_0 > 0$. □

Remark 1. The above non-uniqueness result can be extended to general operators (1) (for instance (3) with $\mu(x) \ge \mu_0 > 0$ and $c^+, f \not\equiv 0$) through a similar in spirit (though more complicated) argument. Note Leray-Schauder degree theory was shown to apply to fully nonlinear equations in [QS2].

Remark 2. When the elliptic operator is in divergence form (as in the above particular case), the problem can also be tackled via variational methods. For nonlinearities similar to $f(v)$ in (21), some related problems have been studied in [J], [JT]. Naturally, the variational approach should permit to obtain more precise results, when it is applicable.

Developments on multiplicity of solutions of non-proper equations (like those pointed out in the preceding remarks) will be given in a future work.

Some simple equations without solutions. Here we list some simple results about existence and non-existence of solutions of

$$\begin{cases} \Delta u + \mu |Du|^2 + c_0 u = -A \text{ in } \Omega \\ u = B \text{ on } \partial\Omega, \end{cases} \tag{23}$$

or equivalently

$$\begin{cases} \Delta u + \mu |Du|^2 + c_0 u = -(A + c_0 B) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $c \equiv c_0, f \equiv A, \psi \equiv B, \mu$ are nonnegative constants. First, if $\mu = 0$ then this problem, for any $A > 0, B \geq 0$, has a positive solution if $c_0 < \lambda_1$, and has no solutions for $c_0 = \lambda_1$ (multiply by the first eigenfunction of the Laplacian). Second, if $\mu > 0$, by the change $v = (1/\mu)(e^{\mu u} - 1)$ problem (23) is equivalent to

$$\Delta v + \mu(A + c_0 B)v + c_0 g_\mu(v) = -(A + c_0 B) \quad \text{in } \Omega \tag{24}$$

and $v = 0$ on $\partial\Omega$, where $g_\mu(v) = (1/\mu)(1 + \mu v)\log(1 + \mu v)$. Now if $c_0 = 0$ then for any $A > 0, B \geq 0$, problem (24) has a positive solution if $A\mu < \lambda_1$ and has no solutions if $A\mu = \lambda_1$.

Finally, for arbitrary small $\mu, c_0 > 0$, if B is such that $\mu(A + c_0 B) = \lambda_1$, we multiply (24) by the first eigenfunction of the Laplacian (normalized so that its L^1 norm is equal to one), integrate, use the fact that $g_\mu(v) \geq -\frac{1}{e\mu}$ for all v, and obtain $c_0 \geq e\lambda_1$, a contradiction.

4. Proof of Theorem 2

For clarity, we are going to start by giving the proof of the interior estimate in the case of the model equation (3), with $\mu = 0, c = f \equiv 0$. The next proposition states that, for any given subdomain $\Omega' \subset\subset \Omega$, if a level set in Ω' of a positive supersolution has sufficiently small measure with respect to $|\Omega'|$, then this supersolution is uniformly positive in Ω'. As usual, constants denoted by C may change from line to line, and depend only on the appropriate quantities.

Proposition 41 *There exist numbers $\delta, \kappa, \rho_0 > 0$ depending only on N, λ, Λ, $\|b\|_{L^p}$, $p > N$, such that if for some $\rho \in (0, \rho_0)$ the ball $B_{2\rho} \subset \Omega$, and $b \in L^p(B_{2\rho})$, $b \geq 0$, $u \in C(B_{2\rho})$ satisfy*

$$\mathcal{L}^-[u] := \mathcal{M}^-_{\lambda,\Lambda}(D^2 u) - b(x)|Du| \leq 0 \qquad \text{in } B_{2\rho}$$
$$u \geq 0 \qquad \text{in } B_{2\rho},$$

then for any $a > 0$

$$\text{meas}\{x \in B_\rho \,:\, u(x) < a\} \leq \delta\,|B_\rho| \qquad \text{implies} \qquad u \geq \kappa a \quad \text{in } B_\rho.$$

Proof. Without restricting the generality we can suppose $a = 1$ (replace u by u/a). Set $v = 1 - (|x|^2/\rho^2)$. Then, by Lemmas 21 and 22, we have in B_ρ

$$\begin{aligned}\mathcal{M}^+_{\lambda,\Lambda}(D^2(v-u)) + b(x)|D(v-u)| &\geq \mathcal{M}^-_{\lambda,\Lambda}(D^2 v) - b(x)|Dv| - \mathcal{L}^-[u] \\ &\geq -\frac{2}{\rho^2}\left(N\Lambda + b(x)|x|\right) \\ &\geq -\frac{C}{\rho^2}\left(1 + \rho b(x)\right),\end{aligned}$$

provided $u \in W^{2,N}(B_{2\rho})$. Extending this inequality to u only continuous is then easy (and very standard, since $v \in C^2$), by using Definition 21 and test functions.

Since $v - u \leq 0$ on ∂B_ρ, by applying Theorem 3 to this inequality we get

$$\sup_{B_\rho}(v-u) \leq C_1\rho^{-1}\|1 + \rho b(x)\|_{L^N(B_\rho\cap\{v-u>0\})}.$$

Note that $\{v - u > 0\} \subset \{u < 1\}$, so $\mathrm{meas}(B_\rho \cap \{v - u > 0\}) \leq \delta\, C\rho^N$, by hypothesis. Then the triangle and Hölder inequalities imply

$$\sup_{B_\rho}(v-u) \leq C\delta^{1/N} + \rho^{\varepsilon_1}\|b\|_{L^p(B_\rho)},$$

where $\varepsilon_1 = (p - N)/Np$. By choosing δ and ρ_0 sufficiently small we get

$$\frac{3}{4} - \inf_{B_{\frac{\rho}{2}}} u = \inf_{B_{\frac{\rho}{2}}} v - \inf_{B_{\frac{\rho}{2}}} u \leq \sup_{B_{\frac{\rho}{2}}}(v-u) \leq \sup_{B_\rho}(v-u) \leq \frac{1}{4}$$

for $\rho \leq \rho_0$, so

$$u \geq \frac{1}{2} \quad \text{in } B_{\frac{\rho}{2}}. \tag{25}$$

Now set, for $s > 0$ and $x \in B_{2\rho} \setminus B_{\frac{\rho}{2}}$,

$$w(x) = \frac{1}{4}\,\frac{|x|^{-s} - (2\rho)^{-s}}{(\rho/2)^{-s} - (2\rho)^{-s}}.$$

It is easy to compute, with the help of Lemma 22, that

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2(|x|^{-s})) - b(x)|D(|x|^{-s})| = s(\lambda(s+1) - \Lambda(N-1) - b(x)|x|)|x|^{-s-2},$$

and hence, fixing s such that $\lambda(s + 1) = \Lambda(N - 1)$,

$$\begin{aligned}\mathcal{M}^+_{\lambda,\Lambda}(D^2(w-u)) + b(x)|D(w-u)| &\geq \mathcal{M}^-_{\lambda,\Lambda}(D^2 w) - b(x)|Dw| - \mathcal{L}^-[u] \\ &\geq -C\rho^s|x|^{-s-2}b(x)|x| \\ &\geq -C\rho^{-1}b(x)\end{aligned}$$

in the set $B_{2\rho} \setminus B_{\frac{\rho}{2}}$. Since $w - u \leq 0$ on $\partial(B_{2\rho} \setminus B_{\frac{\rho}{2}})$, Theorem 3 yields

$$\sup_{B_\rho\setminus B_{\frac{\rho}{2}}}(w-u) \leq \sup_{B_{2\rho}\setminus B_{\frac{\rho}{2}}}(w-u) \leq C\|b\|_{L^N(B_{2\rho})} \leq C\rho^{\varepsilon_1}\|b\|_{L^p(B_\rho)}$$

so, by taking ρ_0 sufficiently small, we have, for $\rho \le \rho_0$,

$$u(x) \ge \inf_{B_\rho \setminus B_{\frac{\rho}{2}}} w - C\rho^{\varepsilon_1} \ge 2^{-s-3} - C\rho^{\varepsilon_1} \ge 2^{-s-4}\,, \qquad \text{for } \; x \in B_\rho \setminus B_{\frac{\rho}{2}}\,,$$

which finishes the proof of Proposition 41. □

We use the following well-known measure theoretic result (the "propagating ink spots" lemma).

Lemma 41 *Let G be a ball and K be some measurable subset of G, such that $|K| \le \eta|G|$, for some $\eta \in (0,1)$. Let $\mathcal{F}$ be the set of all balls B contained in G, and such that $|B \cap K| \ge \eta|B|$. Then there exists $\zeta > 0$ depending only on N and η, such that*

$$\text{meas}(\cup_{B\in\mathcal{F}} B) \ge (1+\zeta)\text{meas}(K).$$

Proof. This is for instance inequality (9.20) from [GT], setting f to be the indicator function of K in the reasoning there. □

With the help of this lemma we can prove the result from Proposition 41 for any $\delta \in (0,1)$.

Proposition 42 *If for some $\rho \in (0, \rho_0)$ (ρ_0 is the number from Proposition 41) the ball $B_{2\rho} \subset \Omega$, and $b \in L^p(B_{2\rho})$ ($p > N$), $b \ge 0$, $u \in C(B_{2\rho})$ satisfy*

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) - b(x)|Du| \le 0 \qquad \text{in } \; B_{2\rho}$$

and $u \ge 0$ in $B_{2\rho}$, then for any $\nu, a > 0$ there exists $\overline{\kappa} > 0$ depending on $\nu, N, \lambda, \Lambda, \|b\|_{L^p}$, $p > N$, such that

$$\text{meas}\,\{x \in B_\rho \,:\, u(x) \ge a\} \ge \nu\,|B_\rho| \qquad \text{implies} \qquad u \ge \overline{\kappa} a \quad \text{in } \; B_\rho.$$

Remark. We will derive the Hölder estimate from Proposition 42. Note this proposition can be viewed as a "very weak" Harnack inequality. The usual weak Harnack inequality $\inf_{B_\rho} u \ge C|B_\rho|^{-1/q}\|u\|_{L^q(B_\rho)}$ contains a stronger statement.

Proof of Proposition 42. Set $K_a = \{x \in B_\rho \,:\, u(x) \ge a\}$. We know that $|K_a| \ge \nu\,|B_\rho|$. If $|K_a| \ge (1-\delta)|B_\rho|$, where δ is the number from Proposition 41 then we conclude, by that proposition.

If, on the other hand, $|K_a| < (1-\delta)|B_\rho|$, we apply Lemma 41, with $\eta = 1-\delta$. By Proposition 41 we have $u \ge \kappa a$ in each ball in $\mathcal{F}$ (defined in Lemma 41), for some $\kappa > 0$, depending on the appropriate quantities. Hence, by Lemma 41,

$$|K_{\kappa a}| \ge (1+\zeta)|K_a| \ge \nu(1+\zeta)\,|B_\rho|.$$

We repeat the same reasoning and get either Proposition 42 or

$$|K_{\kappa^2 a}| \ge \nu(1+\zeta)^2\,|B_\rho|.$$

This process stops after at most n iterations, where n is a number such that $\nu(1+\zeta)^n \geq 1$. □

Proof of the interior C^α-estimate for (3), $\mu = c = f = 0$. We recall we have a solution $u \in C(\Omega)$ of $\mathcal{M}(D^2u) + b(x)|Du| = 0$. Then for any ρ such that $B_{2\rho} \subset \Omega$ the functions $u_1 := u - \inf_{B_{2\rho}} u$ and $u_2 := \sup_{B_{2\rho}} u - u$ satisfy the hypotheses of Proposition 42. In addition,

$$\omega(2\rho) := \underset{B_{2\rho}}{osc}\, u = u_1 + u_2,$$

so at each point of $B_{2\rho}$ one of u_1, u_2 is larger than $\frac{1}{2}\omega(2\rho)$. This implies

$$\text{meas}\left\{x \in B_\rho \,:\, u_i(x) \geq \frac{1}{2}\omega(2\rho)\right\} \geq \frac{1}{2}\,\text{meas}(B_\rho),$$

for one i, say for $i = 1$. Then we can apply Proposition 42 to u_1 and infer

$$u - \inf_{B_{2\rho}} u = u_1 \geq \kappa\omega(2\rho) \qquad \text{in } \ B_\rho,$$

which implies $\inf_{B_\rho} u \geq \kappa \sup_{B_{2\rho}} u + (1-\kappa)\inf_{B_{2\rho}} u$. Hence $\omega(\rho) \leq (1-\kappa)\omega(2\rho)$, for all $\rho \in (0, \rho_0)$. The proof is now standardly finished, with the help of Lemma 8.23 in [GT], which gives

$$\omega(\rho) \leq C\rho^\alpha \rho_0^{-\alpha}\omega(\rho_0) \leq C \sup_{B_{\rho_0}} |u|\, \rho^\alpha,$$

for some α depending on $N, \lambda, \Lambda, \|b\|_{L^p(B_{2\rho})}, p > N$. □.

Next, we give the changes in the proofs of Propositions 41 and 42, which we have to make in order to deal with a nontrivial right-hand side.

Proposition 43 *There exist numbers $\delta, \overline{\kappa}, \rho_0, C_0 > 0$ depending only on $N, \lambda, \Lambda, \|b\|_{L^p}$, $p > N$, such that if for some $\rho \in (0, \rho_0)$ the ball $B_{2\rho} \subset \Omega$ and $f \in L^N(\Omega)$, $b \in L^p(B_{2\rho})$, $b, f \geq 0$, $u \in C(B_{2\rho})$ satisfy $u \geq 0$ in $B_{2\rho}$ and*

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) - b(x)|Du| \leq f(x) \qquad \text{in } \ B_{2\rho},$$

then, for any $a > 0$, $\text{meas}\,\{x \in B_\rho \,:\, u(x) < a\} \leq \delta\,\text{meas}(B_\rho)$ implies

$$\inf_{B_\rho} u \geq \overline{\kappa}a - C_0\rho\|f\|_{L^N(B_{2\rho})}. \tag{26}$$

Proof. The proof goes the same way as the proof of Proposition 41, by adding $-f$ to the right-hand sides of the inequalities to which we apply Theorem 3. We can suppose $\text{meas}\,\{x \in B_\rho \,:\, u(x) < 1\} \leq \delta\,\text{meas}(B_\rho)$, with f replaced by f/a. Then inequality (25) reads

$$u \geq \frac{1}{2} - C_1 a^{-1}\rho\|f\|_{L^N(B_{2\rho})} \qquad \text{in } \ B_{\frac{\rho}{2}}\,, \tag{27}$$

where C_1 is the constant from Theorem 3. We distinguish two cases. First, if $a < 4C_1\rho\|f\|_{L^N(B_\rho)}$ then the conclusion of Proposition 43 trivially holds, with $\overline{\kappa} = 1, C_0 = 4C_1$ (so that the right-hand side of (26) is negative). If not, we have $u \geq \frac{1}{4}$ in $B_{\frac{\rho}{2}}$, and we finish the proof as in Proposition 41. □

Proposition 44 *If for some $\rho \in (0, \rho_0)$ (ρ_0 is the number from Proposition 43) the ball $B_{2\rho} \subset \Omega$ and $f \in L^N(\Omega)$, $b \in L^p(B_{2\rho})$, $b, f \geq 0$, $u \in C(B_{2\rho})$ satisfy*

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) - b(x)|Du| \leq f(x) \qquad \text{in } B_{2\rho}$$
$$u \geq 0 \qquad \text{in } B_{2\rho},$$

then for any $\nu, a > 0$ there exist $\kappa, C > 0$ depending on $\nu, N, \lambda, \Lambda, \|b\|_{L^p}$, $p > N$, such that meas$\{x \in B_\rho : u(x) \geq a\} \geq \nu\,\mathrm{meas}(B_\rho)$ implies

$$\inf_{B_\rho} u \geq \kappa a - C\rho\|f\|_{L^N(B_{2\rho})}. \tag{28}$$

Proof. We have to modify the proof of Proposition 42 as in the proof of the previous proposition. Suppose meas$(K_a) < (1-\delta)$meas(B_ρ). Then in case $a < (2\overline{\kappa})C_0\rho\|f\|_{L^N(B_{2\rho})}$ (the constants $\overline{\kappa}, C_0$ are defined in the Proposition 43) then inequality (28) is trivially true, by choosing κ, C such that its right-hand side is negative. If not, then $u \geq \frac{\overline{\kappa}}{2}a$ in each ball in $\mathcal{F}$ (defined in Lemma 41), by Proposition 43. Then we repeat the same reasoning as in the proof of Proposition 42, distinguishing at each step the cases when a is smaller or larger than $(2\overline{\kappa})^l C_0\rho\|f\|_{L^N(B_{2\rho})}$, $l = 1, \ldots, n$. □

Proof of the interior C^α-estimate for (3), $\mu = c = 0, f \neq 0$. We reason in exactly the same way as in the case $f = 0$, only at the end we get

$$\omega(\rho) \leq (1-\kappa)\omega(2\rho) + \left(C\|f\|_{L^N(B_{2\rho})}\right)\rho,$$

to which Lemma 8.23 of [GT] applies as well : for any $\gamma \in (0,1)$ there exists α depending on $\gamma, N, \lambda, \Lambda, \|b\|_{L^p}$, $p > N$, such that

$$\omega(\rho) \leq C \sup_{B_{\rho_0}} |u|\, \rho^\alpha + C\|f\|_{L^N(B_{2\rho})}\rho^\gamma.$$

Remark. Note that in order to carry out all the above arguments it is actually sufficient to know that

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) + b(x)|Du| \geq -f(x) \quad \text{and} \quad \mathcal{M}^-_{\lambda,\Lambda}(D^2u) - b(x)|Du| \leq f(x).$$

The next proposition deals with the extension of the result to the boundary. It uses the well-known idea of extending the function u as a constant outside the domain (like for example in Theorem 8.26 or 9.27 in [GT]).

Proposition 45 *Suppose for some ball $B \subset \mathbb{R}^N$, and for $f \in L^N(B)$, $b \in L^p(B)$, $b, f \geq 0$, $u \in C(\overline{\Omega})$, $m > 0$, we have*

$$\mathcal{L}^-[u] := \mathcal{M}^-_{\lambda,\Lambda}(D^2u) - b(x)|Du| \leq f(x) \qquad \text{in } \Omega \cap B,$$
$$u \geq 0 \qquad \text{in } \Omega \cap B,$$
$$u \geq 2m \qquad \text{on } \partial\Omega \cap B.$$

Then for any ball $B_{2\rho} \subset \Omega \cup B$, $\rho \le \rho_0$, *and for any* $\nu, a > 0$

$$\text{meas}\,\{x \in B_\rho \;:\; \overline{u}(x) \ge a\} \ge \nu\, \text{meas}(B_\rho) \tag{29}$$

implies

$$\inf_{B_\rho} \overline{u} \ge \kappa a - C\rho \|f\|_{L^N(\Omega)},$$

where κ, C *depend on* $\nu, N, \lambda, \Lambda, \|b\|_{L^p}$, $p > N$, *and* $\overline{u} \in C(B)$ *is defined by*

$$\overline{u}(x) = \begin{cases} m & \text{if } x \in \Omega \setminus B \\ \min\{u(x), m\} & \text{if } x \in \overline{\Omega} \cap B\,. \end{cases}$$

Proof. The function $\overline{u}$ satisfies the hypotheses of Proposition 44 in the ball B – since the minimum of two viscosity supersolutions is a viscosity supersolution, and $\mathcal{L}^-[m] \equiv 0 \le f(x)$ (we have set $f = 0$ outside Ω). □

Proof of the boundary estimate for (3), $\mu = c = 0, f \ne 0$. Let $x_0 \in \partial\Omega$. Then by the uniform cone condition, for some $\rho_1 > 0$ and some $\xi > 0$ (depending on L), the balls B with center x_0 and radii 2ρ, $\rho \le \rho_1$, satisfy $\text{meas}(B \setminus \Omega) \ge \xi \text{meas}(B)$.

We are going to show that for each ball B_ρ with center x_0 and radius $\rho \le \min\{\rho_0, \rho_1\}$ we have

$$\omega(\rho) := \operatorname*{osc}_{\Omega \cap B_\rho} u \le (1-\kappa)\omega(2\rho) + C\|f\|_{L^N(B_{2\rho})}\,\rho + 2 \operatorname*{osc}_{\partial\Omega \cap B_{2\rho}} u. \tag{30}$$

First, if

$$\omega(2\rho) := \operatorname*{osc}_{\Omega \cap B_{2\rho}} u \le 2 \operatorname*{osc}_{\partial\Omega \cap B_{2\rho}} u, \tag{31}$$

inequality (30) is obvious. If (31) doesn't hold, then either

$$\inf_{\partial\Omega \cap B_{2\rho}} u - \inf_{\Omega \cap B_{2\rho}} u \ge \frac{1}{4}\omega(2\rho) \qquad \text{or} \qquad \sup_{\Omega \cap B_{2\rho}} u - \sup_{\partial\Omega \cap B_{2\rho}} u \ge \frac{1}{4}\omega(2\rho).$$

Let's say the first of these holds. Then the function $u_1 = u - \inf_{\Omega \cap B_{2\rho}} u$ satisfies the conditions of Proposition 45, with $a = m = \omega(2\rho)/8$ – note (29) is automatically satisfied thanks to the exterior cone condition. So

$$u(x) - \inf_{\Omega \cap B_{2\rho}} u \ge \kappa\omega(2\rho) - C\|f\|_{L^N(B_{2\rho})}\rho \qquad \text{for each } \; x \in B_\rho.$$

Hence, as before, $\omega(\rho) \le (1-\kappa)\omega(2\rho) + C\|f\|_{L^N(B_{2\rho})}\,\rho$, and (30) holds.

So, by applying Lemma 8.23 of [GT] to (30) for each $\gamma \in (0,1)$ we can find $\alpha_0 \in (0,1)$ such that

$$\omega(\rho) \le C \sup_{\Omega \cap B_{\rho_0}} |u|\, \rho^{\alpha_0} + C\|f\|_{L^p(B_{2\rho})}\rho^\gamma + 2 \operatorname*{osc}_{\partial\Omega \cap B_{h(\rho)}} u,$$

where $h(\rho) = 2\rho_0(\rho/\rho_0)^\gamma$. The boundary estimate in Theorem 2 easily follows from this inequality.

Proof of the global estimate for (3), $\mu = c = 0, f \neq 0$. Putting together the interior and the boundary estimates we already proved is standard, see for example the proof of Theorem 8.29 in [GT] or the proof of Proposition 4.13 in [CC].

Proof of Theorem 2. To get the full strength of Theorem 2 we first use (S), transferring the terms $d(x)h(u,0)$ to the right-hand side of the inequalities, which permits to suppose $d \equiv 0$.

Let us prove the interior estimate. By (S) and Lemma 21 both u and $-u$ are solutions of $\mathcal{M}^-_{\lambda,\Lambda}(D^2u) - \mu|Du|^2 - b(x)|Du| \leq |f(x)|$. Hence, by Lemma 23, the functions

$$w_1 = \frac{1 - e^{-mu_1}}{m}, \qquad w_2 = \frac{1 - e^{-mu_2}}{m}$$

(with $m = \mu/\lambda$, $u_1 = u - \inf_{B_{2\rho}} u$, $u_2 = \sup_{B_{2\rho}} u - u$) satisfy

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2w_i) - b(x)|Dw_i| \leq |f|(1 - mw_i) =: \overline{f}. \tag{32}$$

Since at each point $x \in B_{2\rho}$

$$w_j(x) \geq \frac{1 - e^{-m(\omega(2\rho)/2)}}{m}$$

for one j, say $j = 1$, reasoning as before we get

$$w_1 \geq \kappa \frac{1 - e^{-m(\omega(2\rho)/2)}}{m} - \left(C\|\overline{f}\|_{L^N}\right)\, \rho,$$

for $\rho \leq \rho_0$, the number from Proposition 44. Note that for each t_0 there exists $\xi = \xi(t_0, m)$ such that $t \geq \frac{1-e^{-mt}}{m} \geq \xi t$ for $t \in [0, t_0]$. We apply this with $t_0 = \omega(2\rho_0)/2$ and get $u_1 \geq \kappa\xi\omega(2\rho) - C\|\overline{f}\|_{L^N}\,\rho$ in B_ρ, so again

$$\omega(\rho) \leq C \sup_{B_{\rho_0}} |u|\, \rho^\alpha + C\|f\|_{L^N(B_{2\rho})}\rho^\gamma. \tag{33}$$

for $\rho \in (0, \rho_0)$. Note that here α depends on μ and $\sup u$, because of the choice of ξ, but this dependence can easily be transferred to C, by choosing another α, if necessary. Indeed, ρ_0 is independent of μ and $\sup u$ (ρ_0 comes from the applications of the ABP inequality to (32), as in the proofs of Propositions 41 and 42). Then we can choose $\rho_1 \leq \rho_0$ so small that, by (33), $\mathrm{osc}_{B_{2\rho_1}} u$ is so small that if we repeat the above argument with ρ_0 replaced by ρ_1, we get $\xi \geq 1/2$ – since obviously $\xi \to 1$ as $t_0 \to 0$. This implies (33) holds for $\rho \leq \rho_1$, with a different α_1, which is independent of μ and $\sup u$, the dependence of these now being in the constants C and ρ_1. But since $\rho^\alpha \leq C\rho^{\alpha_1}$ for $\rho \in [\rho_1, \rho_0]$, $C = C(\rho_1, \rho_0)$, we see that we have (33) for α replaced by α_1, and all $\rho \leq \rho_0$.

The boundary estimate is proved similarly. □

References

[A] A. Alexandrov, Uniqueness conditions and estimates for the solution of the Dirichlet problem, Trans. Amer. Math. Soc. Transl. 68 (1968), 89-119.

[AGP] A. Dall'Aglio, D. Giachetti, J.-P. Puel, Nonlinear elliptic equations with natural growth in general domains. Ann. Mat. Pura Appl. 181(4) (2002), 407–426.

[BBGK] G. Barles, A. Blanc, C. Georgelin, M. Kobylanski, Remarks on the maximum principle for nonlinear elliptic PDEs with quadratic growth conditions. Ann. Sc. Norm. Sup. Pisa 28(3) (1999), 381–404.

[BM] G. Barles, F. Murat, Uniqueness and the maximum principle for quasilinear elliptic equations with quadratic growth conditions. Arch. Rat. Mech. Anal. 133(1) (1995), 77–101.

[BP] G. Barles, A. Porretta, Uniqueness for unbounded solutions to stationary viscous Hamilton-Jacobi equations. Ann. Sc. Norm. Super. Pisa Cl. Sci. 5(1) (2006), 107–136.

[Be] H. Berestycki, On some nonlinear Sturm-Liouville problems, J. Diff. Eq. 26 (1977), 375-390.

[BMP1] L. Boccardo, F. Murat, J.-P. Puel, Résultats d'existence pour certains problèmes elliptiques quasilinéaires. (French) Ann. Scuola Norm. Sup. Pisa Cl. Sci. 11(2) (1984), 213–235.

[BMP2] L. Boccardo, F. Murat, J.-P. Puel, Existence de solutions faibles pour des équations elliptiques quasi-linéaires à croissance quadratique. Res. Notes in Math. 84, Pitman, 1983, 19–73.

[BEQ] J. Busca, M. Esteban, A. Quaas, Nonlinear eigenvalues and bifurcation problems for Pucci's operator, Ann. Inst. H. Poincaré, Anal. Nonl. 22(2) (2005), 187-206.

[Ca] X. Cabre, Elliptic PDEs in probability and geometry, Discr. Cont. Dyn. Syst. A, 20(3) (2008), 425-457.

[C] L.A. Caffarelli, Interior estimates for fully nonlinear elliptic equations, Ann. Math. 130 (1989), 189-213.

[CC] L.A. Caffarelli, X. Cabre, Fully Nonlinear Elliptic Equations, A.M.S. Coll. Publ. Vol 43, Providence (1995).

[CCKS] L.A. Caffarelli, M.G. Crandall, M.Kocan, A. Świech, On viscosity solutions of fully nonlinear equations with measurable ingredients, Comm. Pure Appl. Math. 49 (1996), 365-397.

[Ck] K.C. Chang, Methods in nonlinear analysis. Springer Monographs in Mathematics (2005).

[CKLS] M.G. Crandall, M. Kocan, P.L. Lions, A. Świech, Existence results for boundary problems for uniformly elliptic and parabolic fully nonlinear equations, Elec. J. Diff. Eq. 24 (1999), 1-20.

[CIL] M.G. Crandall, H. Ishii, P.-L. Lions, User's guide to viscosity solutions of second-order partial differential equations, Bull. Amer. Math. Soc. 27(1) (1992), 1-67.

[FQ] P. Felmer, A. Quaas, Positive solutions to 'semilinear' equation involving the Pucci's operator, J. Diff. Eq. 199 (2) (2004), 376-393.

[FM] V. Ferone, F. Murat, Nonlinear problems having natural growth in the gradient: an existence result when the source terms are small. Nonl. Anal. 42(7) (2000), 1309–1326.

[F1] K. Fok, Some Maximum Principles and continuity estimates for fully nonlinear elliptic equations of second order, Ph.D. thesis, University of California at Santa Barbara, 1996.

[F2] K. Fok, A nonlinear Fabes-Stroock result. Comm. Part. Diff. Eq. 23(5&6) (1998), 967-983.

[GS] B. Gidas, J. Spruck, A priori bounds for positive solutions of nonlinear elliptic equations. Comm. Part. Diff. Eq. (6) (1981), 883-901.

[GT] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, 2nd edition, Springer Verlag.

[GMP] N. Grenon, F. Murat, A. Porretta, Existence and a priori estimate for elliptic problems with subquadratic gradient dependent terms. C. R. Math. Acad. Sci. Paris 342(1) (2006), 23–28.

[IL] H. Ishii, P.-L. Lions, Viscosity solutions of fully nonlinear second-order elliptic partial differential equations, J. Diff. Eq. 83 (1990), 26-78.

[J] L. Jeanjean, On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer type problem set on R^N, Proc. Roy. Soc. Edinburgh A, 129(4) (1999), 787-809.

[JT] L. Jeanjean, K. Tanaka, A positive solution for an asymptotically linear elliptic problem on R^N autonomous at infinity , ESAIM Control Optim. Calc. Var. 17 (2002), 597-614.

[K1] N.V. Krylov, Nonlinear elliptic and parabolic equations of second order. Mathematics and its Applications, Reidel, 1987.

[K2] N.V. Krylov, Fully nonlinear second order elliptic equations : recent development, Ann. Sc. Norm. Pisa 25(3-4) (1997), 569-595.

[KSa] N.V. Krylov, M. Safonov, A property of the solutions of parabolic equations with measurable coefficients, Math USSR-Izvestia 19 (1980), 151-164.

[KS1] S. Koike, A. Swiech, Maximum principle and existence of L^p-viscosity solutions for fully nonlinear uniformly elliptic equations with measurable and quadratic term, NoDEA Appl. 11 (2004), 491-509.

[KS2] S. Koike, A. Swiech, Maximum principle for fully nonlinear equations via the iterated function method, Math. Ann. 339(2) (2007), 461-484.

[KS3] S. Koike, A. Swiech, Weak Harnack inequality for fully nonlinear uniformly elliptic PDE with unbounded ingredients, preprint.

[LU1] O. Ladizhenskaya, N. Uraltseva, Linear and quasilinear equations of elliptic type, Academic Press, New York-London, 1968.

[LU2] O. Ladizhenskaya, N. Uraltseva, A survey of results on solvability of boundary-value problems for uniformly elliptic and parabolic quasilinear equations, Russian Math. Surveys, 41(5) (1986) 1-31.

[Li1] P.L. Lions, Bifurcation and optimal stochastic control, Nonl. Anal. Theory and Appl. 7(2) (1983), 177-207.

[Li2] P.-L. Lions, Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations. Comm. Part. Diff. Eq. 8(1983), 1101–1174 and 1229-1276.

[MPS] C. Maderna, C. Pagani, S. Salsa, Quasilinear elliptic equations with quadratic growth in the gradient. J. Diff. Eq. 97(1) (1992), 54–70.

[Na] N. Nadirashvili, Nonuniqueness in the martingale problem and the Dirichlet problem for uniformly elliptic operators. Ann. Scuola Norm. Sup. Pisa Cl. Sci. 24(3) (1997), 537–549.

[Ni] M. Nisio, Stochastic differential games and viscosity solutions of Isaacs equations. Nagoya Math. J. 110 (1988), 163–184.

[QS1] A. Quaas, B. Sirakov, Principal eigenvalues and the Dirichlet problem for fully nonlinear elliptic operators, Adv. Math. 218 (2008) 105-135.

[QS2] A. Quaas, B. Sirakov, Existence results for nonproper elliptic equations involving the Pucci operator, Comm. Part. Diff. Eq. 31 (2006), 987-1003.

[Sa] M. Safonov, Nonuniqueness for second-order elliptic equations with measurable coefficients. SIAM J. Math. Anal. 30(4) (1999), 879–895.

[S] A. Świech, $W^{1,p}$-estimates for solutions of fully nonlinear uniformly elliptic equations, Adv. Diff. Eq. 2(6) (1997), 1005-1027.

[T] N. Trudinger, On regularity and existence of viscosity solutions of nonlinear second order, elliptic equations, PDE and the calculus of variations, Vol. II (Progr. Nonl. Diff. Eq. Appl. 2, Birkhäuser Boston) (1989), 939-957.

[W] L. Wang, On the regularity theory of fully nonlinear parabolic equations: I, Comm. Pure Appl. Math. 45(1) (1992), 27-76.

UFR SEGMI, Université Paris 10, 92001 NanterreCedex,France
and CAMS, EHESS, 54 bd Raspail, 75270 ParisCedex 06, France
E-mail : sirakov@ehess.fr