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# Projecting the Fokker-Planck Equation onto a finite dimensional exponential family

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**Abstract**

In the present paper we discuss problems concerning evolutions of densities related to Itô diffusions in the framework of the statistical exponential manifold. We develop a rigorous approach to the problem, and we particularize it to the orthogonal projection of the evolution of the density of a diffusion process onto a finite dimensional exponential manifold. It has been shown by D. Brigo (1996) that the projected evolution can always be interpreted as the evolution of the density of a different diffusion process. We give also a compactness result when the dimension of the exponential family increases, as a first step towards a convergence result to be investigated in the future. The infinite dimensional exponential manifold structure introduced by G. Pistone and C. Sempi is used and some examples are given.

**Keywords** Nonlinear diffusions, Fokker–Planck equation, finite dimensional families, exponential families, stochastic differential equations, Fisher metric, differential geometry and statistics, convergence.

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# 1 Introduction

This paper moves both from the differential geometric approach to nonlinear filtering as developed by Brigo, Hanzon and LeGland [9] and from the rigorous approach to the construction of a differential geometric structure in the infinite dimensional space of probability measures given in Pistone and Sempi [24], see also Pistone and Rogantin [23]. The solution of the filtering problem is a stochastic PDE which can be seen as a generalization of the Fokker–Planck equation (FPE) expressing the density of a diffusion process. This filtering equation is called the Kushner–Stratonovich equation. In [9] the Fisher metric is used to project the Kushner–Stratonovich equation onto a finite dimensional exponential manifold of probability densities. This method can be used also for the simpler FPE. In the present paper we discuss the geometric approach to problems concerning finite dimensionality of densities related to stochastic differential equations given by Itô diffusions. Part of these results were already given in [13], and we shortly present them here in the framework of Pistone and Sempi [24]. This approach is different from the one adopted for example in Brigo, Hanzon and LeGland [9] or in Brigo [13], since it uses the exponential manifold structure rather than the $L_2$ derivation. The $L_2$ structure is obtained by mapping densities into their square roots. We show that this map yields a regular $C^\infty$ parametrization but it is not a chart for the infinite dimensional manifold of densities. In the present paper we consider the projection in Fisher metric of the density–evolution of a diffusion process onto an exponential manifold. Such projection is obtained via the projected FPE. We examine the projected density–evolution and discuss problems related to finite dimensionality, giving some examples. We recall from Brigo [13] that the projected density–evolution can always be interpreted as the density–evolution of a different diffusion process. We conclude by giving a first step for future investigations on the following convergence problem: is it possible to prove that the projected density converges to the original one when the dimension of the exponential manifold on which we project tends to infinity?

# 2 The exponential statistical manifold of positive probability densities

In the present section we give a summary of the construction of the non-parametric exponential statistical manifold as developed in [24] and [23]. In those papers it is shown that the definition of statistical manifold as introduced by Dawid, Efron, Amari and others, and systematically presented in [19], can be given in a non parametric setting using the framework of the theory of manifolds modeled on Banach spaces, as introduced for example in Lang, [18].

We consider a measure space $(X, \mathcal{X}, \mu)$, where $\mu$ is a reference measure, and the set $\mathcal{M}(X, \mathcal{X}, \mu)$ of the a.s. strictly positive densities w. r. t. some measure equivalent to $\mu$. We define on the set $\mathcal{M}(X, \mathcal{X}, \mu)$ a topology such that $\mathcal{M}(X, \mathcal{X}, \mu)$ is an Hausdorff space (i.e. points can be separated by open sets). Then we shall construct a covering of $\mathcal{M}(X, \mathcal{X}, \mu)$ with open sets $\mathcal{U}_p$, $p \in \mathcal{U}_p$, $p \in \mathcal{M}(X, \mathcal{X}, \mu)$, and a corresponding family of Banach spaces $B_p$, with norms $||\cdot||_p$, $p \in \mathcal{M}(X, \mathcal{X}, \mu)$, such that each density $q \in \mathcal{U}_p$ is represented by a coordinate $s_p(q) \in B_p$.

We shall use the notations

$$s_p \;:\; \mathcal{U}_p \to \mathcal{V}_p \subset B_p \tag{1}$$

$$e_p \;:\; \mathcal{V}_p \to \mathcal{U}_p \subset \mathcal{M}(X, \mathcal{X}, \mu) \tag{2}$$

to denote respectively the *charts*, i.e. the mappings from points to coordinates, and the *patches*, i.e. the mappings from coordinates to points.

Following the use in differential geometry, we say that $\{(\mathcal{U}_p, s_p) \;:\; p \in \mathcal{M}(X, \mathcal{X}, \mu)\}$ is an *atlas* if all the space is covered by charts; if moreover each of the *change of coordinates*

$$s_{p_2} \circ e_{p_1} : s_{p_1}\left(\mathcal{U}_{p_1} \cap \mathcal{U}_{p_2}\right) \to e_{p_2}\left(\mathcal{U}_{p_1} \cap \mathcal{U}_{p_2}\right)$$

is a diffeomorphism of some regularity between open sets, the atlas has that regularity. In such a case the atlas, augmented with all the compatible charts, defines the manifold, see Lang [18].

In our case we shall introduce a very special manifold, such that the change of coordinates are actually affine functions —i.e. they differ from a linear function by a constant—, but we will keep a weaker regularity, namely the $C^\infty$–regularity (differentiability of any order) for compatible charts.

We shall denote by $\mathrm{E}_{p\cdot\mu}[\cdot]$ the expectation w.r.t. the probability measure $p \cdot \mu$ (where $p \cdot \mu(dx) = p(x)\mu(dx)$); if there is no ambiguity we will use the notation $\mathrm{E}_p[\cdot]$.

First we define the topology as follows. For simplicity we give only the definition of convergence of sequences. The sequence $(p_n)_{n\in\mathbb{N}}$ in $\mathcal{M}(X, \mathcal{X}, \mu)$ is *e*–convergent (exponentially convergent) to $p$ if $(p_n)_{n\in\mathbb{N}}$ tends to $p$ in $\mu$–probability as $n \to \infty$ and moreover the sequences $(p_n/p)_{n\in\mathbb{N}}$ and $(p/p_n)_{n\in\mathbb{N}}$ are eventually bounded in each $L^\alpha(p)$, $\alpha > 1$, i.e.

$$\forall \alpha > 1 \quad \limsup_{n\to\infty} E_p\left[\left(\frac{p_n}{p}\right)^\alpha\right] < +\infty, \quad \limsup_{n\to\infty} E_p\left[\left(\frac{p}{p_n}\right)^\alpha\right] < +\infty.$$

Now we shall introduce the Banach spaces on which the statistical manifold is modeled. We give a definition that shows how they are connected with well-known statistical objects. For each density $p \in \mathcal{M}(X, \mathcal{X}, \mu)$, the *Cramer class at* $p$ is the set of all random variables $u$ on $X$ such that the *moment generating function*

$$\hat{u}_p(t) = \int e^{tu} p \, d\mu = \mathrm{E}_p\left[e^{tu}\right], \quad t \in \mathbb{R}$$

is finite in a neighborhood of the origin 0. If moreover the expectation of $u$ is zero (the previous condition implies the existence of a finite expectation), then we shall call the set the *centered Cramer class at* $p$.

The centered Cramer class at $p$ is a vector space, and it shall be denoted by $B_p$, i.e.

$$B_p = \{u \in L^1(p \cdot \mu) : 0 \in Dom(\hat{u}_p)^\circ, \mathrm{E}_p[u] = 0\}.$$

It is a Banach space for the norm defined by:

$$\|u\|_p = \inf\left\{r : \mathbf{E}_p\left[\cosh\left(\frac{u}{r}\right) - 1\right] \leq 1\right\} \tag{3}$$

In the previous formula the function $x \mapsto \cosh(x) - 1$ is a convex function that plays in the theory of the spaces $B_p$ the same role as the function $x \mapsto |x|^\alpha/\alpha$ in the theory of Lebesgue spaces $L^\alpha$, $\alpha > 1$. We cite [17] and [25] as general references.

We will denote by ${}^*B_p$ the Banach space of centered random variables of the so called $x \log x$-class. A random variable $u$ belongs to the $x \log x$-class ${}^*B_p$ if and only if it is centered and $(1 + u)\log(1 + u)$ is $(p \cdot \mu)$-integrable.

Now we give some details about the Banach spaces $B_p$ and ${}^*B_p$ which will be useful in the construction of the statistical manifold.

**Proposition 1** 1. *The dual space of the Banach space ${}^*B_p$ is isomorphic to $B_p$, i.e. if $T$ is a continuous linear operator on ${}^*B_p$ then there exists a unique $u \in B_p$ such that $T(k) = \mathrm{E}_p[ku]$, $k \in {}^*B_p$; that is $({}^*B_p)^\star \ni T \leftrightarrow u \in B$.*

2. *All the elements $k$ in ${}^*B_p$ are identified with an element of the dual space $B_p^*$ of $B_p$ with the identification $S(u) = \mathbf{E}_p[ku]$, but ${}^*B_p$ is strictly smaller than $B_p^*$ unless the sample space has a finite number of atoms.*

3. *Denoting with sub-0 the spaces of centered random variables, the following continuous inclusions hold true:*

$$L_0^\infty(p \cdot \mu) \subset B_p \subset \bigcap_{\alpha>1} L_0^\alpha(p \cdot \mu) \subset L_0^\alpha(p \cdot \mu) \subset {}^*B_p \subset B_p^*.$$

The patches of the atlas will be defined on the open ball of radius 1:

$$\mathcal{V}_p = \left\{u \in B_p : \|u\|_p < 1\right\};$$

remark that the condition $\|u\|_p < 1$ is equivalent to the existence of an $\alpha > 1$ such that $\mathrm{E}_p[\cosh(\alpha u) - 1] \leq 1$, which in turn implies $\mathrm{E}_p[e^u] < 4$, see Prop. 2 below.

The *moment generating functional* $G_p : L^{(\cosh \cdot -1)}(p \cdot \mu) \to \overline{R}_+ = [0, +\infty]$ is defined by

$$G_p(u) = \mathbf{E}_p[e^u] \quad .$$

The *cumulant generating functional* $K_p : B_p \to [0, +\infty]$ is defined by

$$K_p(u) = \log G_p(u).$$

**Proposition 2 (Properties of the CGF)** *The cumulant generating functional $K_p$ has proper domain $Dom(G_p) \cap B_p$. If $\mathcal{V}_p$ denotes a subset of the proper domain then $K_p$ satisfies the following properties*

1. *$K_p$ is 0 at 0, otherwise is strictly positive; is convex and infinitely Fréchet differentiable on $\mathcal{V}_p$. The value at 0 of the differential of order $n$ is the value of the $n$-th cumulant under $p$ of the random variable $u$.*
2. *$\forall u \in \mathcal{V}_p$, $q = e^{u-K_p(u)} \cdot p$ is a probability density in $\mathcal{M}(X, \mathcal{X}, \mu)$ and the value of the $n$-th differential at $u$ in the direction $v$ of $K_p$ is the $n$-th cumulant of $v$ under $q$:*

$$\mathrm{D}^n \, K_p\,(u)\ \ v^n = \frac{d^n}{dt^n} \log \mathrm{E}_q \left[e^{tv}\right]\Big|_{t=0} .$$

3. *In particular $\frac{q}{p} - 1 \in {}^*B_p$ and*

$$\begin{array}{r} \mathrm{D}\, K_p(u)\ \ v = \mathrm{E}_q\,[v] = \mathrm{E}_p\left[\left(\frac{q}{p} - 1\right) v\right] \\ \mathrm{D}^2\, K_p(u)\ \ v_1 v_2 = \mathrm{E}_q\,[v_1 v_2] - \mathrm{E}_q\,[v_1]\,\mathrm{E}_q\,[v_2] \end{array} . \tag{4}$$

Using the definitions introduced so far, it is possible to give a definition of the non-parametric exponential model as follows. For each $p$ in $\mathcal{M}(X, \mathcal{X}, \mu)$ *the maximal exponential model* at $p$ is the statistical model

$$\mathcal{E}_p = \left\{e^{u-K_p(u)} \cdot p : u \in \mathrm{Dom}\,(K_p)^\circ, \mathrm{E}_p\,[u] = 0\right\} .$$

The function

$$B_p \supset \mathrm{Dom}\,(K_p)^\circ \ni u \mapsto e^{u-K_p(u)} \cdot p \in \mathcal{M}(X, \mathcal{X}, \mu)$$

is the likelihood function when the 'model parameter' is $u$.

We now have all the elements for the definition of the atlas. Let us consider the following map defined on a subset $\mathcal{V}_p$ of the proper domain $K_p$:

$$e_p : \mathcal{V}_p \ni u \mapsto q = e^{u-K_p(u)} \cdot p \in \mathcal{M}(X, \mathcal{X}, \mu), \tag{5}$$

where $K_p(u) = \log \mathrm{E}_p\,[e^u] = \log G_p(u)$ is the cumulant generating functional computed at $u$.

This mapping is one–to–one because $u$ is centered. According to (1) and (2) we shall denote by $\mathcal{U}_p$ the image of the mapping and by $s_p$ its inverse on $\mathcal{U}_p$. Such an inverse, $s_p : \mathcal{U}_p \to \mathcal{V}_p$ is easily computed, for $q \in \mathcal{U}_p$, as

$$s_p(q) = \log \frac{q}{p} - \mathrm{E}_p\left[\log \frac{q}{p}\right] . \tag{6}$$

*The functions $s_p$, $p \in \mathcal{M}(X, \mathcal{X}, \mu)$, will be the coordinate mappings of our manifold in the sense that, locally around each $p \in \mathcal{M}(X, \mathcal{X}, \mu)$, each $q \in \mathcal{U}_p$ will be "parameterized" by its* centered log–likelihood.

Let us compute now the change-of-coordinates formula: if $p_1$ and $p_2$ are two points in $\mathcal{M}(X, \mathcal{X}, \mu)$ such that $\mathcal{U}_{p_1} \cap \mathcal{U}_{p_2} \neq \emptyset$, then the composite (transition) mapping

$$s_{p_2} \circ e_{p_1} : s_{p_1}(\mathcal{U}_{p_1} \cap \mathcal{U}_{p_2}) \to s_{p_2}(\mathcal{U}_{p_1} \cap \mathcal{U}_{p_2})$$

simplifies to

$$s_{p_2} \circ e_{p_1}(u) = u + \log \frac{p_1}{p_2} - \mathrm{E}_{p_2}\left[u + \log \frac{p_1}{p_2}\right]$$

where the algebraic computations are done in the space of $\mu$–classes of measurable functions and the expectation is well defined as long as $\mathcal{U}_{p_1} \cap \mathcal{U}_{p_2} \neq \emptyset$ because this implies $u + \log \frac{p_1}{p_2} \in \mathcal{V}_{p_2}$.

**Theorem 3** *The collection of pairs $\{(\mathcal{U}_p, s_p) : p \in \mathcal{M}(X, \mathcal{X}, \mu)\}$ is an affine $C^\infty$–atlas on $\mathcal{M}(X, \mathcal{X}, \mu)$. The induced topology on sequences is equivalent to e–convergence.*

**Definition 4 (Exponential manifold)** *The exponential manifold is the manifold defined by the property in theorem 3 on the set $\mathcal{M}(X, \mathcal{X}, \mu)$.*

The manifold structure we have defined is a special one: many other types of atlases have been suggested in the literature, in particular the mixture coordinates and the so-called Amari's imbeddings described in [1].

In the infinite dimensional case those different geometric structures are not equivalent to the exponential manifold, but in some restricted sense they are, because they induce the same manifold structure on finite dimensional sub-manifolds (i.e. parametric statistical manifolds).

The maximal exponential model already defined has a precise place in the general framework. In fact the maximal exponential model $\mathcal{E}_p$ is the connected component containing $p$ of the exponential manifold $\mathcal{M}(X, \mathcal{X}, \mu)$.

In the previous works on the differential geometric approach to nonlinear filtering and to the finite dimensional approximation of the Fokker–Planck equation (Brigo, Hanzon and LeGland [9], [10] and Brigo [13]) we used the $L_2$ structure to project the Kushner–Stratonovich or the Fokker–Planck equation onto a finite dimensional exponential manifold of densities. This procedure uses the map $p \mapsto \sqrt{p}$ from positive densities to their square roots as a tool which allows the $L_2$ structure to enter the picture. Although this is useful to perform computations, and even if this approach yields the same finite dimensional approximation as in the case where one projects according to the exponential manifold structure discussed here (compare formulae given in section 4 with formulae obtained via the $L_2$ structure given in [13]), we notice that this map cannot be used to define a manifold structure. It does not yield charts. This is due to the fact that any open set of $L_2$ contains functions which are negative in a set with positive measure. Then we see that a chart should map open sets in the manifold onto open sets in $L_2$, but these open sets would contain the functions described above, and hence they could not be contained in any set of square roots of densities (which are positive everywhere). This is why the space of square roots of densities cannot have a manifold structure based on $L_2$. In [10] this problem is bypassed by defining a parametric exponential enveloping manifold. Here we use the exponential manifold structure to render the procedure rigorous in an infinite dimensional context.

Now we show the properties of the map from $\mathcal{M}$ to $L_2$ defined by $R: \ p \mapsto \sqrt{p}$.

**Proposition 5** *The mapping*

$$R : \mathcal{M} \ni p \mapsto \sqrt{p} \in L^2(\mu)$$

*is $C^\infty$. If the tangent space is identified with $B_p$ then its tangent map is*

$$T_p R(v) = \frac{1}{2} R(p) v.$$

*In particular the tangent map is surjective at any $p$.*

*Proof.* Let us fix a density $p_0$ and consider the coordinate form of the map $R$; it is defined from $\mathcal{V}_{p_0}$ to $L_2(p_0)$ by $H_{p_0}(u) := \sqrt{e_{p_0}(u)}$. By direct computation one obtains the form of the directional derivative

$$\frac{d}{dv} H_{p_0}(u) = H_{p_0}(u)(\tfrac{1}{2}[v - DK_{p_0}(u)v]) = H_{p_0}(u)\tfrac{1}{2}[v - E_{e_{p_0}(u)}v],$$

and the norm of the differential operator

$$\|D\ H_{p_0}(u)\ v\|_2^2 = \tfrac{1}{4} E_{e_{p_0}(u)}\{(v - E_{e_{p_0}(u)}v)^2\} = \tfrac{1}{4} D^2 K_{p_0}(u)(v, v),$$

which shows that $H_{p_0}$ is differentiable at $u$ since $K_{p_0}$ is infinitely Fréchet differentiable. The coordinate-free form of the differential is $T_p R v = \frac{1}{2}\sqrt{p} v$ if we identify the tangent space at $p$ with $B_p$. Note that the $L_2$ norm of this first derivative represents a variance, so that the derivative operator is one-to-one.

In a similar way the other derivatives of $H_{p_0}$ can be computed as:

$$D^n\ H_{p_0}(u)(v_1, \ldots, v_n) = 2^{-n} H_{p_0}(u)(v_1 - E_{e_{p_0}(u)}v_1, \ldots, v_n - E_{e_{p_0}(u)}v_n).$$

If one computes the $L_2$ norm of this derivative one finds easily that the norm is bounded and the differentiability of any order follows. □

Hence the mapping $p \mapsto \sqrt{p}$ is what we call a regular parametrization. From the exponential coordinates $u$ we deduce $H_{p_0}(u)$ which can be differentiated. Yet $H_{p_0}$, although constituting a parametrization, does not define coordinates. This is due to the fact that we are working in infinite dimension and a regular parametrization is not necessarily a chart.

A basic object of the manifolds theory is the tangent bundle. In the case of the exponential manifold it has been remarked from the very beginning of the geometrical theory that there is a very natural identification between the tangent vectors and the exponential one-dimensional models around a point $p$. In fact each differentiable curve in $\mathcal{M}(X, \mathcal{X}, \mu)$, i.e. each one-dimensional statistical model $p(t)$, $t \in I \subseteq R$, such that $p(0) = p$, has a tangent model of the exponential form $e^{tu - K_p(tu)} \cdot p$. This can be rephrased by saying that these exponential one-dimensional models seem to play the role of straight lines.

**Definition 6 (Tangent space)** *The tangent space $T_p$ at $p$ of the exponential manifold on $\mathcal{M}(X, \mathcal{X}, \mu)$ is the set (indexed by $u$) of the one-dimensional exponential models*

$$e^{tu - K_p(tu)} \cdot p, \quad t \in I\!R, \quad u \in B_p \quad .$$

*Usually we will identify the tangent exponential model $e^{tu-K_p(tu)} \cdot p$ with its* score

$$\left. \frac{d}{dt} \left( tu - K_p(tu) \right) \right|_{t=0} = u \in B_p \quad .$$

*The tangent space inherits the structure of Banach space from $B_p$.*

**Definition 7 (Sub-manifold, sub-model)** *Let $\mathcal{N}$ be a subset of the exponential manifold $\mathcal{M}(X, \mathcal{X}, \mu)$ and, for each density $p \in \mathcal{N}$, let $V_p^1$ and $V_p^2$ be closed subsets of $B_p$, such that there exist:*

1. *a linear invertible and bi-continuous mapping between $B_p$ and some direct sum $V_p^1 + V_p^2$. That is $V_p^1$ and $V_p^2$* split *in $B_p$.*

2. *a chart on a neighbourhood $\mathcal{W}_p$ of $p$:*

$$\sigma_p : \mathcal{W}_p \to V_p^1 + V_p^2,$$

*where $\sigma_p$ maps $\mathcal{W}_p$ onto the product of to open sets $\mathcal{V}_p^1 \times \mathcal{V}_p^2$ and $\mathcal{N} \cap \mathcal{W}_p$ onto $\mathcal{V}_p^1 \times \{0\}$.*

*We will say that $\mathcal{N}$ is a* sub-model *or a* sub-manifold *of the exponential manifold $\mathcal{M}(X, \mathcal{X}, \mu)$.*

A sub-manifold is a manifold defined by the restricted maps. For a list of examples see [22]. Our basic example is a finite dimensional exponential family

$$EM(c) = \{p(\cdot, \theta), \theta \in \Theta\},$$
$$p(\cdot, \theta) := \exp[\theta^T c(\cdot) - \psi(\theta)],$$

where $c = (c_1, \ldots, c_n)$, and $\Theta$ is a convex open set in $I\!R^n$. In this case the local representation at $p(\cdot, \theta_0)$ is

$$p(\cdot, \theta) = \exp[(\theta - \theta_0)^T [c(\cdot) - \psi'(\theta_0)] - (\psi(\theta) - \psi(\theta_0)) + (\theta - \theta_0)^T \psi'(\theta_0)] p(\cdot, \theta_0),$$

and the relevant splitting is

$$\begin{aligned} V^1_{p(\theta_0)} &= \text{span}\left\{ c_i - \frac{\partial}{\partial \theta_i} \psi(\theta_0) \right\} \\ V^2_{p(\theta_0)} &= \left\{ u \in B_{p(\theta_0)} : \mathrm{E}_{p(\theta_0)}\left[ u c_i \right] = 0, i = 1, \ldots, n \right\}. \end{aligned}$$

# 3 Evolution of marginal laws of a diffusion process

On the complete probability space $(\Omega, \mathcal{F}, P)$ let us consider a stochastic process $\{X_t, t \geq 0\}$ of diffusion type. Let the dynamic equation describing $X$ be of the following form

$$dX_t = f_t(X_t) dt + \sigma_t(X_t) dW_t,$$

where $\{W_t, t \geq 0\}$ is a standard Brownian motion independent of the initial condition $X_0$. The equation above is an Itô stochastic differential equation. In the following derivation, we treat the scalar case. The following set of assumptions will be in force throughout the paper.

(A) Initial condition: We assume that the initial state $X_0$ has a density $p_0$ w.r.t. the Lebesgue measure on $I\!R$, with $p_0$ almost surely positive.

(B) Local strong existence: $f \in C^{1,0}$, $a \in C^{2,0}$, which means that $f$ is once continuously differentiable wrt $x$ and continuous wrt $t$ and $a$ is twice continuously differentiable wrt $x$ and continuous wrt $t$. This assumptions imply in particular local Lipschitz continuity.

(C) Non–explosion : there exists $K > 0$ such that

$$2 x f_t(x) + a_t(x) \leq K\,(1 + |x|^2),$$

for all $t \geq 0$, and for all $x \in I\!R$.

Under assumptions (A), (B) and (C) there exists a unique solution $\{X_t\,,\, t \geq 0\}$ to the state equation, see [27], theorem 10.2.1 with $\phi(x, t) = x^2$.

(D) We assume that the law of $X_t$ is absolutely continuous and its density $p_t(x)$ has regularity $C^{2,1}$ and satisfies the Fokker–Planck equation (FPE):

$$\frac{\partial p_t}{\partial t} = \mathcal{L}^*_t p_t, \tag{7}$$

where the backward diffusion operator $\mathcal{L}_t$ is defined by

$$\mathcal{L}_t = f_t \frac{\partial}{\partial x} + \tfrac{1}{2} a_t \frac{\partial^2}{\partial x^2},$$

and its dual (forward) operator is given by

$$\mathcal{L}_t^* p = -\frac{\partial}{\partial x}(f_t p) + \tfrac{1}{2}\frac{\partial^2}{\partial x^2}(a_t p).$$

We assume also $p_t(x)$ to be positive for all $t \geq 0$ and almost all $x \in \mathbb{R}$.

Assumption (D) holds for example under conditions given by boundedness of the coefficients $f$ and $a$ plus uniform ellipticity of $a_t$, see [27] theorem 9.1.9. Different conditions are also given in [16], theorem 6.4.7. Now we rewrite equation (7) in the exponential coordinates (6). Consider as local reference density the solution $p_t$ of FPE at time $t$. We are now working around $p_t$. Consider a curve around $p_t$ corresponding to the solution of FPE around time $t$ expressed in $B_{p_t}$ coordinates:

$$\begin{array}{rcl} (-\epsilon, \epsilon) & \to & \mathcal{V}_{p_t} \\ h & \mapsto & s_{p_t}(p_{t+h}) =: u_h. \end{array}$$

The function $u_h$ represents the expression in coordinates of the density

$$p_{t+h} = \exp[u_h - K_{p_t}(u_h)] p_t =: e_h p_t. \tag{8}$$

Now consider FPE around $t$, i.e.

$$\frac{\partial p_{t+h}}{\partial h} = \mathcal{L}_{t+h}^* p_{t+h}.$$

Substitute (8) in this last equation in order to obtain

$$\frac{\partial e_h p_t}{\partial h} = \mathcal{L}_{t+h}^*(e_h p_t).$$

Write

$$\frac{\partial e_h}{\partial h} = \frac{\mathcal{L}_{t+h}^*(e_h p_t)}{p_t}$$

and set $h = 0$, since we are concerned with the behaviour in $t$. Notice that $e_0 = \exp[u_0 - K_{p_t}(u_0)] = \exp(0) = 1$, and that

$$\frac{\partial e_h}{\partial h}|_{h=0} = \{e_h \frac{\partial [u_h - K_{p_t}(u_h)]}{\partial h}\}|_{h=0} = \frac{\partial [u_h - K_{p_t}(u_h)]}{\partial h}|_{h=0}.$$

Moreover, by straightforward computations (write explicitly the map $K_{p_t}$, use $u_h = s_{p_t}(p_{t+h})$ and differentiate wrt $h$ under the expectation $E_{p_t}$) one verifies

$$\left.\frac{\partial K_{p_t}(u_h)}{\partial h}\right|_{h=0} = 0,$$

so that

$$\left.\frac{\partial u_h}{\partial h}\right|_{h=0} = \frac{\mathcal{L}_t^* p_t}{p_t} \tag{9}$$

is the formal representation in exponential coordinates of the tangent vector in $p_t$. Notice that, again by straightforward computations,

$$\begin{aligned} \alpha_t := \alpha_t(p_t) = \frac{\mathcal{L}_t^* p_t}{p_t} &= -f_t \frac{\partial}{\partial x}(\log p_t) - \frac{\partial f_t}{\partial x} + \\ &+ \tfrac{1}{2}[\, a_t \frac{\partial^2}{\partial x^2}(\log p_t) + a_t \,(\frac{\partial}{\partial x}(\log p_t))^2 + \\ &+ 2\frac{\partial a_t}{\partial x}\frac{\partial}{\partial x}(\log p_t) + \frac{\partial^2 a_t}{\partial x^2}\,]\ . \end{aligned} \tag{10}$$

Summarizing: consider the curve expressing FPE around $p_t$ in $B_{p_t}$ coordinates. Its tangent vector is given by $\alpha_t$. Under suitable assumptions on the coefficients $f_t$ and $a_t$ the function $\alpha_t$ belongs to $B_{p_t}$, according to the convention that locally identifies the tangent space of normed spaces with the normed space itself. To render the computation not only formal we need $\alpha_t$ to be really a tangent vector for our manifold structure. This in turn requires the curve $t \mapsto p_t$ to be differentiable. Below we give a regularity result expressing a condition under which this happens and whose proof is immediate. Moreover, we give a condition which can be used to check whether the evolution stays in a given submanifold.

**Proposition 8 (Regularity and finite dimensionality of the solution of FPE)**

*(i)* *If the map $t \mapsto p_t$ is differentiable in the manifold $\mathcal{M}$ then $\alpha_t$ given in eq. (10) is a tangent vector.*

*(ii)* *If the map $t \mapsto \alpha_t$ is continuous at $t_0$ into $L^{\cosh \cdot -1}$, then $t \mapsto p_t$ is differentiable at $t_0$ as a map into $\mathcal{M}$.*

*(iii)* *Let be given a submanifold $\mathcal{N}$ such that $p_0 \in \mathcal{N}$. If the previous condition is satisfied and*

$$\frac{\mathcal{L}_t^* p}{p}$$

*is tangent to $\mathcal{N}$ at $p$ for all $p \in \mathcal{N}$, then $p_t$ evolves in $\mathcal{N}$.*

Sufficient conditions under which condition (ii) in the proposition happens to be true are given by boundedness for all possible $T > 0$ of $f$, $\partial_x f$, $a$, $\partial_x a$, $\partial^2_{xx} a$ in $[0 \;\; T] \times \mathbb{R}$ plus classical assumptions given in Stroock and Varadhan [27] theorem 9.1.9, or Friedman [16], theorem 6.4.7, ensuring existence of a regular solution or Fokker–Planck equation (as required in (D)). This follows from the fact that if $\alpha_t(x)$ is continuous and bounded in both $t$ and $x$, then it is continuous as a map $t \mapsto \alpha_t$ from $[0 \;\; T]$ to $L^{\cosh \cdot -1}$.

In the following we give examples where this proposition applies. Some of them are obtained from [13] where the detailed derivation is given.

*Example 9.* [Linear case] If $f_t(x) = F_t x$ for all $t \geq 0,\; x \in \mathbb{R}$ ($f$ linear in $x$) and if $a_t(x) = A_t$ for all $t \geq 0,\; x \in \mathbb{R}$ ($a$ does not depend on $x$) and if finally $p_0 \sim \mathcal{N}(m_0, Q_0)$ then it is known that $p_t \sim \mathcal{N}(m_t, Q_t)$ where $m_t = m_0 \exp \int_0^t F_s ds$ and $Q_t$ is the (unique) positive solution of the (scalar Lyapunov) equation

$$\dot{Q}_t = 2F_t Q_t + A_t,$$

with initial condition $Q_0$ given. Consider now a generic Gaussian density $p \sim \mathcal{N}(m, Q)$ and compute

$$(\frac{\mathcal{L}_t^* p}{p})(x) = (\frac{F_t}{Q} + \frac{A_t}{2Q^2})x^2 - (\frac{F_t m}{Q} + \frac{A_t m}{Q^2})x + \frac{A_t m^2}{2Q^2} - F_t - \frac{A_t}{2Q}. \tag{11}$$

When applied to $p_t$, the previous formula yields $\alpha_t$:

$$\alpha_t = (\frac{F_t}{Q_t} + \frac{A_t}{2Q_t^2})x^2 - (\frac{F_t m_t}{Q_t} + \frac{A_t m_t}{Q_t^2})x + \frac{A_t m_t^2}{2Q_t^2} - F_t - \frac{A_t}{2Q_t},$$

where $m_t$ and $Q_t$ have been defined above.

In this case the previous proposition applies. First, one sees that $t \mapsto \alpha_t$ is indeed continuous at any $t_0$ in $L^{\cosh \cdot -1}$. Secondly, one can deduce already from (11) *without solving the Fokker–Planck equation* that the solution will have a Gaussian density. Indeed, one can easily check that the tangent space to the Gaussian submanifold of $\mathcal{M}$ expressed in $B$ coordinates contains the function space span$\{1, x, x^2\}$. Since by expression (11) we see that $(\mathcal{L}_t^* p)/p$ lies in span$\{1, x, x^2\}$ for all $p$ in the Gaussian submanifold, we deduce that the solution of the Fokker–Planck equation will evolve in the Gaussian submanifold.

*Example 10.* [Nonlinear diffusions with unit variance Gaussian law] Let be given a diffusion coefficient $\sigma_t(x)$ satisfying assumptions (B) and assumption (C) when the drift vanishes, i.e. when $f = 0$ (we set as usual $a := \sigma^2$). In [13] it is shown that by defining the drift

$$f_t(x) := \tfrac{1}{2}\frac{\partial a_t}{\partial x}(x) + \tfrac{1}{2}a_t(x)[kt - x] + k,$$

the Fokker–Planck equation for the density of the solution of the stochastic differential equation

$$dX_t = f_t(X_t)dt + \sigma_t(X_t)dW_t \quad X_0 \sim \mathcal{N}(0, 1),$$

is solved by $p_t \sim \mathcal{N}(kt, 1)$ for all possible diffusion coefficients $\sigma_t(x)$. Here the solution of the Fokker–Planck equation evolves in a submanifold of $\mathcal{M}$ given by Gaussian densities with unit variance. Actually, the mean of $p_t$ evolves linearly in time and the variance is fixed to one. Note that in this case

$$\alpha_t = \partial_t \, \log p_t = k(x - kt),$$

and the curve $t \mapsto \alpha_t$ is clearly continuous at any $t_0$ in $L^{\cosh \cdot - 1}$. One might check a priori that if a given $p$ belongs to the submanifold of Gaussian densities with unit variance, then $(\mathcal{L}_t^* p)/p$ belongs to the tangent space of this submanifold if the mean is given by $kt$. Indeed, we are considering the family

$$p(x, \theta) = \frac{1}{\sqrt{2\pi}} \exp[-\tfrac{1}{2}(x-\theta)^2] \sim \mathcal{N}(\theta, 1), \quad \theta \in I\!R,$$

and its tangent space expressed in $B_{p(\cdot,\theta)}$ coordinates, span$\{x - \theta\}$. Let us compute

$$\alpha_{t,\theta}(x) = \tfrac{1}{2}(\partial_x a_t(x)) \; (kt - \theta) + \; \tfrac{1}{2} a_t(x)(x - \theta)(kt - \theta) \; + \; k(x - \theta).$$

Under reasonable assumptions on $a$, this function belongs to the tangent space span$\{x - \theta\}$ if and only if $\theta = kt$. We have been able to check that the density of the diffusion $X$ evolves according to $p_t \sim \mathcal{N}(kt, 1)$ without solving the Fokker–Planck equation.

¿From examples given in [13] one can construct other nonlinear cases where the above proposition applies.

# 4 Projection of the Fokker–Planck equation

In reaching equation (9) we assumed implicitly a few facts. We are assuming that there always exists a neighborhood of $h = 0$ such that in this neighborhood $p_{t+h} \in \mathcal{U}_{p_t}$. Conditions under which this happens will be examined in the future. We only remark that when projecting on a finite dimensional exponential manifold, these conditions are not necessary for the projected equation to exist and make sense, see below. Neither we need equation (9) to have a solution to obtain existence of the solutions of the projected equation. Now we shall project this equation on a finite dimensional parametrized exponential manifold. First notice that according to proposition 8 the law $p_t$ remains finite dimensional if there exists a finite dimensional parametrized submanifold of $\mathcal{M}$,

$$S = \{p(\cdot, \theta), \theta \in \Theta\},$$

such that the corresponding tangent vectors $\alpha_t(p(\cdot, \theta))$ of the FPE are in the tangent space of this finite dimensional submanifold. We take the set $\Theta$ open in $I\!R^m$. If we look for a finite dimensional *exponential* family, we can select a submanifold

$$EM(c) = \{p(\cdot,\theta), \theta \in \Theta\},$$

$$p(\cdot,\theta) := \exp[\theta^T c(\cdot) - \psi(\theta)].$$

We will assume the following on the family $EM(c)$ (see [9], [10] for other more specific assumptions):

(E) We assume $c \in C^2$.

Notice that tangent vectors around a point $p(\cdot,\theta_t)$ of a generic curve $h \mapsto p(\cdot,\theta_{t+h})$ on $EM(c)$, are now obtained according to (after straightforward computations and the chain rule)

$$\frac{\partial s_{p(\cdot,\theta_t)}(p(\cdot,\theta_{t+h}))}{\partial h}|_{h=0} = \sum_{i=1}^{m}[c_i(\cdot) - E_{\theta_t} c_i]\dot{\theta}_t^i. \tag{12}$$

As a consequence, the tangent space at $\theta$ is given by

$$T_\theta EM(c) = \text{span}\{c_1(\cdot) - E_\theta c_1, \ldots, c_m(\cdot) - E_\theta c_m\},$$

where $E_\theta\{\phi\} := \int \phi(x) p(x,\theta) dx$. Consider the following inner product in $T_\theta EM(c)$ (and more in general in $B_{p(\cdot,\theta)}$):

$$\langle v_1, v_2 \rangle_\theta := E_\theta[v_1 \ v_2], \qquad v_1, v_2 \in B_{p(\cdot,\theta)}.$$

Consider the quantities

$$g(\theta)_{ij} := \langle c_i(\cdot) - E_\theta c_i, \ c_j(\cdot) - E_\theta c_j \rangle_\theta, \quad i,j = 1,\ldots,m.$$

Notice that the matrix $g(\theta)$, expressing the inner products of tangent vectors in $T_\theta EM(c)$, is nothing else than the traditional Fisher information matrix

$$( \ E_\theta \ [\partial_{\theta_i} \log p(\cdot,\theta) \ \ \partial_{\theta_j} \log p(\cdot,\theta)] \ )_{i,j=1,\ldots,m}$$

for the family $EM(c)$ (see also [9], [10]). Now define for all $\theta \in \Theta$ the orthogonal projection

$$\Pi_\theta : B_{p(\cdot,\theta)} \longrightarrow T_\theta EM(c)$$
$$\Pi_\theta[v] := \sum_{i=1}^{m}[\sum_{j=1}^{m} g^{ij}(\theta) \ \langle v, c_j(\cdot) - E_\theta c_j \rangle_\theta] \ (c_i(\cdot) - E_\theta c_i).$$

A rapid computation involving duality between $\mathcal{L}$ and $\mathcal{L}^*$ and standard results on the normalization constant $\psi(\theta)$ of exponential families (such as $\partial_{\theta_i}\psi(\theta) = E_\theta c_i$) yields

$$\mathcal{P}_{t,\theta} \ := \ \Pi_\theta[\frac{\mathcal{L}_t^* p(\cdot,\theta)}{p(\cdot,\theta)}] = E_\theta[\mathcal{L}_t c]^T \ g^{-1}(\theta) \ [c(\cdot) - E_\theta c],$$

where integrals of vector functions are meant to be applied to their components. Note that this map is regular in $\theta$ under reasonable assumptions on $f, a$ and $c$. At this point we project equation (9) via this projection. By remembering expression (12) for tangent vectors and the above formula for the projection we obtain the following ($m$–dimensional) ordinary differential equation (in vector form) on the manifold $EM(c)$:

$$[c(\cdot) - E_{\theta_t}c]^T \dot{\theta}_t = E_{\theta_t}[\mathcal{L}_t c]^T \; g^{-1}(\theta_t) \; [c(\cdot) - E_{\theta_t}c]. \tag{13}$$

It follows immediately the following ordinary differential equation for the parameters:

$$\dot{\theta}_t = g^{-1}(\theta_t) \; E_{\theta_t}\{\mathcal{L}_t \, c\}. \tag{14}$$

Notice that, as anticipated above, equation (14) is well defined and admits locally a unique solution if the following condition (ensuring existence of the norm of $\alpha_t(p(\cdot, \theta_t))$ associated to the inner product $\langle \cdot \, , \, \cdot \rangle_{\theta_t}$) holds:

$$(F) \qquad\qquad E_\theta\{\alpha_{t,\theta}^2\} < \infty \;\; \forall \theta \in \Theta, \tag{15}$$

$$\begin{aligned}\alpha_{t,\theta} := \frac{\mathcal{L}_t^* p(\cdot, \theta)}{p(\cdot, \theta)} &= -f_t \, \frac{\partial}{\partial x}(\theta^T c) - \frac{\partial f_t}{\partial x} + \\ &+ \tfrac{1}{2}[\, a_t \, \frac{\partial^2}{\partial x^2}(\theta^T c) + a_t \, (\frac{\partial}{\partial x}(\theta^T c))^2 \; + \\ &+ 2 \, \frac{\partial a_t}{\partial x} \, \frac{\partial}{\partial x}(\theta^T c) + \frac{\partial^2 a_t}{\partial x^2} \,] \; .\end{aligned}$$

We will assume such condition to hold in the following. Notice that this is a condition on the coefficients $f, a, c$. We have thus proven the following

**Proposition 11 ( Projected evolution of the density of an Itô diffusion)** *Assume assumptions (A), (B),(C), (E) and (F) on the coefficients $f, a$, on the initial condition $X_0$ of the Itô diffusion $X$, and on the sufficient statistics $c_1, \ldots, c_n$ of the exponential family $EM(c)$ are satisfied. Then the projection of Fokker–Planck equation describing the evolution of $p_t = p_{X_t}$ onto $EM(c)$ reads, in $B_{p_t}$ coordinates:*

$$[c(\cdot) - E_{\theta_t}c]^T \dot{\theta}_t = E_{\theta_t}[\mathcal{L}c]^T \; g^{-1}(\theta_t) \; [c(\cdot) - E_{\theta_t}c],$$

*and the differential equation describing the evolution of the parameters for the projected density–evolution is*

$$\dot{\theta}_t = g^{-1}(\theta_t) \; E_{\theta_t}\{\mathcal{L}_t \, c\}.$$

Notice that the projected equations exist under conditions which are more general than conditions for existence of the solution of the original Fokker–Planck equation. For more details see Brigo [13].

# 5 Interpretation of the projected density as density of a different diffusion

In this section we shortly expose a problem which was treated in [13]. Consider the projected density $p(\cdot,\theta_t)$, expressing the projection of the density–evolution of the one dimensional diffusion $X$ onto the exponential manifold $EM(c)$. The question is: Can we define a diffusion $Y_t$ whose density is the projected density $p(\cdot,\theta_t)$ ? If the answer is yes, $Y_t$ is a diffusion whose density evolves in a finite dimensional exponential manifold assigned a priori (for example Gaussian). In order to proceed, define a diffusion

$$dY_t = u_t(Y_t)dt + \sigma_t(Y_t)dW_t, \quad Y_0 = X_0, \tag{16}$$

with the same diffusion coefficient as $X_t$. We shall try to define the drift $u$ in such a way that the density–evolution of $Y_t$ coincides with $p(\cdot,\theta_t)$. Call $\mathcal{T}_t$ the backward differential operator of $Y_t$:

$$\mathcal{T}_t = u_t\frac{\partial}{\partial x} + \tfrac{1}{2}a_t\frac{\partial^2}{\partial x^2}.$$

Consider the right hand sides of (9) and (13). Clearly, the density of $Y_t$ coincides with $p(\cdot,\theta_t)$ if

$$\frac{\mathcal{T}^* p(\cdot,\theta_t)}{p(\cdot,\theta_t)} = E_{\theta_t}[\mathcal{L}c]^T\ g^{-1}(\theta_t)\ [c(\cdot) - E_{\theta_t}c]$$

which we can rewrite as

$$\mathcal{T}^* p(\cdot,\theta_t) = \mathcal{P}_{t,\theta_t}\ p(\cdot,\theta_t).$$

By simple calculations one can rewrite the above equation as the following PDE for $u$, where we do not expand the second partial derivative of $a_t p(\cdot,\theta)$:

$$\frac{\partial u_t}{\partial x} + \theta_t^T\frac{\partial c}{\partial x}u_t = \frac{1}{2\,p(\cdot,\theta_t)}\frac{\partial^2}{\partial x^2}(a_t p(\cdot,\theta_t)) - \mathcal{P}_{t,\theta_t}$$

Call $\mathcal{B}_{t,\theta_t}$ the right hand side of such equation. A solution is given by

$$u_t^*(x) := \exp[-\theta_t^T c(x)]\int_{-\infty}^x \mathcal{B}_{t,\theta_t}(y)\exp[\theta_t^T c(y)]dy,$$

as one can verify immediately by substitution. Straightforward calculations yield

$$\begin{aligned} u_t^*(x) \ &:= \ \frac{1}{p(x,\theta_t)}\int_{-\infty}^x [\frac{\partial^2_{xx}(a_t(y)p(y,\theta_t))}{p(y,\theta_t)} - \Pi_{\theta_t}\{\frac{\partial^2_{xx}(a_t(y)p(y,\theta_t))}{p(y,\theta_t)}\} + \\ &\qquad +\Pi_{\theta_t}\{\frac{\partial_x(f_t(y)p(y,\theta_t))}{p(y,\theta_t)}\}]p(y,\theta_t)dy = \\ &= \ \tfrac{1}{2}\frac{\partial a_t}{\partial x}(x) + \tfrac{1}{2}a_t(x)\theta_t^T\frac{\partial c}{\partial x}(x) + \\ &\qquad -E_{\theta_t}\{\mathcal{L}_t c\}^T g^{-1}(\theta_t)\int_{-\infty}^x (c(y) - E_{\theta_t}c)\ \exp[\theta_t^T(c(y) - c(x))]dy. \end{aligned} \tag{17}$$

¿From this last equation one sees that under condition (15) and under the assumption that densities of $EM(c)$ are integrable, the above integral always exists.

We have thus proven the following

**Proposition 12 (Interpretation of the projected density–evolution)** *Assume assumptions (A), (B), (C), (E) and (F) on the coefficients $f, a$ and on the initial condition $X_0$ of the Itô diffusion $X$ and on the sufficient statistics $c$ of the exponential family $EM(c)$ are satisfied. Let $p(\cdot, \theta_t)$ be the projected density evolution, according to proposition 11. Define*

$$\begin{aligned} dY_t &= u_t^*(Y_t)dt + \sigma_t(Y_t)dW_t, \\ u_t^*(x) &:= \tfrac{1}{2}\frac{\partial a_t}{\partial x}(x) + \tfrac{1}{2}a_t(x)\theta_t^T \frac{\partial c}{\partial x}(x) + \\ & \qquad -E_{\theta_t}\{\mathcal{L}_t c\}^T g^{-1}(\theta_t) \int_{-\infty}^{x} (c(y) - E_{\theta_t} c)\ \exp[\theta_t^T (c(y) - c(x))]dy. \end{aligned}$$

*Then $Y$ is an Itô diffusion whose density–evolution coincides with the projected density–evolution $p(\cdot, \theta_t)$ of $X_t$ onto $EM(c)$.*

# 6 Further research on convergence of the projected density towards the original one

Now we consider the problem of the convergence of the projected density–evolution to the evolution of the limit diffusion. The idea is to consider a sequence of nested finite dimensional families and to check what happens when the dimension of the family on which the equation is projected tends to infinity. The problem we shall investigate in the future is the following. Suppose we can define a sequence of families in the following way:

**(G)** we are given a sequence of functions $(c_j)_{j\in I\!N}$. Call $c^m := \{c_1\ c_2\ \ldots c_m\}$, and assume that for all $m$ the family $EM(c^m)$ is a finite dimensional exponential manifold satisfying assumptions (E) and (F). Call $p(\cdot, \theta_t^m)$ the density coming from projection of Fokker–Planck equation onto $EM(c^m)$.

As we saw in the preceding section (see (17)) this is also the density of a diffusion process $Y^m$ with the same initial condition, the same diffusion coefficient and drift given by

$$\begin{aligned} u_t^m(x) &:= \frac{1}{p(x,\theta_t^m)} \int_{-\infty}^{x} [\frac{\partial_{xx}^2 (a_t(y)p(y,\theta_t^m))}{p(y,\theta_t^m)} - \Pi_{\theta_t^m}\{\frac{\partial_{xx}^2 (a_t(y)p(y,\theta_t^m))}{p(y,\theta_t^m)}\} + \\ & \qquad + \Pi_{\theta_t^m}\{\frac{\partial_x (f_t(y)p(y,\theta_t^m))}{p(y,\theta_t^m)}\}]p(y,\theta_t^m)dy. \end{aligned} \tag{18}$$

Hence, if we prove that $Y^m$ converges in law towards the original diffusion $X$, we have a first convergence result of the projected evolution towards the original one. Now, it is well known that since all the diffusions $Y^m$ share the same diffusion coefficient, under reasonable assumptions on $\sigma$ the sequence of the laws of $(Y^m)_m$ is relatively compact in the space of processes with continuous trajectories (see for example Stroock and Varadhan [27] or Bafico and Pistone [3]). Moreover, if the drifts $u^m$ weakly converge to $f$, the law of $Y^m$ will converge to the law of $X$. By looking at expression (18), one sees intuitively that this should happen. Indeed, assume $p(\cdot, \theta^m)$ admits a limit $\bar{p}(\cdot)$ when $m$ tends to infinity, and suppose the projection tends to be exact when $m$ tends to infinity. Then in formula (18) replace $p(\cdot, \theta^m)$ by $\bar{p}$ and eliminate the projection operators. The expression for $u_t(x)$ simplifies to $f_t(x)$, so that we have pointwise convergence of the drifts $u^m$ towards $f$ and we are done. Of course, one needs to make the above idea precise, to show reasonable choices of $c^m$ and to prove the result rigorously. These problems will be investigated in the next future.

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