CROSS-CURRENCY SMILE CALIBRATION

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ABSTRACT
We document the numerical aspects of the calibration of cross-currency options on the local volatility framework. We consider the partial differential equation satisfied by the price of the cross-currency option and see that the most important specifications to set are the boundary conditions. We explain how these conditions can be approximated and test the validity of the approximation on simple cases. .

KEY WORDS
cross-currency options, calibration, local volatility, implied volatility, Dupire formula, adjoint, boundary conditions

1 Motivation

We investigate in this paper the situation of an agent that uses as reference some domestic currency (e.g. USD) and trades options on two currency pairs, e.g. USD/JPY and USD/EUR. Starting from quotes of vanilla options on each pair (e.g. european calls see [9]) one can recover the local volatility surface for USD/JPY and USD/EUR: this is a well known inverse problem in mathematical finance called *calibration of local volatility*, (cf. [13, 7, 6, 7, 9, 8, 1, 4, 5, 10, 11] for some references).

Then supposing that the correlation is known, one can deduce the price of cross-options i.e. EUR/JPY by a direct Black-Scholes model ([3, 9]) as we show latter in Eqn. (2-3).

The focus of this paper is the following: supposing that local volatilities are known how can one solve the equation for the cross-option price. We will see that the problem resides rather in coherent approximations of the boundary conditions.

2 Equation for the price

We describe our situation in the following form: consider two securities S_1 and S_2 whose prices follow, under their respective risk-neutral measure [12, 9], the stochastic differential equations (we do not write the time dependence)

$$dS_i/S_i = \mu_i dt + \sigma_i dW_i, \; i = 1, 2. \quad (1)$$

Here r is the risk-free rate of the domestic currency and r_i the risk-free rate of the i-th foreign currency; $\mu_i = r - r_i$ is the drift; r_i can be seen as a 'dividend' rate); e.g. for the FOREX examples above if S_1 is the price in USD of one unit of EUR then $\mu_1 = r_{USD} - r_{EUR}$.

Further $\sigma_i = \sigma_i(t, S_i)$ are the local volatilities and W_t^i are Brownian motions; we take here constant correlation $< dW_t^1, dW_t^2 >= \rho_{1,2} dt$.

Les us consider (for now) plain vanilla call options contingent on S_1/S_2; by delta-hedging arguments [9] one can show that the price c of a plain vanilla call of strike K on the cross-security S_1/S_2 follows the 2D Black-Scholes [3] equation:

$$\partial_t c + S_1\mu_1\partial_{S_1} c + S_2\mu_2\partial_{S_2} c + \\ + \frac{\sigma_1^2(S_1)^2}{2}\partial_{S_1S_1} c + \frac{\sigma_2^2(S_2)^2}{2}\partial_{S_2S_2} c \\ +\rho_{1,2} S_1 S_2 \sigma_1\sigma_2 \frac{\partial^2 c}{\partial S_1 \partial S_2} - rc = 0 \quad (2)$$

$$c(T, S_1, S_2) = S_2[S_1/S_2 - K]_+ = [S_1 - KS_2]_+ . \quad (3)$$

Remark 1 *The particular form of the Eqn.* (3) *comes from the fact that the option is settled in the second foreign currency (S_2), while here the price 'c' is expressed in the domestic currency. One can, of course, change c in c/S_2 and obtain the PDE satisfied by the price in currency S_2.*

The equation can be then solved by any standard means (see [9, 2, 1]) e.g. through a Crank-Nicholson finite-difference scheme or finite element schemes , etc ...

3 Boundary conditions

We explain in this section what are the boundary conditions to impose on the Eqn. (2-3).

A naive approach would be to take a (large) domain and hope that the conditions outside the domain do not count when computing the price. This works e.g. for basket options with payoff $(S_1 + S_2 - K)_+$ as large S_1 and S_2 naturally induce a null price for the put (payoff $(S_1+S_2-K)_-$) and the conditions result from the put-call parity. In this case however, large values for S_1 and S_2 are significant because their quotient may be close to strike K.

We follow the Fig.1 and describe the four different regions where the boundary conditions are set.

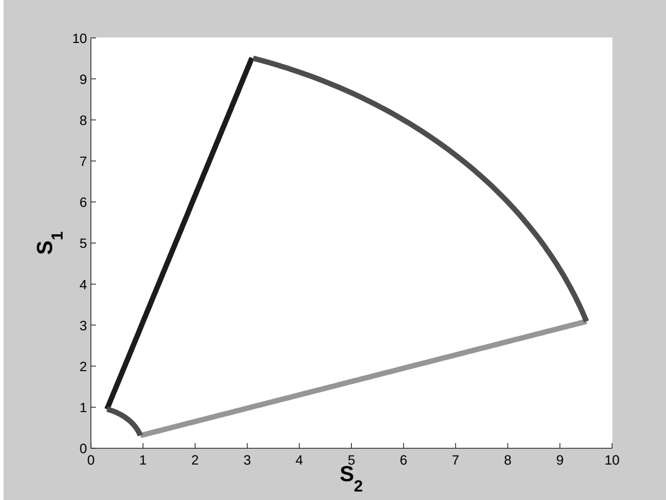

Figure 1. The domain in S_1/S_2. We note four different regions: the line S_1/S_2 constant and small, the line S_1/S_2 constant and large, the inner circular region S_1, S_2 small but with constant volatility and the outer circular region with S_1, S_2 large but with constant volatility.

3.1 On the line $S_1/S_2 = c_0$, with $c_0 << K$

In this case the option is initially out of the money and the underlying far below the strike; therefore it has extremely small probability to finish in the money. The call price will be zero. $c(t, S_1, S_2) = 0$ for $S_1/S_2 = c_0 >> K$.

3.2 On the line $S_1/S_2 = c_\infty$, with $c_\infty >> K$

In this situation the option is initially in the money and the underlying far above the strike; therefore it has extremely small probability to finish out of the money. The **put** price will be zero and the call price will be set from the call-put parity [9]

$$c(t, S_1, S_2) = S_1 e^{-r_1(T-t)} - K S_2 e^{-r_2(T-t)} \quad (4)$$
$$\text{for } S_1/S_2 = c_\infty >> K.$$

3.3 In the region of both S_1, S_2 are small and in the region where both S_1, S_2 are large

The market will not quote extremely in/out of the money options so there is no way to recover the local volatilities σ_1 and σ_2 in these regions; therefore one can set them to any quantity; we set them constant. Denoting now $S_{1:2} = S_1/S_2$ one obtains by the Ito formula

$$dS_{1:2}/S_{1:2} = (\mu_1 - \mu_2 + \sigma_2(\sigma_2 - \rho_{1,2}\sigma_1))dt \quad (5)$$
$$+\sigma_1 dW_1 - \sigma_2 dW_2.$$

Note that, because here σ_1 and σ_2 can be supposed constant, $\sigma_1 dW_1 - \sigma_2 dW_2$ can be written $\sigma_1 dW_1 - \sigma_2 dW_2 = \sigma_{1:2} dW$ with W a Brownian motion and $\sigma_{1:2}^2 = \sigma_2^2 + \sigma_1^2 - 2\rho_{1,2}\sigma_1\sigma_2$. The equation is written in the domestic *numeraire* (the same as for S_1 and S_2). This *numeraire* is not well adapted (the quantity S_1/S_2 units of domestic currency does not mean anything) so one rather wants to work with the risk-neutral probability of the numeraire S_2; a convexity adjustment [9, Chap 25.7 and 27] shows that the risk-neutral evolution equation is

$$dS_{1:2}/S_{1:2} = (r_2 - r_1)dt + \sigma_{1:2} dW. \quad (6)$$

The price of the call is just the price of a plain vanilla option with risk-free rate r_2, dividend rate r_1 and volatility $\sigma_{1:2}$; this price is given by the classic Black & Scholes formula and needs then to be multiplied by S_2:

$$\text{for } S_1, S_2 \text{ both small or large:}$$
$$c(t, S_1, S_2) = e^{-r_1(T-t)} S_1 \Phi(d_1)$$
$$-K S_2 e^{-r_2(T-t)} \Phi(d_2) \quad (7)$$

with

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \exp(-u^2) du \quad (8)$$

the standard normal cumulative distribution function,

$$d_1 = \frac{\ln(S_1/KS_2) + (r_2 - r_1 + \frac{\sigma_{1:2}^2}{2})(T-t)}{\sigma_{1:2}\sqrt{T-t}} \quad (9)$$
$$d_2 = d_1 - \sigma_{1:2}\sqrt{T-t}. \quad (10)$$

Of course, one does not necessarily need to include the region where both S_1 and S_2 are small i.e., one can go with the conical domain till the origin.

4 Discussion and perspectives

We tested the procedure on a standard case (σ_i are time independent, parabolic in S_i) and compared with a Monte-

Carlo approach. The results are confirming the good quality of the approximation. We are currently working on real-life data and will present the results in a future publication.

Depending how many options prices are required, one may also look for a Dupire-like equation for a continuum of option prices (cf. recent works by Pironneau et al.). We are currently working on the implementation of the Dupire-2D equation and on a comparison between the two from a computational point of view.

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