Markov operators in q-Fourier analysis

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Abstract

In this paper there are many important results of q-Fourier analysis proved by new ways and new sets of markov operators acting on the $\mathcal{L}_{q,2,v}$ space of square q-integrable function which were defined.

1 Introduction

In the mathematical literature one finds many articles which deal with the theory of q-Fourier analysis associated with the q-Hankel transform. This theory finds its origins in an important article of H.T. Koornwinder and R.F. Swarttouw [5]. The two authors proved that the q-Bessel functions of Hahn-Exton satisfy an orthogonality relation, which gives a rigorous proof. They noticed that the q-Hankel transform would be an involutive isometric operator of the $\mathcal{L}_{q,2,v}$ space. Our goal is to give a detailed proof of this important result. It should be noticed that in [1] we provided the mains results of q-Fourier analysis in particular that the q-Hankel transform is prolonged on the $\mathcal{L}_{q,2,v}$ space like an isometric involutive operator. But we used the positivity of the q-Bessel translation operator in order to show these results. These property is not ensured for any q in the interval $]0,1[$. Thus, we will prove some main results of q-Fourier analysis without the positivity of the q-Bessel translation operator, which are the following

- Inversion Formula in the $\mathcal{L}_{q,p,v}$ spaces with $p \geq 1$.
- Plancherel Formula in the $\mathcal{L}_{q,p,v} \cap \mathcal{L}_{q,1,v}$ spaces with $p > 2$.
- Plancherel Formula in the $\mathcal{L}_{q,2,v}$ spaces.
- Product formula for the q-Hankel transform .

Note that in a recent paper [2] we have proved that the positivity of the q-Bessel translation operator is ensured in all points of the interval $]0,1[$

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when $v \geq 0$. In this article we will try to show in a clear way the part in which the positivity of the q-Bessel translation operator plays a role in q-Fourier analysis. In particular, when we try to introduce the concept of Markov operators of the $\mathcal{L}_{q,2,v}$ space. Many interesting examples of such operators are the q-Bessel translation operator and the q-heat semigroup.

2 Preliminaries

In the following we will always assume $0 < q < 1$ and $v > -1$. We denote by

$$\mathbb{R}_q = \{\pm q^n, \quad n \in \mathbb{Z}\}, \qquad \mathbb{R}_q^+ = \{q^n, \quad n \in \mathbb{Z}\}.$$

For more information on the q-series theory the reader can see the reference [3,4,6] and the reference [1,5] about the q-bessel Fourier analysis. Also for details of the proof of the following result in this section can be found in [1,5].

Definition 1 *The q-Bessel operator is defined as follows*

$$\Delta_{q,v} f(x) = \frac{1}{x^2}\left[f(q^{-1}x) - (1+q^{2v})f(x) + q^{2v} f(qx)\right].$$

Definition 2 *The eigenfunction of $\Delta_{q,v}$ associated with the eigenvalue $-\lambda^2$ is the function $x \mapsto j_v(\lambda x, q^2)$, where $j_v(., q^2)$ is the normalized q-Bessel function defined by*

$$j_v(x, q^2) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(q^{2v+2}, q^2)_n (q^2, q^2)_n} x^{2n}.$$

Definition 3 *The q-Jackson integral of a function f defined on $\mathbb{R}_q$ is*

$$\int_0^{\infty} f(t) d_q t = (1-q) \sum_{n \in \mathbb{Z}} q^n f(q^n).$$

Definition 4 *We denote by $\mathcal{L}_{q,p,v}$ the space of even functions f defined on $\mathbb{R}_q$ such that*

$$\|f\|_{q,p,v} = \left[\int_0^{\infty} |f(x)|^p x^{2v+1} d_q x\right]^{1/p} < \infty.$$

Definition 5 *We denote by $\mathcal{C}_{q,0}$ the space of even functions defined on $\mathbb{R}_q$ tending to 0 as $x \to \pm\infty$ and continuous at 0 equipped with the topology of uniform convergence. The space $\mathcal{C}_{q,0}$ is complete with respect to the norm*

$$\|f\|_{q,\infty} = \sup_{x\in\mathbb{R}_q} |f(x)|.$$

Proposition 1 *The normalized q-Bessel function $j_v(.,q^2)$ satisfies the orthogonality relation*

$$c_{q,v}^2 \int_0^\infty j_v(xt,q^2) j_v(yt,q^2) t^{2v+1} d_q t = \delta_q(x,y), \qquad \forall x,y \in \mathbb{R}_q.$$

where

$$\delta_q(x,y) = \begin{cases} 0 & \textit{if } x \neq y \\ \frac{1}{(1-q)x^{2(v+1)}} & \textit{if } x = y \end{cases}$$

and

$$c_{q,v} = \frac{1}{1-q} \frac{(q^{2v+2},q^2)_\infty}{(q^2,q^2)_\infty}.$$

Remark 1 *Let f be an even function defined on $\mathbb{R}_q$ then*

$$\int_0^\infty f(y)\delta_q(x,y) y^{2v+1} d_q y = f(x).$$

Proposition 2 *The normalized q-Bessel function $j_v(.,q^2)$ satisfies*

$$|j_v(q^n,q^2)| \leq \frac{(-q^2;q^2)_\infty(-q^{2v+2};q^2)_\infty}{(q^{2v+2};q^2)_\infty} \begin{cases} 1 & \textit{if} \quad n \geq 0 \\ q^{n^2-(2v+1)n} & \textit{if} \quad n < 0 \end{cases}.$$

Definition 6 *The q-Bessel Fourier transform $\mathcal{F}_{q,v}$ is defined by*

$$\mathcal{F}_{q,v} f(x) = c_{q,v} \int_0^\infty f(t) j_v(xt,q^2) t^{2v+1} d_q t, \qquad \forall x \in \mathbb{R}_q.$$

Proposition 3 *Let $f \in \mathcal{L}_{q,1,v}$ then $\mathcal{F}_{q,v} f$ exists and $\mathcal{F}_{q,v} f \in \mathcal{C}_{q,0}$.*

3 Inversion formula in the $\mathcal{L}_{q,p,v}$ space

Theorem 1 *Let f be a function in the $\mathcal{L}_{q,p,v}$ space where $p \geq 1$ then*

$$\mathcal{F}_{q,v}^2 f = f.$$

Proof. If $f \in \mathcal{L}_{q,p,v}$ then $\mathcal{F}_{q,v}f$ exist, and we write

$$\begin{aligned}\mathcal{F}^2_{q,v}f(x) &= c_{q,v}\int_0^\infty \mathcal{F}_{q,v}f(t)j_v(xt,q^2)t^{2v+1}d_qt\\ &= \int_0^\infty f(y)\left[c^2_{q,v}\int_0^\infty j_v(xt,q^2)j_v(yt,q^2)t^{2v+1}d_qt\right]y^{2v+1}d_qy\\ &= \int_0^\infty f(y)\delta_q(x,y)y^{2v+1}d_qy\\ &= f(x).\end{aligned}$$

We can permute the two integrals: in fact if $p > 1$ then we use the Hölder's inequality

$$\begin{aligned}&\int_0^\infty |f(y)|\left[\int_0^\infty \left|j_v(xt,q^2)j_v(yt,q^2)\right| t^{2v+1}d_qt\right]y^{2v+1}d_qy\\ &\leq \left(\int_0^\infty |f(y)|^p\, y^{2v+1}d_qy\right)^{1/p} \times \left(\int_0^\infty \sigma(y)^{\overline{p}}y^{2v+1}d_qy\right)^{1/\overline{p}}.\end{aligned}$$

The numbers p and $\overline{p}$ above are conjugated and

$$\sigma(y) = \int_0^\infty \left|j_v(xt,q^2)j_v(yt,q^2)\right| t^{2v+1}d_qt,$$

then

$$\begin{aligned}&\int_0^\infty \sigma(y)^{\overline{p}}y^{2v+1}d_qy\\ &= \int_0^1 \sigma(y)^{\overline{p}}y^{2v+1}d_qy + \int_1^\infty \sigma(y)^{\overline{p}}y^{2v+1}d_qy.\end{aligned}$$

Not that

$$\begin{aligned}\int_0^1 \sigma(y)^{\overline{p}}y^{2v+1}d_qy &\leq \left\|j_v(.,q^2)\right\|^{\overline{p}}_{q,\infty}\int_0^1\left\{\int_0^\infty \left|j_v(xt,q^2)\right| t^{2v+1}d_qt\right\}^{\overline{p}}y^{2v+1}d_qy\\ &\leq \left\|j_v(.,q^2)\right\|^{\overline{p}}_{q,\infty}\left\|j_v(.,q^2)\right\|^{\overline{p}}_{q,1,v}x^{-2(v+1)\overline{p}}\left[\int_0^1 y^{2v+1}d_qy\right] < \infty,\end{aligned}$$

and

$$\begin{aligned}&\int_1^\infty \sigma(y)^{\overline{p}}y^{2v+1}d_qy\\ &\leq \left\|j_v(.,q^2)\right\|^{\overline{p}}_{q,\infty}\left\|j_v(.,q^2)\right\|^{\overline{p}}_{q,1,v}\int_1^\infty \frac{y^{2v+1}}{y^{2(v+1)\overline{p}}}d_qy\\ &\leq \left\|j_v(.,q^2)\right\|^{\overline{p}}_{q,\infty}\left\|j_v(.,q^2)\right\|^{\overline{p}}_{q,1,v}\int_1^\infty \frac{1}{y^{2(v+1)(\overline{p}-1)+1}}d_qy < \infty.\end{aligned}$$

If $p = 1$ then

$$\begin{aligned}&\int_0^\infty |f(y)| \left\{ \int_0^\infty \left| j_v(xt,q^2) j_v(yt,q^2) \right| t^{2v+1} d_q t \right\} y^{2v+1} d_q y \\ &\le \|f\|_{q,1,v} \left\| j_v(.,q^2) \right\|_{q,\infty} \left\| j_v(.,q^2) \right\|_{q,1,v} \times \frac{1}{x^{2(v+1)}},\end{aligned}$$

this finishes the proof. ■

4 Plancherel Formula in the $\mathcal{L}_{q,1,v} \cap \mathcal{L}_{q,p,v}$ space

Theorem 2 *Let f be a function in the $\mathcal{L}_{q,1,v} \cap \mathcal{L}_{q,p,v}$ space, where $p > 2$ then*

$$\|\mathcal{F}_{q,v} f\|_{q,2,v} = \|f\|_{q,2,v}$$

Proof. Let $f \in \mathcal{L}_{q,1,v} \cap \mathcal{L}_{q,p,v}$ then by Theorem 1 we see that

$$\mathcal{F}_{q,v}^2 f = f,$$

this implies

$$\begin{aligned}\int_0^\infty \mathcal{F}_{q,v} f(x)^2 x^{2v+1} d_q x &= \int_0^\infty \mathcal{F}_{q,v} f(x) \left\{ c_{q,v} \int_0^\infty f(t) j_v(xt,q^2) t^{2v+1} d_q t \right\} x^{2v+1} d_q x \\ &= \int_0^\infty f(t) \left\{ c_{q,v} \int_0^\infty \mathcal{F}_{q,v} f(x) j_v(xt,q^2) x^{2v+1} d_q x \right\} t^{2v+1} d_q t \\ &= \int_0^\infty f(t)^2 t^{2v+1} d_q t.\end{aligned}$$

We can permute the two integrals because

$$\begin{aligned}&\int_0^\infty |f(t)| \left\{ c_{q,v} \int_0^\infty |\mathcal{F}_{q,v} f(x)| \left| j_v(xt,q^2) \right| x^{2v+1} d_q x \right\} t^{2v+1} d_q t \\ &\le \left(\int_0^\infty |f(t)|^p t^{2v+1} d_q t \right)^{1/p} \times \left(\int_0^\infty |\phi(t)|^{\overline{p}} t^{2v+1} d_q t \right)^{1/\overline{p}},\end{aligned}$$

where

$$\phi(t) = c_{q,v} \int_0^\infty |\mathcal{F}_{q,v} f(x)| \left| j_v(xt,q^2) \right| x^{2v+1} d_q x,$$

then

$$\begin{aligned} |\mathcal{F}_{q,v}f(x)| &\le c_{q,v}\int_0^\infty |f(y)|\left|j_v(xy,q^2)\right| y^{2v+1}d_qy\\ &\le c_{q,v}\left(\int_0^\infty |f(y)|^p\, y^{2v+1}d_qy\right)^{1/p}\times\left(\int_0^\infty \left|j_v(xy,q^2)\right|^{\overline{p}} y^{2v+1}d_qy\right)^{1/\overline{p}}\\ &\le c_{q,v}\left(\int_0^\infty |f(y)|^p\, y^{2v+1}d_qy\right)^{1/p}\times\left(\int_0^\infty \left|j_v(y,q^2)\right|^{\overline{p}} y^{2v+1}d_qy\right)^{1/\overline{p}} x^{-2(v+1)/\overline{p}}\\ &\le c_{q,v}\left\|f\right\|_{q,p,v}\left\|j_v(.,q^2)\right\|_{q,\overline{p},v} x^{-2(v+1)/\overline{p}}.\end{aligned}$$

This gives

$$\begin{aligned} \phi(t) &\le c_{q,v}^2\left\|f\right\|_{q,p,v}\left\|j_v(.,q^2)\right\|_{q,\overline{p},v}\int_0^\infty \left|j_v(xt,q^2)\right| x^{(2v+1)-2(v+1)/\overline{p}}d_qx\\ &\le c_{q,v}^2\left\|f\right\|_{q,p,v}\left\|j_v(.,q^2)\right\|_{q,\overline{p},v}\left[\int_0^\infty \left|j_v(x,q^2)\right| x^{2(v+1)/p-1}d_qx\right] t^{-2(v+1)/p}\\ &\le C_1 t^{-2(v+1)/p},\end{aligned}$$

and

$$\begin{aligned} \phi(t) &= c_{q,v}\int_0^\infty |\mathcal{F}_{q,v}f(x)|\left|j_v(xt,q^2)\right| x^{2v+1}d_qx\\ &= \left[c_{q,v}\int_0^\infty |\mathcal{F}_{q,v}f(x/t)|\left|j_v(x,q^2)\right| x^{2v+1}d_qx\right] t^{-2(v+1)}\\ &\le c_{q,v}\left\|\mathcal{F}_{q,v}f\right\|_{q,\infty}\times\left\|j_v(.,q^2)\right\|_{q,1,v}\times t^{-2(v+1)}\\ &\le C_2 t^{-2(v+1)}.\end{aligned}$$

Hence,

$$\begin{aligned} \int_0^\infty |\phi(t)|^{\overline{p}}\, t^{2v+1}d_qt &= \int_0^1 |\phi(t)|^{\overline{p}}\, t^{2v+1}d_qt+\int_1^\infty |\phi(t)|^{\overline{p}}\, t^{2v+1}d_qt\\ &\le C_1\int_0^1 t^{-2(v+1)\overline{p}/p}t^{2v+1}d_qt+C_2\int_1^\infty t^{-2(v+1)\overline{p}}t^{2v+1}d_qt<\infty.\end{aligned}$$

In fact

$$\begin{cases} -1<-2(v+1)\frac{\overline{p}}{p}+2v+1\\ -2(v+1)\overline{p}+2v+1<-1\end{cases}\Leftrightarrow\begin{cases} 0<-2(v+1)(\overline{p}-2)\\ -2(v+1)(\overline{p}-1)<0\end{cases}\Leftrightarrow 1<\overline{p}<2\Leftrightarrow p>2,$$

this finishes the proof. ∎

5 Plancherel Formula in the $\mathcal{L}_{q,2,v}$ space

Theorem 3 *Let f be a function in the $\mathcal{L}_{q,2,v}$ space then*

$$\left\|\mathcal{F}_{q,v}f\right\|_{q,2,v} = \left\|f\right\|_{q,2,v}.$$

Proof. We introduce the function ψ_x as follows

$$\psi_x(t) = c_{q,v}j_v(tx,q^2)$$

The inner product $\langle , \rangle$ in the Hilbert space $\mathcal{L}_{q,2,v}$ is defined by

$$f,g \in \mathcal{L}_{q,2,v} \Rightarrow \langle f,g\rangle = \int_0^\infty f(t)g(t)t^{2v+1}d_qt.$$

By Proposition 1 we write

$$x \neq y \Rightarrow \langle \psi_x, \psi_y\rangle = 0$$

$$\left\|\psi_x\right\|_{q,2,v}^2 = \frac{1}{1-q}x^{-2(v+1)}.$$

We have

$$\mathcal{F}_{q,v}f(x) = \langle f, \psi_x\rangle,$$

and by Theorem 1

$$f \in \mathcal{L}_{q,2,v} \Rightarrow \mathcal{F}_{q,v}^2 f = f,$$

then

$$\langle f, \psi_x\rangle = 0, \forall x \in \mathbb{R}_q^+ \Rightarrow \mathcal{F}_{q,v}f = 0 \Rightarrow f = 0.$$

Hence, $\left\{\psi_x, x \in \mathbb{R}_q^+\right\}$ form an orthogonal basis of the Hilbert space $\mathcal{L}_{q,2,v}$ and we have

$$\overline{\left\{\psi_x, x \in \mathbb{R}_q^+\right\}} = \mathcal{L}_{q,2,v} \tag{1}$$

Now

$$f \in \mathcal{L}_{q,2,v} \Rightarrow f = \sum_{x\in\mathbb{R}_q^+} \frac{1}{\left\|\psi_x\right\|_{q,2,v}^2}\langle f, \psi_x\rangle \psi_x,$$

and then

$$\left\|f\right\|_{q,2,v}^2 = \sum_{x\in\mathbb{R}_q^+} \frac{1}{\left\|\psi_x\right\|_{q,2,v}^2}\langle f, \psi_x\rangle^2 = (1-q)\sum_{x\in\mathbb{R}_q^+} x^{2(v+1)}\mathcal{F}_{q,v}f(x)^2 = \left\|\mathcal{F}_{q,v}f\right\|_{q,2,v}^2,$$

this finishes the proof. ■

6 q-translation operator and q-convolution product

Definition 7 *The q-translation operator is given as follows (see [1])*

$$T^v_{q,x}f(y) = c_{q,v}\int_0^\infty \mathcal{F}_{q,v}f(t)j_v(yt,q^2)j_v(xt,q^2)t^{2v+1}d_qt.$$

Proposition 4 *If $f \in \mathcal{L}_{q,1,v}$ then $T^v_{q,x}f$ exists and*

$$T^v_{q,x}f(y) = \int_0^\infty f(z)D_v(x,y,z)z^{2v+1}d_qz,$$

where

$$D_v(x,y,z) = c^2_{q,v}\int_0^\infty j_v(xs,q^2j_v(ys,q^2j_v(zs,q^2)s^{2v+1}d_qs.$$

Proof. In fact for all $x \in \mathbb{R}_q$

$$\begin{aligned}\left|T^v_{q,x}f(y)\right| &\leq \int_0^\infty \left|\mathcal{F}_{q,v}f(t)\right|\left|j_v(yt,q^2)\right|\left|j_v(xt,q^2)\right|t^{2v+1}d_qt\\ &\leq \left\|\mathcal{F}_{q,v}f\right\|_{q,\infty}\left\|j_v(.,q^2)\right\|_{q,\infty}\left\|j_v(.,q^2)\right\|_{q,1,v}\frac{1}{x^{2(v+1)}} < \infty.\end{aligned}$$

On the other hand

$$\begin{aligned}T^v_{q,x}f(y) &= c_{q,v}\int_0^\infty \mathcal{F}_{q,v}f(t)j_v(yt,q^2)j_v(xt,q^2)t^{2v+1}d_qt\\ &= c_{q,v}\int_0^\infty\left[c_{q,v}\int_0^\infty f(z)j_v(zt,q^2)z^{2v+1}d_qz\right]j_v(yt,q^2)j_v(xt,q^2)t^{2v+1}d_qt\\ &= \int_0^\infty f(z)D_v(x,y,z)z^{2v+1}d_qz.\end{aligned}$$

We can permute the two integrals because

$$\begin{aligned}&\int_0^\infty\left[\int_0^\infty |f(z)|\left|j_v(zt,q^2)\right|z^{2v+1}d_qz\right]\left|j_v(yt,q^2)\right|\left|j_v(xt,q^2)\right|t^{2v+1}d_qt\\ &\leq \|f\|_{q,1,v}\left\|j_v(.,q^2)\right\|^2_{q,\infty}\left\|j_v(.,q^2)\right\|_{q,1,v}\times\frac{1}{x^{2(v+1)}} < \infty,\end{aligned}$$

which finishes the proof. ■

Definition 8 *The q-convolution product is defined by*

$$f *_q g(x) = c_{q,v} \int_0^\infty T^v_{q,x} f(y) g(y) y^{2v+1} d_q y.$$

Proposition 5 *Let* $f, g \in \mathcal{L}_{q,1,v}$ *then* $f *_q g$ *exists and we have*

$$f *_q g = g *_q f.$$

Proof. Let $f, g \in \mathcal{L}_{q,1,v}$ then

$$\begin{aligned} f *_q g(x) &= c_{q,v} \int_0^\infty T^v_{q,x} f(y) g(y) y^{2v+1} d_q y \\ &= c_{q,v} \int_0^\infty \left[c_{q,v} \int_0^\infty \mathcal{F}_{q,v} f(t) j_v(yt, q^2) j_v(xt, q^2) t^{2v+1} d_q t \right] g(y) y^{2v+1} d_q y \\ &= c_{q,v} \int_0^\infty \mathcal{F}_{q,v} f(t) \left[c_{q,v} \int_0^\infty g(y) j_v(yt, q^2) y^{2v+1} d_q y \right] j_v(xt, q^2) t^{2v+1} d_q t \\ &= c_{q,v} \int_0^\infty \mathcal{F}_{q,v} f(t) \mathcal{F}_{q,v} g(t) j_v(xt, q^2) t^{2v+1} d_q t \\ &= g *_q f(x), \end{aligned}$$

The exchange of the integral signs has a valid sense because

$$\begin{aligned} &\int_0^\infty \int_0^\infty |\mathcal{F}_{q,v} f(t)| \times \left| j_v(yt, q^2) \right| \times \left| j_v(xt, q^2) \right| \times |g(y)| \, y^{2v+1} t^{2v+1} d_q t d_q y \\ &\leq \|\mathcal{F}_{q,v} f\|_{q,\infty} \left\| j_v(., q^2) \right\|_{q,\infty} \|g\|_{q,1,v} \left\| j_v(., q^2) \right\|_{q,1,v} \times x^{-2(v+1)}, \end{aligned}$$

which prove the result. ■

Proposition 6 *Let* $f \in \mathcal{L}_{q,1,v}$ *and* $g \in \mathcal{L}_{q,2,v}$ *then*

$$\mathcal{F}_{q,v}(f *_q g) = \mathcal{F}_{q,v}(f) \times \mathcal{F}_{q,v}(g).$$

Proof. In fact we have

$$\mathcal{F}_{q,v} : \mathcal{L}_{q,2,v} \to \mathcal{L}_{q,2,v}$$

and

$$\mathcal{F}_{q,v} : \mathcal{L}_{q,1,v} \to \mathcal{C}_{q,0}.$$

If $f \in \mathcal{L}_{q,1,v}$ and $g \in \mathcal{L}_{q,2,v}$ then

$$\omega = \mathcal{F}_{q,v} f \times \mathcal{F}_{q,v} g \in \mathcal{L}_{q,2,v}$$

On the other hand

$$\mathcal{F}_{q,v}\omega(x) = c_{q,v}\int_0^\infty \mathcal{F}_{q,v}f(t)\mathcal{F}_{q,v}g(t)j_v(xt,q^2)t^{2v+1}d_qt.$$

From the inversion formula in Theorem 1 we have

$$\mathcal{F}_{q,v}\left(\mathcal{F}_{q,v}\omega\right) = \mathcal{F}_{q,v}f \times \mathcal{F}_{q,v}g.$$

Now we write

$$\begin{aligned}\mathcal{F}_{q,v}\omega(x) &= c_{q,v}\int_0^\infty \mathcal{F}_{q,v}f(t)\mathcal{F}_{q,v}g(t)j_v(xt,q^2)t^{2v+1}d_qt\\ &= c_{q,v}\int_0^\infty \mathcal{F}_{q,v}f(t)\left[c_{q,v}\int_0^\infty g(y)j_v(ty,q^2)y^{2v+1}d_qy\right]j_v(xt,q^2)t^{2v+1}d_qt\\ &= c_{q,v}\int_0^\infty g(y)\left[c_{q,v}\int_0^\infty \mathcal{F}_{q,v}f(t)j_v(ty,q^2)j_v(xt,q^2)t^{2v+1}d_qt\right]y^{2v+1}d_qy\\ &= c_{q,v}\int_0^\infty g(y)T^v_{q,x}f(y)y^{2v+1}d_qy\\ &= f *_q g(x).\end{aligned}$$

This implies that

$$\mathcal{F}_{q,v}\left(f *_q g\right) = \mathcal{F}_{q,v}f \times \mathcal{F}_{q,v}g.$$

The exchange of the integral signs can be justified, indeed let

$$\begin{aligned}\phi(y) &= \int_0^\infty \left|\mathcal{F}_{q,v}f(t)\right| \times \left|j_v(xt,q^2)\right| \times \left|j_v(yt,q^2)\right| t^{2v+1}d_qt\\ &= \int_0^\infty \varrho(t) \times \left|j_v(yt,q^2)\right| t^{2v+1}d_qt\end{aligned}$$

where

$$\varrho(t) = \left|\mathcal{F}_{q,v}f(t)\right| \times \left|j_v(xt,q^2)\right| \Rightarrow \varrho \in \mathcal{L}_{q,1,v}$$

then

$$\lim_{y\downarrow 0}\phi(y) < \infty,$$

and

$$\begin{aligned}\phi(y) &= \int_0^\infty \varrho(t) \times \left|j_v(yt,q^2)\right| t^{2v+1}d_qt\\ &= \left[\int_0^\infty \varrho\left(\frac{t}{y}\right) \times \left|j_v(t,q^2)\right| t^{2v+1}d_qt\right] y^{-2(v+1)}\\ &\leq \left[\left\|\mathcal{F}_{q,v}f\right\|_{q,\infty}\left\|j_v(.,q^2)\right\|_{q,\infty}\left\|j_v(.,q^2)\right\|_{q,1,\infty}\right] y^{-2(v+1)},\end{aligned}$$

which prove that $\phi \in \mathcal{L}_{q,2,v}$. This leads to the result. $\blacksquare$

7 Positivity of the q-translation operator

Definition 9 *The operator $T^v_{q,x}$ is said positive if $T^v_{q,x}f \geq 0$ when $f \geq 0$ for all $x \in \mathbb{R}_q$. We denote by Q_v the domain of positivity of $T^v_{q,x}$*

$$Q_v = \left\{q \in]0,1[, \qquad T^v_{q,x} \text{ is positive}\right\}.$$

In a recent paper [2] the authors prove that if $v \geq 0$ then $Q_v =]0,1[$. In the following we assume that $q \in Q_v$.

Proposition 7 *The operator $T^v_{q,x}$ is positive if and only if*

$$D_v(x,y,z) \geq 0, \qquad \forall x,y,z \in \mathbb{R}_q.$$

Proof. If we take $f(z) = \delta_a(z)$ then

$$T^v_{q,x}f(y) = (1-q)a^{2(v+1)}D_v(x,y,a),$$

which prove the result. ■

Proposition 8 *If $f \in \mathcal{L}_{q,1,v}$ then*

$$\int_0^\infty T^v_{q,x}f(y)y^{2v+1}d_qy = \int_0^\infty f(y)y^{2v+1}d_qy.$$

Proof. In fact

$$\begin{aligned}\int_0^\infty T^v_{q,x}f(y)y^{2v+1}d_qy &= \int_0^\infty \left[\int_0^\infty f(z)D_v(x,y,z)z^{2v+1}d_qz\right] y^{2v+1}d_qy \\ &= \int_0^\infty \left[\int_0^\infty D_v(x,y,z)y^{2v+1}d_qy\right] f(z)z^{2v+1}d_qz.\end{aligned}$$

On the other hand

$$D_v(x,y,z) = \mathcal{F}_{q,v}\phi(y)$$

where

$$\phi(t) = c_{q,v}j_v(tx,q^2)j_v(tz,q^2),$$

then

$$\begin{aligned}\int_0^\infty D_v(x,y,z)y^{2v+1}d_qy &= \int_0^\infty \mathcal{F}_{q,v}\phi(y)y^{2v+1}d_qy \\ &= \frac{1}{c_{q,v}}\mathcal{F}^2_{q,v}\phi(0) = \frac{1}{c_{q,v}}\phi(0) = 1.\end{aligned}$$

This leads to the result. ■

Proposition 9 *Let $f, g \in \mathcal{L}_{q,1,v}$ then*

$$\mathcal{F}_{q,v}(f *_q g) = \mathcal{F}_{q,v}(f) \times \mathcal{F}_{q,v}(g).$$

Proof. In fact

$$\begin{aligned}
&\mathcal{F}_{q,v}(f *_q g)(t)\\
&= c_{q,v} \int_0^\infty f *_q g(x) j_v(xt, q^2) x^{2v+1} d_q x\\
&= c_{q,v} \int_0^\infty \left[\int_0^\infty T^v_{q,x} f(y) g(y) y^{2v+1} d_q y\right] j_v(xt, q^2) x^{2v+1} d_q x\\
&= c_{q,v} \int_0^\infty \left(\int_0^\infty g(y) \left[\int_0^\infty f(z) D_v(x,y,z) z^{2v+1} d_q z\right] y^{2v+1} d_q y\right) j_v(xt, q^2) x^{2v+1} d_q x\\
&= c_{q,v} \int_0^\infty f(z) \left(\int_0^\infty g(y) \left[\int_0^\infty D_v(x,y,z) j_v(xt, q^2) x^{2v+1} d_q x\right] y^{2v+1} d_q y\right) z^{2v+1} d_q z\\
&= c^2_{q,v} \int_0^\infty f(z) \left(\int_0^\infty g(y) j_v(zt, q^2) j_v(yt, q^2) y^{2v+1} d_q y\right) z^{2v+1} d_q z\\
&= \mathcal{F}_{q,v}(f)(t) \times \mathcal{F}_{q,v}(g)(t).
\end{aligned}$$

One can see that

$$\int_0^\infty D_v(x,y,z) j_v(xt, q^2) x^{2v+1} d_q x = c_{q,v} j_v(zt, q^2) j_v(yt, q^2),$$

and

$$\int_0^\infty D_v(x,y,z) x^{2v+1} d_q x = 1.$$

We can permute the two integral signs because

$$\begin{aligned}
&\int_0^\infty \left(\int_0^\infty |g(y)| \left[\int_0^\infty |f(z)| \left|D_v(x,y,z)\right| z^{2v+1} d_q z\right] y^{2v+1} d_q y\right) \left|j_v(xt, q^2)\right| x^{2v+1} d_q x\\
&\leq \left\|j_v(., q^2)\right\|_{q,\infty} \int_0^\infty |f(z)| \left(\int_0^\infty |g(y)| \left[\int_0^\infty \left|D_v(x,y,z)\right| x^{2v+1} d_q x\right] y^{2v+1} d_q y\right) z^{2v+1} d_q z\\
&\leq \left\|j_v(., q^2)\right\|_{q,\infty} \int_0^\infty |f(z)| \left(\int_0^\infty |g(y)| \left[\int_0^\infty D_v(x,y,z) x^{2v+1} d_q x\right] y^{2v+1} d_q y\right) z^{2v+1} d_q z\\
&\leq \left\|j_v(., q^2)\right\|_{q,\infty} \|f\|_{q,1,v} \|g\|_{q,1,v} < \infty.
\end{aligned}$$

Note that if $q \in Q_v$ then $|D_v(x,y,z)| = D_v(x,y,z)$. ■

8 Markov operators on the $\mathcal{L}_{q,2,v}$ space

Definition 10 *A linear bounded mapping* $T : \mathcal{L}_{q,2,v} \to \mathcal{L}_{q,2,v}$ *is called a Markov operator if*

- T *is positive, that is* $f \geq 0$ *implies* $Tf \geq 0$.
- *The constante function* $f = 1$ *is a fixed point of* T.
- *For all function* $f, g \in \mathcal{L}_{q,2,v}$ *we have*

$$\langle Tf, g\rangle = \langle f, Tg\rangle.$$

Lemma 1 *Let* μ *be a probability measure on* $\mathbb{R}_q^+$. *If* f *is a positive function in* $\mathcal{L}_{q,v,1}$ *and* g *is convex on* $\mathbb{R}^+$ *then*

$$g\left(\int_0^\infty f d\mu(x)\right) \leq \int_0^\infty g \circ f(x) d\mu(x).$$

In particular

$$\left[\int_0^\infty f(z) D_v(x,y,z) z^{2v+1} d_q z\right]^2 \leq \int_0^\infty f(z)^2 D_v(x,y,z) z^{2v+1} d_q z.$$

Proof. There exists $c \in \mathbb{R}$ such that for all x, y in a convex set of $\mathbb{R}^+$ it holds

$$g(x) \geq g(y) + c(x-y).$$

Let $t = \int_0^\infty f(x) d\mu(x)$. Then $t \in \mathbb{R}^+$. Now let $x \in \mathbb{R}_q^+$ then

$$g\left(f(x)\right) \geq g(t) + c(f(x) - t).$$

Integrating both sides and using the special value of t gives

$$\int_0^\infty g\left(f(x)\right) d\mu(x) \geq \int_0^\infty [g(t) + c(f(x)-t)] d\mu(x) = g(t).$$

In particular for $g(x) = x^2$ and $d\mu(z) = D_v(x,y,z) z^{2v+1} d_q z$ which is a probability measure on $\mathbb{R}_q^+$ the result follows immediately. ■

Proposition 10 $T_{q,x}^v$ *is a Markov operator of* $\mathcal{L}_{q,2,v}$.

Proof. We will prove that $T^v_{q,x}$ sends $\mathcal{L}_{q,2,v}$ to $\mathcal{L}_{q,2,v}$. Let $f \in \mathcal{L}_{q,2,v}$ then

$$\begin{aligned}\left[T^v_{q,x}f(y)\right]^2 &= \left[\int_0^\infty f(z)D_v(x,y,z)z^{2v+1}d_qz\right]^2\\ &\le \int_0^\infty f(z)^2D_v(x,y,z)z^{2v+1}d_qz = T^v_{q,x}f^2(y).\end{aligned}$$

$$\int_0^\infty \left[T^v_{q,x}f(y)\right]^2 y^{2v+1}d_qy \le \int_0^\infty T^v_{q,x}f^2(y)y^{2v+1}d_qy.$$

$$f \in \mathcal{L}_{q,2,v} \Rightarrow f^2 \in \mathcal{L}_{q,1,v}$$

Hence

$$\int_0^\infty T^v_{q,x}f^2(y)y^{2v+1}d_qy = \int_0^\infty f^2(y)y^{2v+1}d_qy$$

which gives

$$\left\|T^v_{q,x}f\right\|_{q,2,v} \le \|f\|_{q,2,v}\,.$$

The operator $T^v_{q,x}$ is said to be a contraction of the $\mathcal{L}_{q,2,v}$ space. Now we will prove that $T^v_{q,x}$ is symmetric. Let $f,g \in \mathcal{L}_{q,2,v}$ then

$$\left|\left\langle T^v_{q,x}f,g\right\rangle\right| = \int_0^\infty \left|T^v_{q,x}f(y)\right|\left|g(y)\right| y^{2v+1}d_qy \le \left\|T^v_{q,x}f\right\|_{q,2,v}\|g\|_{q,2,v} \le \|f\|_{q,2,v}\|g\|_{q,2,v}$$

$$\begin{aligned}\left\langle T^v_{q,x}f,g\right\rangle &= \int_0^\infty T^v_{q,x}f(y)g(y)y^{2v+1}d_qy = \int_0^\infty\left[\int_0^\infty f(z)D_v(x,y,z)z^{2v+1}d_qz\right]g(y)y^{2v+1}d_qy\\ &= \int_0^\infty\left[\int_0^\infty g(z)D_v(x,y,z)z^{2v+1}d_qz\right]f(y)y^{2v+1}d_qy\\ &= \left\langle f,T^v_{q,x}g\right\rangle.\end{aligned}$$

The exchange of the signs integrals is valid sense

$$\begin{aligned}&\int_0^\infty\left[\int_0^\infty |g(z)|\,D_v(x,y,z)z^{2v+1}d_qz\right]|f(y)|\,y^{2v+1}d_qy\\ &\le \left(\int_0^\infty\left[\int_0^\infty |g(z)|\,D_v(x,y,z)z^{2v+1}d_qz\right]^2 y^{2v+1}d_qy\right)^{1/2}\left(\int_0^\infty |f(y)|^2\,y^{2v+1}d_qy\right)^{1/2}\\ &\le \left(\int_0^\infty\left[\int_0^\infty |g(z)|^2\,D_v(x,y,z)z^{2v+1}d_qz\right]y^{2v+1}d_qy\right)^{1/2}\|f\|_{q,2,v} = \|g\|_{q,2,v}\|f\|_{q,2,v}\,.\end{aligned}$$

On the other hand

$$T^v_{q,x}1 = \int_0^\infty D_v(x,y,z)z^{2v+1}d_qz = 1.$$

Also

$$f \geq 0 \Rightarrow T^v_{q,x} f \geq 0.$$

Not that if f is bounded then

$$\begin{aligned} \left| T^v_{q,x} f(x) \right| &\leq \int_0^\infty |f(z)| \left| D_v(x,y,z) \right| z^{2v+1} d_q z \\ &\leq \|f\|_{q,\infty} \left[\int_0^\infty D_v(x,y,z) z^{2v+1} d_q z \right] = \|f\|_{q,\infty}. \end{aligned}$$

which finishes the proof. ■

Theorem 4 *Let $c_{q,v} \varrho(t) t^{2v+1} d_q t$ be a probability measure on $\mathbb{R}^+_q$. Then*

$$K : \mathcal{L}_{q,2,v} \longrightarrow \mathcal{L}_{q,2,v}, \quad f \longmapsto f *_q \varrho$$

is a Markov operator.

Proof. From Proposition 6 we see that K send $\mathcal{L}_{q,2,v}$ to $\mathcal{L}_{q,2,v}$. On the other hand

$$Kf(x) = f *_q \varrho(x) = c_{q,v} \int_0^\infty T_{q,x} f(y) \varrho(y) y^{2v+1} d_q y,$$

which implies that $K1 = 1$ and if $f \geq 0$ then $Kf \geq 0$. Given two functions $f, g \in \mathcal{L}_{q,2,v}$ we write

$$\begin{aligned} \langle Kf, g \rangle &= \int_0^\infty Kf(x) g(x) x^{2v+1} d_q x \\ &= \int_0^\infty \left[c_{q,v} \int_0^\infty T_{q,x} f(y) \varrho(y) y^{2v+1} d_q y \right] g(x) x^{2v+1} d_q x \\ &= \int_0^\infty \left[c_{q,v} \int_0^\infty T_{q,y} f(x) g(x) x^{2v+1} d_q x \right] \varrho(y) y^{2v+1} d_q y \\ &= \int_0^\infty \left[c_{q,v} \int_0^\infty f(x) T_{q,y} g(x) x^{2v+1} d_q x \right] \varrho(y) y^{2v+1} d_q y \\ &= \int_0^\infty \left[c_{q,v} \int_0^\infty T_{q,x} g(y) \varrho(y) y^{2v+1} d_q y \right] f(x) x^{2v+1} d_q x \\ &= \langle f, Kg \rangle. \end{aligned}$$

The exchange of the integral signs has a valid sense

$$\begin{aligned} &\int_0^\infty \left[\int_0^\infty \left| T_{q,y} f(x) \right| |g(x)| \, x^{2v+1} d_q x \right] \varrho(y) y^{2v+1} d_q y \\ &\leq \|f\|_{q,2,v} \|g\|_{q,2,v} \|\varrho\|_{q,1,v}. \end{aligned}$$

Not that if f is bonded then

$$|Kf(x)| \leq \left[c_{q,v}\int_0^{\infty}\varrho(y)y^{2v+1}d_qy\right]\|f\|_{q,\infty} = \|f\|_{q,\infty}$$

which finishes the proof. ■

Proposition 11 *The eigenfunctions of the operator $T^v_{q,x}$ are in the following form*

$$f_n(t) = \frac{\psi_{q^n}(t)}{\|\psi_{q^n}\|_{q,2,v}}, \quad \forall n \in \mathbb{Z}$$

associated to the eigenvalues

$$\lambda_n = \frac{f_n(x)}{f_n(0)}, \quad \forall n \in \mathbb{Z}.$$

Proof. From (1), the sequence $\{f_n\}_{n\in\mathbb{Z}}$ forme a normal orthogonal basis of the Hilbert space $\mathcal{L}_{q,2,v}$. On the other hand

$$T^v_{q,x}j_v(q^ny,q^2) = j_v(q^nx,q^2)j_v(q^ny,q^2),$$

then

$$T^v_{q,x}f_n(y) = \frac{f_n(x)}{f_n(0)}f_n(y),$$

which leads to the result. ■

Remark 2 *Note that the kernel of $T^v_{q,x}$ can be written as follows*

$$D_v(x,y,z) = \sum_{n\in\mathbb{Z}}\frac{f_n(x)f_n(y)f_n(z)}{f_n(0)} \geq 0,$$

then the sequence $\{f_n\}$ satisfies the hypergroup property at the point 0 with measure $x^{2v+1}d_qx$.

Proposition 12 *A linear mapping*

$$K : \mathcal{L}_{q,2,v} \to \mathcal{L}_{q,2,v}, \quad f_n \mapsto c_nf_n$$

is a Markov operator if and only if

$$c_n = c_{q,v}\int_0^{\infty}\frac{f_n(y)}{f_n(0)}\varrho(y)y^{2v+1}d_qy$$

where $c_{q,v}\varrho(y)y^{2v+1}d_qy$ is a probability measure on $\mathbb{R}^+_q$.

Proof. In fact

$$\begin{aligned} f_n *_q \varrho(x) &= c_{q,v}\int_0^\infty T_{q,x}f_n(y)\varrho(y)y^{2v+1}d_qy \\ &= \left[c_{q,v}\int_0^\infty \frac{f_n(y)}{f_n(0)}\varrho(y)y^{2v+1}d_qy\right]f_n(x) = c_nf_n(x), \end{aligned}$$

Then

$$K(f) = f *_q \varrho, \quad \forall f \in \mathcal{L}_{q,2,v}.$$

Theorem 4 leads to the result. ■

9 The q-heat semigroup

Definition 11 *The q-exponential function is defined by*

$$e(z,q) = \sum_{n=0}^{\infty} \frac{z^n}{(q,q)_n} = \frac{1}{(z;q)_\infty}, \quad |z| < 1.$$

Lemma 2 *The unique solution of the following q-difference equation*

$$\begin{cases} -x^2\psi(t) = (1-q^2)D_{q^2,t}\psi(t) \\ \psi(0) = c(x) \end{cases}$$

is in the following form

$$\psi(t) = c(x)e(-tx^2, q^2).$$

Lemma 3 *Let $f \in \mathcal{L}_{q,2,v}$ such that $\Delta_{q,v}f \in \mathcal{L}_{q,2,v}$ then*

$$\mathcal{F}_{q,v}\left[\Delta_{q,v}f\right](x) = -x^2\mathcal{F}_{q,v}f(x).$$

Proof. Let $g = \mathcal{F}_{q,v}f$. From the inversion formula we obtain

$$f = \mathcal{F}_{q,v}^2 f = \mathcal{F}_{q,v}g.$$

Then

$$\Delta_{q,v}f(x) = \mathcal{F}_{q,v}\Big[y \mapsto -y^2g(y)\Big](x)$$

Again, we use the inversion formula we obtain the result. ■

Definition 12 *The q-heat semigroup is introduced in [1] as follows:*

$$P^v_{q,t}f(x) = G^v(.,t,q^2) *_q f(x), \quad t > 0$$

where $G^v(.,t,q^2)$ *is the q-Gauss kernel*

$$G^v(x,t,q^2) = \mathcal{F}_{q,v}\Big[y \mapsto e(-ty^2,q^2)\Big](x) = A(t)\ e\left(-\frac{q^{-2v}}{t}x^2, q^2\right),$$

and

$$A(t) = \frac{(-q^{2v+2}t, -q^{-2v}/t; q^2)_\infty}{(-t, -q^2/t; q^2)_\infty}.$$

Note that $A(t^2) = t^{-2(v+1)}A(1)$.

Proposition 13 *The unique solution of the q-heat equation*

$$\begin{cases} \Delta_{q,v}u(x,t) = (1-q^2)D_{q^2,t}u(x,t), \quad \forall x \in \mathbb{R}_q, \quad \forall t > 0 \\ u(x,0) = f(x), \quad f \in \mathcal{L}_{q,2,v} \\ x \mapsto u(x,t) \in \mathcal{L}_{q,2,v}, \quad \forall t > 0. \end{cases}$$

is in the following form

$$u(x,t) = P^v_{q,t}f(x).$$

Proof. Let

$$\psi(t) = \mathcal{F}_{q,v}\Big[u(.,t)\Big](x).$$

From the following equation

$$\Delta_{q,v}u(x,t) = (1-q^2)D_{q^2,t}u(x,t)$$

we see that

$$\Delta_{q,v}u(.,t) \in \mathcal{L}_{q,2,v},$$

then from Lemma 3 we have

$$\mathcal{F}_{q,v}\Big[\Delta_{q,v}u(.,t)\Big](x) = -x^2\mathcal{F}_{q,v}\Big[u(.,t)\Big](x)$$

and

$$\mathcal{F}_{q,v}\Big[(1-q^2)D_{q^2,t}u(.,t)\Big](x) = (1-q^2)D_{q^2,t}\mathcal{F}_{q,v}\Big[u(.,t)\Big](x),$$

which prove that ψ satisfies

$$(1-q^2)D_{q^2,t}\psi(t) = -x^2\psi(t), \quad \psi(0) = \mathcal{F}_{q,v}f(x).$$

From Lemma 1 and Proposition 5 we see that

$$\psi(t) = e(-tx^2, q^2)\mathcal{F}_{q,v}f(x) = \mathcal{F}_{q,v}\Big[G^v(.,t,q^2)\Big](x)\times\mathcal{F}_{q,v}f(x) = \mathcal{F}_{q,v}\Big[G^v(.,t,q^2)*_q f\Big](x).$$

From Theorem 1 we obtain

$$u(x,t) = G^v(.,t,q^2) *_q f(x).$$

This finishes the proof. ■

Proposition 14 *The q-heat semigroup* $P^v_{q,t}$ *is a Markov operator of* $\mathcal{L}_{q,2,v}$.

Proof. In [1] it is proved that

$$\left\|G^v(.,t,q^2)\right\|_{q,1,v} = \frac{1}{c_{q,v}},$$

then

$$y \mapsto c_{q,v}G^v(y,t,q^2)y^{2v+1}d_qy$$

is a probability measure on $\mathbb{R}^+_q$. Theorem 4 leads to the result. ■

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