On a parabolic logarithmic Sobolev inequality

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Abstract

In order to extend the blow-up criterion of solutions to the Euler equations, Kozono and Taniuchi [12] have proved a logarithmic Sobolev inequality by means of isotropic (elliptic) BMO norm. In this paper, we show a parabolic version of the Kozono-Taniuchi inequality by means of anisotropic (parabolic) BMO norm. More precisely we give an upper bound for the L^∞ norm of a function in terms of its parabolic BMO norm, up to a logarithmic correction involving its norm in some Sobolev space. As an application, we also explain how to apply this inequality in order to establish a long time existence result for a class of nonlinear parabolic problems.

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1 Introduction and main results

In [12], Kozono and Taniuchi showed an L^∞ estimate of a given function by means of its BMO norm (space of functions of bounded mean oscillation) and the logarithm of its norm in some Sobolev space. In fact, they proved that for $f \in W_p^s(\mathbb{R}^n)$, $1 < p < \infty$, the following estimate holds (with $\log^+ x = \max(\log x, 0)$):

$$\|f\|_{L^\infty(\mathbb{R}^n)} \leq C(1 + \|f\|_{BMO(\mathbb{R}^n)}(1 + \log^+ \|f\|_{W_p^s(\mathbb{R}^n)})), \quad sp > n, \tag{1.1}$$

for some constant $C = C(n,p,s) > 0$. The main advantage of the above estimate is that it was successfully applied (see [12, Theorem 2]) to extend the blow-up criterion of solutions to the Euler equations which was originally given by Beale, Kato and Majda in [1]. Inequality (1.1), as well as some variants of it, are shown (see [12, 14, 11]) using harmonic analysis on isotropic functional spaces of the Lizorkin-Triebel and Besov type. However, as is well known, it is important, say for parabolic partial differential equations to consider spaces that are anisotropic.

Motivated by the study of the long time existence of a certain class of singular parabolic coupled systems (see [8, 9]), we show in this paper an analogue of the Kozono-Taniuchi inequality (1.1) but of the parabolic (anisotropic) type. Due to the parabolic anisotropy, we consider functional spaces on $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ with the generic variable $z = (x,t)$, where each coordinate x_i, $i = 1 \cdots n$ is given

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the weight 1, while the time coordinate t is given the weight 2. We now state the main results of this paper. The first result concerns a Kozono-Taniuchi parabolic type inequality on the entire space $\mathbb{R}^{n+1}$. Introducing parabolic bounded mean oscillation BMO_p spaces, and parabolic Sobolev spaces $W_2^{2m,m}$ (for the definition of these spaces, see Definitions 2.1 and 2.2), we present our first theorem.

Theorem 1.1 (Parabolic logarithmic Sobolev inequality on $\mathbb{R}^{n+1}$)
Let $u \in W_2^{2m,m}(\mathbb{R}^{n+1})$, $m > \frac{n+2}{4}$. Then there exists a constant $C = C(m,n) > 0$ such that:

$$\|u\|_{L^\infty(\mathbb{R}^{n+1})} \leq C \left(1 + \|u\|_{BMO_p(\mathbb{R}^{n+1})} \left(1 + \log^+ \|u\|_{W_2^{2m,m}(\mathbb{R}^{n+1})}\right)\right). \tag{1.2}$$

The proof of Theorem 1.1 will be given in Section 2, and is based on an approach developed by Ogawa [14]. Let us mention that our proof in this paper is self-contained. The second result of this paper concerns a Kozono-Taniuchi parabolic type inequality on the bounded domain

$$\Omega_T = (0,1)^n \times (0,T) \subset \mathbb{R}^{n+1}, \quad T > 0.$$

More precisely, our next theorem reads:

Theorem 1.2 (Parabolic logarithmic Sobolev inequality on a bounded domain)
Let $u \in W_2^{2m,m}(\Omega_T)$ with $m > \frac{n+2}{4}$. Then there exists a constant $C = C(m,n,T) > 0$ such that:

$$\|u\|_{L^\infty(\Omega_T)} \leq C \left(1 + \|u\|_{\overline{BMO}_p(\Omega_T)} \left(1 + \log^+ \|u\|_{W_2^{2m,m}(\Omega_T)}\right)\right), \tag{1.3}$$

where $\|\cdot\|_{\overline{BMO}_p(\Omega_T)} = \|\cdot\|_{BMO_p(\Omega_T)} + \|\cdot\|_{L^1(\Omega_T)}$.

The proof of Theorem 1.2 will be given in Section 3.

1.1 Brief review of the literature

The brief review presented here only concerns logarithmic Sobolev inequalities of the elliptic type. Up to our knowledge, logarithmic Sobolev inequalities of the parabolic type have not been treated elsewhere in the literature.

The original type of the logarithmic Sobolev inequalities was found in Brezis-Gallouet [3] and Brezis-Wainger [4] where the authors investigated the relation between L^∞, W_r^k and W_p^s and proved that there holds the embedding:

$$\|f\|_{L^\infty(\mathbb{R}^n)} \leq C(1 + \log^{\frac{r-1}{r}}(1 + \|f\|_{W_p^s(\mathbb{R}^n)})), \quad sp > n \tag{1.4}$$

provided $\|f\|_{W_r^k(\mathbb{R}^n)} \leq 1$ for $kr = n$. The estimate (1.4) was applied to prove global existence of solutions to the nonlinear Schrödinger equation (see [3, 7]). Similar embedding for $f \in (W_p^s(\mathbb{R}^n))^n$ with $\mathrm{div} f = 0$ was investigated by Beale-Kato-Majda in [1]. The authors showed that:

$$\|\nabla f\|_{L^\infty} \leq C(1 + \|\mathrm{rot} f\|_{L^\infty}(1 + \log^+ \|f\|_{W_p^{s+1}}) + \|\mathrm{rot} f\|_{L^2}), \quad sp > n, \tag{1.5}$$

where they made use of this estimate in order to give a blow-up criterion of solutions to the Euler equations (see [1]). In [12], Kozono and Taniuchi showed their inequality (1.1) in order to extend the blow-up criterion of solutions to the Euler equations given a in [1] (see [12, Theorem 2]). A generalized version of (1.1) in Besov spaces was given in Kozono-Ogawa-Taniuchi [11]. Finally, a sharp version of the logarithmic Sobolev inequality of the Beale-Kato-Majda and the Kozono-Taniuchi type in the Lizorkin-Triebel spaces was showed by Ogawa in [14].

1.2 Organization of the paper

This paper is organized as follows. In Section 2, we recall basic tools used in our analysis, and give the proof of Theorem 1.1. In Section 3, we present the proof of Theorem 1.2, and as an application, we explain how to use the parabolic Kozono-Taniuchi inequality in order to prove the long time existence of certain parabolic equations.

2 A parabolic Kozono-Taniuchi inequality on $\mathbb{R}^{n+1}$

This section is devoted to the proof of Theorem 1.1.

2.1 Preliminaries and basic tools

2.1.1 Parabolic BMO_p and Sobolev spaces

We start by recalling some definitions and introducing some notations. A generic point in $\mathbb{R}^{n+1}$ will be denoted by $z = (x,t) \in \mathbb{R}^n \times \mathbb{R}$, $x = (x_1, \dots, x_n)$. Let $S(\mathbb{R}^{n+1})$ be the usual Schwarz space, and $S'(\mathbb{R}^{n+1})$ the corresponding dual space. Let $u \in S'(\mathbb{R}^{n+1})$, for $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ and $\tau \in \mathbb{R}$ we denote by $\mathcal{F}u(\xi,\tau) \equiv \hat{u}(\xi,\tau)$, and $\mathcal{F}^{-1}u(\xi,\tau) \equiv \check{u}(\xi,\tau)$ the Fourier, and the inverse Fourier transform of u respectively. We also denote $D_t^r = \frac{\partial^r}{\partial t^r}$, $r \in \mathbb{N}$, and D_x^s, $s \in \mathbb{N}$, any derivative with respect to x of order s. The parabolic distance from $z = (x,t)$ to the origin is defined by:

$$\|z\| = \max\left\{|x_1|, \dots, |x_n|, |t|^{1/2}\right\}. \tag{2.1}$$

Let $\mathcal{O} \subseteq \mathbb{R}^{n+1}$ be an open set. The parabolic bounded mean oscillation space BMO_p and the parabolic Sobolev space $W_2^{2m,m}$ are now recalled.

Definition 2.1 (Parabolic bounded mean oscillation spaces)
A function $u \in L^1_{loc}(\mathcal{O})$ is said to be of parabolic bounded mean oscillation, $u \in BMO_p(\mathcal{O})$ if we have:

$$\|u\|_{BMO_p(\mathcal{O})} = \sup_{Q \subset \mathcal{O}} \frac{1}{|Q|}\int_Q |u - u_Q| < +\infty. \tag{2.2}$$

Here Q denotes an arbitrary parabolic cube

$$Q = Q_r = Q_r(z_0) = \{z \in \mathbb{R}^{n+1};\ \|z - z_0\| < r\}, \tag{2.3}$$

and

$$u_Q = \frac{1}{|Q|}\int_Q u. \tag{2.4}$$

The functions in BMO_p are defined up to an additive constant. We also define the space $\overline{BMO}_p$ as:

$$\overline{BMO}_p(\mathcal{O}) = BMO_p(\mathcal{O}) \cap L^1(\mathcal{O}) \quad \textit{with} \quad \|\cdot\|_{\overline{BMO}_p} = \|\cdot\|_{BMO_p} + \|\cdot\|_{L^1}.$$

Definition 2.2 (Parabolic Sobolev spaces)
Let m be a non-negative integer. We define the parabolic Sobolev space $W_2^{2m,m}(\mathcal{O})$ as follows:

$$W_2^{2m,m}(\mathcal{O}) = \{u \in L^2(\mathcal{O}); D_t^r D_x^s u \in L^2(\mathcal{O}),\ \forall r,s \in \mathbb{N} \textit{ such that } 2r + s \le 2m\}.$$

The norm of $u \in W_2^{2m,m}(\mathcal{O})$ is defined by: $\|u\|_{W_2^{2m,m}(\mathcal{O})} = \sum_{j=0}^{2m}\sum_{2r+s=j}\|D_t^r D_x^s u\|_{L^2(\mathcal{O})}$.

The next lemma concerns a Sobolev embedding of $W_2^{2m,m}$.

Lemma 2.3 (Sobolev embedding, [13, Lemma 3.3])
Let m be a non-negative integer satisfying $m > \frac{n+2}{4}$. Then there exists a positive constant C depending on m and n such that for any $u \in W_2^{2m,m}(\mathcal{O})$, the function u is continuous and bounded on $\mathcal{O}$, and satisfies

$$\|u\|_{L^\infty(\mathcal{O})} \leq C\|u\|_{W_2^{2m,m}(\mathcal{O})}. \tag{2.5}$$

2.1.2 Parabolic Lizorkin-Triebel and Besov spaces

Here we give the definition of Lizorkin-Triebel spaces. These spaces are constructed out of the parabolic Littlewood-Paley decomposition that we recall here. Let $\psi_0(z) \in C_0^\infty(\mathbb{R}^{n+1})$ be a function such that

$$\psi_0(z) = 1 \text{ if } \|z\| \leq 1 \quad \text{and} \quad \psi_0(z) = 0 \text{ if } \|z\| \geq 2. \tag{2.6}$$

For such a function ψ_0, we may define a smooth, anisotropic dyadic partition of unity $(\psi_j)_{j\in\mathbb{N}}$ by letting

$$\psi_j(z) = \psi_0(2^{-ja}z) - \psi_0(2^{-(j-1)a}z) \quad \text{if} \quad j \geq 1.$$

Here $a = (1,\ldots,1,2) \in \mathbb{R}^{n+1}$, and for $\eta \in \mathbb{R}$, $b = (b_1,\ldots,b_n,b_{n+1}) \in \mathbb{R}^{n+1}$, the dilatation $\eta^b z$ is defined by $\eta^b z = (\eta^{b_1}z_1,\ldots,\eta^{b_n}z_n,\eta^{b_{n+1}}z_{n+1})$. It is clear that

$$\sum_{j=0}^{\infty} \psi_j(z) = 1 \quad \text{for} \quad z \in \mathbb{R}^{n+1},$$

and

$$\operatorname{supp}\psi_j \subset \{z;\, 2^{j-1} \leq \|z\| \leq 2^{j+1}\}, \quad j \geq 1.$$

Define ϕ_j, $j \geq 0$ as the inverse Fourier transform of ψ_j, i.e. $\hat{\phi}_j = \psi_j$. It is worth noticing that

$$\phi_j(z) = 2^{(n+2)(j-1)}\phi_1(2^{(j-1)a}z) \quad \text{for} \quad j \geq 1, \tag{2.7}$$

and that for any $u \in S'(\mathbb{R}^{n+1})$,

$$u = (2\pi)^{-\frac{(n+1)}{2}} \sum_{j=0}^{\infty} \phi_j * u, \text{ with convergence in } S'(\mathbb{R}^{n+1}).$$

We now give the definition of the anisotropic Besov and Lizorkin-Triebel spaces.

Definition 2.4 (Anisotropic Besov spaces)
The anisotropic Besov space $B^s_{p,q}(\mathbb{R}^{n+1}) = B^s_{p,q}$, $s \in \mathbb{R}$, $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$ is the space of functions $u \in S'(\mathbb{R}^{n+1})$ with finite quasi norms

$$\|u\|_{B^s_{p,q}} = \left(\sum_{j=0}^{\infty} 2^{sqj}\|\phi_j * u\|^q_{L^p(\mathbb{R}^{n+1})}\right)^{1/q} \tag{2.8}$$

and the natural modification for $q = \infty$, i.e.

$$\|u\|_{B^s_{p,\infty}} = \sup_{j\geq 0} 2^{sj}\|\phi_j * u\|_{L^p(\mathbb{R}^{n+1})}. \tag{2.9}$$

Definition 2.5 (Anisotropic Lizorkin-Triebel spaces)
The anisotropic Lizorkin-Triebel space $F^s_{p,q}(\mathbb{R}^{n+1}) = F^s_{p,q}$, $s \in \mathbb{R}$, $1 \le p < \infty$ *and* $1 \le q \le \infty$ *(or* $1 \le q < \infty$ *and* $p = \infty$*) is the space of functions* $u \in S'(\mathbb{R}^{n+1})$ *with finite quasi norms*

$$\|u\|_{F^s_{p,q}} = \left\| \left(\sum_{j=0}^{\infty} 2^{sqj} |\phi_j * u|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^{n+1})} \tag{2.10}$$

and the natural modification for $q = \infty$*, i.e.*

$$\|u\|_{F^s_{p,\infty}} = \| \sup_{j \ge 0} 2^{sj} |\phi_j * u| \|_{L^p(\mathbb{R}^{n+1})}. \tag{2.11}$$

A very useful space throughout our analysis will be the truncated anisotropic (parabolic) Lizorkin-Triebel space $\widetilde{F}^s_{p,q}$ that we define here.

Definition 2.6 (Truncated anisotropic Lizorkin-Triebel space)
The truncated anisotropic Lizorkin-Triebel space $\widetilde{F}^s_{p,q}(\mathbb{R}^{n+1}) = \widetilde{F}^s_{p,q}$, $s \in \mathbb{R}$, $1 \le p < \infty$ *and* $1 \le q \le \infty$ *(*$1 \le q < \infty$ *if* $p = \infty$*) is the space of functions* $u \in S'(\mathbb{R}^{n+1})$ *with finite quasi norms*

$$\|u\|_{\widetilde{F}^s_{p,q}} = \left\| \left(\sum_{j=1}^{\infty} 2^{sqj} |\phi_j * u|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^{n+1})} \tag{2.12}$$

and the natural modification for $q = \infty$*, i.e.*

$$\|u\|_{\widetilde{F}^s_{p,\infty}} = \| \sup_{j \ge 1} 2^{sj} |\phi_j * u| \|_{L^p(\mathbb{R}^{n+1})}. \tag{2.13}$$

The basic difference between $F^s_{p,q}$ and $\widetilde{F}^s_{p,q}$ is that in $\widetilde{F}^s_{p,q}$ we omit the term $\phi_0 * u$ and only take in consideration the terms $\phi_j * u$, $j \ge 1$. Sobolev embeddings of parabolic Lizorkin-Triebel and Besov spaces are shown by the next two lemmas.

Lemma 2.7 (Embeddings of Besov spaces, [10, Theorem 7])
Let $s, t \in \mathbb{R}$, $s > t$, *and* $1 \le p, r \le \infty$ *satisfy:* $s - \frac{n+2}{p} = t - \frac{n+2}{r}$. *Then for any* $1 \le q \le \infty$ *we have the following continuous embedding*

$$B^s_{p,q}(\mathbb{R}^{n+1}) \hookrightarrow B^t_{r,q}(\mathbb{R}^{n+1}). \tag{2.14}$$

Lemma 2.8 (Sobolev embeddings, [15, Proposition 2])
Take an integer $m \ge 1$. *Then we have*

$$B^{2m}_{2,1} \hookrightarrow W^{2m,m}_2 \hookrightarrow B^{2m}_{2,\infty}. \tag{2.15}$$

2.2 Basic logarithmic Sobolev inequality

In this subsection we show a basic logarithmic Sobolev inequality. In particular, we show the following lemma.

Lemma 2.9 (Basic logarithmic Sobolev inequality)
Let $u \in W_2^{2m,m}(\mathbb{R}^{n+1})$ for some $m \in \mathbb{N}$, $m > \frac{n+2}{4}$. Then there exists some constant $C = C(m,n) > 0$ such that

$$\|u\|_{\widetilde{F}^0_{\infty,1}} \leq C\left(1 + \|u\|_{\widetilde{F}^0_{\infty,2}}\left(1 + (\log^+ \|u\|_{W_2^{2m,m}})^{1/2}\right)\right). \tag{2.16}$$

Proof. First, let us mention that the ideas of the proof of this lemma are inspired from the proof of Ogawa [14, Corollary 2.4]. The proof is divided into three steps, and the constants in the proof may vary from line to line.

Step 1. (Estimate of $\|u\|_{\widetilde{F}^0_{\infty,1}}$).

Let $\gamma > 0$, and $N \in \mathbb{N}$ be two arbitrary variables. We compute:

$$\begin{aligned}
\|u\|_{\widetilde{F}^0_{\infty,1}} &\leq \left\|\sum_{1\leq j<N} |\phi_j * u|\right\|_{L^\infty} + \left\|\sum_{j\geq N} 2^{-\gamma j} 2^{\gamma j}|\phi_j * u|\right\|_{L^\infty}\\
&\leq N^{1/2}\left\|\left(\sum_{1\leq j<N} |\phi_j * u|^2\right)^{1/2}\right\|_{L^\infty} + C_\gamma 2^{-\gamma N}\left\|\left(\sum_{j\geq N}(2^{\gamma j}|\phi_j * u|)^2\right)^{1/2}\right\|_{L^\infty}\\
&\leq C_\gamma\left(N^{1/2}\|u\|_{\widetilde{F}^0_{\infty,2}} + 2^{-\gamma N}\|u\|_{F^\gamma_{\infty,2}}\right),
\end{aligned}$$

where $C_\gamma > 0$ is a positive constant.

Step 2. (Optimization in N).

We optimize the previous inequality in N by setting:

$$N = 1 \quad \text{if} \quad \|u\|_{F^\gamma_{\infty,2}} \leq 2^\gamma \|u\|_{\widetilde{F}^0_{\infty,2}}.$$

In this case we can easily check that:

$$\|u\|_{\widetilde{F}^0_{\infty,1}} \leq C_\gamma \|u\|_{\widetilde{F}^0_{\infty,2}}\left(1 + \left(\log^+ \frac{\|u\|_{F^\gamma_{\infty,2}}}{\|u\|_{\widetilde{F}^0_{\infty,2}}}\right)^{1/2}\right). \tag{2.17}$$

In the case where $\|u\|_{F^\gamma_{\infty,2}} > 2^\gamma \|u\|_{\widetilde{F}^0_{\infty,2}}$, we choose $1 \leq \beta < 2^\gamma$ such that

$$N = \log^+_{2^\gamma}\left(\beta\frac{\|u\|_{F^\gamma_{\infty,2}}}{\|u\|_{\widetilde{F}^0_{\infty,2}}}\right) \in \mathbb{N}.$$

We then compute:

$$\begin{aligned}
N^{1/2}\|u\|_{\widetilde{F}^0_{\infty,2}} + 2^{-\gamma N}\|u\|_{F^\gamma_{\infty,2}} &\leq \|u\|_{\widetilde{F}^0_{\infty,2}}\left(\frac{1}{\beta} + \left[\log^+_{2^\gamma}\left(\beta\frac{\|u\|_{F^\gamma_{\infty,2}}}{\|u\|_{\widetilde{F}^0_{\infty,2}}}\right)\right]^{1/2}\right)\\
&\leq \|u\|_{\widetilde{F}^0_{\infty,2}}\left(\frac{1}{\beta} + \left[\frac{2}{\log 2^\gamma}\log^+ \frac{\|u\|_{F^\gamma_{\infty,2}}}{\|u\|_{\widetilde{F}^0_{\infty,2}}}\right]^{1/2}\right)\\
&\leq C_\gamma \|u\|_{\widetilde{F}^0_{\infty,2}}\left(1 + \left(\log^+ \frac{\|u\|_{F^\gamma_{\infty,2}}}{\|u\|_{\widetilde{F}^0_{\infty,2}}}\right)^{1/2}\right),
\end{aligned}$$

hence we also have (2.17) with a different constant C_γ.

Step 3. (Estimate of $\|u\|_{F^\gamma_{\infty,2}}$ and conclusion).

Noting the inequality

$$x\left(\log\left(e+\frac{y}{x}\right)\right)^{1/2} \le \begin{cases} C\left(1+x(\log(e+y))^{1/2}\right) & \text{for} \quad 0<x\le 1 \\ Cx(\log(e+y))^{1/2} & \text{for} \quad x>1, \end{cases}$$

we deduce from (2.17) that:

$$\|u\|_{\widetilde{F}^0_{\infty,1}} \le C\left(1+\|u\|_{\widetilde{F}^0_{\infty,2}}\left(1+\left(\log^+\|u\|_{F^\gamma_{\infty,2}}\right)^{1/2}\right)\right), \tag{2.18}$$

where the constant C depends also on γ. We now estimate the term $\|u\|_{F^\gamma_{\infty,2}}$. Choose γ such that

$$0<\gamma<2m-\frac{n+2}{2}.$$

Call $\alpha = 2m - \frac{n+2}{2}$, we compute:

$$\begin{aligned} \|u\|_{F^\gamma_{\infty,2}} &= \left\| \left(\sum_{j\ge 0} 2^{2j\gamma}|\phi_j * u|^2\right)^{1/2}\right\|_{L^\infty} \\ &\le \left(\sum_{j\ge 0} 2^{2j(\gamma-\alpha)}\right)^{1/2} \left\|\sup_{j\ge 0} 2^{\alpha j}|\phi_j * u|\right\|_{L^\infty} \\ &\le C\|u\|_{B^\alpha_{\infty,\infty}}. \end{aligned} \tag{2.19}$$

It is easy to check (see (2.14), Lemma 2.7, and (2.15), Lemma 2.8) that we have the continuous embeddings

$$W^{2m,m}_2 \hookrightarrow B^{2m}_{2,\infty} \hookrightarrow B^\alpha_{\infty,\infty}.$$

Therefore (from inequality (2.19)) we get:

$$\|u\|_{F^\gamma_{\infty,2}} \le C\|u\|_{W^{2m,m}_2},$$

hence the result directly follows from (2.18). ■

2.3 Proof of Theorem 1.1

In this subsection we present the proof of several lemmas leading to the proof the Theorem 1.1. We start with the following lemma concerning mean estimates of functions on parabolic cubes. Call $Q_{2^j}\subset\mathbb{R}^{n+1}$, $j\ge 0$, any arbitrary parabolic cube of radius 2^j (see (2.3) for the definition of parabolic cubes). For the sake of simplicity, we denote

$$Q^j = Q_{2^j} \quad \text{for all} \quad j\in\mathbb{Z}. \tag{2.20}$$

Our next lemma reads:

Lemma 2.10 (Mean estimates on parabolic cubes)
Let $u \in BMO_p(\mathbb{R}^{n+1})$. Take $Q^j \subset Q^{j+1}$, $j \geq 0$ (Q^j and Q^{j+1} do not necessarily have the same center). Then we have (with the notation (2.4)):

$$|u_{Q^{j+1}} - u_{Q^j}| \leq \left(1 + 2^{n+2}\right) \|u\|_{BMO_p}. \tag{2.21}$$

More generally, we have for any $Q^j \subseteq Q^k$, $j,k \in \mathbb{Z}$:

$$|u_{Q^k} - u_{Q^j}| \leq (k-j) \left(1 + 2^{n+2}\right) \|u\|_{BMO_p}. \tag{2.22}$$

Proof. We easily remark that:

$$|Q^{j+1}| = 2^{n+2}|Q^j|.$$

We compute:

$$\begin{aligned} |u_{Q^{j+1}} - u_{Q^j}| &= \frac{1}{|Q^j|}\int_{Q^j} |u_{Q^{j+1}} - u_{Q^j}| \\ &\leq \frac{1}{|Q^j|}\int_{Q^j} |u - u_{Q^j}| + \frac{1}{|Q^j|}\int_{Q^j} |u - u_{Q^{j+1}}| \\ &\leq \|u\|_{BMO_p} + \frac{2^{n+2}}{|Q^{j+1}|}\int_{Q^{j+1}} |u - u_{Q^{j+1}}| \\ &\leq \|u\|_{BMO_p} + 2^{n+2}\|u\|_{BMO_p} \leq \left(1 + 2^{n+2}\right)\|u\|_{BMO_p}, \end{aligned}$$

which immediately gives (2.21), and consequently (2.22). ∎

The following two lemmas are of notable importance for the proof of the logarithmic Sobolev inequality (1.2). In the first lemma we bound the terms $\phi_j * u$ for $j \geq 1$, while, in the second lemma, we give a bound on $\phi_0 * u$.

Lemma 2.11 (Estimate of $\|\phi_j * u\|_{L^\infty(\mathbb{R}^{n+1})}$ for $j \geq 1$)
Let $u \in BMO_p(\mathbb{R}^{n+1})$. Then there exists a constant $C = C(n) > 0$ such that:

$$\|u * \phi_j\|_{L^\infty(\mathbb{R}^{n+1})} \leq C\|u\|_{BMO_p(\mathbb{R}^{n+1})} \quad \textit{for any} \quad j \geq 1, \tag{2.23}$$

where $(\phi_j)_{j\geq 1}$ is the sequence of functions given in (2.7).

Proof. We will show that

$$|(\phi_j * u)(z)| \leq C\|u\|_{BMO_p} \quad \text{for} \quad z = 0. \tag{2.24}$$

The general case with $z \in \mathbb{R}^{n+1}$ could be deduced from (2.24) by translation. Throughout the proof, we will sometimes omit (when there is no confusion) the dependence of the norm on the space $\mathbb{R}^{n+1}$. The proof is divided into three steps.

Step 1. (Decomposition of $(\phi_j * u)(0)$ on parabolic cubes).

Since $\hat{\phi}_j$ is supported in $\{z \in \mathbb{R}^{n+1};\, 2^{j-1} \leq \|z\| \leq 2^{j+1}\}$ then $\hat{\phi}_j(0) = 0 = \int_{\mathbb{R}^{n+1}} \phi_j$. Using this equality, we can write:

$$(\phi_j * u)(0) = \int_{\mathbb{R}^{n+1}} \phi_j(-z)(u(z) - u_{Q^{1-j}})dz$$

where Q^{1-j} is the parabolic cube defined by (2.20) and centered at 0. This implies that

$$|(\phi_j * u)(0)| \leq \overbrace{\int_{Q^{1-j}} |\phi_j(-z)||u(z) - u_{Q^{1-j}}|dz}^{A_1} + \overbrace{\int_{\mathbb{R}^{n+1}\setminus Q^{1-j}} |\phi_j(-z)||u(z) - u_{Q^{1-j}}|dz}^{A_2}. \qquad (2.25)$$

Step 1.1. (Estimate of A_1).

From (2.7), the term A_1 can be estimated as follows:

$$\begin{aligned} A_1 &\leq 2^{(n+2)(j-1)}\|\phi_1\|_{L^\infty} \int_{Q^{1-j}} |u(z) - u_{Q^{1-j}}|dz \\ &\leq 2^{(n+2)(j-1)}|Q^{1-j}|\, \|\phi_1\|_{L^\infty}\|u\|_{BMO_p} \\ &\leq |Q_1|\|\phi_1\|_{L^\infty}\|u\|_{BMO_p}, \end{aligned}$$

hence

$$A_1 \leq C_0\|u\|_{BMO_p} \quad \text{with} \quad C_0 = |Q_1|\|\phi_1\|_{L^\infty(\mathbb{R}^{n+1})}. \qquad (2.26)$$

Step 2. (Estimate of A_2).

We rewrite A_2 as the following series:

$$A_2 = 2^{(n+2)(j-1)} \sum_{-\infty<k\leq j} \int_{Q^{2-k}\setminus Q^{1-k}} \left|\phi_1\left(-2^{(j-1)a}z\right)\right| |u(z) - u_{Q^{1-j}}|dz. \qquad (2.27)$$

Since ϕ_1 is the inverse Fourier transform of a compactly supported function then we have:

$$\forall \overline{m} \in \mathbb{N}^*,\ \exists C_1 > 0,\ |\phi_1(z)| \leq \frac{C_1}{\|z\|^{\overline{m}}} \quad \text{for all} \quad \|z\| \geq 1. \qquad (2.28)$$

The asymptotic behavior of ϕ_1 shown by (2.28) leads to the following decomposition of the term A_2:

$$\begin{aligned} A_2 \;&\leq\; \overbrace{C_1 2^{(n+2)(j-1)} \sum_{-\infty<k\leq j} \int_{Q^{2-k}\setminus Q^{1-k}} \frac{1}{\|2^{(j-1)a}z\|^{\overline{m}}}|u(z) - u_{Q^{2-k}}|dz}^{A_3} \\ &+\; \overbrace{C_1 2^{(n+2)(j-1)} \sum_{-\infty<k\leq j} \int_{Q^{2-k}\setminus Q^{1-k}} \frac{1}{\|2^{(j-1)a}z\|^{\overline{m}}}|u_{Q^{2-k}} - u_{Q^{1-j}}|dz}^{A_4}. \end{aligned}$$

Step 2.1. (Estimate of A_3).

Since the integral appearing in A_3 is done over $Q^{2-k} \setminus Q^{1-k}$, we obtain

$$\|2^{(j-1)a}z\|^{\overline{m}} \geq 2^{\overline{m}(j-k)}.$$

Using this inequality together with the fact that

$$\int_{Q^{2-k}\setminus Q^{1-k}} |u(z) - u_{Q^{2-k}}|dz \leq 2^{(n+2)(2-k)}|Q_1|\|u\|_{BMO_p},$$

we can estimate the term A_3 as follows:

$$A_3 \le C_1 2^{n+2} \left(\sum_{-\infty<k\le j} 2^{-(\overline{m}-(n+2))(j-k)} \right) |Q_1| \|u\|_{BMO_p}, \tag{2.29}$$

where the above series converges for $\overline{m} > n+2$.

Step 2.2. (Estimate of A_4).

Using Lemma 2.10, and the fact that $\|2^{(j-1)a}z\|^{\overline{m}} \ge 2^{\overline{m}(j-k)}$ on $Q^{2-k} \setminus Q^{1-k}$, the term A_4 can be estimated as follows:

$$\begin{aligned} A_4 &\le C_1 2^{(n+2)(j-1)} \left(\sum_{-\infty<k\le j} 2^{-\overline{m}(j-k)}(1+j-k)|Q^{2-k}| \right) \|u\|_{BMO_p} \\ &\le C_1 2^{n+2} \left(\sum_{-\infty<k\le j} 2^{-(\overline{m}-(n+2))(j-k)}(1+j-k) \right) |Q_1| \|u\|_{BMO_p}, \end{aligned} \tag{2.30}$$

where the above series also converges for $\overline{m} > n+2$.

Step 3. (Conclusion).

From (2.26), (2.29) and (2.30), inequality (2.23) directly follows with a constant $C>0$ independent of j. ■.

Lemma 2.12 (Estimate of $\|\phi_0 * u\|_{L^\infty(\mathbb{R}^{n+1})}$)
Let $u \in W_2^{2m,m}(\mathbb{R}^{n+1})$ with $m > \frac{n+2}{4}$. Then there exists a constant $C = C(m,n) > 0$ such that we have:

$$\|\phi_0 * u\|_{L^\infty} \le C \left(1 + \|u\|_{BMO_p} \left(1 + \log^+ \|u\|_{W_2^{2m,m}} \right) \right). \tag{2.31}$$

Proof. The constants that will appear may differ from line to line, but only depend on n and m. The proof of this lemma combines somehow the proof of Lemmas 2.9 and 2.11. We write down u_{Q^1} as a finite sum of a telescopic sequence for $N \ge 1$:

$$u_{Q^1} = (u_{Q^1} - u_{Q^2}) + \cdots + (u_{Q^{N-1}} - u_{Q^N}) + u_{Q^N}.$$

From Lemma 2.10, we deduce that:

$$|u_{Q^1}| \le C(N-1)\|u\|_{BMO_p} + |u_{Q^N}|.$$

Remark that applying Cauchy-Schwarz inequality, we get

$$|u_{Q^N}| \le \frac{1}{|Q^N|} \int_{Q^N} |u| \le \left(\int_{Q^N} u^2 \right)^{1/2} \left(\int_{Q^N} 1^2 \right)^{1/2},$$

then we obtain

$$|u_{Q^1}| \le C \left(N\|u\|_{BMO_p} + 2^{-\gamma N} \|u\|_{W_2^{2m,m}} \right) \quad \text{with} \quad \gamma = \frac{n+2}{2}. \tag{2.32}$$

Following similar arguments as the proof of Lemma 2.9, we may optimize (2.32) in N, we finally get:

$$|u_{Q^1}| \le C\left(1 + \|u\|_{BMO_p}\left(1 + \log^+ \|u\|_{W_2^{2m,m}}\right)\right). \tag{2.33}$$

We now estimate $|(\phi_0 * u)(z)|$ for $z = 0$. Again, the same estimate could be obtained for any $z \in \mathbb{R}^{n+1}$ by translation. We write

$$\begin{aligned}
(\phi_0 * u)(0) &= \int_{\mathbb{R}^{n+1}} \phi_0(-z)u(z) \\
&= \int_{\mathbb{R}^{n+1}} \phi_0(-z)(u(z) - u_{Q^1}) + \int_{\mathbb{R}^{n+1}} \phi_0(-z)u_{Q^1} \\
&= \overbrace{\int_{Q_1} \phi_0(-z)(u(z) - u_{Q^1})}^{B_1} + \overbrace{\int_{\mathbb{R}^{n+1}\backslash Q^1} \phi_0(-z)(u(z) - u_{Q^1})}^{B_2} + \overbrace{\int_{\mathbb{R}^{n+1}} \phi_0(-z)u_{Q^1}}^{B_3},
\end{aligned}$$

where

$$|B_1| \le C\|u\|_{BMO_p}, \tag{2.34}$$

and, from (2.33),

$$|B_3| \le C\left(1 + \|u\|_{BMO_p}\left(1 + \log^+ \|u\|_{W_2^{2m,m}}\right)\right). \tag{2.35}$$

In order to estimate B_2, we argue as in Step 2 of Lemma 2.11. In fact we have:

$$\begin{aligned}
|B_2| &\le \sum_{k\ge 1}\int_{Q^{k+1}\backslash Q^k} |\phi_0(-z)||u(z) - u_{Q^{k+1}}| + \sum_{k\ge 1}\int_{Q^{k+1}\backslash Q^k} |\phi_0(-z)||u_{Q^{k+1}} - u_{Q^1}| \\
&\le \left(\sum_{k\ge 1}(\sup_{Q^{k+1}\backslash Q^k} |\phi_0(-z)|)|Q^{k+1}|(1+k)\right)\|u\|_{BMO_p} \\
&\le 2^{n+2}\left(\sum_{k\ge 1} 2^{-(\overline{m}-(n+2))}(1+k)\right)|Q_1|\|u\|_{BMO_p},
\end{aligned} \tag{2.36}$$

where for the last line we have used the fact that $|\phi_0(z)| \le \frac{C}{\|z\|^{\overline{m}}}$ for $\|z\| \ge 1$. Of course the above series converges if we choose $\overline{m} > n+2$. From (2.34), (2.35) and (2.36), the result follows. ■

Corollary 2.13 (A control of $\|u\|_{\widetilde{F}^0_{\infty,2}}$)
Let $u \in BMO_p(\mathbb{R}^{n+1}) \cap \widetilde{F}^0_{\infty,1}(\mathbb{R}^{n+1})$, then $u \in \widetilde{F}^0_{\infty,2}(\mathbb{R}^{n+1})$ and we have:

$$\|u\|_{\widetilde{F}^0_{\infty,2}} \le C\|u\|^{1/2}_{BMO_p}\|u\|^{1/2}_{\widetilde{F}^0_{\infty,1}}, \tag{2.37}$$

where $C = C(n) > 0$ is a positive constant.

Proof. Using (2.23), we compute:

$$\|u\|_{\widetilde{F}^0_{\infty,2}} = \left\|\left(\sum_{j\ge 1}|\phi_j * u|^2\right)^{1/2}\right\|_{L^\infty} \le \left\|\left(\sup_{j\ge 1}\|\phi_j * u\|_{L^\infty}\sum_{j\ge 1}|\phi_j * u|\right)^{1/2}\right\|_{L^\infty} \le C\|u\|^{1/2}_{BMO_p}\|u\|^{1/2}_{\widetilde{F}^0_{\infty,1}},$$

which terminates the proof. ■

Remark 2.14 *From [2], it seems that BMO_p spaces can be characterized in terms of parabolic Lizorkin-Triebel spaces. In the case of elliptic spaces, it is a well-known result (see [16, 6]) which allows to simplify the proof of the Kozono-Taniuchi inequality.*

We can now give the proof of our first main result (Theorem 1.1).

Proof of Theorem 1.1. Using (2.16) and (2.37), we obtain:

$$\|u\|_{\widetilde{F}^0_{\infty,1}} \leq C\left(1+\|u\|^{1/2}_{BMO_p}\|u\|^{1/2}_{\widetilde{F}^0_{\infty,1}}\left(1+(\log^+\|u\|_{W_2^{2m,m}})^{1/2}\right)\right). \tag{2.38}$$

Notice that the constant C can always be chosen such that $C \geq 1$. If $\|u\|_{\widetilde{F}^0_{\infty,1}} \leq 1$, we evidently have:

$$\|u\|_{\widetilde{F}^0_{\infty,1}} \leq C \leq C\left(1+\|u\|_{BMO_p}\left(1+\log^+\|u\|_{W_2^{2m,m}}\right)\right). \tag{2.39}$$

If $\|u\|_{\widetilde{F}^0_{\infty,1}} > 1$, then, dividing (2.38) by $\|u\|^{1/2}_{\widetilde{F}^0_{\infty,1}}$, we can easily deduce inequality (2.39). Using the fact that

$$\|u\|_{L^\infty} \leq C\sum_{j\geq 0}\|\phi_j * u\|_{L^\infty} \leq C\left(\|\phi_0 * u\|_{L^\infty}+\|u\|_{\widetilde{F}^0_{\infty,1}}\right),$$

and using inequalities (2.31) and (2.39), we directly get into the result. ■

3 A parabolic Kozono-Taniuchi inequality on a bounded domain

The goal of this section is to present, on the one hand, the proof of Theorem 1.2. On the other hand, at the end of this section, we give an application where we show how to use inequality (1.3) in order to maintain the long time existence of solutions to some parabolic equations. Let us indicate that throughout this section, the positive constant $C = C(T) > 0$ may vary from line to line.

3.1 Proof of Theorem 1.2

In order to simplify the arguments of the proof, we first show Theorem 1.2 in the special case when $n = m = 1$. Then we give the principal ideas how to prove the result in the general case. Call

$$I = (0,1) \quad \text{and} \quad \Omega_T = I \times (0,T),$$

we first show the following proposition:

Proposition 3.1 (Theorem 1.2, case: $n = m = 1$)
Let $u \in W_2^{2,1}(\Omega_T)$. Then there exists a constant $C = C(T) > 0$ such that:

$$\|u\|_{L^\infty(\Omega_T)} \leq C\left(1+\|u\|_{\overline{BMO}_p(\Omega_T)}\left(1+\log^+\|u\|_{W_2^{2,1}(\Omega_T)}\right)\right). \tag{3.1}$$

As a similar inequality of (3.1) is already shown on the $\mathbb{R}^2$ (see inequality (1.2)), the idea of the proof of (3.1) lies in using (1.2) for a special extension of the function $u \in W_2^{2,1}(\Omega_T)$ to the entire space $\mathbb{R}^2$. For this reason, we demand that the extended function stays in $W_2^{2,1}(\mathbb{R}^2)$ which is done via the following arguments. Remark first that the function u can be extended by continuity to the boundary $\partial\Omega_T$ of Ω_T. Take $\tilde{u}$ as the function defined over

$$\widetilde{\Omega}_T = (-1,2) \times (-T,2T)$$

as follows:

$$\tilde{u}(x,t) = \begin{cases} -3u(-x,t) + 4u\left(-\dfrac{x}{2}, t\right) & \text{for} \quad -1 < x < 0,\ 0 \le t \le T, \\ -3u(2-x,t) + 4u\left(\dfrac{3-x}{2}, t\right) & \text{for} \quad 1 < x < 2,\ 0 \le t \le T, \end{cases} \tag{3.2}$$

and

$$\tilde{u}(x,t) = \begin{cases} u(x,-t) & \text{for} \quad -T < t \le 0 \\ u(x, 2T-t) & \text{for} \quad T \le t < 2T. \end{cases} \tag{3.3}$$

A direct consequence of this extension is the following lemma.

Lemma 3.2 **(L^1 estimate of $\tilde{u}$)**
Let $\tilde{u}$ be the function defined by (3.2) and (3.3). Then there exists a constant $C = C(T) > 0$ such that:

$$\|\tilde{u}\|_{L^1(\widetilde{\Omega}_T)} \le C\|u\|_{L^1(\Omega_T)}. \tag{3.4}$$

Proof. The proof of this lemma is direct by the extension. ■

Another important consequence of the extension (3.2) and (3.3) is the fact that $\tilde{u} \in W_2^{2,1}(\widetilde{\Omega}_T)$, and that we have (see for instance [5])

$$\|\tilde{u}\|_{W_2^{2,1}(\widetilde{\Omega}_T)} \le C\|u\|_{W_2^{2,1}(\Omega_T)}, \quad C = C(T) > 0. \tag{3.5}$$

Let $\mathcal{Z}_1 \subset \mathcal{Z}_2$ be the two subsets of $\widetilde{\Omega}_T$ defined by:

$$\mathcal{Z}_1 = \{(x,t);\ -1/4 < x < 5/4 \quad \text{and} \quad -T/4 < t < 5T/4\},$$

and

$$\mathcal{Z}_2 = \{(x,t);\ -3/4 < x < 7/4 \quad \text{and} \quad -3T/4 < t < 7T/4\}.$$

Taking the cut-off function $\Psi \in C_0^\infty(\mathbb{R}^2)$, $0 \le \Psi \le 1$ satisfying:

$$\Psi(x,t) = \begin{cases} 1 & \text{for} \quad (x,t) \in \mathcal{Z}_1 \\ 0 & \text{for} \quad (x,t) \in \mathbb{R}^2 \setminus \mathcal{Z}_2, \end{cases} \tag{3.6}$$

we can easily deduce from (3.5) that $\Psi\tilde{u} \in W_2^{2,1}(\mathbb{R}^2)$, and

$$\|\Psi\tilde{u}\|_{W_2^{2,1}(\mathbb{R}^2)} \le C\|u\|_{W_2^{2,1}(\Omega_T)}. \tag{3.7}$$

Since $\Psi\tilde{u} \in W_2^{2,1}(\mathbb{R}^2)$, we can apply inequality (1.2) to the function $\Psi\tilde{u}$, and, having (3.7) in hands, the proof of Proposition 3.1 directly follows if we can show that

$$\|\Psi\tilde{u}\|_{BMO_p(\mathbb{R}^2)} \le C\|u\|_{\overline{BMO}_p(\Omega_T)}, \tag{3.8}$$

and this will be done in the forthcoming arguments.

3.1.1 Proof of Proposition 3.1

In all what follows, it will be useful to deal with an equivalent norm of the BMO_p space. This norm is given by the following lemma.

Lemma 3.3 (Equivalent BMO_p norms)
Let $u \in BMO_p(\mathcal{O})$, $\mathcal{O} \subseteq \mathbb{R}^{n+1}$ is an open set. The parabolic BMO_p norm of u given by (2.2) is equivalent to the following norm, that we keep give it the same notation:

$$\|u\|_{BMO_p}(\mathcal{O}) = \sup_{Q\subset\mathcal{O}} \left(\inf_{c\in\mathbb{R}} \frac{1}{|Q|} \int_Q |u-c| \right), \quad Q \text{ given by (2.3)}. \tag{3.9}$$

Proof. The proof of this lemma is direct. It suffices to see that for any $c \in \mathbb{R}$, we have:

$$|u-u_Q| \leq |u-c| + |c-u_Q| \leq |u-c| + \frac{1}{|Q|}\int_Q |u-c|,$$

which immediately gives:

$$\int_Q |u-u_Q| \leq 2\int_Q |u-c|,$$

hence

$$\frac{1}{2|Q|}\int_Q |u-u_Q| \leq \inf_{c\in\mathbb{R}} \frac{1}{|Q|}\int_Q |u-c| \leq \frac{1}{|Q|}\int_Q |u-u_Q|, \tag{3.10}$$

and the equivalence of the two norms follows. ■

From now on, and for the sake of simplicity, we will denote:

$$⨍_Q u = \frac{1}{|Q|}\int_Q u.$$

The following lemma gives an estimate of $\inf_{c\in\mathbb{R}} ⨍_Q |u-c|$ on small parabolic cubes.

Lemma 3.4 *Let $f \in L^1_{loc}(\mathbb{R}^2)$. Take $Q_r \subseteq Q_{2r}$ two parabolic cubes of $\mathbb{R}^2$. We do not require that the cubes have the same center. Then we have:*

$$\inf_{c\in\mathbb{R}} ⨍_{Q_r} |f-c| \leq 8 \inf_{c\in\mathbb{R}} ⨍_{Q_{2r}} |f-c|. \tag{3.11}$$

Proof. For $c \in \mathbb{R}$, we compute:

$$⨍_{Q_r} |f-c| \leq \frac{|Q_{2r}|}{|Q_r|} ⨍_{Q_{2r}} |f-c| \leq 8 ⨍_{Q_{2r}} |f-c|.$$

Taking the infimum of both sides we arrive to the result. ■

The next lemma gives an estimate of $\inf_{c\in\mathbb{R}} ⨍_{Q_r} |\tilde{u}-c|$ on small parabolic cubes in

$$\widehat{\Omega}_T = (-1,2) \times (0,T).$$

Define the term $r_0 > 0$ as the greatest positive real number such that there exists $Q_{r_0} \subseteq \Omega_T$, i.e.,

$$r_0 = \sup\{r > 0;\ r \leq 1/2 \text{ and } r^2 \leq T/2\}. \tag{3.12}$$

We show the following:

Lemma 3.5 (Estimates on small parabolic cubes in $\widehat{\Omega}_T$)
Let $\tilde{u}$ be the function defined by (3.2) and (3.3). Take any parabolic cube Q_r satisfying:

$$Q_r \subseteq \widehat{\Omega}_T, \quad \text{with} \quad r \leq r_1 \text{ and } 2r_1 = r_0, \tag{3.13}$$

where r_0 is given by (3.12). Then there exists a universal constant $C > 0$ such that:

$$\inf_{c\in\mathbb{R}} \fint_{Q_r} |\tilde{u} - c| \leq C\|u\|_{BMO_p(\Omega_T)}. \tag{3.14}$$

Proof. Call Ω_T^d and Ω_T^g the right and the left neighbor sets of Ω_T defined respectively by:

$$\Omega_T^d = (-1, 0) \times (0, T) \quad \text{and} \quad \Omega_T^g = (1, 2) \times (0, T).$$

First let us mention that if the cube Q_r lies in Ω_T then inequality (3.14) is evident (see the equivalent definition (3.9) of the parabolic BMO_p norm). Two remaining cases are to be considered: either Q_r intersects the set $\{x = 0\}\cup\{x = 1\}$, or Q_r lies in $\Omega_T^d \cup \Omega_T^g$. Our assumption (3.13) on the radius of the parabolic cube makes it impossible that the cube Q_r meets Ω_T^d and Ω_T^g at the same time. Therefore, and in order to make the proof simpler, we only consider the following cases: either Q_r intersects the set $\{x = 0\}$, or Q_r lies in Ω_T^g. The proof is then divided into three main steps:

Step 1. (Q_r intersects the line $\{x = 0\}$).

Step 1.1. (First estimate).

Again the assumption (3.13) imposed on the radius r makes it possible to embed Q_r in a larger parabolic cube $Q_{2r} \subseteq \widehat{\Omega}_T$ of radius $2r$, which is symmetric with respect to the line $\{x = 0\}$ (see Figure 1). Then the center of the cube Q_{2r} should be also on the same line, but we do not require

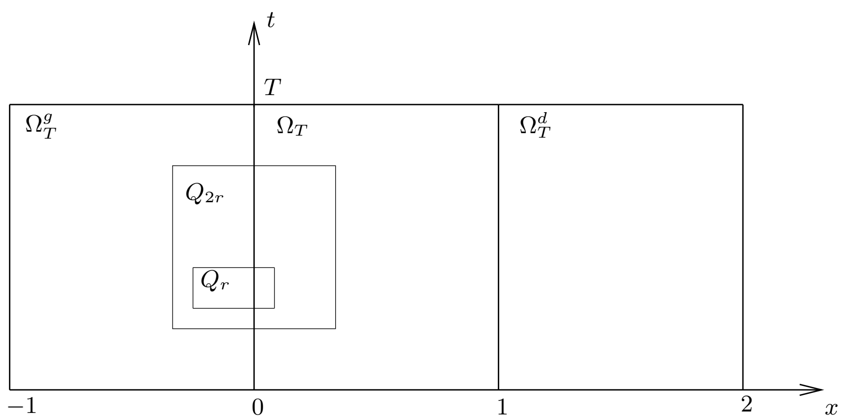

Figure 1: Analysis on cubes intersecting $\{x = 0\}$

that the two cubes Q_r and Q_{2r} have centers with the same ordinate t. Now, using Lemma 3.4, we deduce that:

$$\inf_{c\in\mathbb{R}} \fint_{Q_r} |\tilde{u} - c| \leq 8 \inf_{c\in\mathbb{R}} \fint_{Q_{2r}} |\tilde{u} - c|, \tag{3.15}$$

and hence in order to conclude, we need to estimate the right hand side of the above inequality with respect to $\|u\|_{BMO_p}(\Omega_T)$. Call Q_{2r}^d and Q_{2r}^g the right and the left sides of Q_{2r} defined respectively by:

$$Q_{2r}^d = Q_{2r} \cap \Omega_T \quad \text{and} \quad Q_{2r}^g = Q_{2r} \cap \Omega_T^g.$$

Also call $Q_{2r}^{trans} \subseteq \Omega_T$, the translation of the cube Q_{2r} by the vector $(2r, 0)$, i.e.

$$Q_{2r}^{trans} = (2r, 0) + Q_{2r}.$$

For $c \in \mathbb{R}$, we compute:

$$\begin{aligned} \int_{Q_{2r}} |\tilde{u} - c| &= \int_{Q_{2r}^g} |\tilde{u} - c| + \int_{Q_{2r}^d} |u - c|, \\ &\leq \int_{Q_{2r}^g} |\tilde{u} - c| + \int_{Q_{2r}^{trans}} |u - c|, \end{aligned} \tag{3.16}$$

where we have used the fact that $\tilde{u} = u$ on Ω_T, and that $Q_{2r}^d \subseteq Q_{2r}^{trans}$.

Step 1.2. (Estimate of $\int_{Q_{2r}^g} |\tilde{u} - c|$).

We compute (using the definition (3.2) of the function $\tilde{u}$ on Ω_T^g):

$$\begin{aligned} \int_{Q_{2r}^g} |\tilde{u}(x,t) - c| dxdt &= \int_{Q_{2r}^g} |-3u(-x,t) + 4u(-x/2,t) - c| dxdt \\ &\leq 3 \int_{Q_{2r}^g} |u(-x,t) - c| dxdt + 4 \int_{Q_{2r}^g} |u(-x/2,t) - c| dxdt \\ &\leq 3 \int_{Q_{2r}^d} |u(x,t) - c| dxdt + 8 \int_{Q_{2r}^{\bar{d}}} |u(x,t) - c| dxdt, \end{aligned} \tag{3.17}$$

where

$$Q_{2r}^{\bar{d}} = \{(x/2, t);\ (x,t) \in Q_{2r}^d\} \subseteq Q_{2r}^d \subseteq Q_{2r}^{trans}.$$

From (3.17) we easily deduce that:

$$\int_{Q_{2r}^g} |\tilde{u} - c| \leq 11 \int_{Q_{2r}^{trans}} |u - c|,$$

and hence (using (3.16)), we finally get:

$$\int_{Q_{2r}} |\tilde{u} - c| \leq 12 \int_{Q_{2r}^{trans}} |u - c|. \tag{3.18}$$

Since $|Q_{2r}| = |Q_{2r}^{trans}|$, inequality (3.18) gives

$$⨍_{Q_{2r}} |\tilde{u} - c| \leq 12 ⨍_{Q_{2r}^{trans}} |u - c|.$$

Since Q_{2r}^{trans} is a parabolic cube in Ω_T, taking the infimum over $c \in \mathbb{R}$ of the above inequality, we obtain:

$$\inf_{c \in \mathbb{R}} ⨍_{Q_{2r}} |\tilde{u} - c| \leq 12 \|u\|_{BMO_p(\Omega_T)}. \tag{3.19}$$

From (3.15) and (3.19), we deduce (3.14).

Step 2. ($Q_r \subseteq \Omega_T^g$).

Let $0 < a_0 < b_0 < 1$ and $0 < a_1 < b_1 < T$ be such that

$$Q_r = (-b_0, -a_0) \times (a_1, b_1).$$

For any $c \in \mathbb{R}$, we compute:

$$\begin{aligned} \int_{Q_r} |\tilde{u}(x,t) - c| dxdt &= \int_{Q_r} |-3u(-x,t) + 4u(-x/2,t) - c| dxdt \\ &\leq 3 \int_{Q_r^s} |u(x,t) - c| dxdt + 8 \int_{Q_r^{\bar{s}}} |u(x,t) - c| dxdt \end{aligned} \tag{3.20}$$

with (see Figure 2),

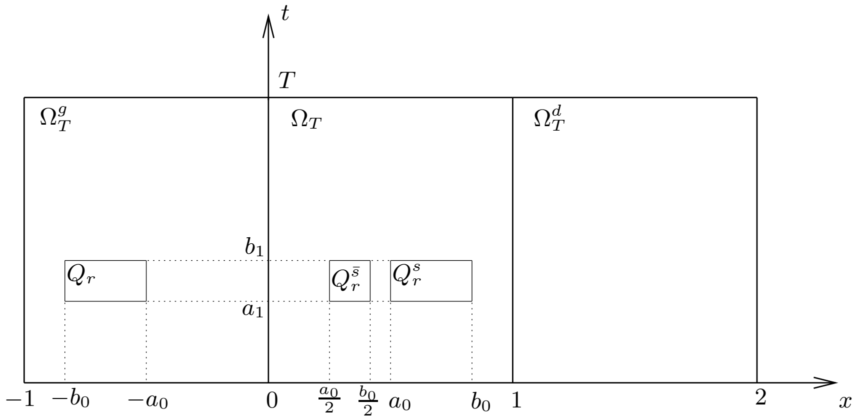

Figure 2: Analysis on cubes $Q_r \subseteq \Omega_T^g$

$$Q_r^s = (a_0, b_0) \times (a_1, b_1) \subseteq \Omega_T \quad \text{and} \quad Q_r^{\bar{s}} = \left(\frac{a_0}{2}, \frac{b_0}{2}\right) \times (a_1, b_1) \subseteq \Omega_T.$$

We remark that Q_r^s is a parabolic cube in Ω_T, while $Q_r^{\bar{s}}$ is not (his aspect ratio is different). In fact $Q_r^{\bar{s}}$ could be embedded in a parabolic cube $Q_r^{\bar{s}} \subseteq Q_r^{\bar{\bar{s}}} \subseteq \Omega_T$, where $Q_r^{\bar{\bar{s}}}$ is simply a space translation of Q_r^s. In particular we have:

$$|Q_r| = |Q_r^s| = |Q_r^{\bar{\bar{s}}}|. \tag{3.21}$$

The above arguments, together with (3.20) give:

$$\int_{Q_r} |\tilde{u} - c| \leq 3 \int_{Q_r^s} |u - c| + 8 \int_{Q_r^{\bar{\bar{s}}}} |u - c|. \tag{3.22}$$

Taking the infimum in $c \in \mathbb{R}$ for both sides of inequality (3.22), leads to

$$\inf_{c \in \mathbb{R}} \fint_{Q_r} |\tilde{u} - c| \leq 11 \|u\|_{BMO_p(\Omega_T)}, \tag{3.23}$$

which implies (3.14).

Step 3. (Conclusion).

As it was already mentioned at the beginning of the proof, the case where the parabolic cube Q_r meets the line $\{x = 1\}$ or lies completely in Ω_T^d, could be treated using identical arguments. Therefore, for all small parabolic cubes Q_r satisfying (3.13), inequality (3.14) is always valid, and this terminates the proof of Lemma 3.5. ■

A generalization of Lemma 3.5 is now given.

Lemma 3.6 (Estimates on small parabolic cubes in $\widetilde{\Omega}_T$)
Let $\tilde{u}$ be the function defined by (3.2) and (3.3). Take any parabolic cube $Q_r \subseteq \widetilde{\Omega}_T$ satisfying:

$$r \leq r_2 \quad \textit{with} \quad r_2\sqrt{2} = r_1, \tag{3.24}$$

where r_1 is given by (3.13). Then there exists a universal constant $C > 0$ such that:

$$\inf_{c\in\mathbb{R}} \fint_{Q_r} |\tilde{u} - c| \leq C\|u\|_{BMO_p(\Omega_T)}. \tag{3.25}$$

Sketch of the proof. The arguments leading to the proof of this lemma are already contained in the proof of Lemma 3.5. First notice that if $Q_r \subseteq \widehat{\Omega}_T$, we enter directly (since $r \leq r\sqrt{2} \leq r_1$) to the framework of Lemma 3.5, and hence (3.25) is direct. Because $r \leq r_1$, remark that there exists a cube $Q_r^{'}$ obtained by a time translation of Q_r such that $Q_r^{'} \subseteq \widehat{\Omega}_T$. Therefore it is impossible that Q_r meets at the same time $(-1, 2) \times (T, 2T)$ and $(-1, 2) \times (-T, 0)$. For this reason, we either consider parabolic cubes intersecting $\{t = T\}$ (see Figure 3), or parabolic cubes in $(-1, 2) \times (T, 2T)$ (see Figure 4).

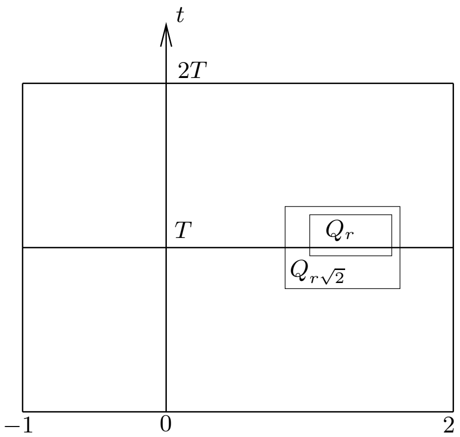

Figure 3: $Q_r \cap \{t = T\} \neq \emptyset$

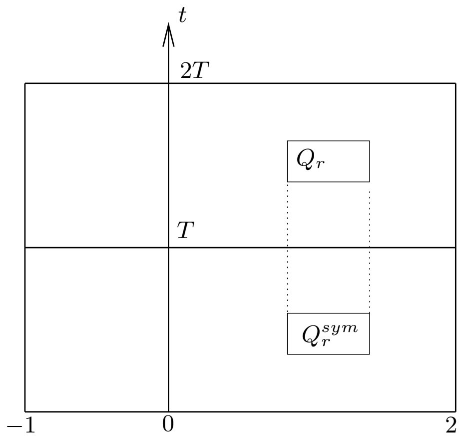

Figure 4: $Q_r \cap \{t = T\} = \emptyset$

Case $Q_r \cap \{t = T\} \neq \emptyset$. In this case, we first embed Q_r in a larger parabolic cube $Q_{r\sqrt{2}}$ which is symmetric with respect to the line $\{t = T\}$, so the center of this cube lies in $\{t = T\}$. We now repeat the same arguments as in Step 1 of Lemma 3.5, using in particular the symmetry (3.3) of the function $\tilde{u}$ with respect to $\{t = T\}$, and the fact that we can consider the cube

$$Q_{r\sqrt{2}}^{trans'} = (0, -2r^2) + Q_{r\sqrt{2}}$$

such that

$$Q_{r\sqrt{2}}^{trans'} \subseteq Q_{r_1} \subseteq \widehat{\Omega}_T,$$

for some cube Q_{r_1}. Indeed, estimates on all such cubes $Q^{trans'}_{r\sqrt{2}}$ are already controlled by (3.14).

Case $Q_r \cap \{t = T\} = \emptyset$. In this case we repeat the same arguments as in Step 2 of Lemma 3.5. Indeed, in the present case, it is even simpler since the function $\tilde{u}$ is symmetric with respect to $\{t = T\}$. ■

We now show how to prove estimate (3.8).

Proof of estimate (3.8). The parabolic BMO_p norm (3.9) of $\Psi\tilde{u}$ could be estimated taking the supremum of $⨍_{Q_r} |\Psi\tilde{u} - (\Psi\tilde{u})_{Q_r}|$, $Q_r \subseteq \mathbb{R}^2$, over small parabolic cubes (Q_r with $r \leq r_2/2$), and big parabolic cubes (Q_r with $r > r_2/2$). The proof is then divided into two steps.

Step 1. (Analysis on big parabolic cubes Q_r, $r > r_2/2$).

We compute, using the fact that $\Psi = 0$ on $\mathbb{R}^2 \setminus \mathcal{Z}_2$, and $\Psi \leq 1$ on $\mathbb{R}^2$ (see (3.6)):

$$\begin{aligned} ⨍_{Q_r} |\Psi\tilde{u} - (\Psi\tilde{u})_{Q_r}| &\leq 2 ⨍_{Q_r} |\Psi\tilde{u}| \leq \frac{2}{|Q_r|}\int_{Q_r\cap\mathcal{Z}_2} |\tilde{u}| \\ &\leq \frac{2^2}{r_2^3}\int_{Q_r\cap\mathcal{Z}_2} |\tilde{u}| \leq \frac{2^2}{r_2^3}\int_{\widetilde{\Omega}_T} |\tilde{u}| \leq C\|u\|_{L^1(\Omega_T)}. \end{aligned} \tag{3.26}$$

Step 2. (Analysis on small parabolic cubes Q_r, $r \leq r_2/2$).

From the definition (3.24) of r_2, and the construction (3.6) of the function Ψ, we deduce that if Q_r intersects $\mathcal{Z}_2$ then forcely $Q_r \subseteq \widetilde{\Omega}_T$. If not, i.e. $Q_r \cap \mathcal{Z}_2 = \emptyset$ then $\Psi = 0$ on Q_r, and therefore:

$$⨍_{Q_r} |\Psi\tilde{u} - (\Psi\tilde{u})_{Q_r}| = 0. \tag{3.27}$$

Then we have only to consider $Q_r \subseteq \widetilde{\Omega}_T$.

Step 2.1. (First estimate).

Using (3.10), we get

$$⨍_{Q_r} |\Psi\tilde{u} - (\Psi\tilde{u})_{Q_r}| \leq 2\inf_{c\in\mathbb{R}} ⨍_{Q_r} |\Psi\tilde{u} - c| \leq 2 ⨍_{Q_r} |\Psi\tilde{u} - c_0\Psi_{Q_r}|, \tag{3.28}$$

for any fixed constant $c_0 \in \mathbb{R}$. Remark that we can write:

$$\Psi\tilde{u} - c_0\Psi_{Q_r} = (\Psi - \Psi_{Q_r})\tilde{u} + (\tilde{u} - c_0)\Psi_{Q_r}. \tag{3.29}$$

Hence, we deduce that

$$\begin{aligned} ⨍_{Q_r} |\Psi\tilde{u} - (\Psi\tilde{u})_{Q_r}| &\leq Cr ⨍_{Q_r} |\tilde{u}| + 2\inf_{c_0\in\mathbb{R}} ⨍_{Q_r} |\tilde{u} - c_0| \\ &\leq Cr ⨍_{Q_r} |\tilde{u}| + 2C\|u\|_{BMO_p(\Omega_T)}, \end{aligned} \tag{3.30}$$

where for the first line we have used that fact that $\Psi \leq 1$ and that Ψ is Lipschitz, and for the second line we have used (3.25).

Step 2.2. (Estimate of $\fint_{Q_r} |\tilde{u}|$).

We have

$$\begin{aligned}
\fint_{Q_r} |\tilde{u}| &\leq |\tilde{u}_{Q_r}| + \fint_{Q_r} |\tilde{u} - \tilde{u}_{Q_r}| \\
&\leq |\tilde{u}_{Q_r}| + 2 \inf_{c\in\mathbb{R}} \fint_{Q_r} |\tilde{u} - c| \\
&\leq |\tilde{u}_{Q_r}| + 2C\|u\|_{BMO_p(\Omega_T)}, \qquad (3.31)
\end{aligned}$$

where for the second line, we have used (3.10), while for the third line, we have used (3.25). Remark that from the proof of Lemma 2.10 with $n = 1$, we have for $Q_{2^j r} \subseteq Q_{2^{j+1} r} \subseteq \widetilde{\Omega}_T$:

$$\begin{aligned}
|\tilde{u}_{Q_{2^j r}} - \tilde{u}_{Q_{2^{j+1} r}}| &\leq \fint_{Q_{2^j r}} |\tilde{u} - \tilde{u}_{Q_{2^j r}}| + 2^3 \fint_{Q_{2^{j+1} r}} |\tilde{u} - \tilde{u}_{Q_{2^{j+1} r}}| \\
&\leq 2(1+2^3) \sup_{Q_\rho \subseteq \widetilde{\Omega}_T,\, \rho \leq 2^{j+1} r} \left(\inf_{c\in\mathbb{R}} \fint_{Q_\rho} |\tilde{u} - c| \right) \\
&\leq 2C(1+2^3)\|u\|_{BMO_p(\Omega_T)},
\end{aligned}$$

where we have used (3.10) for the second line, and, for the third line, we have used (3.25) assuming $2^{j+1} r \leq r_2$. Defining

$$j_0 = \min\{j \in \mathbb{N};\ r_2/2 \leq 2^j r < r_2\},$$

and using a telescopic sequence, we can deduce that

$$\begin{aligned}
|\tilde{u}_{Q_r} - \tilde{u}_{Q_{2^{j_0} r}}| &\leq j_0(2C(1+2^3))\|u\|_{BMO_p(\Omega_T)} \\
&\leq C(1 + |\log r|)\|u\|_{BMO_p(\Omega_T)}. \qquad (3.32)
\end{aligned}$$

Moreover, we have

$$|\tilde{u}_{Q_{2^{j_0} r}}| \leq \frac{1}{|Q_{r_2/2}|} \int_{\widetilde{\Omega}_T} |\tilde{u}| \leq C\|u\|_{L^1(\Omega_T)}, \qquad (3.33)$$

where we have used (3.4) for the second inequality. From (3.30), (3.32) and (3.33), we get:

$$\fint_{Q_r} |\tilde{u}| \leq C\left(\|u\|_{L^1(\Omega_T)} + (1 + |\log r|)\|u\|_{BMO_p(\Omega_T)}\right) \qquad (3.34)$$

for some constant $C > 0$.

Step 2.3. (Conclusion for $r \leq r_2/2$).

Finally, putting together (3.30) and (3.34), we deduce that

$$\begin{aligned}
\fint_{Q_r} |\Psi\tilde{u} - (\Psi\tilde{u})_{Q_r}| &\leq C\left\{(r|\log r| + 1)\|u\|_{BMO_p(\Omega_T)} + \|u\|_{L^1(\Omega_T)}\right\} \\
&\leq C\left(\|u\|_{BMO_p(\Omega_T)} + \|u\|_{L^1(\Omega_T)}\right), \qquad (3.35)
\end{aligned}$$

where in the second line, we have used that $r \in (0, 1)$, and that $r|\log r|$ is bounded.

Step 3 (General conclusion).

Putting together (3.26), (3.27) and (3.35), we get (3.8). ■

We are now ready to show the proof of Proposition 3.1.

Proof of Proposition 3.1. Applying estimate (2.37), with $m = n = 1$, to the function $\Psi\tilde{u} \in W_2^{2,1}(\mathbb{R}^2) \subseteq L^\infty(\mathbb{R}^2)$, we get:

$$\|u\|_{L^\infty(\Omega_T)} = \|\Psi\tilde{u}\|_{L^\infty(\Omega_T)} \leq \|\Psi\tilde{u}\|_{L^\infty(\mathbb{R}^2)} \leq C\left(1 + \|\Psi\tilde{u}\|_{BMO_p(\mathbb{R}^2)}\left(1 + \log^+\|\Psi\tilde{u}\|_{W_2^{2,1}(\mathbb{R}^2)}\right)\right).$$

Here, we have also used the fact that $\Psi = 1$ over Ω_T (see (3.6)). Using (3.7), (3.8) and the above inequality, we directly get (3.1). ■

3.1.2 Ideas of the proof of Theorem 1.2

One of the main motivations for starting with the detailed proof of Proposition 3.1 (a simplified version of Theorem 1.2) is that it was used to show [8, Theorem 1.1]. The other motivation is that the arguments of the proof of Theorem 1.2 are all contained in the proof of Proposition 3.1. It suffices to make the following generalizations that we list below.

Extension of $\tilde{u}$. In order to extend the function $u \in W_2^{2m,m}(\Omega_T)$ to the function $\tilde{u} \in W_2^{2m,m}(\widetilde{\Omega}_T)$ with $\widetilde{\Omega}_T = (-1,2)^n \times (-T, 2T)$, we first make the extension separately and successively with respect to the spatial variables x_i, with $i = 1 \cdots n$. Then we make the extension with respect to the time variable that is treated somehow differently. Fix $(x_2, \ldots, x_n, t) \in (0,1)^{n-1} \times (0,T)$, the spatial extension of u in x_1 is as follows:

$$\tilde{u}(x_1, \ldots) = \begin{cases} \displaystyle\sum_{j=0}^{2m-1} c_j u(-\lambda_j x_1, \ldots) & \text{for} \quad -1 < x_1 < 0, \\ \displaystyle\sum_{j=0}^{2m-1} c_j u(1 + \lambda_j(1 - x_1), \ldots) & \text{for} \quad 1 < x_1 < 2, \end{cases} \tag{3.36}$$

with $\lambda_j = \frac{1}{2^j}$, and where we require that:

$$\sum_{j=0}^{2m-1} c_j(-\lambda_j)^k = 1 \quad \text{for} \quad k = 0 \cdots 2m-1.$$

The above inequalities can be regarded as a linear system whose associated matrix is of the Vandermonde type and hence invertible. This ensures the existence of the constants c_j, $j = 0 \cdots 2m-1$, and therefore the above extension (3.36) gives sense.

After doing the extension with respect to x_1, the extension with respect to x_2 is done in the same way by varying the x_2 and fixing all other variables. This is repeated successively until the x_n variable.

For the time variable, we also use the same extension (3.36). Indeed, in this case, we may only sum up to $m-1$ in (3.36).

<u>*The cut-off function* Ψ</u>. For the definition of the cut-off function Ψ, we first define the two sets:

$$\mathcal{Z}_1 = \{(x_1, \ldots, x_n, t);\ \forall i = 1 \cdots n,\ -1/4 < x_i < 5/4 \quad \text{and} \quad -T/4 < t < 5T/4\}$$

and

$$\mathcal{Z}_2 = \{(x_1, \ldots, x_n, t);\ \forall i = 1 \cdots n,\ -3/4 < x_i < 7/4 \quad \text{and} \quad -3T/4 < t < 7T/4\}.$$

The function Ψ is then defined as $\Psi \in C_0^\infty(\mathbb{R}^{n+1})$ with $0 \le \Psi \le 1$ and

$$\Psi(x,t) = \begin{cases} 1 & \text{for} \quad (x,t) \in \mathcal{Z}_1 \\ 0 & \text{for} \quad (x,t) \in \mathbb{R}^2 \setminus \mathcal{Z}_2. \end{cases} \tag{3.37}$$

<u>*Generalization of Lemma 3.6*</u>. An analogue estimate of (3.25) could be obtained for $n+1$-dimensional parabolic cubes $Q_r \subseteq \widetilde{\Omega}_T = (-1,2)^n \times (-T, 2T)$. It suffices to replace r_2 satisfying (3.24), by the radius

$$r_{n+1} = \frac{r_n}{\sqrt{2}},$$

where r_n is defined recursively as follows: $r_{j+1} = r_j/2$ for $0 \le j \le n-1$.

Using the above generalizations, the proof of Theorem 1.2 follows, line by line, the proof of Proposition 3.1. ■

3.2 Application of the parabolic Kozono-Taniuchi inequality

In this subsection, we show how to apply the parabolic Kozono-Taniuchi inequality in order to give some *a priori* estimates for the solution of certain parabolic equations. These *a priori* estimates provide a well control on the solution in order to avoid singularities at a finite time, and hence serve for the long-time existence. The application that will be given here deals with a model that can be considered as a toy model. Indeed, this is a simplification of the one treated in [8], where a rigorous proof of the long-time existence of solutions of a singular parabolic coupled system was shown (see [8, Theorem 1.1]). Consider, for $0 < a < 1$, the following parabolic equation:

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = \sin(u_x(x,t)u_x(x+a,t)) + \sin(\log u_x(x,t)) \quad \text{on} \quad \mathbb{R} \times (0,\infty), \\ u(x+1,t) = u(x,t) + 1 \quad \text{on} \quad \mathbb{R} \times (0,\infty), \\ u_x(x,0) \ge \delta_0 > 0 \quad \text{on} \quad \mathbb{R}, \end{cases} \tag{3.38}$$

the following proposition could be shown:

Proposition 3.7 (Gradient estimate)
Let $v = u_x$ and $m(t) = \min_{x \in \mathbb{R}} v(x,t)$. If $u \in C^\infty(\mathbb{R} \times [0,\infty))$ is a smooth solution of (3.38), then, for some constant $C = C(t) > 0$ we have:

$$m_t \ge -Cm\left(|\log m| + 1\right), \quad \forall t \ge 0. \tag{3.39}$$

Remark 3.8 *Inequality (3.39) directly implies that for every $t \ge 0$ we have $m(t) > 0$. This is important to avoid the logarithmic singularity in (3.38) when $v = u_x = 0$.*

Remark 3.9 *The proof of the above proposition walks in parallel with the proof of [8, Theorem 1.1]. For this reason we only present a heuristic proof explaining only the basic ideas. The interested reader could see the full details in [8].*

Ideas of the proof of Proposition 3.7. Heuristically, the proof breaks into the following four steps. In what follows all the constants can depend on the time t, but are bounded for any finite t.

Step 1. (First estimate from below on the gradient).

Writing down the equation satisfied by v:

$$\begin{cases} v_t(x,t) - v_{xx}(x,t) = \cos(v(x,t)v(x+a,t))\left\{v_x(x,t)v(x+a,t) + v(x,t)v_x(x+a,t)\right\} \\ \qquad\qquad + \cos(\log v(x,t))\dfrac{v_x(x,t)}{v(x,t)} \quad \text{on} \quad \mathbb{R}\times(0,\infty), \\ v(x+1,t) = v(x,t) \quad \text{on} \quad \mathbb{R}\times(0,\infty) \\ v(x,0) \geq \delta_0 > 0 \quad \text{on} \quad \mathbb{R}, \end{cases} \tag{3.40}$$

we can show that for every $t \geq 0$:

$$m_t \geq -mG \quad \text{with} \quad G(t) = \max_{x\in\mathbb{R}} |v_x(x,t)|. \tag{3.41}$$

Step 2. (Estimate of $\|v_x\|_{BMO_p}$).

Using the fact that $u(x+1,t) = u(x,t)+1$, and that the right hand term of the first equation of (3.38) is bounded, we apply the BMO theory for parabolic equation to (3.38) and hence we obtain, for some positive constant $c_1 > 0$:

$$\|v_x\|_{BMO_p((0,1)\times(0,t))} \leq c_1 \quad \text{for any} \quad t>0.$$

However, the L^p theory for parabolic equation applied to (3.38) gives, for some positive constant $c_2 > 0$:

$$\|v_x\|_{L^1((0,1)\times(0,t))} \leq c_2 \quad \text{for any} \quad t>0.$$

Finally, the above two inequalities give:

$$\|v_x\|_{\overline{BMO}_p((0,1)\times(0,t))} \leq c_1 + c_2 \quad \text{for any} \quad t>0. \tag{3.42}$$

Step 3. (Estimate of $\|v_x\|_{W_2^{2,1}}$).

Let $w = v_x$, we write down the equation satisfied by w:

$$\begin{cases} w_t(x,t) - w_{xx}(x,t) \\ = -\sin(v(x,t)v(x+a,t))\left\{v(x+a,t)v_x(x,t) + v(x,t)v_x(x+a,t)\right\}^2 \\ + \cos(v(x,t)v(x+a,t))\left\{v(x+a,t)v_{xx}(x,t) + 2v_x(x,t)v_x(x+a,t) + v(x,t)v_{xx}(x+a,t)\right\} \\ - \sin(\log v(x,t))\dfrac{v_x^2(x,t)}{v^2(x,t)} + \cos(\log v(x,t))\left\{\dfrac{v_{xx}(x,t)}{v(x,t)} - \dfrac{v_x^2(x,t)}{v^2(x,t)}\right\} \quad \text{on} \quad \mathbb{R}\times(0,\infty), \\ w(x+1,t) = w(x,t) \quad \text{on} \quad \mathbb{R}\times(0,\infty) \\ w(x,0) = v_x(x,0) \quad \text{on} \quad \mathbb{R}. \end{cases} \tag{3.43}$$

Using the L^p theory for parabolic equations (with various values of p) to (3.38), (3.40) and (3.43), we deduce, for some other positive constant $c > 0$, that:

$$\|v_x\|_{W_2^{2,1}((0,1)\times(0,t))} \leq \frac{c}{m^2(t)} \quad \text{for any} \quad t>0. \tag{3.44}$$

Step 4. (Conclusion).

Applying parabolic Kozono-Taniuchi inequality (3.1) to the function v_x, using particularly (3.42) and (3.44), we deduce that:

$$G \leq C(1 + |\log m|),$$

which, joint to (3.43), directly give the result. ■

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