GEOMETRIC QUANTIZATION FOR PROPER MOMENT MAPS

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Abstract. We establish a geometric quantization formula for Hamiltonian actions of a compact Lie group acting on a non-compact symplectic manifold such that the associated moment map is proper. In particular, we resolve the conjecture of Vergne in this non-compact setting.

0. Introduction

The famous geometric quantization conjecture of Guillemin and Sternberg [9] states that for a compact pre-quantizable symplectic manifold admitting a Hamiltonian action of a compact Lie group, the principle of "quantization commutes with reduction" holds. This conjecture was first proved independently by Meinrenken [14] and Vergne [23] for the case where the Lie group is abelian, and then by Meinrenken [15] in the general case. The singular reduction case was proved by Meinrenken-Sjamaar in [16]. There are also an analytic approach to the original conjecture developed by Tian and Zhang [20] as well as a proof developed by Paradan [17] by making use of the theory of transversally elliptic operators, see also [24] for an excellent survey.

It is natural to consider the generalizations of the above results to actions on non-compact spaces. One of the aspects of this issue has been considered by Weitsman in [26], where the properness of the associated moment map is assumed. In her ICM 2006 Plenary lecture [25], Vergne made a quantization conjecture (under the assumption that the zero point set of the vector field generated by the moment map is compact), which generalizes the original Guillemin-Sternberg conjecture to this non-compact setting. A special case of this conjecture had indeed been verified already by Paradan in [18] where he proved a quantization formula valid for the case where a maximal compact subgroup of a non-compact real semi-simple Lie group acts on the co-adjoint orbits of the real semi-simple Lie group itself.

The purpose of this paper is to establish a general quantization formula in this framework of a compact group acting on a non-compact space with proper moment map. As we will see, our result could be viewed as an extended version of the conjecture of Vergne, in the sense that we do not make any extra assumptions beside the properness of the moment map.

To be more precise, let (M,ω) be a non-compact symplectic manifold with symplectic form ω. We assume that (M,ω) is prequantizable, that is, there exists a complex line bundle L (called a prequantized line bundle) carrying a Hermitian metric h^L and a

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Hermitian connection ∇^L such that

$$\frac{\sqrt{-1}}{2\pi}\left(\nabla^L\right)^2 = \omega. \tag{0.1}$$

We also assume that there exists an almost complex structure J on TM such that

$$g^{TM}(u,v) = \omega(u,Jv), \quad u,v \in TM \tag{0.2}$$

defines a Riemannian metric on TM.

Let G be a compact connected Lie group with Lie algebra denoted by $\mathfrak{g}$. We assume that G acts on the left on M and this action can be lifted to an action on L. Moreover, we assume that G preserves g^{TM}, J, h^L and ∇^L.

For any $K \in \mathfrak{g}$, let K^M be the vector field generated by K over M.

Let $\mu : M \to \mathfrak{g}^*$ be defined by the Kostant formula [10]

$$2\pi\sqrt{-1}\,\mu(K) := \mu^L(K) = \nabla^L_{K^M} - L_K, \quad K \in \mathfrak{g}. \tag{0.3}$$

Then μ is the corresponding moment map, i.e. for any $K \in \mathfrak{g}$,

$$d\mu(K) = i_{K^M}\omega. \tag{0.4}$$

We call the G action with a moment map $\mu : M \to \mathfrak{g}^*$ verifying (0.4) a Hamiltonian action.

From now on, we assume that the following fundamental assumption holds.

Fundamental Assumption. The moment map $\mu : M \to \mathfrak{g}^*$ is proper, in the sense that the inverse image of a compact subset is compact.

Let T be a maximal torus of G, $C_G \subset \mathfrak{t}^*$ be a Weyl chamber associated to T, $\Lambda \subset \mathfrak{t}^*$ be the weight lattice, and $\Lambda^G_+ = \Lambda \cap C_G$ be the set of dominate weights. Then the ring of characters $R(G)$ of G has a $\mathbb{Z}$-basis V^G_γ, $\gamma \in \Lambda^G_+$: V^G_γ is the irreducible G-representation with highest weight γ.

Take any $\gamma \in \Lambda^G_+$. If γ is a regular value of the moment map μ, then one can construct the Marsden-Weinstein symplectic reduction $(M_\gamma, \omega_\gamma)$, where $M_\gamma = \mu^{-1}(G\cdot\gamma)/G$ is a compact (as μ is proper) orbifold. Moreover, the line bundle L (resp. the almost complex structure J) induces a prequantized line bundle L_γ (resp. an almost complex structure J_γ) over $(M_\gamma, \omega_\gamma)$. One can then construct the associated Spin^c-Dirac operator (twisted by L_γ), $D^{L_\gamma}_+ : \Omega^{0,\mathrm{even}}(M_\gamma, L_\gamma) \to \Omega^{0,\mathrm{odd}}(M_\gamma, L_\gamma)$ (cf. Section 1.6, (1.5)) on M_γ whose index

$$Q(L_\gamma) := \dim \mathrm{Ker}\, D^{L_\gamma}_+ - \dim \mathrm{Coker}\, D^{L_\gamma}_+ \in \mathbb{Z}; \tag{0.5}$$

is well-defined. If $\gamma \in \Lambda^G_+$ is not a regular value of μ, then by proceeding as in [16] (cf. [17, §7.4]), one still gets a well-defined quantization number $Q(L_\gamma)$ extending the above definition[1].

On the other hand, let $\mathfrak{g}$ be equipped with an Ad_G-invariant metric. Set $H = |\mu|^2$. Then since μ is proper, for any $c > 0$, $U_c := H^{-1}([0,c]) = \{x \in M : H(x) \leq c\}$ is a compact subset of M.

[1] See also Section 2.1 for a standard perturbative definition.

Recall that by Sard's theorem, the set of critical values of the function $\mathcal{H}: M \to \mathbb{R}$ is of measure zero.

Let $X^{\mathcal{H}} = -J(d\mathcal{H})$ be the Hamiltonian vector field associated to $\mathcal{H}$.

For any regular value $c > 0$ of $\mathcal{H}$, one knows $X^{\mathcal{H}}$ is nowhere zero on $\partial U_c = \mathcal{H}^{-1}(c)$. Thus, according to Atiyah [1, §1, §3] and Paradan [17, §3] (cf. also Vergne [23]), the triple $(U_c, X^{\mathcal{H}}, L)$ defines a transversally elliptic symbol (corresponding to the Spin^c-Dirac operator (twisted by L) on M) associated to the G-action on U_c. And according to Atiyah [1, §1], it admits a well-defined transversal index whose character is a distribution on G.

For any $\gamma \in \widehat{G}$, let $Q(L)_c^{\gamma} \in \mathbb{Z}$ denote the γ-component of this transversal index.

Theorem 0.1. *For any $\gamma \in \widehat{G}$, there exists $c_\gamma > 0$ such that $Q(L)_c^{\gamma} \in \mathbb{Z}$ does not depend on $c > c_\gamma$, with c a regular value of $\mathcal{H}$. And $Q(L)_c^{\gamma=0} \in \mathbb{Z}$ does not depend on $c > 0$, with c a regular value of $\mathcal{H}$.*

According to Theorem 0.1, for any $\gamma \in \widehat{G}$, we have a well-defined integer $Q(L)_c^{\gamma}$ not depending on the regular value $c \gg 0$. From now on we denote it by $Q(L)^{\gamma}$.

We can now state our main result as follows.

Theorem 0.2. *For any $\gamma \in \widehat{G}$, the following identity holds,*

$$Q(L)^{\gamma} = Q(L_\gamma). \tag{0.6}$$

Remark 0.3. If the zero set of $X^{\mathcal{H}}$ is compact, then Theorem 0.1 was already known in [17] and [25], while Theorem 0.2 was conjectured by Vergne in [25, §4.3]. Thus Theorem 0.2 can be thought of as an extended version of the Vergne conjecture.

If we set

$$Q_G(L)^{-\infty} = \bigoplus_{\gamma \in \widehat{G}} Q(L_\gamma)\, V_\gamma^G, \tag{0.7}$$

then by Theorem 0.2, $Q_G(L)^{-\infty}$ equals to the formal geometric quantization in the sense of [26, Definition 4.1] and [19, Definition 1.2]. In particular, it verifies the functorial quantization property described as follows.

It is clear that if M is compact, then Theorem 0.1 holds tautologically and Theorem 0.2 is the Guillemin-Sternberg conjecture proved in [16]. Let (N, ω^N) be such a pair with N being compact, F the notation for the prequantized line bundle over N, etc. Combining Theorem 0.2 with the result [26, Theorem 1] (cf. also [19, Theorem 1.5]) one gets the following functorial quantization result.

Let $L \otimes F$ be the prequantized line bundle over $M \times N$ obtained by the tensor product of the natural liftings of L and F to $M \times N$.

Theorem 0.4. *For the induced action of G on $(M \times N, \omega^M \oplus \omega^N)$ and $L \otimes F$, the following identity holds,*

$$Q\left(L \otimes F\right)^{\gamma=0} = \sum_{\gamma \in \widehat{G}} Q(L_\gamma)\, Q(F)^{\gamma^*}. \tag{0.8}$$

By taking N to be the orbits of the co-adjoint action of G on $\mathfrak{g}^*$, one recovers Theorem 0.2 from Theorem 0.4 immediately. Thus Theorem 0.2 and 0.4 are actually equivalent. In this paper we will write out our proof for Theorem 0.4 directly. In this way we also get a new proof of the following functorial quantization result due to Weitsman [26, Theorem 1] and Paradan [19, Theorem 1.5], without using the symplectic cut techniques there.

Corollary 0.5. Under the assumptions in Theorem 0.4, the following identity holds,

$$Q(L \otimes F)_{=0} = \sum_{\in\widehat{G}} Q(L)\, Q(F). \tag{0.9}$$

Now let K be a compact subgroup of G such that the moment map of the induced Hamiltonian action of K on M also verifies the fundamental assumption of being proper. Then by combining Theorem 1.2, (0.7) with [19, Theorem 1.3], one gets the following consequence on the relation between $Q_G(L)^{\infty}$ and $Q_K(L)^{\infty}$.

Corollary 0.6. Any irreducible representation of K has a finite multiplicity in $Q_G(L)^{\infty}$. Moreover, the following identity holds, when both sides are viewed as virtual representation spaces of K,

$$Q_G(L)^{\infty}\big|_K = Q_K(L)^{\infty}. \tag{0.10}$$

Our proof of Theorems 0.1 and 0.4 is analytic. It makes use of Braverman's analytic interpretation [5, §14] of the transversal index of a wide class of transversally elliptic operators covering the ones mentioned above. The main idea is that through the analytic interpretation of Braverman, one can further express this transversal index by using the Atiyah-Patodi-Singer type index[2] for Dirac type operators on manifolds with boundary [2]. One can then apply the analytic methods developed in [20] and [22] to study the corresponding quantization problem.

Indeed, after we interpret the transversal index by the APS type index, it is almost direct to prove Theorem 0.1 by applying the analytic techniques in [20] and [22].

The proof of Theorem 0.4 needs more effort. Tautologically, one would adapt the idea of two steps deformations appeared in [18, §3] to the current situation. The main point comes from the fact that, after choosing the suitable (already non-trivial) first step deformation, in order to pass from the boundaries of submanifolds of $M \times N$ to boundaries of specific forms obtained by the product of a boundary in M with N, one may encounter a lot of zero points of the vector fields used in the deformation. It is then necessary to eliminate the potential contributions caused by these possible zero points.[3]

Recall that in the analytic proof of the original Guillemin-Sternberg conjecture developed in [20], one already encounters the local pointwise estimates around the zero points of the Hamiltonian vector fields used in the deformation there. It is remarkable

[2] We will briefly call it as APS index in what follows.

[3] The situation considered in [18] is much simpler as there is no zero point of the vector fields used in the corresponding deformation, he can then apply directly the homotopy invariance of the transversal index.

that such kind of local estimates still hold in the current (apparently more sophisticated) non-compact situation.

We would like to point out that one may indeed formulate Theorems 0.1, 0.2 and 0.4 directly using the APS type index which is more intrinsic (as the definition of the transversal index involved depends on certain extra structures which go beyond the symplectic structures involved).

It would also be interesting to give a direct analytic proof of Corollary 0.6 without using the symplectic cut techniques in [19].

The rest of this paper is organized as follows. In Section 1, we first apply the results of Braverman [5] to interpret certain transversal index as a kind of APS type index. We then apply the analytic approach of the quantization formulas developed in [20], [22] and prove Theorem 0.1. In Section 2, we present our proof of Theorem 0.4 modulo a vanishing result, Theorem 2.4, whose proof will be carried out in Section 3.

Some results of this paper have been announced in [13].

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1. Transversal index and quantization for proper moment maps

In this section, we prove Theorem 0.1. In doing so, we first express the transversal index appearing in the context as certain APS type index, with the help of a result by Braverman [5], then we apply the analytic methods developed in [20] and [22] to complete the proof of Theorem 0.1.

This section is organized as follows. In Section 1.1, we recall the definition certain transversal index in the sense of Atiyah [1] and Paradan [17] for group actions on manifolds with boundary. In Section 1.2 we consider certain APS type index in the same framework as in Section 1.1. In Section 1.3 we prove an invertibility result for some induced boundary operator. In Section 1.4 we introduce the idea of spectral flow in our context and use it to complete the proof of a result stated in Section 1.2. In Section 1.5, we give an APS index interpretation of the transversal index in Section 1.1 by applying a result of Braverman [5]. In Section 1.6, we prove Theorem 0.1 by applying the analytic method developed in [20] and [22].

1.1. Transversal index. Let M be an even dimensional compact oriented Spin^c-manifold with non-empty boundary ∂M. Let E be a complex vector bundle over M.

Let G be a compact connected Lie group with Lie algebra denoted by $\mathfrak{g}$. We assume that G acts on the left on M and that this action can be lifted to an action of G on E.

Let the tangent vector bundle $\pi : TM \to M$ carry a G-invariant metric g^{TM}. Then one identifies TM and T^*M via g^{TM}.

For $K \in \mathfrak{g}$, we denote by $K^M_x = \frac{\partial}{\partial t} e^{tK} x|_{t=0}$ the corresponding vector field on M.

Following [1, p. 7] (cf. [17, §3]), set

$$T_G M = \left\{ v \in TM : \langle v(x), K^M(x)\rangle = 0 \text{ for all } K \in \mathfrak{g} \right\}. \tag{1.1}$$

Let $S(TM) = S_+(TM) \oplus S_-(TM)$ be the bundle of spinors associated to the spin^c-structure on TM and g^{TM}. For any $V \in TM$, the Clifford action $c(V)$ exchanges $S_\pm(TM)$, the $\mathbb{Z}_2$-grading on $S(TM)$.

Let $\mu : M \to \mathfrak{g}$ be a G-equivariant map. Let μ^M denote the vector field over M such that $\mu^M(x)$ equals to the value at x of the vector field generated by $\mu(x) \in \mathfrak{g}$ over M.

We make the assumption that μ^M is nowhere zero on ∂M.

Let $\sigma^M_{E,\mu} \in \mathrm{Hom}\left(\pi^*(S_+(TM) \otimes E), \pi^*(S_-(TM) \otimes E)\right)$ denote the symbol defined by

$$\sigma^M_{E,\mu}(x, v(x)) = \sqrt{-1}\, c(v + \mu^M) \otimes \mathrm{Id}_{\pi^* E}\big|_{(x,v(x))} \quad \text{for } (x, v(x)) \in T_x M. \tag{1.2}$$

Since μ^M is nowhere zero on ∂M, by (1.1) and (1.2) one sees that the zero set of the restriction of $\sigma^M_{E,\mu}$ on $T_G M$ is contained in a compact subset of $\mathring{M} \cap T_G \mathring{M}$ (where $\mathring{M} = M \setminus \partial M$ is the interior of M). Thus, it defines a G-transversally elliptic symbol on $T_G \mathring{M}$ in the sense of Atiyah [1, §1, §3] and Paradan [17, §3] (see also Vergne [23]), which in turn determines a transversal index[4]

$$\mathrm{Ind}\left(\sigma^M_{E,\mu}\right) = \sum_{\gamma \in \widehat{G}} \mathrm{Ind}_\gamma\left(\sigma^M_{E,\mu}\right) \cdot V^G_\gamma, \tag{1.3}$$

with each $\mathrm{Ind}_\gamma(\sigma^M_{E,\mu}) \in \mathbb{Z}$.

1.2. The Atiyah–Patodi–Singer (APS) index. We continue the setting in Section 1.1.

Let h^E be a G-invariant Hermitian metric on E, ∇^E a G-invariant Hermitian connection with respect to h^E. Let $h^{S(TM)\otimes E}$ be the metric on $S(TM) \otimes E$ induced by the metrics on $S(TM)$ and on E.

Let $\nabla^{S(TM)}$ be the Clifford connection on $S(TM)$ induced by the Levi-Civita connection ∇^{TM} of g^{TM} and a Hermitian connection on the line bundle defining the spin^c structure. Let $\nabla^{S(TM)\otimes E}$ be the Hermitian connection on $S(TM) \otimes E$ obtained by the tensor product of the connections $\nabla^{S(TM)}$ and ∇^E.

Let dv_M be the Riemannian volume form on (M, g^{TM}). For $s \in \mathscr{C}^\infty(M, S(TM) \otimes E)$, its L_2-norm $\|s\|_0$ is defined by

$$\|s\|_0^2 = \int_M |s|^2_{S(TM)\otimes E}(x)\, dv_M(x). \tag{1.4}$$

Let $\langle \ ,\ \rangle$ be the inner product on $\mathscr{C}^\infty(M, S(TM) \otimes E)$ corresponding to $\|\ \|_0^2$.

One can construct now the Spin^c-Dirac operator (twisted by E) (cf. [11, Appendix D]) by

$$D^E = \sum_{i=1}^{\dim M} c(e_i) \nabla^{S(TM)\otimes E}_{e_i} : \mathscr{C}^\infty(M, S(TM) \otimes E) \to \mathscr{C}^\infty(M, S(TM) \otimes E), \tag{1.5}$$

where $\{e_i\}$ is an orthonormal basis of TM. Then D^E is G-equivariant and formally self-adjoint. Denote by $D^E_\pm$ the restrictions of D^E on $\mathscr{C}^\infty(M, S_\pm(TM) \otimes E)$ respectively.

[4] We will not exploit the distribution nature of this index here.

Following [8, p. 139], let $\varepsilon > 0$ be less than the injectivity radius of g^{TM}. We use the inward geodesic flow to identify a neighborhood of the boundary ∂M with the collar $\partial M \times [0,\varepsilon]$. Let $e_{\dim M}$ be the inward unit normal vector field perpendicular to ∂M. Let $e_1, \cdots, e_{\dim M-1}$ be an oriented orthonormal basis of $T\partial M$ so that $e_1, \cdots, e_{\dim M-1}, e_{\dim M}$ is an oriented orthonormal basis of $TM|_{\partial M}$. By using parallel transport with respect to ∇^{TM} along the unit speed geodesics perpendicular to ∂M, $e_1, \cdots, e_{\dim M}$ forms an oriented orthonormal basis of TM over $\partial M \times [0,\varepsilon]$.

Let $D^E_{\partial M} : \mathscr{C}^\infty(\partial M, (S(TM)\otimes E)|_{\partial M}) \to \mathscr{C}^\infty(\partial M, (S(TM)\otimes E)|_{\partial M})$ be the induced (by D^E) Dirac operator on ∂M defined by (cf. [8, p. 142])

$$D^E_{\partial M} = -\sum_{i=1}^{\dim M-1} c(e_{\dim M})c(e_i)\nabla^{S(TM)\otimes E}_{e_i} + \frac{1}{2}\sum_{i=1}^{\dim M-1}\pi_{ii}, \tag{1.6}$$

where

$$\pi_{ij} = \left\langle \nabla^{TM}_{e_i} e_j, e_{\dim M}\right\rangle\Big|_{\partial M}, \quad 1 \leqslant i,j \leqslant \dim M - 1, \tag{1.7}$$

is the second fundamental form of the isometric embedding $\iota_{\partial M} : \partial M \hookrightarrow M$. Let $D^E_{\partial M,+}$ be the restriction of $D^E_{\partial M}$ to $\mathscr{C}^\infty(\partial M, (S_+(TM)\otimes E)|_{\partial M})$.

As in (1.4), we define the inner product $\langle\cdot,\cdot\rangle_{\partial M,0}$ and the L_2-norm $\|\cdot\|_{\partial M,0}$ on $\mathscr{C}^\infty(\partial M, (S(TM)\otimes E)|_{\partial M})$.

By [8, Lemma 2.2], $D^E_{\partial M}$ is a formally self-adjoint first order elliptic operator intrinsically defined on ∂M [5].

On the other hand, since G acts on M and thus preserves ∂M, one has

$$\mathcal{X}^M|_{\partial M} \in \mathscr{C}^\infty(\partial M, T\partial M). \tag{1.12}$$

[5] In fact, one has for any $s \in \mathscr{C}^\infty(M, S(TM)\otimes E)$, the following identities hold on ∂M,

$$\begin{aligned} D^E s &= c(e_{\dim M})\nabla^{S(TM)\otimes E}_{e_{\dim M}} s + c(e_{\dim M})\left(D^E_{\partial M} - \frac{1}{2}\sum_{i=1}^{\dim M-1}\pi_{ii}\right)(s|_{\partial M}); \\ c(e_{\dim M})D^E_{\partial M}(s|_{\partial M}) &= -D^E_{\partial M}\, c(e_{\dim M})(s|_{\partial M}). \end{aligned} \tag{1.8}$$

In particular, let $\nabla^{T\partial M}$ be the Levi-Civita connection on $(T\partial M, g^{T\partial M})$, with $g^{T\partial M}$ the metric on $T\partial M$ induced by g^{TM}, we define $\tilde{c}(X) = -c(e_{\dim M})c(X)$, for any $X \in T\partial M$, then $\tilde{c}(\cdot)$ induces a Clifford action of the Clifford algebra $C(T\partial M)$ on $(S(TM)\otimes E)|_{\partial M}$, and the Hermitian connection

$$\nabla^{(S(TM)\otimes E)|_{\partial M}}_X = \nabla^{S(TM)\otimes E}_X + \frac{1}{2}\sum_{i,j=1}^{\dim M-1}\pi_{ij}\, c(e_{\dim M})c(e_j)\langle X, e_i\rangle, \tag{1.9}$$

is the $\tilde{c}(\cdot)$-Clifford connection on $(S(TM)\otimes E)|_{\partial M}$, verifying for example that for any $X, Y \in \mathscr{C}^\infty(\partial M, T\partial M)$,

$$\left[\nabla^{(S(TM)\otimes E)|_{\partial M}}_X, \tilde{c}(Y)\right] = \tilde{c}\left(\nabla^{T\partial M}_X Y\right). \tag{1.10}$$

Thus $D^E_{\partial M}$ is the Dirac operator on $(S(TM)\otimes E)|_{\partial M}$ associated to $\nabla^{(S(TM)\otimes E)|_{\partial M}}$, i.e.

$$D^E_{\partial M} = \sum_{j=1}^{\dim M-1}\tilde{c}(e_j)\nabla^{(S(TM)\otimes E)|_{\partial M}}_{e_j}. \tag{1.11}$$

From (1.11), we understand that $D^E_{\partial M}$ is formally self-adjoint and intrinsically defined on ∂M.

By (1.12) and following [8, Lemma 2.2] and [22, §1c)] (see also [5, §2.8]), set for $T \in \mathbb{R}$,

(1.13)
$$D^E_T = D^E + \frac{\sqrt{-1}T}{2} c(V^M) : \mathscr{C}^\infty(M, S(TM)\otimes E) \to \mathscr{C}^\infty(M, S(TM)\otimes E);$$
$$D^E_{\pm,T} = D^E_{\pm} + \frac{\sqrt{-1}T}{2} c(V^M) : \mathscr{C}^\infty(M, S_\pm(TM)\otimes E) \to \mathscr{C}^\infty(M, S_\mp(TM)\otimes E);$$

and

(1.14)
$$\begin{aligned} D^E_{\partial M,T} = D^E_{\partial M} - \frac{\sqrt{-1}T}{2} c(e_{\dim M})\, c(V^M) &: \mathscr{C}^\infty(\partial M, (S(TM)\otimes E)|_{\partial M}) \to \mathscr{C}^\infty(\partial M, (S(TM)\otimes E)|_{\partial M});\\ D^E_{\partial M,\pm,T} = D^E_{\partial M,T}|_{\mathscr{C}^\infty(\partial M,(S_\pm(TM)\otimes E)|_{\partial M})}&. \end{aligned}$$

Then $D^E_{\partial M,T}$ is also formally self-adjoint and preserves $\mathscr{C}^\infty(\partial M, (S_\pm(TM)\otimes E)|_{\partial M})$. For any $\lambda \in \mathrm{Spec}\{D^E_{\partial M,\pm,T}\}$, the spectrum of $D^E_{\partial M,\pm,T}$, let $E_{\lambda,\pm,T}$ be the corresponding eigenspace. Let $P_{\geq 0,\pm,T}$ (resp. $P_{>0,\pm,T}$) be the orthogonal projection from the L^2-completions of $\mathscr{C}^\infty(\partial M, (S_\pm(TM)\otimes E)|_{\partial M})$ onto $\oplus_{\lambda\geq 0} E_{\lambda,\pm,T}$ (resp. $\oplus_{\lambda>0} E_{\lambda,\pm,T}$).

For any $T \in \mathbb{R}$, let $(D^E_{+,T}, P_{\geq 0,+,T})$ (resp. $(D^E_{-,T}, P_{>0,-,T})$) denote the corresponding Atiyah-Patodi-Singer boundary value problem [2]. More precisely, the boundary condition of $D^E_{+,T}$ is $P_{\geq 0,+,T}(s|_{\partial M}) = 0$ for $s \in \mathscr{C}^\infty(M, S_+(TM)\otimes E)$ (resp. of $D^E_{-,T}$ is $P_{>0,-,T}(s|_{\partial M}) = 0$ for $s \in \mathscr{C}^\infty(M, S_-(TM)\otimes E)$).

Both $(D^E_{+,T}, P_{\geq 0,+,T})$ and $(D^E_{-,T}, P_{>0,-,T})$ are elliptic, and $(D^E_{-,T}, P_{>0,-,T})$ is the adjoint of $(D^E_{+,T}, P_{\geq 0,+,T})$ (cf. (1.8), [8, Theorem 2.3]). Moreover, it is clear that both $(D^E_{+,T}, P_{\geq 0,+,T})$ and $(D^E_{-,T}, P_{>0,-,T})$ are G-equivariant.

Let $Q^M_{APS,T}(E,\gamma) \in \mathbb{Z}$, $\gamma \in \widehat{G}$, be defined by

(1.15)
$$\begin{aligned} \bigoplus_{\gamma\in\widehat{G}} Q^M_{APS,T}(E,\gamma)\, V_\gamma &= \mathrm{Ind}\left(D^E_{+,T}, P_{\geq 0,+,T}\right)\\ &:= \mathrm{Ker}\left(D^E_{+,T}, P_{\geq 0,+,T}\right) - \mathrm{Ker}\left(D^E_{-,T}, P_{>0,-,T}\right). \end{aligned}$$

The following important result will be proved in the next two subsections.

Proposition 1.1. *For any $\gamma \in \widehat{G}$, there exists $T_\gamma > 0$ such that $Q^M_{APS,T}(E,\gamma)$ does not depend on $T > T_\gamma$.*

1.3. An estimate on the boundary.

The proof of Proposition 1.1 consists two steps. In the first step, we show that the boundary operator $D^E_{\partial M,T}$, when restricted to the γ-component, is invertible when $T > 0$ is large. Then in the second step, we apply the spectral flow idea to complete the proof of Proposition 1.1.

In this subsection, we carry out the first step.

For any $\gamma \in \widehat{G}$, let $\mathscr{C}^\infty(\partial M, (S(TM)\otimes E)|_{\partial M})^\gamma$ denote the γ-component of $\mathscr{C}^\infty(\partial M, (S(TM)\otimes E)|_{\partial M})$. We denote the restriction of $D^E_{\partial M,T}$ on $\mathscr{C}^\infty(\partial M, (S(TM)\otimes E)|_{\partial M})^\gamma$ by $D^E_{\partial M,T}(\gamma)$.[6]

Proposition 1.2. *For any $\gamma \in \widehat{G}$, there exists $T_\gamma > 0$ such that for any $T > T_\gamma$, $D^E_{\partial M,T}(\gamma)$ is invertible.*

[6] Since $D^E_{\partial M,T}$ is G-equivariant, $D^E_{\partial M,T}(\gamma)$ is well-defined.

Proof. Let $\mathfrak{g}$ (thus $\mathfrak{g}^*$) be equipped with an Ad_G-invariant metric. Let $h_1, \cdots, h_{\dim G}$ be an orthonormal basis of $\mathfrak{g}^*$ with its dual basis V_i of $\mathfrak{g}$. Then one has

$$\mu(x) = \sum_{i=1}^{\dim M} \mu_i(x) V_i, \tag{1.16}$$

where μ_i, $1 \leq i \leq \dim G$, are smooth functions on M. From (1.16), one gets

$$\mathcal{H}^M(x) = \sum_{i=1}^{\dim M} \mu_i(x) V_i^M(x). \tag{1.17}$$

Recall that from (1.12), $\mathcal{H}^M|_{\partial M} \in \mathscr{C}^\infty(\partial M, T\partial M)$. From (1.6), (1.14) and (1.17), one finds,

$$\left(D^E_{\partial M, T}\right)^2 = \left(D^E_{\partial M}\right)^2 - \frac{\sqrt{-1}T}{2}\sum_{i=1}^{\dim M - 1} c(e_i) c(e_{\dim M}) c\left(\nabla_{e_i}\mathcal{H}^M\right) \tag{1.18}$$
$$+ \frac{\sqrt{-1}T}{2}\sum_{i=1}^{\dim M - 1} c(e_{\dim M}) c(e_i) \left[\nabla^{S(TM)\otimes E}_{e_i}, c(e_{\dim M}) c\left(\mathcal{H}^M\right)\right]$$
$$- \sqrt{-1}T\nabla^{S(TM)\otimes E}_{\mathcal{H}^M} + \frac{T^2}{4}\left|\mathcal{H}^M\right|^2.$$

For any $1 \leq i \leq \dim G$, let L_{V_i} denote the Lie derivative of V_i acting on $\mathscr{C}^\infty(M, S(TM)\otimes E)$ and thus also on $\mathscr{C}^\infty(\partial M, (S(TM)\otimes E)|_{\partial M})$. Then

$$\nabla^{S(TM)\otimes E}_{V_i^M} - L_{V_i} \in \mathscr{C}^\infty(M, \mathrm{End}(S(TM)\otimes E))$$

is a bounded operator.

By (1.17), we write

$$\nabla^{S(TM)\otimes E}_{\mathcal{H}^M} = \sum_{i=1}^{\dim G} \mu_i L_{V_i} + \sum_{i=1}^{\dim G} \mu_i \left(\nabla^{S(TM)\otimes E}_{V_i^M} - L_{V_i}\right). \tag{1.19}$$

It is clear that when restricted on $\mathscr{C}^\infty(\partial M, (S(TM)\otimes E)|_{\partial M})^\gamma$, each L_{V_i} is bounded. On the other hand, since $\mathcal{H}^M$ is nowhere zero on ∂M, there exists $C > 0$ such that

$$\left|\mathcal{H}^M\right|^2 \geq 4C \quad \text{on } \partial M. \tag{1.20}$$

From (1.18)–(1.20), there exists a positive constant $C' > 0$ such that for any $s \in \mathscr{C}^\infty(\partial M, (S(TM)\otimes E)|_{\partial M})^\gamma$, one has

$$\left\|D^E_{\partial M, T} s\right\|^2_{\partial M, 0} \geq \left\|D^E_{\partial M} s\right\|^2_{\partial M, 0} - TC'\|s\|^2_{\partial M, 0} + T^2 C\|s\|^2_{\partial M, 0}. \tag{1.21}$$

From (1.21), one sees easily that Proposition 1.2 holds for $T \geq 2C'/C$.

1.4. Spectral flow and a proof of Proposition 1.1. It is an easy matter to extend the idea of spectral flow ([3, §7]) to the current γ-component situation. Recall that such an extension to the G-invariant case has already been considered in [22, §4a)].

Let $\{D_t; 0 \leq t \leq 1\}$ be a one parameter smooth family of self-adjoint G-equivariant Dirac type operators acting on $\mathscr{C}^\infty(\partial M, (S_+(TM)\otimes E)|_{\partial M})$. We define the γ-component spectral flow of $\{D_t; 0 \leq t \leq 1\}$, denoted by $\mathrm{sf}_\gamma\{D_t; 0 \leq t \leq 1\}$, to be the spectral flow of the family of self-adjoint Fredholm operators $\{D_t(\gamma); 0 \leq t \leq 1\}$ in the sense of

Atiyah-Patodi-Singer [3, §7], where for any $t \in [0,1]$, $D_t(\gamma)$ is the restriction of D_t on $\mathscr{C}^\infty(\partial M, (S_+(TM)\otimes E)|_{\partial M})^\gamma$.

In view of (1.15), one can then proceed as in [22, Theorem 4.2] (cf. also [6, Theorem 1.1]) to get that for any $T > T' \geq T_0$,

$$Q^M_{APS,T}(E,\gamma) - Q^M_{APS,T'}(E,\gamma) = -\operatorname{Sf}\left\{ D^E_{\partial M,+,t}, T' \leqslant t \leqslant T \right\}. \tag{1.22}$$

By (1.22) and Proposition 1.2, we get Proposition 1.1.

Remark 1.3. Indeed, since the induced boundary operators are invertible for $T > T_0$, the APS boundary problems involved form a continuous family of Fredholm operators, which implies the constancy of the associated index. By this one avoids the consideration of spectral flow in the proof of Proposition 1.1.

1.5. Transversal index and the APS index. Denote by $Q^M_{APS}(E,\gamma)$ the quantization number $Q^M_{APS,T}(E,\gamma)$ for $T > T_0$ appearing in Proposition 1.1.

The following main result of this subsection identifies the transversal index in Section 1.1 with the above APS type index.

Theorem 1.4. *The following identity holds for $\gamma \in \widehat{G}$,*

$$\operatorname{Ind}\left(\sigma^M_{E,\gamma}\right) = Q^M_{APS}(E,\gamma). \tag{1.23}$$

Proof. We can enlarge the manifold M a little bit to get a compact Spin^c-manifold U with boundary such that $M \subset U \setminus \partial U = \widehat{U}$, the interior of U, and that everything extends in a G-invariant way to U from M, moreover, the corresponding vector field X^U is nowhere zero on $U \setminus \mathring{M}$, with $\mathring{M}$ the interior part of M. The existence of U is clear.

Then by proceeding as in [5, §14], one can construct a complete metric $g^{T\widehat{U}}$ on $\widehat{U}$ such that $g^{T\widehat{U}}|_{TM} = g^{TM}$, as well as an admissible (G-invariant) function f on $\widehat{U}$ in the sense of [5, Definition 2.6] such that $f|_M \equiv 1$, and $f > \frac{1}{2}$ on $\widehat{U} \setminus \mathring{M}$, $f \to +1$ near ∂U.

Let

$$D^E_{\widehat{U},T} = D^E + \frac{\sqrt{-1}T}{2} f c(X^U) \tag{1.24}$$

be the corresponding (G-equivariant) Dirac type operator on $\widehat{U}$. Then it is shown in [5, §14] that when restricted to a fixed γ-component, for any $T > 0$, the restricted operator $D^E_{\widehat{U},T}(\gamma)$ is a Fredholm operator, moreover, one has

$$\operatorname{Ind}\left(D^E_{\widehat{U},+,T}(\gamma)\right) = \operatorname{Ind}\left(\sigma^U_{E,\gamma}\right), \tag{1.25}$$

where $D^E_{\widehat{U},+,T}(\gamma)$ is the restriction of $D^E_{\widehat{U},T}(\gamma)$ on the γ-component of $\mathscr{C}^\infty(\widehat{U}, S_+(T\widehat{U})\otimes E)$ as in (1.13).

As X^U is nowhere zero on $U \setminus \mathring{M}$, by the excision formula of the transversal index [1, §3],

$$\operatorname{Ind}\left(\sigma^M_{E,\gamma}\right) = \operatorname{Ind}\left(\sigma^U_{E,\gamma}\right). \tag{1.26}$$

Now, the boundary ∂M cuts $\widehat{U}$ into two manifolds with boundary ∂M: M and $\widehat{U} \setminus \mathring{M}$. Let $D^E_{\widehat{U}\setminus\mathring{M},+,T,APS}$ (resp. $D^E_{M,+,T,APS}$) be the corresponding APS type operators (i.e. the

APS type boundary value problems) on $\widehat{U}\setminus\widetilde{M}$ (resp. M) respectively as in Section 1.2. Let $D^{E}_{\widehat{U}\setminus\widetilde{M},+,T,APS}(\ )$ (resp. $D^{E}_{M,+,T,APS}(\ )$) be the restriction onto the corresponding -components.

By proceeding as in [7, §3], which applies to the Fredholm operators here (one may deform the metrics and operators involved in a G-invariant way to the situation of product structure near ∂M, which does not alter the restriction of $D^{E}_{\widehat{U},T}$ on ∂M (thus it does not alter the indices considered), if necessary), one deduces that if $T > T$ so that Proposition 1.2 holds, then one has the following splitting formula, in which each index does not depend on $T > T$,

$$\text{(1.27)} \qquad \operatorname{Ind} D^{E}_{\widehat{U},+,T}(\ ) = \operatorname{Ind} D^{E}_{\widehat{U}\setminus\widetilde{M},+,T,APS}(\ ) + \operatorname{Ind} D^{E}_{M,+,T,APS}(\ ):$$

Lemma 1.5. The following identity holds for $T > T$,

$$\text{(1.28)} \qquad \operatorname{Ind} D^{E}_{\widehat{U}\setminus\widetilde{M},+,T,APS}(\ ) = 0:$$

Proof. Let U' be a G-invariant open subset of $U\setminus M$ such that $U\setminus U'$ is compact in $\widehat{U}\setminus M$. Let U'' be a G-invariant open subset of $(U\setminus M)\cup\partial M$ such that $U'\cup U'' = U\setminus\widetilde{M}$ and $\overline{U''}$ is compact in $\widehat{U}\setminus\widetilde{M}$. Let f_1, f_2 be two G-invariant functions on U with $\operatorname{supp}(f_1)\subset U'$, $\operatorname{supp}(f_2)\subset U''$ such that $f_1; f_2$ forms a partition of unity associated with the open covering $U'; U''$ of $U\setminus\widetilde{M}$. The existence of $f_1; f_2$ is clear.

As the vector field ^{U} is nowhere zero on $U\setminus\widetilde{M}$, and $f > 1=2$ on $U\setminus\widetilde{M}$, by proceeding as in [5, §9.3] and [22, §2] respectively, one knows that for any smooth section s lying in the -component of $\mathscr{C}^1(\widehat{U}\setminus\widetilde{M}; S(T\widehat{U})\otimes E|_{\widehat{U}\setminus\widetilde{M}})$ with compact support and verifying the APS boundary condition on ∂M as in Section 1.2, one has

$$\text{(1.29)} \qquad \left\| D^{E}_{\widehat{U},T}(f_1 s)\right\|_0^2 > C\left\| D^{E}_{\widehat{U}}(f_1 s)\right\|_0^2 + (T - b)\|f_1 s\|_0^2 \ ;$$

$$\text{(1.30)} \qquad \left\| D^{E}_{\widehat{U},T}(f_2 s)\right\|_0^2 > C'\left\| D^{E}_{\widehat{U}}(f_2 s)\right\|_0^2 + (T - b')\|f_2 s\|_0^2$$

respectively, for some positive constants $C; C'; b'$, and b.

From (1.29) and (1.30) and by taking $T > T$ large enough and proceeding the gluing arguments as in [4, pp. 115–116], one gets Lemma 1.5. □

On the other hand, since $f|_M \equiv 1$, one sees directly, in view of Proposition 1.2, that when $T > T$,

$$\text{(1.31)} \qquad \operatorname{Ind} D^{E}_{M,+,T,APS}(\ ) = Q^{M}_{APS}(E;\ ):$$

From (1.25)–(1.31), we get (1.23). □

Remark 1.6. Theorem 1.4 allows us to use analytic method to deal with the transversal index problems. This is the viewpoint we adopt in this paper.

1.6. A proof of Theorem 0.1.

We now apply Theorem 1.4 to the setting considered in Introduction. We have the canonical splitting $TM\otimes_{\mathbb{R}}\mathbb{C} = T^{(1,0)}M\oplus T^{(0,1)}M$, for the complexification of TM, with

$$\begin{aligned} T^{(1,0)}M &= \{u \in TM \otimes_{\mathbb{R}} \mathbb{C}; Ju = \sqrt{-1}\,u\}, \\ T^{(0,1)}M &= \{u \in TM \otimes_{\mathbb{R}} \mathbb{C}; Ju = -\sqrt{-1}\,u\}. \end{aligned} \tag{1.32}$$

Let $T^{*(0,1)}M$ be the dual of $T^{(0,1)}M$. For any α a 1-form on M, we denote by $\alpha^* \in TM$ its metric dual.

The almost complex structure J on M determines a canonical Spin^c-structure on TM with the associated Hermitian line bundle $\det(T^{(1,0)}M)$ and we have

$$S(TM) = \Lambda T^{*(0,1)}M; \quad S^{\pm}(TM) = \Lambda^{\substack{\text{even}\\ \text{odd}}} T^{*(0,1)}M. \tag{1.33}$$

For any $W \in TM$, we write its complexification as $W = w + \overline{w} \in T^{(1,0)}M \oplus T^{(0,1)}M$, let $\overline{w}^* \in T^{*(0,1)}M$ be the metric dual of w (cf. [4, §5]). Then

$$c(W) = \sqrt{2}(\overline{w}^* \wedge - i_w) \tag{1.34}$$

defines the canonical Clifford action of W on $\Lambda(T^{*(0,1)}M)$. It exchanges $\Lambda^{\text{even}}(T^{*(0,1)}M)$ and $\Lambda^{\text{odd}}(T^{*(0,1)}M)$.

The Levi-Civita connection ∇^{TM} together with the almost complex structure J induces via projection a canonical Hermitian connection $\nabla^{T^{(1,0)}M}$ on $T^{(1,0)}M$. This induces a Hermitian connection $\nabla^{\det}$ on $\det(T^{(1,0)}M)$. The Clifford connection $\nabla^{\Lambda(T^{*(0,1)}M)}$ on $\Lambda T^{*(0,1)}M$ is induced by the Levi-Civita connection ∇^{TM} and the connection $\nabla^{\det}$ (cf. [20, §1a)], [12, §1.3], [11, Appendix D]).

We take $E = L$ and denote $\Omega^{0,\bullet}(M;L) = \mathscr{C}^{\infty}(M, \Lambda(T^{*(0,1)}M) \otimes L)$. We denote by $L_2(\Omega^{0,\bullet}(M;L))$ the L_2-completion of the elements in $\Omega^{0,\bullet}(M;L)$ with compact support.

Recall that we assume that the moment map $\mu : M \to \mathfrak{g}^*$ is proper. For a regular value $c > 0$ of $\mathcal{H} = |\mu|^2$, denote by M_c the G-invariant manifold with boundary

$$M_c = \{x \in M : \mathcal{H}(x) \leqslant c\}. \tag{1.35}$$

Also recall that $X^{\mathcal{H}}$ is the Hamiltonian vector field of $\mathcal{H}$, i.e. $i_{X^{\mathcal{H}}}\omega = d\mathcal{H}$. As in (1.16), we can write

$$\mu = \sum_{i=1}^{\dim G} \mu_i h_i, \tag{1.36}$$

under the identification of $\mathfrak{g}$ with $\mathfrak{g}^*$, as

$$\mu = \sum_{i=1}^{\dim G} \mu_i V_i, \tag{1.37}$$

and from (0.4), one has (cf. [20, (1.19)]),

$$X^{\mathcal{H}} = -J(d\mathcal{H}) = -2J \sum_{i=1}^{\dim G} \mu_i (d\mu_i) = 2 \sum_{i=1}^{\dim G} \mu_i V_i^M = 2\mu^M. \tag{1.38}$$

Since $c > 0$ is a regular value of $\mathcal{H}$, by (1.38) one knows that $X^{\mathcal{H}} = 2\mu^M$ is nowhere zero on $\partial M_c = \mathcal{H}^{-1}(c)$.

We can now restate Theorem 0.1 as follows.

Theorem 1.7. *For any $\gamma \in \widehat{G}$, there exists $c_\gamma > 0$ such that for any regular values c', c of $\mathcal{H}$ with $c', c > c_\gamma$, the following identity for transversal indices holds,*

$$\mathrm{Ind}\left(\sigma^{M_c}_{L,\gamma}\right) = \mathrm{Ind}\left(\sigma^{M_{c'}}_{L,\gamma}\right). \tag{1.39}$$

If $\gamma = 0$, we can take $c_\gamma = 0$.

Proof. Let $c' > c > 0$ be two regular values of $\mathcal{H}$. Let $M_{c,c'}$ denote the manifold with boundary $M_{c,c'} = \overline{M_{c'} \setminus M_c}$.

By the additivity of the transversal index (cf. [1, Theorem 3.7, §6] and [17, Prop. 4.1]), one has

$$\mathrm{Ind}\left(\sigma^{M_{c'}}_{L,\gamma}\right) - \mathrm{Ind}\left(\sigma^{M_c}_{L,\gamma}\right) = \mathrm{Ind}\left(\sigma^{M_{c,c'}}_{L,\gamma}\right). \tag{1.40}$$

Thus (1.39) is equivalent to say that when $c > 0$ is large enough, one has $\mathrm{Ind}\left(\sigma^{M_{c,c'}}_{L,\gamma}\right) = 0$; which by Theorem 1.4 is equivalent to say that

$$Q^{M_{c,c'}}_{APS}(L,\gamma) = 0. \tag{1.41}$$

If $\gamma = 0$, by [22, Theorem 4.3], we get (1.41) for $c' > c > 0$. Thus we get (1.39) when $\gamma = 0$.

From (1.13), we see that in the current situation, one has

$$D^L_T = D^L + \frac{\sqrt{-1}T}{2} c\left(X^{\mathcal{H}}\right) : \Omega^{0,*}(M_{c,c'}, L) \to \Omega^{0,*}(M_{c,c'}, L). \tag{1.42}$$

Let $e_1, \dots, e_{\dim M}$ be an orthonormal basis of $TM_{c,c'}$. By [20, Theorem 1.6], one has the following Bochner type formula,

$$\begin{aligned}\left(D^L_T\right)^2 = \left(D^L\right)^2 &+ \frac{\sqrt{-1}T}{4}\sum_{j=1}^{\dim M} c(e_j)\, c\left(\nabla^{TM}_{e_j} X^{\mathcal{H}}\right) \\ &- \frac{\sqrt{-1}T}{2}\mathrm{Tr}\left[\nabla^{T^{(1,0)}M} X^{\mathcal{H}}\big|_{T^{(1,0)}M}\right] + \frac{T}{2}\sum_{i=1}^{\dim G}\left(\sqrt{-1}c\left(JV_i^M\right)c\left(V_i^M\right) + \left|V_i^M\right|^2\right) \\ &+ 4\pi T\mathcal{H} - 2\sqrt{-1}T\sum_{i=1}^{\dim G}\mu_i L_{V_i} + \frac{T^2}{4}\left|X^{\mathcal{H}}\right|^2.\end{aligned} \tag{1.43}$$

Let $\Omega^{0,*}_\gamma(M_{c,c'}, L)$ denote the γ-component of $\Omega^{0,*}(M_{c,c'}, L)$. Then each L_{V_i}, $1 \leq i \leq \dim G$, acts on $\Omega^{0,*}_\gamma(M_{c,c'}, L)$ as a linear bounded operator $A_\gamma(V_i)$ and we denote its operator norm by $\|A_\gamma(V_i)\|$. Set

$$F^L_T = \left(D^L_T\right)^2 + 2\sqrt{-1}T\sum_{i=1}^{\dim G}\mu_i L_{V_i} - 2\sqrt{-1}T\sum_{i=1}^{\dim G}\mu_i A_\gamma(V_i). \tag{1.44}$$

Lemma 1.8. *There exists $c_\gamma > 0$ such that for any $c > c_\gamma$, and any $s \in L_2(\Omega^{0,*}_\gamma(M, L))$, $\mathrm{supp}(s) \subset \overline{M \setminus M_c}$, one has*

$$\left\langle \sum_{i=1}^{\dim G}\mu_i A_\gamma(V_i) s, s \right\rangle \leq \langle \mathcal{H}s, s\rangle. \tag{1.45}$$

Proof. By (1.37), one verifies that

$$\left\langle \sum_{i=1}^{\dim G} \sqrt{-1}\,A^{\gamma}(V_i)\, s, s \right\rangle \leq \frac{1}{2}\sum_{i=1}^{\dim G}\left(\|\nabla_i s\|_0^2 + \|A^{\gamma}(V_i)\, s\|_0^2 \right)$$
$$(1.46) \qquad \leq \frac{1}{2}\langle H s, s\rangle + \frac{1}{2}\sum_{i=1}^{\dim G} \|A^{\gamma}(V_i)\|^2 \|s\|_0^2 .$$

From (1.46), one then verifies easily that Lemma 1.8 holds for $c_\gamma = \sum_{i=1}^{\dim G} \|A^{\gamma}(V_i)\|^2$. Note that c_γ is bigger than the absolute value of the eigenvalue of the Casimir operator of G acting on the irreducible G-representation with highest weight γ, and $c_\gamma = 0$ iff $\gamma = 0$.

By (1.43)–(1.45) and proceeding in exactly the same way as in the proof of [20, Proposition 2.2], one verifies that the following result holds.

Proposition 1.9. *Let $c' > c > c_\gamma$ be two regular values of H. Then for any $x \in M_{c,c'} \setminus (\partial M_c \cup \partial M_{c'})$, there exists an open neighborhood U_x of x in $M_{c,c'} \setminus (\partial M_c \cup \partial M_{c'})$ such that there exist $C_x > 0$, $b_x > 0$ such that for any $T \geq 1$ and any $s \in \Omega^{0,\bullet}(M, L)^{\gamma}$ with $\mathrm{supp}(s) \subset U_x$, one has*

$$(1.47) \qquad \mathrm{Re}\left\langle F_T^L s, s\right\rangle \geq C_x \left(\left\| D^L s\right\|_0^2 + (T - b_x)\|s\|_0^2 \right).$$

On the other hand, since X^H is nowhere zero on $\partial M_{c,c'}$, the estimates described in [22, Proposition 2.4] holds near $\partial M_{c,c'}$ for the γ-component here as well. By this and by Proposition 1.9, one can then proceed the gluing argument as in [22, Theorem 2.6], which goes back to [4, pp. 115–117], to the the similar estimates for the γ-component (instead of the G-invariant component there), to see that the similar estimates in [22, Theorem 2.6] still holds here on $M_{c,c'}$ for the γ-component.

To be more precise, by Proposition 1.2, one knows that there exists $T_\gamma > 0$ such that for any $T \geq T_\gamma$, $D^L_{\partial M_{c,c'},T}$ is invertible.

From (1.21) for $D^L_{\partial M_{c,c'},T}$ and (1.43), and by the same argument as in the proof of [22, Proposition 2.4], we get : there exists an open neighborhood U of $\partial M_{c,c'}$ and constants $T_0 > 0$, C, $C_1 > 0$ such that for any $T \geq T_0$ and $s \in \Omega^{0,\bullet}(M_{c,c'}, L)^{\gamma}$, with $\mathrm{supp}(s) \subset U$ and $P_{>0,\gamma,T}\, s|_{\partial M_{c,c'}} = 0$, the following inequality holds,

$$(1.48) \qquad \|D_T^L s\|_0^2 \geq C \left(\|D^L s\|_0^2 + (T - C_1)\|s\|_0^2 \right).$$

On the other hand, by (1.44) one knows that for any $s \in \Omega^{0,\bullet}(M_{c,c'}, L)^{\gamma}$, one has

$$(1.49) \qquad \left(D_T^L\right)^2 s = F_T^L s .$$

By these and by the above discussions one sees that there exist $C' > 0$, $b > 0$ such that for any $T \geq T_\gamma$ and any $s \in \Omega^{0,\bullet}(M_{c,c'}, L)^{\gamma}$ such that $P_{>0,\gamma,T}(s|_{\partial M_{c,c'}}) = 0$, one has

$$(1.50) \qquad \left\| D_T^L s\right\|_0^2 \geq C' \left(\left\| D^L s\right\|_0^2 + (T - b)\|s\|_0^2 \right).$$

By taking $T \geq T_\gamma$ sufficiently large in (1.50), one gets (1.41), which completes the proof of Theorem 1.7 and thus Theorem 0.1.

Remark 1.10. If the zero set of $X^{\mathcal{H}} = 2\mu^M$ is compact, then Theorem 0.1 is known in [17] and [25] already.

Remark 1.11. Our proof of Theorem 0.4 in the following two sections follows the similar line of the arguments as in the above proof of Theorem 0.1. However, we have to deal with more subtle deformations and estimates, some of which are highly non-trivial. See the next two sections for more details.

2. Quantization for proper moment maps: a proof of Theorem 0.4

In this section, we prove Theorem 0.4 modulo a vanishing result Theorem 2.4 which will be proved in Section 3.

This section is organized as follows: In Section 2.1, we give a reformulation of Theorem 0.4 as Theorem 2.3. In Section 2.2, we state a vanishing result Theorem 2.4 away from $\mu^{-1}(0)$. In Section 2.3, we give a proof of Theorem 2.3 by using Theorem 2.4.

2.1. A reformulation of Theorem 0.4.
For convenience, we recall the basic setting.

Let (M,ω), (N,ω^N) be two symplectic manifolds with symplectic forms ω, ω^N respectively. We assume that N is compact.

Let (L, h^L) be a Hermitian line bundle on M with Hermitian connection ∇^L, and (F, h^F) a Hermitian line bundle on N with Hermitian connection ∇^F. Let $R^L = (\nabla^L)^2$, $R^F = (\nabla^F)^2$ be the associated curvatures.

We suppose that

$$\omega = \frac{\sqrt{-1}}{2\pi} R^L, \quad \omega^N = \frac{\sqrt{-1}}{2\pi} R^F. \tag{2.1}$$

Let J^M, J^N be almost complex structures on TM, TN respectively such that $\omega(\cdot, J^M\cdot)$ defines a metric g^{TM} on TM, and $\omega^N(\cdot, J^N\cdot)$ defines a metric g^{TN} on TN.

Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$, and $\mathfrak{g}$ admits an Ad_G-invariant metric. Suppose that G acts (by left) on M, N and its actions on M, N lift on L and F respectively. Moreover, we assume that the G-action preserves the above metrics and connections on TM, TN, L, F and J^M, J^N.

For any $K \in \mathfrak{g}$, let $K^M \in \mathscr{C}^\infty(M, TM)$ denote the vector field generated by K on M. Recall that the moment map $\mu : M \to \mathfrak{g}^*$ has been defined in (0.3). Let $\mu' : N \to \mathfrak{g}^*$ be the moment map defined in the same way for (N, ω^N) and (F, h^F, ∇^F).

We will use the same notation for the natural extension of the objects on M, N to $M \times N$. In particular, $L \otimes F$ is the Hermitian line bundle on $M \times N$ induced by L and F with the Hermitian connection induced by ∇^L, ∇^F.

The G-action on $M \times N$ is defined by $g \cdot (x, y) = (gx, gy)$ for $(x, y) \in M \times N$. We define the symplectic form $\tilde\omega$ and the almost complex structure J on $M \times N$ by

$$\tilde\omega_{(x,y)} = \omega_x + \omega^N_y, \quad J = (J^M, J^N). \tag{2.2}$$

The induced moment map $\tilde\mu : M \times N \to \mathfrak{g}^*$ is given by

$$\tilde\mu(x, y) = \mu(x) + \mu'(y). \tag{2.3}$$

Recall that we have assumed that the moment map $\mu : M \to \mathfrak{g}^*$ is proper. Since N is compact, one sees that the moment map $\tilde\mu$ is also proper.

Recall that for any $\gamma \in \Lambda^*_+$, the index $Q(L\otimes F)^{\gamma}$ has been defined by applying Theorem 0.1 on $M\times N$, while $Q((L\otimes F)_\gamma)$ is the quantization number defined in [16] which by [17, §7.4], can be defined as follows.

If γ is a regular value of the moment map μ, then one can construct the Marsden–Weinstein symplectic reduction $((M\times N)_\gamma, \omega_\gamma)$, where $(M\times N)_\gamma = \mu^{-1}(G\cdot\gamma)/G$ is a compact orbifold. Moreover, $L\otimes F$ (resp. J) induces a prequantized line bundle $(L\otimes F)_\gamma$ (resp. an almost complex structure J_γ) over $((M\times N)_\gamma, \omega_\gamma)$. One then constructs the associated Spin^c-Dirac operator (twisted by $(L\otimes F)_\gamma$) on $(M\times N)_\gamma$ whose index $Q((L\otimes F)_\gamma)$ as in (0.5) is well-defined.

If $\gamma\in\Lambda^*_+$ is not a regular value of μ, then take $a\in\mathfrak{g}^*$ sufficiently close to γ such that a is a regular value of μ, and by replacing γ by a, we get the index $Q((L\otimes F)_a)$. For a a regular value of μ and close enough to γ, $Q((L\otimes F)_a)$ does not depend on a and we denote it as $Q((L\otimes F)_\gamma)$.

The following result can be viewed as a quantization formula for the $\gamma=0$ (or G-invariant) component.

Theorem 2.1. *The following identity holds,*

$$Q(L\otimes F)^{\gamma=0} = Q((L\otimes F)_{\gamma=0}). \tag{2.4}$$

Proof. If M is compact, this is the Guillemin–Sternberg conjecture proved in [15] and [16].

In the general case, for any two regular values $c'>c>0$ of $|\mu|^2$, let $Q_{APS}^{(M\times N)_{c,c'}}(L\otimes F,\mu)^{\gamma=0}$ be the APS type index defined in Section 1.5 for the current situation, where $(M\times N)_{c,c'}=\{x\in M\times N : c\le|\mu(x)|^2\le c'\}$. Then by [22, Theorem 4.3], one finds

$$Q_{APS}^{(M\times N)_{c,c'}}(L\otimes F,\mu)^{\gamma=0}=0. \tag{2.5}$$

By Theorems 0.1, 1.4, (1.40) and (2.5), and by taking $c'>0$ large enough, one sees that for any regular value $c>0$ of $|\mu|^2$, one has

$$\mathrm{Ind}_{\gamma=0}\left(\sigma^{(M\times N)_c}_{L\otimes F,\mu}\right)=\mathrm{Ind}_{\gamma=0}\left(\sigma^{(M\times N)_{c'}}_{L\otimes F,\mu}\right)=Q(L\otimes F)^{\gamma=0}; \tag{2.6}$$

where $(M\times N)_c=\{(x,y)\in M\times N : |\mu|^2\le c\}$, and similarly for $(M\times N)_{c'}$.

If 0 is a regular value of μ, then from [22, Theorem 4.3], $Q_{APS}^{(M\times N)_c}(L\otimes F,\mu)^{\gamma=0}=Q((L\otimes F)_{\gamma=0})$. From Theorem 1.4, (2.6), we get (2.4).

On the other hand, if we take $c>0$ small enough so that there is no critical point of $|\mu|^2$ besides $\mu^{-1}(0)$, then by [17, Prop. 7.10] and [18] one knows that even if 0 is a singular value of μ,

$$\mathrm{Ind}_{\gamma=0}\left(\sigma^{(M\times N)_c}_{L\otimes F,\mu}\right)=Q((L\otimes F)_{\gamma=0}). \tag{2.7}$$

From (2.6) and (2.7), one gets (2.4) which completes the proof of Theorem 2.1.

Remark 2.2. See Remark 2.8 for an outline of an analytic proof of (2.7).

By Theorem 2.1, one can reformulate Theorem 0.4 as follows.

Theorem 2.3. The following identity holds,

$$Q(L\otimes F)^{\gamma=0} = \sum_{\gamma\in\widehat{G}} Q(L)^{\gamma}\, Q(F)^{\gamma^*} \; . \tag{2.8}$$

In the rest of this section, we will present a proof of Theorem 2.3 modulo a vanishing result away from $\mu^{-1}(0)$, which will be stated in the next subsection and proved in the next section.

2.2. A vanishing result away from $\mu^{-1}(0)$.

In this section, we state a vanishing result which can be thought of the first step in the two steps deformation proof of Theorem 2.3.

We use the Ad_G-invariant metric on $\mathfrak{g}$ to identify $\mathfrak{g}$ and $\mathfrak{g}^*$. Let V_i, $1 \leqslant i \leqslant \dim G$, be an orthonormal basis of $\mathfrak{g}$, then we can write the moment maps μ^M and μ^N as

$$\mu^M = \sum_{i=1}^{\dim G} \mu^M_i V_i; \qquad \mu^N = \sum_{i=1}^{\dim G} \mu^N_i V_i. \tag{2.9}$$

For any $1 \leqslant i \leqslant \dim G$, denote by $V_i^M, V_i^N, V_i^{M\times N}$ the Killing vector fields on $M, N, M\times N$ induced by V_i respectively. Then one verifies easily from (0.4) that

$$\begin{aligned} V_i^{M\times N} &= V_i^M + V_i^N; \\ d\mu^M_i &= J^M V_i^M; \quad d\mu^N_i = J^N V_i^N. \end{aligned} \tag{2.10}$$

Let $D^{L\otimes F}: \Omega^{0,\bullet}(M\times N, L\otimes F) \to \Omega^{0,\bullet}(M\times N, L\otimes F)$ be the Spin^c Dirac operator on $M\times N$ (cf. Section 1.6, (1.5)).

By Sard's theorem, the set of critical values of the functions $|\mu|^2$ and $|\mu^M|^2$ on $M\times N$ has measure zero in $\mathbb{R}$. Especially, for any $C > 0$, there exists $C' > C$ such that C' is a regular value for the functions $|\mu^M|^2$ and $\frac{1}{2}|\mu|^2$ on $M\times N$.

As N is a compact manifold, there exists $C_0 > 0$ such that $|\mu^N| \leqslant C_0$ on N.

For $A \geqslant 6C_0^2$ large enough which is a regular value of the functions $|\mu^M|^2$ and $\frac{1}{2}|\mu|^2$ on $M\times N$, we define

$$\begin{aligned} \mathcal{M} &= \{(x,y)\in M\times N; |\mu^M(x)|^2 \geqslant A; |\mu(x,y)|^2 \leqslant 2A\} \subset M\times N; \\ M_A &= \{x\in M; |\mu^M(x)|^2 \leqslant A\}; \quad \partial M_A = \{x\in M; |\mu^M(x)|^2 = A\}; \\ \mathcal{M}_1 &= \{(x,y)\in M\times N; |\mu^M(x)|^2 = A\}; \\ \mathcal{M}_2 &= \{(x,y)\in M\times N; |\mu(x,y)|^2 = 2A\}. \end{aligned} \tag{2.11}$$

By our choice of A, we know that $\mathcal{M}_1$, $\mathcal{M}_2$ are smooth sub-manifolds of $M\times N$. Moreover, as

$$|\mu|^2 = |\mu^M|^2 + |\mu^N|^2 + 2\langle \mu^M, \mu^N\rangle \tag{2.12}$$

and $A \geqslant 6C_0^2$, we know that $\mathcal{M}_1 \cap \mathcal{M}_2 = \emptyset$.[7] Thus $\mathcal{M}$ is a smooth manifold with boundary $\partial\mathcal{M}$ and

$$\partial\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2; \quad \mathcal{M}_1 = \partial M_A \times N. \tag{2.13}$$

[7] Indeed, by (2.12), one gets $|\mu^M| \geqslant |\mu| - C_0$. Thus, if $|\mu|^2 = 2A$, then $|\mu^M| \geqslant A^{1/2} + ((\sqrt{2}-1)A^{1/2} - C_0) > A^{1/2}$ when $A \geqslant 6C_0^2$.

We will introduce a deformation of the vector field $\sum_{j=1}^{\dim G} \theta_j (V_j^M + V_j^N)$ on M_1 to the vector field $\sum_{j=1}^{\dim G} \psi_j (V_j^M + V_j^N)$ on M_2, where $\theta_j = \mu_j + \eta_j$, $j = 1, \cdots, \dim G$, are the coordinates of θ associated to the basis V_j, $j = 1, \cdots, \dim G$, of $\mathfrak{g}$.

Let ψ_j, $j = 1, \cdots, \dim G$, be smooth functions on M such that there exists a strictly positive function $\varphi \in \mathscr{C}^\infty(\partial M)$ such that

$$\psi_j(x,y) = \begin{cases} \varphi(x,y)\,\theta_j(x,y) & \text{on } M_1; \\ \varphi(x,y)\,\mu_j(x,y) & \text{on } M_2. \end{cases} \tag{2.14}$$

Let X^ψ be the vector field on M defined by

$$X^\psi = \sum_{j=1}^{\dim G} \psi_j \left(V_j^M + V_j^N \right). \tag{2.15}$$

As $|\mu|^2$ and $|\theta|^2$ are G-invariant functions on $M \times N$, M_1 and M_2 are G-submanifolds with boundary of $M \times N$, in particular,

$$V_j^{M \times N}(x,y) \in T\partial M, \quad X^\psi(x,y) \in T\partial M, \quad \text{for } (x,y) \in \partial M. \tag{2.16}$$

As A is a regular value of the functions $|\mu|^2$ and $\frac{1}{2}|\theta|^2$ on $M \times N$, we know that

$$X^\psi = \begin{cases} \varphi \sum_{j=1}^{\dim G} \theta_j \left(V_j^M + V_j^N \right) \neq 0 & \text{pointwise over } M_1; \\ \varphi \sum_{j=1}^{\dim G} \mu_j \left(V_j^M + V_j^N \right) \neq 0 & \text{pointwise over } M_2. \end{cases} \tag{2.17}$$

Thus X^ψ is nowhere zero on ∂M.

Clearly, X^ψ is induced on M by the G-equivariant map $\widetilde{X}^\psi : M \to \mathfrak{g}$ defined by

$$\widetilde{X}^\psi = \sum_{j=1}^{\dim G} \psi_j V_j. \tag{2.18}$$

We can now state the main result of this subsection as follows.

Theorem 2.4. When $A > 0$ is large enough, there exist functions ψ_j, $1 \leqslant j \leqslant \dim G$, verifying the above properties, such that the following identity for the APS type index holds,

$$Q^M_{APS}\left(L, F, \widetilde{X}^\psi\right)^{\gamma = 0} = 0. \tag{2.19}$$

The proof of Theorem 2.4 will be given in the next section.

2.3. A proof of Theorem 2.3. We continue the discussion in the previous subsection.

Denote by $\mathscr{M}_1 = M_A \times N = \{(x,y) \in M \times N : |\mu(x)|^2 \leqslant A\}$, $\mathscr{M}_2 = \{(x,y) \in M \times N : |\mu(x)|^2 \leqslant 2A\}$. Then $\mathscr{M}_1$, $\mathscr{M}_2$ are submanifolds with boundary of $M \times N$ such that $\partial \mathscr{M}_i = M_i$, $i = 1, 2$. Recall also $\mathscr{M}_2 = \mathscr{M}_1 \cup M$.

Let $\Psi : \mathscr{M}_2 \to \mathfrak{g}$ be a G-equivariant map such that the induced vector field $\Psi^{\mathscr{M}_2}$ on $\mathscr{M}_2$ verifies $\Psi^{\mathscr{M}_2}|_M = X^\psi$. The existence of Ψ is clear.

From (2.17), the positivity of φ on ∂M and the additivity of the transversal index (cf. [1, Theorem 3.7, §6] and [17, Prop. 4.1]), one has,

$$\mathrm{Ind}_{\gamma=0}\left(\sigma^{\mathscr{M}_2}_{L \otimes F, \Psi}\right) = \mathrm{Ind}_{\gamma=0}\left(\sigma^{M}_{L \otimes F, \widetilde{X}}\right) + \mathrm{Ind}_{\gamma=0}\left(\sigma^{\mathscr{M}_1}_{L \otimes F, \theta}\right). \tag{2.20}$$

From Theorems 1.4, 2.4 and (2.20), one gets

$$\operatorname{Ind}_{\tau=0}\left(\sigma^{\mathscr{M}_2}_{L\otimes F,\mu}\right) = \operatorname{Ind}_{\tau=0}\left(\sigma^{\mathscr{M}_1}_{L\otimes F,\mu}\right). \tag{2.21}$$

Lemma 2.5. *The following identities hold*

$$\begin{aligned} \operatorname{Ind}_{\tau=0}\left(\sigma^{\mathscr{M}_1}_{L\otimes F,\mu}\right) &= \operatorname{Ind}_{\tau=0}\left(\sigma^{\mathscr{M}_1}_{L\otimes F,\mu'}\right);\\ \operatorname{Ind}_{\tau=0}\left(\sigma^{\mathscr{M}_2}_{L\otimes F,\mu}\right) &= \operatorname{Ind}_{\tau=0}\left(\sigma^{\mathscr{M}_2}_{L\otimes F,\mu'}\right). \end{aligned} \tag{2.22}$$

Proof. For any $t\in[0,1]$, set $\mu_t = (1-t)\mu + t\mu'$, $\nu_t = (1-t)\nu + t\nu'$. Let $X^{\mathscr{M}_1}_t$, $X^{\mathscr{M}_2}_t$ be the induced vector fields on $\mathscr{M}_1$, $\mathscr{M}_2$ respectively. Then by (2.17) and the positivity of φ', one sees that $X^{\mathscr{M}_1}_t$ (resp. $X^{\mathscr{M}_2}_t$) is nowhere zero on $\partial\mathscr{M}_1 = M_{1}$ (resp. $\partial\mathscr{M}_2 = M_{2}$). Formula (2.22) then follows from the homotopy invariance of the transversal index (cf. [1, Theorems 2.6, 3.7] and [17, §3]).

Recall that by our assumption, the induced vector field X^M of $\mu: M\to\mathfrak{g}$ is nowhere zero on ∂M_A.

Proposition 2.6. *The following identity holds,*

$$\operatorname{Ind}_{\tau=0}\left(\sigma^{\mathscr{M}_1}_{L\otimes F,\mu}\right) = \sum_{\gamma\in\widehat{G}} \operatorname{Ind}\left(\sigma^{M_A}_{L,\mu}\right)\cdot Q(F). \tag{2.23}$$

Proof. Proposition 2.6 can be proved by using the homotopy invariance of the transversal index. Here, we will develop an analytic proof.

Recall that $X^{M\times N}$ denotes the vector field generated by μ on $M\times N$. One has

$$X^{M\times N} = \sum_{j=1}^{\dim G} \mu_j \left(V_j^M + V_j^N\right). \tag{2.24}$$

For any $t\in[0,1]$, we define the deformation

$$X^{\mathscr{M}_1}(t) = \sum_{j=1}^{\dim G} \mu_j \left(V_j^M + (1-t)V_j^N\right). \tag{2.25}$$

For any $t\in[0,1]$, $T>0$, we define the following Dirac type operators as in (1.13),

$$\begin{aligned} D_T^{L\otimes F}(t) &= D^{L\otimes F} + \frac{\sqrt{-1}T}{2}c\left(X^{\mathscr{M}_1}(t)\right)\\ &: \Omega^{0,\bullet}(\mathscr{M}_1, L\otimes F) \to \Omega^{0,\bullet}(\mathscr{M}_1, L\otimes F);\\ D_{\pm,T}^{L\otimes F}(t) &= D_T^{L\otimes F}(t)|_{\Omega^{0,\frac{\text{even}}{\text{odd}}}(\mathscr{M}_1, L\otimes F)}. \end{aligned} \tag{2.26}$$

Let $(D_{+,T}^{L\otimes F}(t), P_{>0,+,T}(t))$ denote the canonically associated (elliptic) APS boundary value problem defined similarly as in Section 1.2. Let $\operatorname{Ind}_{\tau=0}(D_{+,T}^{L\otimes F}(t), P_{>0,+,T}(t))$ denote the G-invariant part of the corresponding index.

Lemma 2.7. *There exists $T_0>0$ such that for any $T>T_0$, $\operatorname{Ind}_{\tau=0}(D_{+,T}^{L\otimes F}(t), P_{>0,+,T}(t))$ does not depend on $t\in[0,1]$.*

Proof. For any $T>0$, $t\in[0,1]$, let $D^{L\otimes F}_{\partial\mathscr{M}_1,T}(t)$ be the Dirac type operator on the boundary $\partial\mathscr{M}_1$ induced by $D^{L\otimes F}_T(t)$ (cf. (1.14)).

In view of Section 1.4, in order to prove Lemma 2.7, we need only to show that there exists $T_0>0$ such that when restricted to the subspace of G-invariant sections of $\Omega^{0,*}(\mathscr{M}_1, L\otimes F)|_{\partial\mathscr{M}_1}$, $D^{L\otimes F}_{\partial\mathscr{M}_1,T}(t)$ is invertible for any $T>T_0$, $t\in[0,1]$.

On ∂M_1, set

$$X(t)=\frac{\left\langle {}^{\mathscr{M}_1}(t),\ ^{M,N}\right\rangle\ ^{M,N}}{|\ ^{M,N}|^2}\,;\quad Y(t)={}^{\mathscr{M}_1}(t)-X(t). \tag{2.27}$$

Then $X(t)$, $Y(t)$ are perpendicular to each other. Moreover, one verifies that the following identity holds on ∂M_1,

$$|X(t)|=\frac{\left|\sum_{i=1}^{\dim G}\ {}_iV_i^M\right|^2+(1-t)\left|\sum_{i=1}^{\dim G}\ {}_iV_i^N\right|^2}{\left(\left|\sum_{i=1}^{\dim G}\ {}_iV_i^M\right|^2+\left|\sum_{i=1}^{\dim G}\ {}_iV_i^N\right|^2\right)^{1/2}}\geq c_A>0 \tag{2.28}$$

for some positive constant c_A which might depend on A.

Now we write $D^{L\otimes F}_{\partial M_1,T}(t)$ explicitly as, in view of (1.14),

$$\begin{aligned} D^{L\otimes F}_{\partial\mathscr{M}_1,T}(t) &= D^{L\otimes F}_{\partial\mathscr{M}_1}-\frac{\sqrt{-1}T}{2}c(e_{\dim M})\,c\left({}^{\mathscr{M}_1}(t)\right)\\ &= D^{L\otimes F}_{\partial\mathscr{M}_1}-\frac{\sqrt{-1}T}{2}c(e_{\dim M})\,c(Y(t))-\frac{\sqrt{-1}T}{2}c(e_{\dim M})\,c(X(t)). \end{aligned} \tag{2.29}$$

Then as $X(t), Y(t)\in T\partial\mathscr{M}_1$ are perpendicular to each other, one has

$$\begin{aligned} \left(D^{L\otimes F}_{\partial\mathscr{M}_1,T}(t)\right)^2 &= -\frac{\sqrt{-1}T}{2}\left[D^{L\otimes F}_{\partial\mathscr{M}_1}, c(e_{\dim M})\,c(X(t))\right]\\ &\quad+\frac{T^2}{4}|X(t)|^2+\left(D^{L\otimes F}_{\partial\mathscr{M}_1}-\frac{\sqrt{-1}T}{2}c(e_{\dim M})\,c(Y(t))\right)^2. \end{aligned} \tag{2.30}$$

By (2.28) and proceeding similarly as in the proof of Proposition 1.2, especially (1.18), (1.19), one gets that there exist $T_0>0$ and $C>0$ such that for any $T>T_0$, $t\in[0,1]$ and any G-invariant section s of $\Omega^{0,*}(\mathscr{M}_1, L\otimes F)|_{\partial\mathscr{M}_1}$, one has

$$\left\langle\left(-\frac{\sqrt{-1}}{2}\left[D^{L\otimes F}_{\partial\mathscr{M}_1}, c(e_{\dim M})\,c(X(t))\right]+\frac{T}{4}|X(t)|^2\right)s,s\right\rangle\geq C\langle s,s\rangle. \tag{2.31}$$

From (2.31), one sees that for any $T>T_0$, $t\in[0,1]$, when restricted to the subspace of G-invariant sections of $\Omega^{0,*}(\mathscr{M}_1, L\otimes F)|_{\partial\mathscr{M}_1}$, $D^{L\otimes F}_{\partial\mathscr{M}_1,T}(t)$ is invertible, thus $(D^{L\otimes F}_{+,T}(t), P_{\geq 0,+,T})$ form a continuous family of Fredholm operators, which implies the constancy of the associated index. The proof of Lemma 2.7 is thus completed.

Coming back to the proof of Proposition 2.6. Let ${}^{M_A}=\sum_{i=1}^{\dim G}\ {}_iV_i^{M_A}$ be the vector field on M_A induced by $:M\to g$. By (2.25), one has

$${}^{\mathscr{M}_1}(1)=\sum_{i=1}^{\dim G}\ {}_iV_i^M={}^{M_A}\,;\quad\text{on } \mathscr{M}_1. \tag{2.32}$$

Now we deform inside M_A (leaving the data on ∂M_A unchanged) in G-invariant manner the metrics and connections as well as Φ^{M_A} to the situation that everything is of product nature near ∂M_A. We denote also the spinor $S(TM_A)$ of M_A (associated to the product metric near the boundary now) obtained from $\Lambda(T^{*(0,1)}M)$. Then when taking product with N, we also deform things on $\mathscr{M}_1$ to a situation which is of product nature near $\partial\mathscr{M}_1 = M_1 = \partial M_A \times N$.

We then attach an infinite cylinder $\partial M_A \times [0,+\infty)$ along the boundary ∂M_A and extend everything in M_A to the now complete manifold $\widetilde{M}_A$ with cylindrical end, and $\Phi^{\widetilde{M}_A}$ is constant along the direction $[0,+\infty)$. By taking product with N, we get similar construction $\widetilde{\mathscr{M}}_1$ for $\mathscr{M}_1$. We denote the extended subjects on the obtained manifolds with cylindrical end by a "$\sim$" modifying notation. Then $S(T\widetilde{\mathscr{M}}_1) = S(T\widetilde{M}_A)\widehat{\otimes}\Lambda(T^{*(0,1)}N)$.

Now since for $T > T_0$, $D^{L\otimes F}_{\partial\mathscr{M}_1,T}(1)$ is invertible on the subspace of G-invariant subspace, the standard arguments in [2, Prop. 3.11] show that

$$\begin{aligned}\widetilde{D}^{\widetilde{L}\otimes F}_{+,T} &= \widetilde{D}^{\widetilde{L}\otimes F}_{+} + \frac{\sqrt{-1}T}{2}c\left(\Phi^{\widetilde{M}_A}\right)\\ &: L^2\left(\widetilde{\mathscr{M}}_1, S_+(T\widetilde{\mathscr{M}}_1)\widehat{\otimes}\widetilde{L}\otimes F\right)^G \to L^2\left(\widetilde{\mathscr{M}}_1, S_-(T\widetilde{\mathscr{M}}_1)\widehat{\otimes}\widetilde{L}\otimes F\right)^G;\end{aligned} \tag{2.33}$$

where we use the superscript G to denote the subspace of G-invariant sections, is a Fredholm operator. Moreover, its index equals to $\mathrm{Ind}_{\tau=0}(D^{L\otimes F}_{+,T}(1), P_{\geq 0,+,T}(1))$.

Let $e_1,\cdots,e_{\dim M}$ (resp. $f_1,\cdots,f_{\dim N}$) denote the orthonormal basis of TM_A (resp. TN). Then we can write

$$\widetilde{D}^{\widetilde{L}\otimes F}_T = \sum_{i=1}^{\dim M} c(e_i)\nabla^{S(T\widetilde{\mathscr{M}}_1)\otimes\widetilde{L}\otimes F}_{e_i} + \frac{\sqrt{-1}T}{2}c\left(\Phi^{\widetilde{M}_A}\right) + \sum_{j=1}^{\dim N} c(f_j)\nabla^{S(T\widetilde{\mathscr{M}}_1)\otimes\widetilde{L}\otimes F}_{f_j}; \tag{2.34}$$

where $\sum_{i=1}^{\dim M} c(e_i)\nabla^{S(T\widetilde{\mathscr{M}}_1)\otimes\widetilde{L}\otimes F}_{e_i} + \frac{\sqrt{-1}T}{2}c(\Phi^{\widetilde{M}_A})$ (resp. $\sum_{j=1}^{\dim N} c(f_j)\nabla^{S(T\widetilde{\mathscr{M}}_1)\otimes\widetilde{L}\otimes F}_{f_j}$) is lifted from a corresponding operator on $\widetilde{M}_A$ (resp. N) denoted by $\widetilde{D}^{\widetilde{L}}_T$ (resp. D^F).

By (2.34), one has

$$\left(\widetilde{D}^{\widetilde{L}\otimes F}_T\right)^2 = \left(\widetilde{D}^{\widetilde{L}}_T\right)^2 + \left(D^F\right)^2, \tag{2.35}$$

from which one gets

$$\mathrm{Ker}_{L^2}\left(\widetilde{D}^{\widetilde{L}\otimes F}_T\right) = \mathrm{Ker}_{L^2}\left(\widetilde{D}^{\widetilde{L}}_T\right)\otimes \mathrm{Ker}\, D^F. \tag{2.36}$$

Since N is compact, $\dim(\mathrm{Ker}\, D^F)$ is of finite dimension. From (2.33), (2.36) and the standard arguments in [2, Prop. 3.11], one then deduces that when $T > 0$ is large enough,

$$\mathrm{Ind}_{\tau=0}\left(D^{L\otimes F}_{+,T}(1), P_{\geq 0,+,T}(1)\right) = \sum_{\gamma\in\widehat{G}} Q^{M_A}_{APS}(L,\gamma)\cdot Q(F). \tag{2.37}$$

By taking $A > 0$ large enough, from Theorem 1.4, Lemma 2.7 and (2.37), one gets (2.23), which completes the proof of Proposition 2.6.

From (2.21)–(2.23) and by taking $A > 0$ large enough, in view of Theorems 1.4 and 1.7, one gets (2.8) which completes the proof of Theorem 2.3.

The proof of Theorem 0.4 is thus also completed.

Remark 2.8. By combining the argument in this subsection with those of [21] and [22], one is able to get a new proof of (2.7). Indeed, if $c > 0$ is a regular value of $|\mu|^2$, for $a \in \mathfrak{g}$ close enough to $0 \in \mathfrak{g}$ which is a regular value for μ, let O_a be the co-adjoint orbit of a and consider the moment map $\mu_a : (M \setminus N)_c \times O_a \to \mathfrak{g}$ defined by $\mu_a(x, y) = \mu(x) - y$ for $x \in (M \setminus N)_c$ and $y \in O_a$. Then as $|a|$ is small enough, $d|\mu_{ta}|^2$, $t \in [0,1]$, are nowhere zero on $\partial(M \setminus N)_c \times O_a$. One can then use the argument in this subsection to relate the G-invariant APS type index on $(M \setminus N)_c$ to that on $(M \setminus N)_c \times O_a$, which in turn leads to the contribution on $\mu_a^{-1}(0)/G = \mu^{-1}(G\,a)/G$, by combining the arguments in [22] and [21].

3. Vanishing result: a proof of Theorem 2.4

In this section, we will establish our vanishing result, Theorem 2.4. Assume that X is a suitable vector field on M (depend on the parameter A) in (2.15) deforming $\sigma^{M \setminus N}$ on M_1 to $\sigma^{M \setminus N}$ on M_2, we need to prove that when $T > 0$ is large enough, the G-invariant component of the APS type index of the deformed Dirac operator D_T^M in (3.9) on M is zero. In fact, we will prove that after we fix A large enough, the restriction of D_T^M with APS boundary condition on the G-invariant sections is invertible for T large enough.

As X is nowhere zero on ∂M, by the argument of (1.48) (cf. [22, Prop. 2.4]), we have an estimate (3.117) for D_T^M on an open neighborhood of ∂M similar to (1.48). By the gluing argument as in [4, p. 115–116], if we establish the local estimate (3.94) inside M, then we can conclude Theorem 2.4.

As we are interested in the restriction of $D_T^{M,2}$ to the G-invariant subspace of $\Omega^{0,\bullet}(M, L \otimes F)$, we will take the term $\mu_j L_{V_j}$ in (3.11) as zero. If $z \notin \mathrm{zero}(X) = \{z \in M; X(z) = 0\}$, then from the Bochner type formula (3.11), the term $\frac{T^2}{4}|X|^2$ is the leading term and we get easily (3.94). If $z \in \mathrm{zero}(X)$, then we hope to adapt the argument in [20, §2b] to get the local estimate around z. Basically, we hope $\mu_j \mu_j$ will be a positive term, and it will control all the tensors in (3.11). As there is no assumption on the vector fields V_j^M and their covariant derivatives on M, we can not expect to get our estimate by a simple argument. To control locally the covariant derivatives of V_j^M in (3.11) near $z \in \mathrm{zero}(X)$, we will use the harmonic oscillator argument from [20, §2b)], thus we like to impose that the vector filed $\mu_j V_j^M$ is a Hamiltonian vector field along the direction M.

As N is compact, the tensor V_j^N and their derivatives are bounded. Thus if on $\mathrm{zero}(X)$, we can control $\sqrt{-1}(I_1 + I_2)$ which involve V_j^M and $(d^M \mu_j)$ in (3.10) by $\mu_j \mu_j$, then basically, we can achieve our local estimate. What we prove in Proposition 3.11 (whose proof occupies from Section 3.3 until Section 3.7) is that for our choice μ_j in (3.4), $\sqrt{-1}(I_1 + I_2)$ is bounded from below uniformly on $A > A_0$, any regular value of $|\mu|^2$ and $\frac{1}{2}|\mu|^2$, and on $\mathrm{zero}(X) \cap M$. Moreover $\mu_j \mu_j \simeq |\mu|^2 > A$ on M, thus when $A > 0$ is large enough, the term $2\mu_j \mu_j$ will really control the tensors which do not involve the derivative of V_j^M in (3.11). In this way, we can obtain the local estimate (3.94) around each $z \in \mathrm{zero}(X)$. Now from the gluing argument of [4, p. 115–116], (3.94) and (3.117), we get (3.118), which imply the invertibility of the restriction of the operator D_T^M with APS boundary condition to the G-invariant part. Especially we establish Theorem 2.4.

This section is organized as follows. In Section 3.1, we propose a construction of the deformed vector field X in (2.15) which depends on two functions ρ, ψ. In Section 3.2, we establish a Bochner type formula (3.11) for the Dirac operator deformed by X. In Section 3.3, we study the relation on the zero set of X in M of the vector fields generated by the group action, and in Section 3.4, we study the asymptotics of the functions appeared in the above relations when the parameter $A \to +\infty$. In Section 3.5, we compute precisely the tensors involving the vector field V_j^M and $(d^M \nu_j)$ on the zero set of X in the Bochner formula (3.11). In Section 3.6, we study the coefficients appeared in the computation in Section 3.5. In Section 3.7, we prove in Proposition 3.11 that the sum of the tensors involving the vector field V_j^M and $(d^M \nu_j)$ on the zero set of X in the Bochner formula is uniformly bounded from below for our choice ν_j. In Section 3.8, under the help of Prop. 3.11, for A large enough fixed, we establish the local estimate around each point inside M. Finally in Section 3.9, we prove Theorem 2.4.

We use the notation as in Section 2, and when a subscript index appears two times in a formula, we sum up with this index.

3.1. The construction of the deformed vector field. We first specify the deformation vector field used in Section 2.2.

Let $e, \tilde{e} \in \mathscr{C}^\infty(\mathbb{R})$ be such that

$$(3.1)\qquad e(t) = \begin{cases} t^2; & \text{for } t \leq \frac{1}{3};\\ 1; & \text{for } t \geq \frac{2}{3}; \end{cases} \qquad \tilde{e}(t) = \begin{cases} 1 - t^2; & \text{for } t \leq \frac{1}{3};\\ 2(1-t); & \text{for } t \geq \frac{2}{3}; \end{cases}$$
$$e(t) + \tilde{e}(t) \geq \frac{29}{27}; \qquad \tilde{e}'(t) < 0; \qquad \text{for } \frac{1}{3} \leq t \leq \frac{2}{3}.$$

The existence of $e, \tilde{e}$ is easy to see. For example, one may set $e_0(t) = t^2$, $\tilde{e}_0(t) = 1 - t^2$ on $t \leq \frac{3}{8}$; $e_0(t) = 1$, $\tilde{e}_0(t) = 2(1-t)$ on $t \geq \frac{5}{8}$; and e_0, $\tilde{e}_0$ are linear on $\frac{3}{8} \leq t \leq \frac{5}{8}$. By an approximation argument from e_0, $\tilde{e}_0$, one gets $e, \tilde{e}$ verifying (3.1).

For $A > 0$, set

$$(3.2)\qquad \rho(t) = e\left(\frac{t}{A} - 1\right); \qquad \psi(t) = \tilde{e}\left(\frac{t}{A} - 1\right):$$

Let $\theta \in \mathscr{C}^\infty(M \times N)$, $\mu : M \times N \to \mathfrak{g}^* \simeq \mathfrak{g}$ be defined by

$$(3.3)\qquad \theta = \rho |\mu^M|^2 + \psi\left(|\mu|^2 - |\mu^M|^2 - |\mu^N|^2\right); \qquad \mu = \ldots(\ldots):$$

Recall that $V_1, \ldots, V_{\dim G}$ is an orthonormal basis of $\mathfrak{g}$, V_i^M, V_i^N are the Killing vector fields on M, N induced by V_i. For any function Q with values in $\mathfrak{g}$, we will denote Q_i its i-component with respect to the base $\{V_i\}$. Set[8]

$$(3.4)\qquad \begin{aligned} \nu_j &= 2\left(1 + \psi'\left(|\mu|^2 - |\mu^M|^2 - |\mu^N|^2\right)\right)\mu_j + 2\rho'(|\mu^M|^2)\mu_j;\\ \xi_j &= 2\mu_j - 2\psi'(\ldots)\mu_i\mu_i\mu_j: \end{aligned}$$

Let X be the vector field on M in (2.11) defined by

$$(3.5)\qquad X = \nu_j V_j^M + \xi_j V_j^N:$$

[8] See Lemma 3.7 for a possible motivation of introducing the ν_j's here.

Then one verifies,

$$
(3.6)\qquad
\begin{aligned}
X^{\mathcal H} &= 2\mu_j\left(V_j^M + V_j^N\right) && \text{on } \partial M_1;\\
X^{\mathcal H} &= \left(2 + \frac{8}{A}\mu_i\mu_i\right)\mu_j\left(V_j^M + V_j^N\right) && \text{on } \partial M_2.
\end{aligned}
$$

We can now state the main result of this section, which precise Theorem 2.4, as follows.

Theorem 3.1. *When $A > 0$ large enough is a regular value for both $|\mu|^2$ and $\frac{1}{2}|\mu|^2$, $X^{\mathcal H}$ is nowhere zero on ∂M_1 and ∂M_2. Moreover, we have*

$$
(3.7)\qquad Q^{\mathcal M}_{APS}(L\otimes F; X^{\mathcal H})^{\gamma=0} = 0.
$$

We devote the rest of this section to the proof of Theorem 3.1.

3.2. A Bochner type formula for the deformation by $X^{\mathcal H}$. We will fix the symplectic form $\omega^{\mathcal M}$ and the almost complex structure $J^{\mathcal M}$ on $\mathcal M$ induced from $M\times N$. Then the associated Riemannian metric $g^{T\mathcal M}$ is also induced by g^{TM} and g^{TN}. Let $\nabla^{T\mathcal M}$ be the Levi-Civita connection on $(\mathcal M, g^{T\mathcal M})$.

For any function ϕ on $M\times N$, we denote by $d^M\phi$, $d^N\phi$ its differential along M, N respectively.

We denote also $V_j^{\mathcal M}$ the Killing vector field on $\mathcal M$ induced by $V_j\in\mathfrak g$, then

$$
(3.8)\qquad V_j^{\mathcal M} = V_j^{M\times N} = V_j^M + V_j^N.
$$

Let $\nabla^{T^{(1,0)}M}$, $\nabla^{T^{(1,0)}N}$ be the connections on $T^{(1,0)}M$, $T^{(1,0)}N$ induced by the Levi-Civita connections ∇^{TM}, ∇^{TN} as explained in Section 1.6. Let $\nabla^{\Lambda^{0,\bullet}}$ be the Hermitian connection on $\Lambda(T^{*(0,1)}(M\times N))\otimes L\otimes F$ induced by the Clifford connections on $\Lambda(T^{*(0,1)}M)$ and on $\Lambda(T^{*(0,1)}N)$ and the connections ∇^L, ∇^F.

Definition 3.2. For $T\in\mathbb R$, let $D^{\mathcal M}_T$ be the operator defined by

$$
(3.9)\qquad D^{\mathcal M}_T = D^{L\otimes F} + \frac{\sqrt{-1}T}{2}c(X^{\mathcal H}) : \Omega^{0,\bullet}(\mathcal M, L\otimes F)\to\Omega^{0,\bullet}(\mathcal M, L\otimes F).
$$

Recall that when a subscript index appears two times in a formula, we sum up with this index. Set

$$
(3.10)\qquad
\begin{aligned}
I_1 &= \frac{1}{4}c\left(d^M\mu_j\right)c\left(V_j^M + 2V_j^N\right) + \frac{1}{2}c\left(d^N\mu_j\right)c\left(V_j^M\right);\\
I_2 &= \frac{1}{4}\left\langle\left(1 + \frac{J^M}{\sqrt{-1}}\right)V_j^M, d^M\mu_j\right\rangle;\\
I_3 &= \frac{1}{2}c\left(d^N\mu_j\right)c\left(V_j^N\right).
\end{aligned}
$$

Let $\{e_k\}_{k=1}^{\dim M}$ (resp. $\{f_i\}_{i=1}^{\dim N}$) be an orthonormal frame of TM (resp. TN), then $\{e_a\}_{a=1}^{\dim\mathcal M} := \{e_k\}\cup\{f_i\}$ is an orthonormal frame of $T\mathcal M$.

The following Bochner type formula holds for $(D^{\mathcal M}_T)^2$.

Theorem 3.3. *The following Bochner type formula holds*

$$D_T^{M,2} = D^{L_F,2} + \frac{\sqrt{-1}T}{4}\sum_{k=1}^{\dim M} c(e_k)c\left(\nabla^{TM}_{e_k}\mu_jV_j^M\right) \tag{3.11}$$
$$+\frac{\sqrt{-1}T}{4}\sum_{i=1}^{\dim N} c(f_i)c\left(\nabla^{TN}_{f_i}V_j^N\right)\mu_j - \frac{\sqrt{-1}T}{2}\mathrm{Tr}\left[\nabla^{T^{(1,0)}M}\mu_jV_j^M\big|_{T^{(1,0)}M}\right]$$
$$-\frac{\sqrt{-1}T}{2}\mu_j\mathrm{Tr}\left[\nabla^{T^{(1,0)}N}V_j^N\big|_{T^{(1,0)}N}\right]$$
$$+2\pi T\mu_j\mu_j - \sqrt{-1}T\mu_jL_{V_j} + \frac{T^2}{4}|X^{\mathcal H}|^2 + \sqrt{-1}T(I_1+I_2+I_3).$$

Proof. For any $1\le a\le \dim M$, it is clear that

$$\nabla^{\Omega^{0,\bullet}}_{e_a}c(X^{\mathcal H}) = c(X^{\mathcal H})\nabla^{\Omega^{0,\bullet}}_{e_a} + c\left(\nabla^{TM}_{e_a}X^{\mathcal H}\right). \tag{3.12}$$

By (3.9), (3.12), we have

(3.13)
$$D_T^{M,2} = D^{L_F,2} + \frac{\sqrt{-1}T}{2}\sum_{a=1}^{\dim M}\left(c(e_a)\nabla^{\Omega^{0,\bullet}}_{e_a}c(X^{\mathcal H}) + c(X^{\mathcal H})c(e_a)\nabla^{\Omega^{0,\bullet}}_{e_a}\right) + \frac{T^2}{4}|X^{\mathcal H}|^2$$
$$= D^{L_F,2} + \frac{\sqrt{-1}T}{2}\sum_{a=1}^{\dim M} c(e_a)c\left(\nabla^{TM}_{e_a}X^{\mathcal H}\right) - \sqrt{-1}T\nabla^{\Omega^{0,\bullet}}_{X^{\mathcal H}} + \frac{T^2}{4}|X^{\mathcal H}|^2.$$

By [20, Lemma 1.5],

$$\nabla^{\Omega^{0,\bullet}}_{V_j^M} = L_{V_j} + 2\pi\sqrt{-1}\langle\mu, V_j\rangle \tag{3.14}$$
$$+\frac{1}{4}\sum_{a=1}^{\dim M} c(e_a)c\left(\nabla^{TM}_{e_a}V_j^M\right) + \frac{1}{2}\mathrm{Tr}\left[\nabla^{T^{(1,0)}M}V_j^M\big|_{T^{(1,0)}M}\right].$$

Thus by (3.5), (3.14),

$$\nabla^{\Omega^{0,\bullet}}_{X^{\mathcal H}} = \mu_jL_{V_j} + 2\pi\sqrt{-1}\mu_j\mu_j \tag{3.15}$$
$$+\frac{1}{4}\sum_{k=1}^{\dim M} c(e_k)c\left(\nabla^{TM}_{e_k}V_j^M\right)\mu_j + \frac{1}{4}\sum_{i=1}^{\dim N} c(f_i)c\left(\nabla^{TN}_{f_i}V_j^N\right)\mu_j$$
$$+\frac{1}{2}\mu_j\mathrm{Tr}\left[\nabla^{T^{(1,0)}M}V_j^M\big|_{T^{(1,0)}M}\right] + \frac{1}{2}\mu_j\mathrm{Tr}\left[\nabla^{T^{(1,0)}N}V_j^N\big|_{T^{(1,0)}N}\right].$$

Thus from (3.5), (3.10), (3.14) and (3.15), we get

$$\nabla^{\Omega^{0,\bullet}}_{X^{\mathcal H}} = \mu_jL_{V_j} + 2\pi\sqrt{-1}\mu_j\mu_j + \frac{1}{4}\sum_{k=1}^{\dim M} c(e_k)c\left(\nabla^{TM}_{e_k}\mu_jV_j^M\right) \tag{3.16}$$
$$+\frac{1}{4}\sum_{i=1}^{\dim N} c(f_i)c\left(\nabla^{TN}_{f_i}V_j^N\right)\mu_j - \frac{1}{4}c\left(d^M\mu_j\right)c\left(V_j^M\right)$$
$$+\frac{1}{2}\mathrm{Tr}\left[\nabla^{T^{(1,0)}M}\mu_jV_j^M\big|_{T^{(1,0)}M}\right] + \frac{1}{2}\mu_j\mathrm{Tr}\left[\nabla^{T^{(1,0)}N}V_j^N\big|_{T^{(1,0)}N}\right] - I_2.$$

Also by (3.5),

(3.17)

$$\frac{1}{2}\sum_{a=1}^{\dim M} c(e_a)c\ r^{TM}_{e_a} X \quad = \frac{1}{2}\sum_{k=1}^{\dim M} c(e_k)c\ r^{TM}_{e_k} \quad {}_j V_j^M \quad + \frac{1}{2}c \quad d^M \quad {}_j \quad c\ V_j^N$$
$$+ \frac{1}{2}\sum_{i=1}^{\dim N} c(f_i)c\ r^{TN}_{f_i} V_j^N \quad {}_j + \frac{1}{2}c \quad d^N \quad {}_j \quad c\ V_j^M + V_j^N \quad :$$

By (3.10), (3.13), (3.16) and (3.17), the proof of Theorem 3.3 is completed.

3.3. Relations between terms in (3.11) near zero(X): Part 1. In this subsection, as well as the next a few subsections, we establish certain formulas concerning the relationships between the terms appearing in (3.11), on the zero set of X in M . These relations are crucial for the proof of Theorem 3.1.

Set

(3.18)
$$\begin{aligned} {}_1 &= 1 + \ {}^0(j\ j^2)(j\ j^2 \quad j\ j^2);\\ {}_2 &= 1 \quad 2\ {}^0(\ )\ {}_i\ {}_i\ {}_1;\\ {}_4 &= 1 \quad (\ ) \quad 2^0(\ )\ (j\ j^2)\ {}_i\ {}_i;\\ {}_5 &= (1 \quad (\ ))\ {}_1 \quad (j\ j^2)\\ &= 1 \quad (\ ) \quad (j^2 j) + (1 \quad (\ ))\ {}^0(j\ j^2)(j\ j^2 \quad j\ j^2): \end{aligned}$$

These functions appear naturally on zero(X), as is clear from the following (3.19) and Lemma 3.4.

From (3.4), (3.18), one gets

(3.19)
$${}_j = 2\ {}_1\ {}_j + 2\ (j\ j^2)\ {}_j :$$

From (3.4), (3.18) and (3.19), one sees

(3.20)
$$\begin{aligned} {}_j &= 2\ {}_j + 2(1 \quad (\ ))\ {}_j \quad 2\ {}^0(\ )\ {}_i\ {}_i\ 2\ {}_1\ {}_j + 2\ (j\ j^2)\ {}_j\\ &= 2\ {}_2\ {}_j + 2\ {}_4\ {}_j : \end{aligned}$$

Lemma 3.4. *On zero(X) = fx 2 M : X (x) = 0g, we have*

(3.21)
$${}_2\ {}_j V_j^M = \quad {}_4\ {}_j V_j^M \ ; \quad {}_2\ {}_j V_j^M = \quad 2\ {}_5\ {}_j V_j^M \ ;$$

and

(3.22)
$${}_2\ {}_j V_j^N = \quad {}_4\ {}_j V_j^N \ ; \quad {}_2\ {}_j V_j^N = \quad 2\ {}_5\ {}_j V_j^N \ ; \quad {}_2\ {}_j V_j^N = (\ {}_2 \quad {}_4)\ {}_j V_j^N \ :$$

Proof. By (2.15), one sees that

(3.23)
$$X \quad = 0 \quad \text{if and only if} \quad \sum_j {}_j V_j^M = 0 \quad \text{and} \quad \sum_j {}_j V_j^N = 0 :$$

From (3.20), the equation $\sum_j {}_j V_j^M = 0$ in (3.23) is equivalent to the first equation of (3.21).

By (3.18), (3.19) and the first equation of (3.21), we have on zero(X),

$$(3.24)\quad {}_2\, |V_j^M = 2\, {}_1\, {}_2\, |V_j^M + 2\, (|\ |^2)\, {}_2\, |V_j^M$$
$$= 2\ 1\ (\ )\ 2^0(\ )\ (|\ |^2)\, {}_{i\,i\,1}\ (|\ |^2)(1\ 2^0(\ )\, {}_{i\,i\,1})\ |V_j^M$$
$$= 2\,(1\ (\ ))\, {}_1\ (|\ |^2)\ |V_j^M = 2\, {}_5\, |V_j^M :$$

In the same way, by (3.20), $\sum_j |V_j^N = 0$ in (3.23) is equivalent to the first equation of (3.22).

By (2.3) and the first equation of (3.22), as in (3.24), we have

$$(3.25)\quad \begin{aligned} {}_2\, |V_j^N &= 2\, {}_1\, {}_2\, |V_j^N + 2\, (|\ |^2)\, {}_2\, |V_j^N = 2\, {}_5\, |V_j^N ; \\ {}_2\, |V_j^N &= {}_2\, |V_j^N + {}_2\, |V_j^N = ({}_2\ {}_4)\, |V_j^N : \end{aligned}$$

The proof of Lemma 3.4 is completed.

3.4. Estimates of $_i$; $i = 1;2;4;5$, when $A > 0$ is large. Recall that , have been defined in (3.2), (3.3).

Lemma 3.5. *There exists $A_0 > 0$ such that for $A > A_0$, we have*

$$(3.26)\quad A < \ < 2A ; \quad \text{on } M \ n\, @M :$$

Thus

$$(3.27)\quad 0 < (\ ) < 1 \quad \text{on } M \ n\, @M :$$

Moreover, uniformly on M and the parameter A, we have

$$(3.28)\quad {}_1 = 1 + O(A^{1=2}); \quad {}_2 = 1 + O(A^{1=2});$$

$$(3.29)\quad {}_4 = (1\ (\ ))(1 + O(A^{1=2}));$$

$$(3.30)\quad {}_5 = 1\ (\ )\ (|\ |^2)\ (1 + O(A^{1=2})):$$

Finally, for any $A > A_0$, we have

$$(3.31)\quad \begin{aligned} 1\ (\ )\ (|\ |^2) &< 0 \quad \text{if } (x;y) \in M \ n\, @M ; \\ &= 0 \quad \text{if } (x;y) \in @M : \end{aligned}$$

Proof. As N is compact, there exists $C_0 > 0$ such that

$$(3.32)\quad |\ | \leq C_0:$$

From (2.11), (2.12) and (3.32), one sees that on M, one has

$$(3.33)\quad \begin{aligned} 2A &> |\ |^2 + |\ |^2 + 2h\ ;\ i \\ &> |\ |^2 + |\ |^2\ \ \frac{1}{3}|\ |^2 + 3|\ |^2\ > \frac{2}{3}|\ |^2\ \ 2C_0^2: \end{aligned}$$

From (3.33), one knows that when $A > 0$ is large enough, one has on M that

$$(3.34)\quad A^{1=2} \leq |\ | < \sqrt{3:5}\,A^{1=2}:$$

From (2.3), (2.11), (3.1)–(3.3) and (3.34), one deduces that for $A > 0$ large enough, one also has

$$(3.35)\quad |\ | < 2A^{1=2};\ |\ |^2\ |\ |^2 = O(A^{1=2});\ = |\ |^2 + O(A^{1=2}) \text{ on } M :$$

If $|\,|^2 < 1.3A$, then by (3.35), when $A > 0$ is large enough, one has $\ < \frac{4A}{3}$. Then by (3.1)–(3.3), one has for $\ < \frac{4A}{3}$,

$$(3.36)\qquad \frac{}{A} - 1 = \frac{|\,|^2}{A} - 1 \quad 1 + \frac{1}{A}\;\frac{|\,|^2}{A} - 1\;(|\,|^2 - |\,|^2) = \frac{|\,|^2}{A} - 1\;(1 + O(A^{-1/2})).$$

If $|\,|^2 > 1.3A$, then by (3.35), one has

$$(3.37)\qquad > A + 0.3A + O(A^{1/2}) > 1.2A;$$

when $A > 0$ is large enough.

From (3.1), (3.2), (3.3), (3.35), (3.36) and (3.37), one gets (3.26) and (3.27).

From (2.11), (3.2), (3.3) and (3.32), one gets

$$(3.38)\qquad |\,| < 2A^{1/2} \quad \text{on } M.$$

From (3.2), (3.18), (3.32), (3.35) and (3.38), we get (3.28).

On the other hand, if $|\,|^2 < 1.3A$ and $A > 0$ is large enough so that $\ < \frac{4A}{3}$, then by (3.1) and (3.2), one has

$$(3.39)\qquad 1 - (\ ) = \frac{}{A} - 1^{3};\qquad {}'(\ ) = \frac{3}{A}\,\frac{}{A} - 1^{2};\quad {}''(\ ) = \frac{6}{A^2}\,\frac{}{A} - 1.$$

From (3.1), (3.2), (3.32), (3.38) and (3.39), one finds

$$(3.40)\qquad {}'(\ )\,(|\,|^2) = (1 - (\ ))^{4/3}O(A^{-1});\qquad {}''(\ )\,(|\,|^2)_{ii} = (1 - (\ ))O(A^{-3/2}).$$

By (3.32), (3.38) and (3.40), we see that (3.29) holds for $A > 0$ large enough and $|\,|^2 < 1.3A$.

While if $|\,|^2 > 1.3A$, by (3.37), when $A > 0$ is large enough, $\ > 1.1A$, and thus by (3.1) and (3.2), one has $1 - (\ ) > 1 - (1.1A) > 0$, from which (3.29) holds tautologically.

The proof of (3.29) is thus completed.

If $|\,|^2 > \frac{5A}{3}$, then from (3.1),

$$(3.41)\qquad (|\,|^2) = 1;\quad {}'(|\,|^2) = {}''(|\,|^2) = 0;$$

from which, one gets

$$(3.42)\qquad 1 - (\ )\quad (|\,|^2) = (\ );\quad {}_5 = (\ ).$$

Thus from (3.18) and (3.42), we get (3.30) and (3.31).

Now by (3.1), (3.2), (3.3) and (3.35), it is clear that when $A > 0$ is large enough,

$$(3.43)\qquad (\ ) = |\,|^2 + {}'\,|\,|^2\;O\;A^{1/2}.$$

Let $0 < {}_0 < \frac{1}{9}$ be such that

$$(3.44)\qquad e(t) + e(t) > \frac{28.5}{27};\qquad \text{for } \frac{1 - 2_0}{3} \leqslant t \leqslant \frac{2}{3}.$$

By (3.1), the existence of $_0$ is clear.

If $\frac{(4-\ _0)A}{3} \leqslant |\ |^2 \leqslant \frac{5A}{3}$, then from (3.1), (3.43) and (3.44), one sees that for A large enough,

$$1 - \ (\ ) \ |\ |^2 \leqslant \frac{1}{27} \tag{3.45}$$

thus we get again (3.30) and (3.31) from (3.18).

If $|\ |^2 < \frac{(4-\ _0)A}{3}$, then by (3.35), one sees that when $A > 0$ is large enough, one has $\ < \frac{4A}{3}$. Thus, by (3.1), (3.2), (3.36) and (3.39), one has

$$\begin{aligned} \ '(|\ |^2)(1 - \ (\ )) &= \frac{|\ |^2}{A} - 1 \ \frac{|\ |^2}{A} - 1 \ (1 + O(A^{-1/2}))^3 \ O(A^{-1}) \\ &= \frac{|\ |^2}{A} - 1^4 \ O(A^{-1}); \end{aligned} \tag{3.46}$$

$$1 - \ (\ ) \ \ (|\ |^2) = \frac{|\ |^2}{A} - 1^2 \ 1 \ \frac{|\ |^2}{A} - 1 \ (1 + O(A^{-1/2})) :$$

from which and (3.18) we get again (3.30) and (3.31).

The proof of (3.30) and (3.31) is now complete, as well as Lemma 3.5.

The following Lemma will also be used in the proof of Proposition 3.11.

Lemma 3.6. There exists $A_0 > 0$ such that for any $A > A_0$,

$$1 < \frac{(1 - \ (\ ))^2 \ \ (|\ |^2)}{1 - \ (\ ) \ \ (|\ |^2)} < 12 \quad \text{on } M \setminus \partial M : \tag{3.47}$$

Proof. In fact, by (3.27) and (3.31),

$$(1 - \ (\ ))^2 \ \ (|\ |^2) < 1 - \ (\ ) \ \ (|\ |^2) < 0 \text{ on } M \setminus \partial M : \tag{3.48}$$

Thus we need to prove that

$$11 - 10 \ (\ ) - 11 \ \ (|\ |^2) - \ (\ )^2 < 0 \quad \text{if } (x;y) \in M \setminus \partial M : \tag{3.49}$$

If $|\ |^2 > \frac{5A}{3}$, then $\ (|\ |^2) = 1$, thus (3.49) holds.

Let $_0 > 0$ be defined as in (3.44).

If $A < |\ |^2 < \frac{(4-\ _0)A}{3}$, then by (3.2), (3.36) and (3.39)

$$\begin{aligned} 11 - 10 \ (\ ) - 11 \ \ (|\ |^2) - \ (\ )^2 \\ = \frac{|\ |^2}{A} - 1^2 \ 11 + 12 \frac{|\ |^2}{A} - 1 \ \frac{|\ |^2}{A} - 1^4 + O(A^{-1/2}) : \end{aligned} \tag{3.50}$$

From (3.50), we know (3.49) holds for A large enough.

If $\frac{(4-\ _0)A}{3} \leqslant |\ |^2 \leqslant \frac{5A}{3}$, then by (3.45), as $0 < \ (\ ) < 1$, we get, when $A > 0$ is large enough, that

$$11 - 10 \ (\ ) - 11 \ \ (|\ |^2) - \ (\ )^2 \leqslant \frac{11}{27} + \ (\ ) - \ (\ )^2 < \frac{4}{27}: \tag{3.51}$$

Thus (3.47) holds on $M \setminus \partial M$.

3.5. Relations between terms in (3.11) near zero(X): Part 2. The rst of two relations we would include in this subsections is

Lemma 3.7. *We have*

$$ {}_j V_j^M = J^M (d^M j \, j^2) : \tag{3.52}$$

Proof. By (2.3), (3.3), we have

$$\begin{aligned} d &= 2 \; 1 + {}^0(j\,j^2)(j\,j^2 \quad j\,j^2) \quad (j\,j^2) \quad {}_j d^M \; {}_j + 2 \; (j\,j^2) \; {}_j d \; {}_j \\ &= {}_j d^M \; {}_j + 2 \; (j\,j^2) \; {}_j d^N \; {}_j ; \\ d \; {}_j &= d \; {}_j \quad {}^0(\;) \; {}_j d \quad (\;) d^N \; {}_j : \end{aligned} \tag{3.53}$$

From (2.10) and (3.53), we get

$$\begin{aligned} (d \;\;) &= J^M V_j^M + J^N V_j^N ; \\ (d \;\;) &= {}_k J^M V_k^M + 2 \; (j\,j^2) \; {}_k J^N V_k^N ; \\ (d \; {}_j) &= J^M V_j^M \quad {}^0(\;) \; {}_j \; {}_k J^M V_k^M \\ &\quad + (1 \quad (\;)) J^N V_j^N \quad 2 \; {}^0(\;) \; (j\,j^2) \; {}_j \; {}_k J^N V_k^N : \end{aligned} \tag{3.54}$$

From (3.4), (3.54), we get

$$ {}_j V_j^M = 2 \; {}_j \; V_j^M \quad {}^0(\;) \; {}_j \; {}_k V_k^M = J^M (d^M j\,j^2) : \tag{3.55}$$

Thus we get (3.52).

On the other hand, by Lemma 3.5, we see that when $A > 0$ is large enough, ${}_2 > 1=2$:

To state the second relation we would include in this subsection, we introduce the following two functions,

$$\begin{aligned} I_4 &= \quad 2 \; {}^0(\;) \; {}^{00}(j\,j^2) \; {}_i \; {}_i (j\,j^2 \quad j\,j^2) \; \frac{4}{2}^{2} + 4 \; {}^0(\;) \; {}^0(j\,j^2) \; {}_i \; {}_i \frac{4}{2} \\ &\quad + 2 \quad {}^{00}(\;) \; {}_i \; {}_i + (\; {}^0(\;))^2 \; {}_i^2 \quad \frac{5}{2}^{2} + 2 \; {}^0(\;) \frac{5}{2} ; \\ I_5 &= 2 \; {}^0(\;) \; {}^0(j\,j^2) \; {}_i \; {}_i \frac{(\; 2 \quad 4) \; 4}{{}_2^2} \\ &\quad 2 \; {}^{00}(\;) \; (j\,j^2) \; {}_i \; {}_i + 2(\; {}^0(\;))^2 \; (j\,j^2) \; {}_i^2 \; \frac{(\; 2 \quad 4) \; 5}{{}_2^2} \\ &\quad + \; {}^0(\;) \; \frac{(\; 2 \quad 4) \; 5}{{}_2^2} + \; 1 \quad 2 \; (\;) \; \frac{5}{2} \quad (j\,j^2) \frac{2 \quad 4}{2} \; ; \end{aligned} \tag{3.56}$$

which appear naturally in the following lemma.

Lemma 3.8. On zero(X) M, we have

$$\frac{1}{4}c((d^M\ _j))c(V_j^M) + \frac{1}{4}\ 1 + \frac{J^M}{p\ 1}\ V_j^M ; (d^M\ _j) \tag{3.57}$$

$$= \frac{1}{2}\ _2\ c(J^M V_j^M)c(V_j^M) + \frac{1}{p\ 1} jV_j^M j^2$$

$$+ I_4\ c(\ _k J^M V_k^M)c(\ _j V_j^M) + \frac{1}{p\ 1}\sum_k\ _k V_k^M\ ^2 !$$

and

$$(3.58)\quad \begin{aligned} \frac{1}{4}c((d^M\ _j))c(V_j^N) &= \frac{1}{2}\ _2\ c(J V_j^M)c(V_j^N) + I_4\ c(_k J^M V_k^M)c(\ _j V_j^N);\\ \frac{1}{4}c((d^N\ _j))c(V_j^M) &= \frac{1}{2}\ _4\ c(J V_j^N)c(V_j^M) + I_5\ c(_k J^N V_k^N)c(\ _j V_j^M): \end{aligned}$$

Proof. By (2.3), (3.4), we have

$$(3.59)\quad \begin{aligned} d\ _j = &\ 2\ ^{00}(j\ j^2)(j\ j^2\ \ j\ j^2)\ _j\ \ 2\ ^0(j\ j^2)\ _j + 2\ ^0(j\ j^2)\ _j\ 2\ _k d^M\ _k\\ &+ 4\ ^0(j\ j^2)\ _j\ _k d\ _k + \ 2 + 2\ ^0(j\ j^2)(j\ j^2\ \ j\ j^2)\ d^M\ _j + 2\ (j\ j^2)d^N\ _j: \end{aligned}$$

From (2.10), (3.59), we get

$$(3.60)\quad \begin{aligned} (d\ _j) = &\ 4\ ^{00}(j\ j^2)(j\ j^2\ \ j\ j^2)\ _j\ _k J^M V_k^M\\ &+ 4\ ^0(j\ j^2)(\ _j\ _k + \ _k\ _j)J^M V_k^M\\ &+ \ 2 + 2\ ^0(j\ j^2)(j\ j^2\ \ j\ j^2)\ J^M V_j^M\\ &+ 4\ ^0(j\ j^2)\ _j\ _k J^N V_k^N + 2\ (j\ j^2)J^N V_j^N : \end{aligned}$$

Now from (3.4), (3.54) and (3.60), we get

$$(3.61)\quad \begin{aligned} (d^M\ _j) = &\ \ 2\ ^{00}(\ )\ _i\ _i\ _j(d^M\ )\ \ 2\ ^0(\ )\ _i\ \ _i(d^M\ _j) + \ _j(d^M\ _i)\\ &+ 2(d^M\ _j)\\ = &\ \ 2\ ^{00}(\ )\ _i\ _i\ _j\ _k J^M V_k^M\ \ 2\ ^0(\ )\ _i\ _i\ h\ 4\ ^{00}(j\ j^2)(j\ j^2\ \ j\ j^2)\ _j\ _k J^M V_k^M\\ &+ 4\ ^0(j\ j^2)(\ _j\ _k + \ _k\ _j)J^M V_k^M + \ 2 + 2\ ^0(j\ j^2)(j\ j^2\ \ j\ j^2)\ J^M V_j^M\ i\\ &\ 2\ ^0(\ )\ _i\ _j\ J^M V_i^M\ \ ^0(\ )\ _i\ _k J^M V_k^M\ + 2\ J^M V_j^M\ \ ^0(\ )\ _j\ _k J^M V_k^M\ : \end{aligned}$$

Thus from (3.18) and (3.61),

$$(3.62)\quad \begin{aligned} (d^M\ _j) = &\ 2\ _2 J^M V_j^M\ \ 8\ ^0(\ )\ ^{00}(j\ j^2)\ _i\ _i(j\ j^2\ \ j\ j^2)\ _j\ _k J^M V_k^M\\ &\ 8\ ^0(\ )\ ^0(j\ j^2)\ _i\ _i(\ _j\ _k + \ _k\ _j)J^M V_k^M\\ &+ 2\ \ ^{00}(\ )\ _i\ _i + (\ ^0(\ ))^2\ _i^2\ \ _j\ _k J^M V_k^M\ \ 2\ ^0(\ )(\ _k\ _j + \ _j\ _k)J^M V_k^M : \end{aligned}$$

By (3.62),

(3.63) $c((d^M \quad _j))c(V_j^M) = 2 \; _2 \; c(J^M V_j^M)c(V_j^M)$
$8\,{}^0(\;)\;{}^{00}(j\,j^2)\;_{i\,i}(j\,j^2 \quad j\,j^2)c(\;_kJ^M V_k^M)c(\;_jV_j^M)$
$8\,{}^0(\;)\;{}^0(j\,j^2)\;_{i\,i}\,\big[c(\;_kJ^M V_k^M)c(\;_jV_j^M) + c(\;_jJ^M V_j^M)c(\;_kV_k^M)\big]$
$+\,2 \quad {}^{00}(\;)\;_{i\,i} + (\;{}^0(\;))^2\;{}^2_i\;\; c(\;_kJ^M V_k^M)c(\;_jV_j^M)$
$2\,{}^0(\;)\,\big[c(\;_jJ^M V_j^M)c(\;_kV_k^M) + c(\;_kJ^M V_k^M)c(\;_jV_j^M)\big]$

and by using the fact that J^M is anti-symmetric, $\langle\;\rangle$ on TM is C-bilinear, we get

(3.64) $1 + \frac{J^M}{\sqrt{-1}} \quad V_j^M\,;(d^M \quad _j)) \;=\; 2\;_2\frac{1}{\sqrt{-1}}\,j\,V_j^M\,j^2$
$8\,{}^0(\;)\;{}^{00}(j\,j^2)\;_{i\,i}(j\,j^2 \quad j\,j^2)\frac{1}{\sqrt{-1}}\sum_j \quad _jV_j^M \;{}^2$
$8\,{}^0(\;)\;{}^0(j\,j^2)\;_{i\,i}\frac{2}{\sqrt{-1}} \quad _jV_j^M\,;\;_kV_k^M$
$+\,2 \quad {}^{00}(\;)\;_{i\,i} + (\;{}^0(\;))^2\;{}^2_i\;\frac{1}{\sqrt{-1}}\sum_k \quad _kV_k^M \;{}^2$
$2\,{}^0(\;)\frac{2}{\sqrt{-1}} \quad _jV_j^M\,;\;_kV_k^M \quad :$

For the term $c((d^M \quad _j))c(V_j^N)$, from (3.18), (3.62), we have

(3.65) $c((d^M \quad _j))c(V_j^N) = 2\;_2 \quad c(J^M V_j^M)c(V_j^N)$
$8\,{}^0(\;)\;{}^{00}(j\,j^2)\;_{i\,i}(j\,j^2 \quad j\,j^2)c(\;_kJ^M V_k^M)c(\;_jV_j^N)$
$8\,{}^0(\;)\;{}^0(j\,j^2)\;_{i\,i}\,\big[c(\;_kJ^M V_k^M)c(\;_jV_j^N) + c(\;_jJ^M V_j^M)c(\;_kV_k^N)\big]$
$+\,2 \quad {}^{00}(\;)\;_{i\,i} + (\;{}^0(\;))^2\;{}^2_i\;\; c(\;_kJ^M V_k^M)c(\;_jV_j^N)$
$2\,{}^0(\;)\,\big[c(\;_jJ^M V_j^M)c(\;_kV_k^N) + c(\;_kJ^M V_k^M)c(\;_jV_j^N)\big] \;:$

Moreover, from (2.10), (3.54), (3.60) and $_i = \quad _i \quad (\;)\,_i$, we get

(3.66) $(d^N \quad _j) \;=\; 2\,{}^0(\;)\;_{i\,i}(d^N \quad _j) \quad 2\,{}^0(\;)\;_{i\,j}(d^N \quad _i)$
$2\,{}^{00}(\;)\;_{i\,i\,j}(d^N \quad) \quad 2\,{}^0(\;)\;_{i\,j}(d^N \quad _i) \;+\; 2(d^N \quad _j)$
$= \quad 2\,{}^0(\;)\;_{i\,i}\,\big[4\,{}^0(j\,j^2)\;_{j\,k}J^N V_k^N + 2\;(j\,j^2)J^N V_j^N\big]$
$2\,{}^0(\;)\;_{i\,j}\,\big[(1 \quad (\;))J^N V_i^N \quad 2\,{}^0(\;)\;(j\,j^2)\;_{i\,k}J^N V_k^N\big]$
$4\,{}^{00}(\;)\;(j\,j^2)\;_{i\,i\,j\,k}J^N V_k^N \quad 2\,{}^0(\;)(\;_i \quad (\;)\,_i)\;_jJ^N V_i^N$
$+\,2\,\big[(1 \quad (\;))J^N V_j^N \quad 2\,{}^0(\;)\;(j\,j^2)\;_{j\,k}J^N V_k^N\big] \;:$

From (3.18) and (3.66),

$$(3.67)\quad (d^N \;\; {}_j) = 2\; {}_4 J^N V_j^N \quad 8\;{}^0(\;)\;{}^0(j\;j^2)\;{}_i\;{}_i\;{}_j\;{}_k J^N V_k^N$$
$$+ \quad 4\;{}^{00}(\;)\;(j\;j^2)\;{}_i\;{}_i + 4(\;{}^0(\;))^2\;(j\;j^2)\;{}_i^2 \quad 2\;{}^0(\;)\;\;{}_j\;{}_k J^N V_k^N$$
$$2\;{}^0(\;)(1 \quad 2\;(\;))\;{}_j\;{}_i J^N V_i^N \quad 4\;{}^0(\;)\;(j\;j^2)\;{}_j\;{}_k J^N V_k^N :$$

From (3.67), one has

$$(3.68)\quad c((d^N \;\; {}_j))c(V_j^M) = 2\;{}_4 \quad c(J^N V_j^N)c(V_j^M)$$
$$8\;{}^0(\;)\;{}^0(j\;j^2)\;{}_i\;{}_i c(\;{}_k J^N V_k^N)c(\;{}_j V_j^M)$$
$$+ \quad 4\;{}^{00}(\;)\;(j\;j^2)\;{}_i\;{}_i + 4(\;{}^0(\;))^2\;(j\;j^2)\;{}_i^2 \quad 2\;{}^0(\;)\;c(\;{}_k J^N V_k^N)c(\;{}_j V_j^M)$$
$$2\;{}^0(\;)(1 \quad 2\;(\;))c(\;{}_i J^N V_i^N)c(\;{}_j V_j^M)$$
$$4\;{}^0(\;)\;(j\;j^2)c(\;{}_k J^N V_k^N)c(\;{}_j V_j^M):$$

On zero(X), from (3.21), (3.63) and (3.64), we then get

$$(3.69)\quad \frac{1}{4}c((d^M \;\; {}_j))c(V_j^M) + \frac{1}{4} \quad 1 + \frac{J^M}{\sqrt{1}} \quad V_j^M\,;(d^M \;\; {}_j)$$
$$= \frac{1}{2}\;{}_2 \quad c(J^M V_j^M)c(V_j^M) + \frac{1}{\sqrt{1}} jV_j^M j^2$$
$$+ \frac{1}{4} \quad 8\;{}^0(\;)\;{}^{00}(j\;j^2)\;{}_i\;{}_i(j\;j^2 \quad j\;j^2)\;\frac{4}{2}\;{}^2$$
$$+ 8\;{}^0(\;)\;{}^0(j\;j^2)\;{}_i\;{}_i\frac{2\;{}_4}{2} + 2 \quad {}^{00}(\;)\;{}_i\;{}_i + (\;{}^0(\;))^2\;{}_i^2 \quad \frac{2\;{}_5}{2}\;{}^2$$
$$+ 2\;{}^0(\;)\frac{4\;{}_5}{2} \quad c(\;{}_k J^M V_k^M)c(\;{}_j V_j^M) + \frac{1}{\sqrt{1}} \sum_k \;{}_k V_k^M \;{}^2\Big)\; :$$

From (3.69) we get (3.57).

On zero(X), from (3.21), (3.22) and (3.65), as in (3.69), we get the first equation of (3.58).

On zero(X), from (3.21), (3.22) and (3.68), we get

$$(3.70)\quad \frac{1}{4}c((d^N \;\; {}_j))c(V_j^M) = \frac{1}{2}\;{}_4 \quad c(J^N V_j^N)c(V_j^M)$$
$$+ \frac{1}{4} \quad 8\;{}^0(\;)\;{}^0(j\;j^2)\;{}_i\;{}_i\frac{(\;{}_2 \quad {}_4)\;{}_4}{{}_2^2}$$
$$4\;{}^{00}(\;)\;(j\;j^2)\;{}_i\;{}_i + 4(\;{}^0(\;))^2\;(j\;j^2)\;{}_i^2 \quad 2\;{}^0(\;)\;\frac{2(\;{}_2 \quad {}_4)\;{}_5}{{}_2^2}$$
$$+ 2\;{}^0(\;)(1 \quad 2\;(\;))\frac{2\;{}_5}{2} \quad 4\;{}^0(\;)\;(j\;j^2)\frac{{}_2 \quad {}_4}{2} \quad c(\;{}_k J^N V_k^N)c(\;{}_j V_j^M):$$

From (3.70), we get the second equation of (3.58).

3.6. Estimates of I_4 and I_5. In this section, we establish the following estimate result for the terms I_4, I_5 appeared in Subsection 3.5.

Lemma 3.9. When $A > 0$ is large enough, we have

$$(3.71)\quad \begin{aligned} I_4 &= 2\ {}^0(\ )\ 1\quad (\ )\quad (j\ {}^2j)\ (1 + O(A^{1=2})); \\ I_5 &= {}^0(\ )\ (1\quad (\ ))^2\quad (j\ {}^2j)\ (1 + O(A^{1=2})): \end{aligned}$$

Moreover

$$(3.72)\quad \begin{aligned} I_4 &> 0 \quad \text{if } (x;y) 2 M\ n @M\ ; \\ I_4 &= 0 \quad \text{if } (x;y) 2 @M\ = M_1 [M_2: \end{aligned}$$

Proof. If $j\ j^2 6\ \frac{5A}{3}$, by Lemma 3.5, (3.2), (3.35), (3.36) and (3.38), we get

$$(3.73)\quad \begin{aligned} I_4 = {}^0(\ )O(A^{1})(1\quad (\ ))^2\quad {}^0(\ )O(A^{1=2})(1\quad (\ )) \\ +\ 2\quad {}^{00}(\ )\ {}_i\ {}_i + (\ {}^0(\ ))^2\ {}^2_i\quad 1\quad (\ )\quad (j\ {}^2j)^2\ (1 + O(A^{1=2})) \\ +\ 2\ {}^0(\ )\ 1\quad (\ )\quad (j\ {}^2j)\ (1 + O(A^{1=2})): \end{aligned}$$

By (3.1), (3.36), (3.39), (3.45) and (3.46), there exist $C > 0$ such that for any $A > A_0$, if $j\ j^2 6\ \frac{5A}{3}$, then

$$(3.74)\quad \begin{aligned} &1\quad (\ ) 6\ Cj1\quad (\ )\quad (j\ {}^2)jj;\quad (j\ j^2) 6\ Cj1\quad (\ )\quad (j\ {}^2)j; \\ &jA\ {}^0(\ )j 6\ Cj1\quad (\ )\quad (j\ {}^2)j: \end{aligned}$$

By (3.39), (3.73) and (3.74), we get (3.71) for I_4.

If $j\ j^2 > \frac{5A}{3}$, then by (3.1), (3.41), (3.42) and (3.56),

$$(3.75)\quad \begin{aligned} I_4 &= O(A^{3=2})\ (\ )^2\quad 2\ {}^0(\ )\ (\ )(1 + O(A^{1=2})) \\ &= \quad 2\ {}^0(\ )\ (\ )(1 + O(A^{1=2})): \end{aligned}$$

Thus we get the rst equation of (3.71).

As $e^0(t) < 0$ for $t > 0$, thus ${}^0(\ ) < 0$ on $j\ j^2 > A$, from (3.31), (3.71), we get (3.72).

If $j\ j^2 6\ \frac{5A}{3}$, by (3.1), (3.43), for A large enough, $(\ ) > (\frac{5A}{3}) + O(A^{1=2}) > \frac{1}{3}$, and so by (3.28) and (3.29),

$$(3.76)\quad {}_2\quad {}_4 = \quad (\ ) + O(A^{1=2}):$$

Thus if $j\ j^2 6\ \frac{5A}{3}$, by Lemma 3.5, (3.1), (3.38), (3.40) and (3.76), we have

$$(3.77)\quad \begin{aligned} I_5 = {}^0(\ )(1\quad (\ ))O(A^{1=2}) + (1\quad (\ ))\ 1\quad (\ )\quad (j\ {}^2j)\ O(A^{3=2}) \\ +\ {}^0(\ )\ 1\quad (\ ) + O(A^{1=2})\quad 1\quad (\ )\quad (j\ {}^2j) \\ {}^0(\ )\ (j\ j^2)\quad (\ ) + O(A^{1=2})\ : \end{aligned}$$

Now by (3.39), (3.48), (3.74) and (3.77), we get (3.71) for I_5.

If $j\ j^2 > \frac{5A}{3}$, then by (3.18) and (3.41), we have ${}_1 = 1$ and

$$(3.78)\quad {}_2\quad {}_4 = \quad (\ ):$$

Thus if $|\xi|^2 > \frac{5A}{3}$, by (3.1), (3.2), (3.38), (3.41), (3.42), (3.56) and (3.78), we get

$$(3.79)\quad I_5 = (\psi'(\rho))^2 \,(\rho)^2 O(A^{1/2}) + \psi'(\rho) \Big[\rho\,|\xi|^2 (1 + O(A^{1/2})) - (1 - 2\psi(\rho))\,\rho\,(1 + O(A^{1/2})) - \rho\,(1 + O(A^{1/2})) \Big] :$$

As $0 < \psi(\rho) < 1$ on $M \setminus \partial M$, from (3.2) and (3.79), we get if $|\xi|^2 > \frac{5A}{3}$,

$$(3.80)\qquad I_5 = \psi'(\rho)\,\rho\,(\rho)(2 - \psi(\rho))(1 + O(A^{1/2})):$$

From (3.41), (3.80), we get again (3.71) for I_5.

The proof of Lemma 3.9 is completed.

3.7. Estimates of $I_1 + I_2$. In this subsection, we establish an estimate result on zero$(X^{\mathcal{H}})$. Before doing this, we first prove the following lemma.

For any $x \in M, y \in N$, $W \in T_xM$, let $B(W) \in \mathrm{End}(\Lambda(T^{*(0,1)}(M \times N)))_{(x,y)}$ be defined by

$$(3.81)\qquad B(W) = \sqrt{-1}\,c(J^M W)c(W) + |W|^2 :$$

Lemma 3.10. For any $W \in T_xM$, $V \in T_yN$, the endomorphisms $B(W)$, $\sqrt{-1}c(W)c(V)$ of $\Lambda(T^{*(0,1)}(M \times N))_{(x,y)}$ are self-adjoint and for any $k > 0$,

$$(3.82)\qquad \begin{aligned} &B(W) \geq 0;\\ &\sqrt{-1}c(W)c(V) \geq -\frac{1}{2k}B(W) - k|V|^2 : \end{aligned}$$

Proof. As W, V are orthogonal to each other, $B(W)$, $\sqrt{-1}c(W)c(V)$ are self-adjoint.

For $W \in T_xM$, $V \in T_yN$, we write their complexification as $W = w + \overline{w}$, $V = v + \overline{v}$, with $w \in T^{(1,0)}M$, $\overline{w} \in T^{(0,1)}M$, $v \in T^{(1,0)}N$, $\overline{v} \in T^{(0,1)}N$. Let $\overline{w}^* \in T^{*(0,1)}M$, $\overline{v}^* \in T^{*(0,1)}N$ be the metric duals of w, v. Then we deduce from (1.34) that

$$(3.83)\qquad \begin{aligned} B(W) &= c(\sqrt{-1}w - \sqrt{-1}\overline{w})c(w + \overline{w}) + |W|^2\\ &= -2(\overline{w}^* \wedge + i_{\overline{w}})(\overline{w}^* \wedge - i_{\overline{w}}) + 2|w|^2 = 4\overline{w}^* \wedge i_{\overline{w}} : \end{aligned}$$

Thus we get the first equation of (3.82) which has been obtained in [20, (2.9), (2.13)].

Moreover,

$$(3.84)\qquad \sqrt{-1}c(W)c(V) = 2\sqrt{-1}\,(\overline{w}^* \wedge \overline{v}^* \wedge - \overline{w}^* \wedge i_{\overline{v}} - i_{\overline{w}}\overline{v}^* \wedge + i_{\overline{w}} i_{\overline{v}}):$$

If $\alpha \in \Lambda(T^{*(0,1)}(M \times N))_{(x,y)}$, then we can write

$$(3.85)\qquad \alpha = \overline{w}^* \wedge \overline{v}^* \wedge \alpha_1 + \overline{w}^* \wedge \alpha_2 + \overline{v}^* \wedge \alpha_3 + \alpha_4;$$

and α_i $(i = 1;2;3;4)$ do not contain the terms $\overline{w}^*$ or $\overline{v}^*$. Then we get

$$(3.86)\qquad \begin{aligned} B(W)\alpha &= 4|w|^2\overline{w}^* \wedge \overline{v}^* \wedge \alpha_1 + 4|w|^2\overline{w}^* \wedge \alpha_2;\\ \sqrt{-1}c(W)c(V)\alpha &= 2\sqrt{-1}|w|^2|v|^2\alpha_1 + 2\sqrt{-1}|w|^2\overline{v}^* \wedge \alpha_2\\ &\quad 2\sqrt{-1}|v|^2\overline{w}^* \wedge \alpha_3 + 2\sqrt{-1}\overline{w}^* \wedge \overline{v}^* \wedge \alpha_4 : \end{aligned}$$

From (3.86),

$$(3.87)\qquad \langle B(W)\alpha, \alpha\rangle = 4|w|^4|v|^2|\alpha_1|^2 + 4|w|^4|\alpha_2|^2 = 4|i_{\overline{w}}\alpha|^2 :$$

From (3.86), (3.87), we get for any $k > 0$,

$$(3.88)\qquad \begin{aligned}\langle \sqrt{-1}c(W)c(V)\sigma, \sigma\rangle &= 4|w|^2|v|^2\,\mathrm{Im}\langle \sigma_1, \sigma_4\rangle - 4|w|^2|v|^2\,\mathrm{Im}\langle\sigma_2,\sigma_3\rangle\\ &\geq -\frac{2}{k}|\sigma_{\bar w}|^2 - 2k|v|^2|\sigma|^2.\end{aligned}$$

From (3.87), (3.88), we get the second equation of (3.82).

Recall that the terms I_1, I_2 have been defined in (3.10). We now state the crucial estimate for $I_1 + I_2$ as follows.

Proposition 3.11. *There exist $C > 0$, $A_0 > 0$ such that for any $A > A_0$ and $(x,y) \in \mathrm{zero}(X^H)\times M$, we have*

$$(3.89)\qquad \sqrt{-1}(I_1 + I_2) \geq -C\,\mathrm{Id}.$$

Proof. By (3.10), and apply Lemma 3.10 to (3.57) and (3.58) with $k = 8$, we get

$$(3.90)\qquad \begin{aligned}\sqrt{-1}(I_1 + I_2) \geq{}& \left(\frac12\eta_2 - \frac18\eta_2 - \frac18\eta_4\right)\sum_{j=1}^{\dim G} B(V_j^M) - (8\eta_2 + 8\eta_4)\sum_{j=1}^{\dim G}|V_j^N|^2\\ &+\left(I_4 - \frac18 I_4 - \frac18 I_5\right) B\left(\sum_{j=1}^{\dim G}|V_j^M|\right)^2\\ &-(16I_4 + 16I_5)\left(\sum_{j=1}^{\dim G}|V_j^N|\right)^2.\end{aligned}$$

By Lemma 3.5, $0 < \rho(\cdot) < 1$ on $M\setminus\partial M$, we get for A large enough

$$(3.91)\qquad \eta_2 - \frac18\eta_2 - \frac18\eta_4 = \frac34 + \frac18\rho(\cdot) + O(A^{-1/2}) > 0.$$

From Lemmas 3.6, 3.9, we know that when $A > 0$ is large enough,

$$(3.92)\qquad I_4 - \frac18 I_4 - \frac18 I_5 \geq \frac18 I_4 \geq 0 \quad \text{if } (x,y)\in M\setminus(M_1\cup M_2).$$

As $|V_j^N|$, η_j, I_j have upper bound not depending on $x \in M$, thus $|V_j^N|^2$, $\left(\sum_{j=1}^{\dim G}|V_j^N|\right)^2$ are uniformly bounded on M.

As $\mathrm{zero}(X^H) \times M \subset M\setminus\partial M$, from (3.82), (3.90), (3.91) and (3.92), there exist $A_0 > 0$, $C > 0$ such that for any $A > A_0$, and $(x,y) \in \mathrm{zero}(X^H)\times M$, (3.89) holds.

3.8. A local point estimate around zero(X^H). Recall that D_T^{M} is the deformed Dirac operator on M defined by (3.9) according to the functions defined in (3.1), (3.2) and (3.3). For any $T > 0$, let $F_T^M : \Omega^{0,\bullet}(M, L\otimes F) \to \Omega^{0,\bullet}(M, L\otimes F)$ be defined by

$$(3.93)\qquad F_T^M = D_T^{M,2} + \sqrt{-1}T\sum_j L_{V_j}.$$

The purpose of this subsection is to prove the following local pointwise estimate around each $x \in \mathrm{zero}(X^H)$.

Proposition 3.12. There exists $A_0 > 0$ such that for any $A > A_0$ a regular value of both $|\mu|^2$ and $\frac{1}{2}|\mu|^2$, and that if M_A is constructed as in (2.11), then for any $z \in M_A \setminus \partial M_A$, there exist an open neighborhood U_z of z and $C_z > 0$, $b_z > 0$ such that for any $s \in \Omega^{0,\bullet}(M_A, L \otimes F)$ with $\mathrm{supp}(s) \subset U_z$ and $T > 1$, one has

$$\mathrm{Re}\left\langle F_T^M s, s\right\rangle > C_z\left(\left\|D^{L\otimes F} s\right\|_0^2 + (T - b_z)\|s\|_0^2\right). \tag{3.94}$$

Proof. Recall that $\{f_i\}$ is an orthonormal frame of TN. Set

$$R_j = \frac{\sqrt{-1}}{4}\sum_{i=1}^{\dim N} c(f_i)c(\nabla^{TN}_{f_i} V_j^N) - \frac{\sqrt{-1}}{2}\mathrm{Tr}\left[\nabla^{T^{(1,0)}N} V_j^N|_{T^{(1,0)}N}\right]. \tag{3.95}$$

By (3.11), (3.93) and (3.95),

$$\begin{aligned} F_T^M = D^{L\otimes F,2} &+ \frac{\sqrt{-1}T}{4}\sum_{k=1}^{\dim M} c(e_k)c\left(\nabla^{TM}_{e_k}(\mu_j V_j^M)\right) \\ &- \frac{\sqrt{-1}T}{2}\mathrm{Tr}\left[\nabla^{T^{(1,0)}M}(\mu_j V_j^M)|_{T^{(1,0)}M}\right] \\ &+ T\left(-2\mu_j\mathcal{L}_j + \mu_j R_j + \sqrt{-1}(I_1 + I_2 + I_3)\right) + \frac{T^2}{4}|X^{\mathcal{H}}|^2. \end{aligned} \tag{3.96}$$

By (3.10), (3.18), (3.19), (3.35), (3.67), $= \ +$ and $= \ (\ )$, we know that there exists $C > 0$ such that for any A large enough, on M_A

$$|(d^N\mu_j)| \leqslant C; \quad \text{thus } |I_3| \leqslant C. \tag{3.97}$$

Recall that by (3.20), $\mu_j = 2\varphi_2\mu_j + 2\varphi_4\mu_j$. Thus from Lemma 3.5, (2.3), (2.12), (3.32) and (3.34)

$$\begin{aligned} &|\mu_j| = O(A^{1/2}); \\ &\mu_j\mu_j = 2\varphi_2|\mu|^2 + 2\varphi_4|\mu|^2 + 2(\varphi_2 + \varphi_4)\langle\mu, \mu\rangle \\ &\qquad = 2|\mu|^2(1 + O(A^{-1/2})). \end{aligned} \tag{3.98}$$

By Proposition 3.11, (3.2), (3.38), (3.97) and (3.98), there exists $A_0 > 0$, such that for any $A > A_0$ which is a regular values of the functions $|\mu|^2$ and $\frac{1}{2}|\mu|^2$ on $M \setminus N$, and $(x, y) \in \{X^{\mathcal{H}} = 0\} \cap M_A$, we have

$$\begin{aligned} &-2\mu_j\mathcal{L}_j + \frac{1}{2}\mu_j(R_j + R_j) + \sqrt{-1}(I_1 + I_2 + I_3) > A; \\ &|\mu^0(\ )_i\ _i| < \frac{1}{4}. \end{aligned} \tag{3.99}$$

Let's fix $z = (x_0, y_0) \in M_A \setminus \partial M_A$.

If $X^{\mathcal{H}}(x_0, y_0) \neq 0$, then (3.94) follows trivially from (3.96).

If $X^{\mathcal{H}}(x_0, y_0) = 0$, we will adapt the argument in [20, §2b)].

Let $\{e_1,\ldots,e_{\dim M}\}$ be an orthogonal frame of TM. Set

$$\Delta^{M} = -\sum_{j=1}^{\dim M}\Big((\nabla^{0,\bullet}_{e_j})^2 - \nabla^{0,\bullet}_{\nabla^{TM}_{e_j}e_j}\Big),$$

(3.100)

$$\Delta^{N} = -\sum_{j=1}^{\dim N}\Big((\nabla^{0,\bullet}_{f_j})^2 - \nabla^{0,\bullet}_{\nabla^{TN}_{f_j}f_j}\Big), \quad \Delta^{M\times N} = \Delta^{M} + \Delta^{N},$$

be the Bochner Laplacians acting on $\Omega^{0,\bullet}(M,L)$, $\Omega^{0,\bullet}(N,F)$, $\Omega^{0,\bullet}(M\times N, L\boxtimes F)$. From the Lichnerowicz formula for $D^{L\boxtimes F,2}$ [12, Theorem 1.3.5], [11, Appendix D], one finds on $M\times N$,

$$D^{L\boxtimes F,2} = \Delta^{M\times N} + O(1). \tag{3.101}$$

Let $\{e_1,\ldots,e_{\dim M}\}$ be an orthogonal base of $T_{x_0}M$. Let $(x_1,\ldots,x_{\dim M})$ be the normal coordinate system with respect to $\{e_j\}_{j=1}^{\dim M}$ near x_0. Certainly, we can choose $\{e_1,\ldots,e_{\dim M}\}$ such that the function $|\mu|^2(\cdot,y_0)$ has the following expression near x_0,

$$|\mu|^2(x,y_0) = |\mu|^2(x_0,y_0) + \sum_{j=1}^{\dim M} a_j x_j^2 + O(|x|^3), \tag{3.102}$$

where the a_j's may possibly be zero.

The following Lemma is an analogue of [20, Lemma 2.3].

Lemma 3.13. *The following inequality holds at the point (x_0,y_0),*

$$\frac{\sqrt{-1}}{4}\sum_{k=1}^{\dim M} c(e_k)c\big(\nabla^{TM}_{e_k}(\mu_j V_j^M)\big) - \frac{\sqrt{-1}}{2}\mathrm{Tr}\Big[\nabla^{T^{(1,0)}M}(\mu_j V_j^M)|_{T^{(1,0)}M}\Big] \geqslant \sum_{j=1}^{\dim M}|a_j|, \tag{3.103}$$

where the inequality is strict if at least one of the a_j's is negative.

Proof. Let $\{e_1,\ldots,e_{\dim M}\}$ be an orthonormal frame of TM near x_0 so that at x_0, $e_j = e_j$ for $1\leqslant j\leqslant \dim M$. From Lemma 3.7 and (3.102), we find that

$$(\mu_j V_j^M)_{(x,y_0)} = 2\sum_{j=1}^{\dim M} t_j(x,y_0)J^M e_j, \quad \text{and} \quad t_j(x,y_0) = a_j x_j + O(|x|^2). \tag{3.104}$$

From (3.81), (3.83), (3.104), we deduce that at the point (x_0,y_0),

$$\begin{aligned}
&\frac{\sqrt{-1}}{4}\sum_{k=1}^{\dim M} c(e_k)c\big(\nabla^{TM}_{e_k}(\mu_j V_j^M)\big) - \frac{\sqrt{-1}}{2}\mathrm{Tr}\Big[\nabla^{T^{(1,0)}M}(\mu_j V_j^M)|_{T^{(1,0)}M}\Big]\\
&= \frac{\sqrt{-1}}{2}\sum_{j=1}^{\dim M} a_j c(e_j)c(J^M e_j) - \frac{\sqrt{-1}}{2}\sum_{j=1}^{\dim M}\Big\langle\Big(1+\frac{J^M}{\sqrt{-1}}\Big)(a_j J^M e_j), e_j\Big\rangle\\
&= \frac{1}{2}\sum_{j=1}^{\dim M} a_j\big(B(e_j) - 2\big) \geqslant \sum_{j=1}^{\dim M}|a_j|,
\end{aligned} \tag{3.105}$$

where the last inequality is strict if at least one of the a_j's is negative.

Let $(y_1, \dots, y_{\dim N})$ be a normal coordinate system near y_0 on $U_{y_0} \subset N$. Let F_T^{M*} be formal adjoint of F_T^M. From (3.96), (3.99), (3.101), (3.103) and (3.104), we find that near (x_0, y_0),

$$(3.106)\quad \frac{1}{2}(F_T^M + F_T^{M*}) \geq -\Delta^{M\times N} - T\sum_{j=1}^{\dim M}|a_j| + T^2\sum_{j=1}^{\dim M} t_j(x,y_0)^2 + \frac{T^2}{4}\sum_{j=1}^{\dim G}\left|V_j^N\right|^2 + TA + O(1+T|x|).$$

Following [20, §2b)], let $\varepsilon > 0$, which will be further fixed later, be sufficiently small so that the orthonormal frame $\{e_j\}_{j=1}^{\dim M}$ is well defined over

$$(3.107)\quad B_\varepsilon(x_0) = \{x \in M : d(x,x_0) < \varepsilon\}.$$

Let $(\nabla_{e_j})^*$ be the formal adjoint of $\nabla_{e_j}^{0,\bullet}$ on $B_\varepsilon(x_0)$. Set

$$(3.108)\quad \mathcal{L}_T^M = \sum_{j=1}^{\dim M}\left((\nabla_{e_j})^* + T(\mathrm{sgn}\, a_j)t_j(x,y_0)\right)\left(\nabla_{e_j}^{0,\bullet} + T(\mathrm{sgn}\, a_j)t_j(x,y_0)\right).$$

Clearly, $\mathcal{L}_T^M$ is nonnegative when acting on compactly supported sections over $B_\varepsilon(x_0)$. Moreover, we verify from (3.104) that

$$(3.109)\quad \mathcal{L}_T^M = -\Delta^M - T\sum_{j=1}^{\dim M}|a_j| + T^2\sum_{j=1}^{\dim M} t_j(x,y_0)^2 + O(\partial_x + 1 + T|x|),$$

where by $O(\partial_x + 1 + T|x|)$ we mean a first order differential operator

$$(3.110)\quad \sum_{j=1}^{\dim M} b_j(x)\frac{\partial}{\partial x_j} + d(x) \quad \text{with } b_j(x) = O(1); \quad d(x) = O(1+T|x|).$$

We will also use similar notation for other operators.

From (3.106), (3.108), (3.109), we get, when acting on sections with compact support in $B_\varepsilon(x_0) \times U_{y_0}$, for any $k > 1$,

$$(3.111)\quad \begin{aligned} \frac{1}{2}(F_T^M + F_T^{M*}) &\geq -\Delta^N + \mathcal{L}_T^M + TA + O(\partial_x + 1 + T|x|) \\ &\geq -\frac{1}{k}\Delta^N - \frac{1}{k}\Delta^M - \frac{T}{k}\sum_{j=1}^{\dim M}|a_j| + TA + O(\partial_x + 1 + T|x|). \end{aligned}$$

For any $s \in \Omega^{0,\bullet}(M \times N, L \otimes F)$ with $\mathrm{supp}(s) \subset B_\varepsilon(x_0) \times U_{y_0}$, by (3.101), it is standard that there exist $C_1, C_2, C_3 > 0$ such that

$$(3.112)\quad \begin{aligned} \langle -\Delta^{M\times N}s, s\rangle &\geq C_1\|D^{L\otimes F}s\|_0^2 - C_2\|s\|_0^2, \\ \langle O(1+T|x|)s, s\rangle &\leq C_3(1+T\varepsilon)\|s\|_0^2. \end{aligned}$$

Also by the elliptic estimate and Cauchy inequality, there exist $C_4, C_5 > 0$ such that

$$(3.113)\quad \langle O(\partial_x)s, s\rangle \leq C_4\varepsilon\|D^{L\otimes F}s\|_0^2 + \frac{C_5}{\varepsilon}\|s\|_0^2.$$

From (3.111)–(3.113), we get

$$\text{(3.114)}\quad \operatorname{Re}\langle F_T^M s, s\rangle \geq \left(\frac{C_1}{k} - C_4\varepsilon\right)\|D^{L\otimes F} s\|_0^2 + T\left(A - C_3\varepsilon - \frac{1}{k}\sum_{j=1}^{\dim M}|a_j|\right)\|s\|_0^2 - \left(\frac{C_5}{\varepsilon} + \frac{C_2}{k} + C_3\right)\|s\|_0^2.$$

Now, we take k large enough such that $A - \frac{1}{k}\sum_{j=1}^{\dim M}|a_j| > A/2$, then we choose ε small enough such that

$$\text{(3.115)}\quad \frac{C_1}{k} - C_4\varepsilon > 0; \quad A/2 - C_3\varepsilon > A/4.$$

With this choice of ε, by (3.114)–(3.115), we get Proposition 3.12 at (x_0, y_0).

3.9. Proof of Theorem 3.1. We now assume that $A \geq A_0$ is a regular value for both $|\mu|^2$ and $\frac{1}{2}|\mu|^2$, with A_0 verifying the conditions in Proposition 3.12. In particular, by (3.6), X^H is nowhere zero on $\partial M = M_1 \cup M_2$.

Let e_n be the inward pointing unit normal at any boundary point of M. Set

$$\text{(3.116)}\quad D_{\partial M,T} = D^{L\otimes F}_{\partial M} - \frac{\sqrt{-1}T}{2} c(e_n)c(X^H) : \Omega^{0,*}(M, L\otimes F)|_{\partial M} \to \Omega^{0,*}(M, L\otimes F)|_{\partial M}$$

be the Dirac type operator on ∂M induced by D_T^M.

Since X^H is nowhere zero on ∂M, in view of (3.5), by Proposition 1.2, we know that when restricted to the G-invariant subspace of $\Omega^{0,*}(M, L\otimes F)|_{\partial M}$, there exists $T_1 > 0$ such that for any $T \geq T_1$, $D_{\partial M,T}$ is invertible. Moreover, a similar formula like (1.21) holds.

One can then proceed as in the proof of (1.48), to see that there exists an open neighborhood $U_{\partial M}$ of M and positive constants $C_1 > 0$, $b_1 > 0$ such that for $T \geq T_1$ and any G-invariant element of $\Omega^{0,*}(M, L\otimes F)$ such that $\operatorname{supp}(s) \subset U_{\partial M}$ and $P_{\geq 0,T}(s|_{\partial M}) = 0$, where $P_{\geq 0,T}$ are the APS projections associated to $D_{\partial M,T}$ (cf. Section 1.2), one has

$$\text{(3.117)}\quad \left\|D_T^M s\right\|_0^2 \geq C_1\left\|D^{L\otimes F} s\right\|_0^2 + (T - b_1)\|s\|_0^2.$$

From (3.93), (3.117) and Proposition 3.12, one can then proceed as in the proof of (1.50) to see that there exist $C > 0$, $b > 0$ such that for any $T \geq T_1$ and G-invariant element of $\Omega^{0,*}(M, L\otimes F)$ such that $P_{\geq 0,T}(s|_{\partial M}) = 0$, one has

$$\text{(3.118)}\quad \left\|D_T^M s\right\|_0^2 \geq C\left\|D^{L\otimes F} s\right\|_0^2 + (T - b)\|s\|_0^2.$$

By taking $T \geq T_1$ large enough, we get Theorem 3.1.

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