MAXIMAL SOLUTIONS FOR $-\Delta u + u^q = 0$ IN OPEN AND FINELY OPEN SETS

MOSHE MARCUS AND LAURENT VERON

Abstract. We derive sharp estimates for the maximal solution U of (*) $-\Delta u + u^q = 0$ in an arbitrary open set $D \subset \mathbb{R}^N$ The estimates involve the Bessel capacity $C_{2,q'}$, for q in the supercritical range $q \geq q_c := N/(N-2)$. We provide a pointwise necessary and sufficient condition, via a Wiener type criterion, in order that $U(x) \to \infty$ as $x \to y$ for given $y \in \partial D$. This completes the study of such criterions carried out in [10] and [18]. Further, we extend the notion of solution to $C_{2,q'}$ finely open sets and show that, under very general conditions, a boundary value problem with blow-up on a specific subset of the boundary is well-posed. This implies, in particular, uniqueness of large solutions.

Solutions maximales de $\Delta u = u^q$ dans des ensembles ouverts et finement ouverts

Résumé. Nous démontrons des estimations précises pour la solution maximale U de (*) $-\Delta u + u^q = 0$ dans un domaine arbitraire $D \subset \mathbb{R}^N$. Ces estimations impliquent la capacité de Bessel $C_{2,q'}$, pour q appartenant à l'intervalle sur-critique $q \geq q_c := N/(N-2)$. Nous donnons une condition nécessaire et suffisante ponctuelle, via un critère de type Wiener, pour que $U(x) \to \infty$ quand $x \to y$ pour un $y \in \partial D$ arbitraire. Ce résultat complète l'étude de tels critères menée dans [10] et [18]. En outre, nous étendons la notion de solution à des ensembles finement ouverts pour la topologie $C_{2,q'}$ et montrons que, sous des conditions très générales, un problème aux limites avec explosion sur un sous-ensemble spécifique du bord est bien posé. Cela implique en particulier l'unicité des grandes solutions.

Contents

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1. Introduction

In this paper we study solutions of the equation

$$-\Delta u + |u|^{q-1}u = 0, \tag{1.1}$$

in $\Omega \setminus F$, Ω a smooth domain in $\mathbb{R}^N$, $N \geq 3$ and $F \subset \Omega$, F compact or, more generally, a bounded set, closed in the $C_{2,q'}$ fine topology. Here $q > 1$ and $C_{2,q'}$ refers to the Bessel capacity with the specified indexes. If $1 < q < q_c = N/(N-2)$ then the fine topology is equivalent to the Euclidean topology. Therefore, throughout the paper we shall assume that $q \geq q_c$, in which case the two topologies are different.

If D is an open set and μ is a Radon measure in D, a function $u \in L^q_{\text{loc}}(D)$ is a solution of

$$-\Delta u + |u|^{q-1}u = \mu \ \text{ in } \ D \tag{1.2}$$

if the equation is satisfied in the distribution sense. It is known [6] that (1.2) possesses a solution if and only if μ vanishes on sets of $C_{2,q'}$ capacity zero. When this is the case we say that μ satisfies the $(\text{B-P})_q$ condition (i.e., the Baras-Pierre condition). If $D = \mathbb{R}^N$ and μ is a Radon measure satisfying this condition then (1.2) possesses a *unique* solution.

Further, if D is open, it is known that $C_{2,q'}(\mathbb{R}^N \setminus D) = 0$ if and only if the only solution of (1.1) in D is the trivial solution. In view of the Keller – Osserman estimates, the set of solutions of (1.1) in D (denoted by $\mathcal{U}_D$) is uniformly bounded in compact subsets of D and every sequence of solutions possesses a subsequence which converges to a solution u. Finally the compactness together with the maximum principle imply that $\max \mathcal{U}_D$ is a solution in D. The *maximal solution* in D is denoted by U_F, $F = \mathbb{R}^N \setminus D$.

Now suppose that $F = \cup_{n=1}^{\infty} K_n$ where $\{K_n\}$ is an increasing sequence of compact sets such that

$$C_{2,q'}(F \setminus K_j) \to 0.$$

Then $\{U_{K_n}\}$ is non-decreasing and we denote $V_F := \lim U_{K_n}$. In this case F may not be closed; in fact, it may be dense in $D = F^c$, so that in general we cannot apply the Keller – Osserman estimates. Therefore, on this basis, it is not even clear whether V_F is finite a.e. in D. It will be shown in the course of this paper that this is actually the case.

Naturally, further questions come up: Is V_F, in some sense, a generalized solution of (1.1) in D and, if so, is it the maximal solution? Is it possible to characterize V_F in terms of its behavior at the boundary?

The main objective of this paper is the study of properties of the maximal solution of (1.1) in F^c, *first* in the case that F is compact; *secondly* in the case that F is merely $C_{2,q'}$-finely closed. In the second case we introduce a new notion of solution which we call a $C_{2,q'}$*-strong solution* (see Definition 7.1) and show that V_F is indeed a solution in this sense and that it is the maximal solution. We also show that many of the properties of the set of classical solutions are shared by the class of $C_{2,q'}$-strong solutions.

For F *compact*, the properties of U_F have been intensively investigated, especially in the last twenty years. A question that received special attention was the existence, uniqueness and estimates of solutions of the boundary value problem

$$\begin{aligned} &-\Delta u + |u|^{q-1}u = 0 \quad \text{in} \quad D = F^c, \\ &\lim_{D\ni x\to y} u(x) = \infty \quad \forall y \in \partial D. \end{aligned} \tag{1.3}$$

The question of existence reduces to the question whether U_F blows up everywhere on the boundary.

A solution of (1.3) is called a *large solution* of (1.1) in D. If D is a smooth domain with compact boundary, it is known that a large solution exists and is unique, (see [22], [2], [3], [32]). These results were extended in various ways, weakening the assumptions on the domain, extending it to more general classes of equations and obtaining more information on the asymptotic behavior of solutions at the boundary, (see [4], [23], [21], [5] and references therein).

In the present paper we also consider two related notions:

(a) A solution u is an *almost large solution* of (1.1) in D if

$$\lim_{D\ni x\to y} u(x) = \infty \quad C_{2,q'} \text{ a.e. } y \in \partial F. \tag{1.4}$$

This notion is, in a sense, more natural, because (as we shall show) U_F is invariable with respect to $C_{2,q'}$ equivalence of sets. (Two Borel sets E , F are $C_{2,q'}$ equivalent if $C_{2,q'}(F\triangle E) = 0$.)

(b) A solution u of (1.1) is a ∂_q*-large solution* in D if

$$\lim_{D\ni x\to y} u(x) = \infty \quad C_{2,q'} \text{ a.e. } y \in \partial_q F, \tag{1.5}$$

where $\partial_q F$ denotes the boundary of F in the $C_{2,q'}$-fine topology.

Here is a quick review of results pertaining to the case F compact.

In the *subcritical case*, i.e. $1 < q < q_c := N/(N-2)$, the properties of U_F are well understood. In this case $C_{2,q'}(F) > 0$ for any non-empty set and it is classical that positive solutions may have isolated point singularities of two types: weak and strong. This easily implies that the maximal solution U_F is always a large solution in F^c. Sharp estimates of the large solution where obtained in [28]. In addition it is proved in [33] that the large solution is unique if $\partial F^c \subset \partial \overline{F^c}^c$.

In the subcritical case, solutions with point singularities served as building blocks for solutions with general singularities. In the *supercritical case*, i.e. $q \geq q_c$, the situation is much more complicated, because there are no solutions with point singularities.

Sharp estimates for U_F were obtained by Dhersin and Le Gall [10] in the case $q = 2$, $N \geq 4$. These estimates were expressed in terms of the Bessel capacity $C_{2,2}$ and were used to provide a Wiener type criterion – to which we refer as (WDL; 2) – for the pointwise blow up of U_F, i.e., given $y \in F$,

$$(1.6) \qquad \lim_{F^c \ni x \to y} U_F(x) = \infty \iff \text{the (WDL; 2) criterion is satisfied at y.}$$

These results were obtained by probabilistic tools; hence the restriction to $q = 2$.

Labutin [18] extended the results of [10] in the case $q > q_c$. Specifically, he obtained sharp estimates for U_F similar to those in [10], with $C_{2,2}$ replaced by $C_{2,q'}$. These estimates were used to obtain a Wiener criterion involving $C_{2,q'}$ (we refer to it as (WDL;q)) relative to which the following was proved:

$$(1.7) \qquad U_F \text{ is a large solution} \iff \text{(WDL;q) holds } \textit{everywhere} \text{ in } F.$$

Of course this result is weaker then (1.6). However a careful examination of Labutin's proof reveals that, in the case $q > q_c$, his argument actually proves (1.6). In the case $q = q_c$ Labutin's estimate was not sharp and it did not yield (1.6) although it was sufficient in order to obtain (1.7).

Uniqueness was not discussed in the above papers. Necessary and sufficient conditions are not yet known. Sufficient conditions for uniqueness of large solutions, for arbitrary $q > 1$, can be found in [23], [27] and references therein. Uniqueness will also be one of the main subjects of the present work.

The first part of the present paper (Sections 2-4) is devoted to the study of the maximal solution U_F when F *is compact* and of the almost large solution in bounded open sets. Here is the list of main results obtained in this part of the paper:

I. Sharp capacitary estimates of U_F in the full supercritical range $q \geq q_c$, $N \geq 3$. As a result, we show that a variant of (1.6) holds in the entire supercritical range. Specifically, we show that, for $y \in F$,

$$(1.8) \qquad \lim_{F^c \ni x \to y} U_F(x) = \infty \iff W_F(y) = \infty,$$

where $W_F : \mathbb{R}^N \to [0, \infty]$ is the *capacitary potential of* F, (see (2.2) for its definition).

For $q > q_c$ the condition $W_F(y) = \infty$ is equivalent to the (WDL;q) criterion mentioned before. However our proof does not require separate treatment of the border case $q = q_c$ and is simpler than the proof in [18] even for $q > q_c$.

II. *For every compact set F, U_F is an almost large solution in F^c and U_F is σ-moderate.*

The statement 'U_F is σ-moderate ' means that there exists a monotone increasing sequence of bounded, positive measures concentrated in F, $\{\mu_n\}$, satisfying the (B-P)$_q$ condition, such that $u_{\mu_n} \uparrow U_F$.

Finally we establish an existence and uniqueness result; for its statement we need some additional notation. For any set $E \subset \mathbb{R}^N$

$$\widetilde{E} = \text{closure of } E \text{ in the } C_{2,q'}\text{-fine topology}, \quad \partial_q E := \widetilde{E} \cap \widetilde{E^c}.$$

III. *Let $\Omega = \cup \Omega_n$, where $\{\Omega_n\}$ is an increasing sequence of open sets, and put $D_n = \mathbb{R}^N \setminus \overline{\Omega}_n$. Assume that*

$$C_{2,q'}(\Omega \setminus \Omega_n) \to 0 \ \textit{ and } \ C_{2,q'}(\partial\Omega_n \setminus \widetilde{D}_n) \to 0. \tag{1.9}$$

Then the boundary value problem

$$-\Delta u + u^q = 0 \ \textit{ in } \ \Omega, \qquad \lim_{\Omega \ni x \to y} u(x) = \infty \quad \textit{for } C_{2,q'} \textit{ a.e. } y \in \partial_q \Omega \tag{1.10}$$

possesses exactly one solution.

In other words, an open set Ω as above, possesses exactly one ∂_q-large solution. If $\partial\Omega$ is compact then, this solution is an almost large solution. Indeed, by **II**, the maximal solution $U_{\partial\Omega}$ is an almost large solution in Ω. Since $\partial_q \Omega \subset \partial\Omega$, this implies that $U_{\partial\Omega}$ is a ∂_q-large solution. By **III**, $U_{\partial\Omega}$ is the unique such solution in Ω.

In the second part of the paper (Sections 5-7) we extend our investigation to the case where F is $C_{2,q'}$ finely closed. We introduce the notion of $C_{2,q'}$-strong solution in $D = \mathbb{R}^N \setminus F$, which is now merely $C_{2,q'}$-finely open, and prove that V_F is a $C_{2,q'}$-strong solution. By definition a $C_{2,q'}$-strong solution belongs to a certain type of local Lebesgue space described in Section 6 below. Further we derive integral a-priori estimates which serve to replace the Keller-Osserman estimate in this case. Using them we prove removability and compactness results. In addition we show that the capacitary estimates **I** and the Wiener criterion for pointwise blowup, namely (1.8), persist for V_F. We also establish the following version of **II**:

II'. *For every $C_{2,q'}$-finely closed set F, V_F is the maximal $C_{2,q'}$-strong solution in F^c. V_F is a ∂_q-large solution and it is σ-moderate.*

Finally, we have the following existence and uniqueness result:

III'. *Let Ω be a $C_{2,q'}$-finely open set. Let $\{G_n\}$ be a sequence of open sets such that*

$$C_{2,q'}(G_n \Delta \Omega) \to 0, \qquad C_{2,q'}(\partial G_n \setminus \partial_q \widetilde{G}_n) \to 0. \tag{1.11}$$

Then (1.10) *possesses exactly one $C_{2,q'}$-strong solution. The definition of blow up at the boundary is defined in a manner appropriate for this class of solutions (see Definition 7.2)*

Note that here we do not assume that G_n is contained in Ω or contains Ω. If $\Omega \subset G_n$ for every $n \in \mathbb{N}$ then (1.11) implies (1.9).

This seems to be the first study of the subject in the setting of the $C_{2,q'}$ fine topology, introducing a notion of solution in sets where the classical distribution derivative is not applicable. However the related subject of 'finely harmonic functions' has been studied for a long time (see e.g. [16]). Finely harmonic functions are defined on finely open sets relative to classical $C_{1,2}$-capacity; however their definition depends on specific properties of harmonic functions (e.g. the mean value property).

The framework presented here is particularly suitable for the study or (1.1) and (1.2) because limits of solutions in open domains lead naturally to $C_{2,q'}$ strong solutions in $C_{2,q'}$-finely open sets. The underlying limit is relatively weak, namely, limit in the topology of a local Lebesgue space defined by a family of weighted semi-norms with weights in $W^{2,q'}(\mathbb{R}^N)$ that are *bounded and compactly supported in the finely open set* (see Section 6).

At present this framework is presented mainly in the context of the study of maximal solutions and uniqueness of solutions with blow up on the $C_{2,q'}$ boundary. A more detailed study, including an extension to more general boundary value problems will appear elsewhere.

Partial list of notations

- $[a < f < b]$ means $\{x : a < f(x) < b\}$.
- $A\Delta B = (A \cup B) \setminus (A \cap B)$.
- If f, g are non-negative functions with domain D then $f \sim g$ means that there exits a constant C such that $C^{-1} f \leq g \leq Cf$.
- $A \overset{q}{\sim} B$ means $C_{2,q'}(A\Delta B) = 0$, $\quad A \overset{q}{\subset} B$ means $C_{2,q'}(A \setminus B) = 0$.
- $\widetilde{A}$ means 'the closure of A in the $C_{2,q'}$ fine topology'.
- $\partial_q A$ means 'the boundary of A in the $C_{2,q'}$ fine topology'.
- $\mathrm{int}_q A$ means 'the interior of A in the $C_{2,q'}$ fine topology'.
- $A \Subset B$ means 'A bounded and $\overline{A} \subset B$'.
- $B_r(x_0) = \{x \in \mathbb{R}^N : |x - x_0| < r\}$.
- χ_A denotes the characteristic function of the set A.
- (B-P)$_q$ condition: A measure μ satisfies this condition if $|\mu|(E) = 0$ for every Borel set E such that $C_{2,q'}(E) = 0$.
- u_μ denotes the solution of (1.2) in $\mathbb{R}^N$ when μ is a Radon measure satisfying the (B-P)$_q$ condition.

2. Upper estimate of the maximal solution.

In this section F denotes a non-empty compact set in $\mathbb{R}^N$ and the maximal solution of (1.1) in $\mathbb{R}^N\setminus F$ is denoted by U_F. Further, for $x\in\mathbb{R}^N$, we denote

$$\begin{aligned} T_m(x) &= \{y\in\mathbb{R}^N : 2^{-(m+1)}\le |y-x|\le 2^{-m}\},\\ F_m(x) &= F\cap T_m(x),\quad F^*_m(x)=F\cap \overline{B}_{2^{-m}}(x), \end{aligned} \tag{2.1}$$

$$\begin{aligned} W_F(x) &= \sum_{-\infty}^{\infty} 2^{\frac{2m}{q-1}} C_{2,q'}\left(2^m F_m(x)\right),\\ W^*_F(x) &= \sum_{-\infty}^{\infty} 2^{\frac{2m}{q-1}} C_{2,q'}\left(2^m F^*_m(x)\right). \end{aligned} \tag{2.2}$$

We call W_F the $C_{2,q'}$*-capacitary potential* of F. It is known that the two functions in (2.2) are equivalent, i.e., there exists a constant C depending only on q,N such that

$$W_F(x)\le W^*_F(x)\le CW_F(x) \tag{2.3}$$

see e.g. [29].

If K is a compact subset of a domain Ω put,

$$X_K(\Omega):=\{\eta\in C^2_c(\Omega): 0\le\eta\le 1,\ \eta=1 \text{ on } N^K_\eta\}, \tag{2.4}$$

where N^K_η denotes an open neighborhood of K depending on η.

The following theorem is due to Labutin [18]:

Theorem 2.1. *Let $q\ge q_c$. There exists a constant C depending only on q,N such that, for every compact set F,*

$$U_F(x)\le CW_F(x)\quad \forall x\in D. \tag{2.5}$$

For the convenience of the reader we provide a concise proof; components of this proof will also be used later on in the paper. The main ingredient in the proof is contained in the lemma stated below.

Lemma 2.2. *Let $R>1$ and denote by φ_R the solution of*

$$-\Delta\varphi=\chi_{B_R(0)}\quad \text{in } \mathbb{R}^N,\qquad \lim_{|x|\to\infty}\varphi(x)=0. \tag{2.6}$$

Given $\eta\in W^{2,q'}(\mathbb{R}^N)$, $0\le\eta\le 1$, put

$$\zeta_\eta=\varphi_R(1-\eta)^{2q'}.$$

There exists a constant $\bar c(N,q,R)$ such that, for every compact set $K\subset B_1(0)$,

$$\int_{\mathbb{R}^N\setminus K} U^q_K\zeta_\eta\,dx\le \bar c\,\|\eta\|^{q'}_{W^{2,q'}(\mathbb{R}^N)}\qquad \forall\eta\in X_K(\mathbb{R}^N), \tag{2.7}$$

$$\int_{B_R(0)\setminus K} U_K(1-\eta)^{2q'}dx\le \bar c\,\|\eta\|^{q'}_{W^{2,q'}(\mathbb{R}^N)}\qquad \forall\eta\in X_K(\mathbb{R}^N). \tag{2.8}$$

Proof. For $|x| \geq R+2$,

$$0 < \varphi_R(x) + U_K(x) \leq c|x|^{2-N}, \quad |\nabla \varphi_R(x)| + |\nabla U_K(x)| \leq c|x|^{1-N} \tag{2.9}$$

where $c = c(N,q,R)$. For every $R' > R$ and $\eta \in W^{2,q'}(\mathbb{R}^N)$,

$$\int_{B_{R'}(0)\setminus K} \big(-U_K \Delta \zeta_\eta + U_K^q \zeta_\eta\big)dx = -\frac{1}{R'}\int_{\partial B_{R'}} (U_K \nabla \zeta_\eta - \zeta_\eta \nabla U_K)\cdot x dS. \tag{2.10}$$

By (2.9), the right hand side of (2.10) tends to zero as $R' \to \infty$ and we obtain,

$$\int_D \big(-U_K \Delta \zeta_\eta + U_K^q \zeta_\eta\big)dx = 0, \tag{2.11}$$

where $D := \mathbb{R}^N \setminus K$. Further,

$$\Delta \zeta_\eta = \varphi_R \Delta (1-\eta)^{2q'} - (1-\eta)^{2q'} \chi_{B_2} + 2\nabla \varphi_R \cdot \nabla (1-\eta)^{2q'}$$

so that,

$$\begin{aligned}&\int_D U_K^q \zeta_\eta \, dx + \int_{B_R(0)\setminus K} U_K (1-\eta)^{2q'} dx = \\ &\int_D U_K \big(\varphi_R \Delta((1-\eta)^{2q'}) + 2\nabla \varphi_R \cdot \nabla((1-\eta)^{2q'})\big)dx.\end{aligned} \tag{2.12}$$

Now,

$$\Delta((1-\eta)^{2q'}) = -2q'(1-\eta)^{2q'-1}\Delta \eta + 2q'(2q'-1)(1-\eta)^{2q'-2}|\nabla \eta|^2,$$

so that

$$\int_D U_K \varphi_R \Delta((1-\eta)^{2q'})dx \leq c(I_1 + I_2), \tag{2.13}$$

where

$$I_1 := \int_D U_K \varphi_R (1-\eta)^{2q'-1}|\Delta \eta| dx, \quad I_2 := \int_D U_K \varphi_R (1-\eta)^{2q'-2}|\nabla \eta|^2 dx.$$

The estimate of I_1 is standard.

$$\begin{aligned} I_1 &\leq \Big(\int_D U_K^q \zeta_\eta \, dx\Big)^{1/q} \Big(\int_D \varphi_R (1-\eta)|\Delta \eta|^{q'} dx\Big)^{1/q'} \\ &\leq c\Big(\int_D U_K^q \zeta_\eta \, dx\Big)^{1/q} \|\eta\|_{W^{2,q'}(\mathbb{R}^N)}. \end{aligned} \tag{2.14}$$

To estimate I_2 we consider $\eta \in X_K(B_R(0))$ and use the interpolation inequality

$$\big\||\nabla \eta|^2\big\|_{L^{q'}(D)} \leq c(q,N,R)\, \|\eta\|_{L^\infty(D)} \big\|D^2 \eta\big\|_{L^{q'}(D)}. \tag{2.15}$$

We obtain,

$$
\begin{aligned}
I_2 &\leq \Big(\int_D U_K^q \zeta_\eta \, dx\Big)^{1/q} \Big(\int_D \varphi_R |\nabla\eta|^{2q'} dx\Big)^{1/q'} \\
&\leq c\Big(\int_D U_K^q \zeta_\eta \, dx\Big)^{1/q} \left\| |\nabla\eta|^2 \right\|_{L^{q'}(D)} \qquad (2.16)\\
&\leq c\Big(\int_D U_K^q \zeta_\eta \, dx\Big)^{1/q} \|\eta\|_{W^{2,q'}(\mathbb{R}^N)} .
\end{aligned}
$$

for $\eta \in X_K(B_R(0))$. Next

$$
\begin{aligned}
&\int_D U_K \nabla\varphi_R \cdot \nabla((1-\eta)^{2q'}) dx \leq 2q' \int_D U_K |\nabla\varphi_R||\nabla\eta|(1-\eta)^{2q'-1} dx \\
&\leq c\Big(\int_D U_K^q \zeta_\eta \, dx\Big)^{1/q} \Big(\int_D \varphi_R^{-\frac{q'}{q}} (|\nabla\varphi_R||\nabla\eta|)^{q'} dx\Big)^{1/q'} .
\end{aligned}
\tag{2.17}
$$

In view of the fact that, for $|x| \geq R+2$, $\varphi_R(x) \geq c|x|^{2-N}$, (2.9) implies

$$
\varphi_R^{-\frac{q'}{q}} |\nabla\varphi_R|^{q'} \leq c(N,q,R).
$$

Hence

$$
\int_D U_K \nabla\varphi_R \cdot \nabla((1-\eta)^{2q'}) dx \leq c\Big(\int_D U_K^q \zeta_\eta \, dx\Big)^{1/q} \|\eta\|_{W^{1,q'}(\mathbb{R}^N)} \tag{2.18}
$$

Combining (2.12)–(2.18) we obtain (2.7) and (2.8) for $\eta \in X_K(B_R(0))$.

Pick $\omega \in C_c^\infty(B_R(0)$ such that $0 \leq \omega \leq 1$ and $\omega = 1$ in $B_1(0)$. Given $\eta \in X_K(\mathbb{R}^N)$, (2.7) and (2.8) are valid if η is replaced by $\omega\eta$. However $(1-\eta) \leq (1-\omega\eta)$ and

$$
\|\omega\eta\|_{W^{2,q'}(\mathbb{R}^N)} \leq c(N,q,\omega) \|\eta\|_{W^{2,q'}(\mathbb{R}^N)} .
$$

Therefore (2.7) and (2.8) are valid for every $\eta \in X_K(\mathbb{R}^N)$. □

Corollary 2.3. *Assume that $R > 3/2$. There exists a constant $c_1 = c_1(N,q,R)$ such that, for every compact set $K \subset B_1(0)$*

$$
\int_{[3/2<|x|]} U_K^q \varphi_R \, dx + \int_{[3/2<|x|<R]} U_K dx \leq c_1 C_{2,q'}(K) \tag{2.19}
$$

and

$$
\sup_{[3/2<|x|<R]} U_K \leq c_1 C_{2,q'}(K). \tag{2.20}
$$

Proof. Recall that

$$
C_{2,q'}(K) = \inf\{\|\eta\|^{q'}_{W^{2,q'}(\mathbb{R}^N)} : \eta \in X_K(\mathbb{R}^N)\}. \tag{2.21}
$$

Let $\omega \in C_c^\infty(B_{3/2}(0))$ be a function such that $0 \leq \omega \leq 1$ and $\omega = 1$ on $B_1(0)$. For every compact set $K \subset B_1(0)$ put

$$
C^\omega_{2,q'}(K) = \inf\{\|\omega\eta\|^{q'}_{W^{2,q'}(\mathbb{R}^N)} : \eta \in X_K(\mathbb{R}^N)\}. \tag{2.22}
$$

Clearly $C_{2,q'}(K) \leq C^{\omega}_{2,q'}(K)$ and since

$$\|\omega\eta\|^{q'}_{W^{2,q'}(\mathbb{R}^N)} \leq c(N,q,\omega)\,\|\eta\|^{q'}_{W^{2,q'}(\mathbb{R}^N)}$$

we have

$$C_{2,q'}(K) \leq C^{\omega}_{2,q'}(K) \leq c(N,q,\omega)C_{2,q'}(K). \tag{2.23}$$

Let $\{\eta_n\}$ be a sequence in $X_K(\mathbb{R}^N)$ such that

$$\|\omega\eta_n\|^{q'}_{W^{2,q'}(\mathbb{R}^N)} \to C^{\omega}_{2,q'}(K).$$

For $K \subset B_1(0)$ (2.7) implies that,

$$\begin{aligned}\int_{\mathbb{R}^N\setminus B_{3/2}} U^q_K\varphi_R\,dx &\leq \liminf_{n\to\infty}\int_{\mathbb{R}^N\setminus K} U^q_K\varphi_R(1-\omega\eta_n)^{2q'}\,dx \\ &\leq c(N,q,\omega)C_{2,q'}(K).\end{aligned} \tag{2.24}$$

This proves (2.19). Inequality (2.20) (with the supremum over a slightly smaller annulus, say, $[3/2+\epsilon < |x| < R-\epsilon]$ with $\epsilon > 0$ such that $R > 3/2+2\epsilon$) follows from (2.19) and Harnack's inequality applied as in [28]. □

Proof of Theorem 2.1. Inequality (2.20) implies,

$$U_F(x) \leq c(N,q)\rho_F(x)^{-2/(q-1)}C_{2,q'}\big(F/\rho_F(x)\big) \tag{2.25}$$

for every every compact set $F \subset \mathbb{R}^N$ and every $x \in \mathbb{R}^N \setminus F$ such that $\rho_F(x) \geq (3/2)\mathrm{diam}\,F$. Recall that $\rho_F(x) := \mathrm{dist}\,(x,F)$.

The implication relies on the similarity transformation associated with (1.1). For any $a > 0$, we have

$$U_F(x) = a^{-2/(q-1)}U_{F/a}(x/a) \quad \forall x \in \mathbb{R}^N \setminus F. \tag{2.26}$$

Assume, as we may, that $F \subset B_R(0)$, $R = \mathrm{diam}\,F$. Fix a point $\bar{x} \in \mathbb{R}^N \setminus F$ such that $a := \rho_F(\bar{x}) \geq R$. Applying (2.20) to the set $K = 3F/2a$, we obtain

$$\begin{aligned}U_F(\bar{x}) &= (2a/3)^{-2/(q-1)}U_K(3\bar{x}/2a) \\ &\leq c(N,q)a^{-2/(q-1)}C_{2,q'}(K) \leq c'(N,q)a^{-2/(q-1)}C_{2,q'}(F/a).\end{aligned}$$

Next we show that (2.25) is equivalent to (2.5). Let $x \in D$ and put

$$M(x) := min\{m \in \mathbb{N} : 2^{-m} < \rho_F(x)\}. \tag{2.27}$$

Then $F_k(x) = \emptyset$ for all $k \geq M(x)$ and consequently

$$W_F(x) = \sum_{k=-\infty}^{M(x)} 2^{\frac{2k}{q-1}}C_{2,q'}\left(2^kF_k(x)\right) \leq C2^{\frac{2M(x)}{q-1}}\sup_{k\leq M(x)} C_{2,q'}(2^kF_k(x)).$$

However it is known that there exists a constant A depending only on q, N such that

$$C_{2,q'}(aE) \leq Aa^{N-\frac{2}{q-1}}C_{2,q'}(E) \quad \forall a \in (0,1), \tag{2.28}$$

(see e.g. [29]). In addition, for every $\ell > 1$ there exists a constant A, depending on q, N, ℓ, such that

$$C_{2,q'}(aE) \leq Aa^{N-\frac{2}{q-1}} C_{2,q'}(E) \quad \forall a \in (1, \ell). \tag{2.29}$$

Inequality (2.28) implies that

$$\begin{aligned} W_F(x) &\leq C_1 2^{\frac{2M(x)}{q-1}} C_{2,q'}(2^{M(x)} F) \\ &\leq C_2 \rho_F(x)^{\frac{-2}{q-1}} C_{2,q'}(2F/\rho_F(x)) \leq C_3 \rho_F(x)^{\frac{-2}{q-1}} C_{2,q'}(F/\rho_F(x)), \end{aligned}$$

where C_i are constants depending only on q, N. Thus (2.5) implies (2.25).

To prove the implication in the opposite direction we use the following facts:

For every compact set F there exists a sequence of bounded domains $\{D_n\}$ such that

$$\text{(i) } \cup D_n = D := F^c, \text{ (ii) } \overline{D}_n \subset D_{n+1}, \text{ (iii) } \partial D_n \text{ is Lipschitz.} \tag{2.30}$$

Such a sequence is called a *Lipschitz exhaustion of* D.

If u_n denotes the maximal solution of (1.1) in D_n then u_n is the unique large solution of (1.1) in D_n (see [27]), $u_n > u_{n+1}$ in D_n and $U_F = \lim u_n$.

Let E_i, $i = 1, \dots, k$ be compact sets and $E := \cup_1^k E_i$. One can choose a Lipschitz exhaustion $\{D_{i,n}\}_{n=1}^\infty$ of $D_i := E_i^c$, $i = 1, \dots, k$, such that the sequence $\{D_n\}$, $D_n = \cap_{i=1}^k D_{i,n}$, is a Lipschitz exhaustion of D. Let $u_{i,n}$ be the large solution in $D_{i,n}$. Then $v_n = \max(u_{1,n}, \dots, u_{k,n})$ is a subsolution while $w_n = \sum_{i=1}^k u_{i,n}$ is a supersolution of (1.1) in D_n. Hence u_n, the unique large solution of (1.1) in D_n, satisfies $v_n \leq u_n \leq w_n$. Consequently

$$\max(U_{E_1}, \dots, U_{E_k}) \leq U_E \leq \sum_{i=1}^k U_{E_i}. \tag{2.31}$$

Returning to the notation of Theorem 2.1, fix $\bar{x} \in D$ and put

$$i(\bar{x}) = \max\{i \in \mathbb{Z} : F \subset \overline{B}_{2^{-i}}(\bar{x})\}.$$

Then $F = \cup_{i(\bar{x})}^{M(\bar{x})} F_m(\bar{x})$ and, by (2.31) and (2.5),

$$U_F \leq \sum_{m=i(\bar{x})}^{M(\bar{x})} U_{F_i(\bar{x})} \leq C \sum_{m=i(\bar{x})}^{M(\bar{x})} 2^{\frac{2m}{q-1}} C_{2,q'}\left(2^m F_m(\bar{x})\right).$$

In particular, $U_F(\bar{x}) \leq CW_F(\bar{x})$. Thus (2.25) implies (2.5).

3. Lower estimate of the maximal solution

We need the following well-known result:

Proposition 3.1. *Let μ be a positive measure in $W^{-2,q}_{\text{loc}}(\mathbb{R}^N)$ and let Ω be a smooth domain with compact boundary. Then there exists a unique solution of each of the problems*

$$-\Delta u + u^q = \mu \quad in \quad \Omega, \quad u = 0 \quad on \quad \partial\Omega \tag{3.1}$$

and

$$-\Delta u + u^q = \mu \quad in \quad \Omega, \quad u = \infty \quad on \quad \partial\Omega. \tag{3.2}$$

If Ω is the whole space then there exists a unique solution u_μ of the equation

$$-\Delta u + u^q = \mu \quad in\ \mathbb{R}^N. \tag{3.3}$$

In each case the solution increases monotonically with μ. Finally

$$u_\mu = \lim_{R\to\infty} u^R_{\mu,0} = \lim_{R\to\infty} u^R_{\mu,\infty}$$

where $u^R_{\mu,0}$ and $u^R_{\mu,\infty}$ are the solutions of (3.1) and (3.2) respectively, when $\Omega = B_R(0)$.

When $\mu \in L^1_{loc}(\mathbb{R}^N)$ the result is due to Brezis [8] and Brezis-Strauss [9]. In the case of a smooth bounded domain Ω, with $\mu \in W^{-2,q}(\Omega)$, the result is due to Baras and Pierre [6]. The final observation is easily verified.

In this section, the solution of (3.1) will be denoted by $u_{\mu,\Omega}$.

If F is a compact subset of $\mathbb{R}^N$, we define

$$V_F := \sup\{u_\mu : \mu \in \mathfrak{M}_+(\mathbb{R}^N) \cap W^{-2,q}(\mathbb{R}^N),\ \mu(F^c) = 0\}. \tag{3.4}$$

Then V_F is the maximal σ-moderate solution of (1.1) in $F^c := \mathbb{R}^N \setminus F$. Obviously,

$$V_F \leq U_F. \tag{3.5}$$

We derive a lower estimate for V_F, equivalent to the upper estimate for U_F obtained in the previous section. More precisely:

Theorem 3.2. *Assume that F is a compact subset of $B_a(0)$ and let D be a bounded smooth domain such that $B_{6a}(0) \subset D$. Then, for every $x \in B_{2a}(0) \setminus F$, there exists a positive measure $\mu^x \in W^{-2,q}(\mathbb{R}^N)$, supported in F, such that*

$$cW_F(x) \leq u_{\mu^x,D}(x) \leq V_F(x), \tag{3.6}$$

where c is a positive constant depending only on N, q. In particular,

$$c(N,q)W_F(x) \leq V_F(x) \quad \forall x \in \mathbb{R}^N \setminus F. \tag{3.7}$$

Proof. Let λ be a bounded Borel measure supported in D. We denote by $\mathbb{G}_D[\lambda]$ the Green potential of the measure in D:

$$\mathbb{G}_D[\lambda](\cdot) := \int_D g_D(\cdot,\xi)\, d\lambda(\xi), \tag{3.8}$$

where g_D denotes Green's function in D.

If μ is a positive measure in $W^{-2,q}(D)$ then,

$$u_{\mu,D} \leq \mathbb{G}_D[\mu]$$

and consequently

$$u_{\mu,D} = \mathbb{G}_D[\mu] - \mathbb{G}_D[u^q_{\mu,D}] \geq \mathbb{G}_D[\mu] - \mathbb{G}_D[(\mathbb{G}_D[\mu])^q]. \tag{3.9}$$

Given $x_0 \in B_{2a} \setminus F$ we construct a measure $\mu^{x_0} \in W^{-2,q}(\mathbb{R}^N)$, concentrated on F such that (3.6) holds. By shifting the origin to x_0 we may assume that $x_0 = 0$. We observe that (3.6) is invariant with respect to dilation. Therefore we may assume that $a = 1/2$. Following the shift and the dilation we have

$$F \subset B_1(0), \quad B_2(0) \subset D, \quad 0 \in F^c, \tag{3.10}$$

and we have to prove (3.6), with an appropriate measure μ^0, at $x = 0$. The right inequality in (3.6) is trivial. Therefore we have to prove only that, for some non-negative measure $\mu^0 \in W^{-2,q}(\mathbb{R}^N)$ supported in F,

$$c(N,q)W_F(0) \leq u_{\mu^0,D}(0). \tag{3.11}$$

In view of (3.10),

$$u_{\mu^0,B_2(0)} \leq u_{\mu^0,D}.$$

Therefore it is enough to prove (3.11) for $D = B_2(0)$ which we assume in the rest of the proof.

In what follows we shall freely use the notation introduced in the previous section and write simply F_n, T_n instead of $F_n(0), T_n(0)$ etc.. Observe that in the present case $F_n = \emptyset$ for $n \leq -1$ and $F^*_n = F$ for $n \leq 0$. For every non-negative integer n, let ν_n denote the capacitary measure of $2^n F_n$. Thus, ν_n is a positive measure in $W^{-2,q}(\mathbb{R}^N)$ supported in $2^n F_n$ which satisfies

$$\nu_n(2^n F_n) = C_{2,q'}(2^n F_n) = \|\nu_n\|^q_{W^{-2,q}}. \tag{3.12}$$

Let μ_n, μ be the Borel measures in $\mathbb{R}^N$ given by

$$\mu_n(A) = 2^{-n(N-2q')}\nu_n(2^n A) \quad n = 0,1,2,\dots \quad \mu = \sum_0^\infty \mu_n. \tag{3.13}$$

Thus

$$\operatorname{supp}\mu_n \subset F_n, \qquad \operatorname{supp}\mu \subset F, \tag{3.14}$$

$$\mu_n(F_n) = 2^{-n(N-2q')}C_{2,q'}(2^n F_n), \qquad \mu \in W^{-2,q}(\mathbb{R}^N). \tag{3.15}$$

Observe also that, for $x, \xi \in B_1(0)$,

$$g_D(x,\xi) \approx |x-\xi|^{2-N}. \tag{3.16}$$

The notation $f \approx h$ means that there exists a positive constant c depending only on N, q such that $c^{-1}h \leq f \leq ch$.

The remaining part of the proof consists of a series of estimates of the terms on the right hand side of (3.9) for μ as above.

Lower estimate of $\mathbb{G}_D[\mu]$ *.* Using (3.15) and (3.16) we obtain,

$$c_N 2^{-(n+1)(2-N)} \leq g(0,\xi) \quad \forall \xi \in B_1(0),$$

$$\begin{aligned} \mathbb{G}_D[\mu](0) &= \sum_{n\geq 0} \int_{F_n} g(0,\xi) d\mu_n(\xi) \geq c \sum_{n\geq 0} \int_{F_n} 2^{n(N-2)} \, d\mu_n(\xi) \\ &= \sum_{n\geq 0} c 2^{-2n/(q-1)} C_{2,q'}(2^n F_n) = c W_F(0). \end{aligned} \tag{3.17}$$

Upper estimate of $\mathbb{G}_D[(\mathbb{G}_D[\mu])^q](0)$*.* We prove that

$$\begin{aligned} \mathbb{G}_D[(\mathbb{G}_D[\mu])^q](0) &= \int_D g_D(0,\xi) \mathbb{G}_D[\mu]^q(\xi) d\xi \\ &= \sum_{-1}^{\infty} \int_{T_k} g_D(0,\xi) \Big(\sum_{n\geq 0} \mathbb{G}_D[\mu_n](\xi) \Big)^q d\xi \leq c(N,q) W_F(0). \end{aligned} \tag{3.18}$$

This estimate requires several steps. Denote

$$I_1 = \sum_{k=3}^{\infty} \int_{T_k} g_D(0,\xi) \Big(\sum_{n=0}^{k-3} \mathbb{G}_D[\mu_n](\xi) \Big)^q \, d\xi \tag{3.19}$$

$$I_2 = \sum_{-1}^{\infty} \int_{T_k} g_D(0,\xi) \Big(\sum_{n>k+2} \mathbb{G}_D[\mu_n](\xi) \Big)^q \, d\xi \tag{3.20}$$

$$I_3 = \sum_{-1}^{\infty} \int_{T_k} g_D(0,\xi) \Big(\sum_{n=(k-2)_+}^{k+2} \mathbb{G}_D[\mu_n](\xi) \Big)^q \, d\xi \tag{3.21}$$

Then

$$\mathbb{G}_D[(\mathbb{G}_D[\mu])^q](0) \leq 3^q (I_1 + I_2 + I_3) \tag{3.22}$$

and we estimate each of the terms on the right hand side separately.

Estimate of I_1*.* We start with the following facts:

$$g_D(0,\xi) \leq c_N 2^{k(N-2)} \quad \forall \xi \in T_k$$

and

$$g_D(\xi, z) \leq c_N 2^{-n(2-N)} \quad \forall (\xi, z) \in T_k \times F_n.$$

These inequalities and (3.15) imply, for every $\xi \in T_k$,

$$\begin{aligned} \mathbb{G}_D[\mu_n](\xi) &= \int_{F_n} g_D(\xi, z) d\mu_n(z) \leq c_N 2^{n(N-2)} \mu_n(F_n) \\ &= c_N 2^{n(N-2)} 2^{-n(N-2q')} C_{2,q'}(2^n F_n) = c_N 2^{2n/(q-1)} C_{2,q'}(2^n F_n). \end{aligned}$$

Hence

$$
\begin{aligned}
I_1 &\leq c(N,q)\sum_{k=3}^{\infty} 2^{k(N-2)}\int_{T_k}\Big(\sum_{n=0}^{k-3} 2^{2n/(q-1)}C_{2,q'}(2^nF_n)\Big)^q\,d\xi \\
&\leq \sum_{k=3}^{\infty} 2^{k(N-2))}2^{-kN}\Big(\sum_{n=0}^{k-3} 2^{2n/(q-1)}C_{2,q'}(2^nF_n)\Big)^q \\
&\leq \sum_{k=3}^{M+1} 2^{-2k}\Big(\sum_{n=0}^{k-3} 2^{2n/(q-1)}C_{2,q'}(2^nF^*_n)\Big)^q.
\end{aligned}
\tag{3.23}
$$

where $M = M(0)$ is defined as in (2.27). Further, we claim that,

$$
\begin{aligned}
I'_1 := \sum_{k=3}^{M+1} 2^{-2k}\Big(\sum_{n=0}^{k-3} 2^{2n/(q-1)}C_{2,q'}(2^nF^*_n)\Big)^q \leq \\
c(N,q)\sum_{n=0}^{M+1} 2^{2n/(q-1)}C_{2,q'}(2^nF^*_n).
\end{aligned}
\tag{3.24}
$$

This inequality is a consequence of the following statement proved in [29, App. B]:

Lemma 3.3. *Let K be a compact set in $\mathbb{R}^N$ and let $\alpha > 0$ and $p > 1$ be such that $\alpha p \leq N$. Put*

$$
\phi(t) = C_{\alpha,p}\Big(\frac{1}{t}(K\cap B_t)\Big) = C_{\alpha,p}\Big(\frac{1}{t}K\cap B_1\Big), \quad \forall t > 0. \tag{3.25}
$$

Put $r_m = 2^{-m}$. Then, for every $\gamma\in\mathbb{R}$ and every $k\in\mathbb{N}$,

$$
\frac{1}{c}\sum_{m=i+1}^{k} r_m^{\gamma}\phi(r_m) \leq \int_{r_k}^{r_i} t^{\gamma}\phi(t)\frac{dt}{t} \leq c\sum_{m=i+1}^{k} r_{m-1}^{\gamma}\phi(r_{m-1}). \tag{3.26}
$$

where c is a constant depending only on γ, q, N.

Actually, in [29] this result was proved in the case $\alpha = 2/q$, $p = q'$, in $\mathbb{R}^{N-1}$ assuming $2/(q-1) \leq N-1$. However the proof applies to any α, p such that $\alpha p \leq N$. In particular it applies to the present case, namely, $\alpha = 2$, $p = q'$ with $2q' \leq N$.

We proceed to derive (3.24) from the above lemma. Put $r_m = 2^{-m}$, $\gamma = -\frac{2}{q-1}$ and define ϕ and φ by

$$
\phi(r_m) := C_{2,q'}(r_m^{-1}F^*_m), \quad \varphi(r,s) := \int_r^s t^{\gamma}\phi(t)\frac{dt}{t} \quad 0 < r < s. \tag{3.27}
$$

By Lemma 3.3,

$$
\frac{1}{c}\sum_{m=i+1}^{k} r_m^{\gamma}C_{2,q'}(r_m^{-1}F^*_m) \leq \varphi(r_k,r_i) \leq c\sum_{m=i+1}^{k} r_{m-1}^{\gamma}C_{2,q'}(r_{m-1}^{-1}F^*_{m-1}), \tag{3.28}
$$

for every $i, k \in \mathbb{N}$, $i < k$. The constant c depends only on q, N, Q. Hence (taking into account that $F^*_m = \emptyset$ for $m > M+1$)

$$\begin{aligned}\varphi(0, r_i) &:= \lim_{r\downarrow 0} \varphi(r, r_i) \leq c \sum_{m=i+1}^{\infty} r^{\gamma}_{m-1} C_{2,q'}(r^{-1}_{m-1} F^*_{m-1}) \\ &\leq c \sum_{m=i}^{M+1} r^{\gamma}_m C_{2,q'}(r^{-1}_m F^*_m).\end{aligned} \tag{3.29}$$

Further, by (3.28),

$$I'_1 = \sum_{k=3}^{M+1} r^2_k \Big(\sum_{n=0}^{k-3} r^{\gamma}_n C_{2,q'}(r^{-1}_n F^*_n)\Big)^q \leq \sum_{k=3}^{M+1} r^2_k \varphi^q(r_{k-3}, 1). \tag{3.30}$$

Since $\varphi(\cdot, s)$ is non-increasing,

$$\sum_{k=3}^{M+1} r^2_k \varphi^q(r_{k-3}, 1) \leq c \int_{r_{M-2}}^{1} t^2 \varphi^q(t, 1) \frac{dt}{t} \leq c \int_0^1 t\varphi^q(t, 1)\, dt. \tag{3.31}$$

By (3.29)

$$\begin{aligned}\int_0^1 t\varphi^q(t, 1)\, dt &\leq -c \int_0^1 t^2 \varphi^{q-1}(t, 1)\dot{\varphi}(t, 1) dt \\ &\leq -c \int_0^1 \dot{\varphi}(t, 1) dt \leq c\varphi(0, 1) \leq c\Big(\sum_{m=0}^{M+1} r^{\gamma}_m C_{2,q'}(r^{-1}_m F^*_m)\end{aligned} \tag{3.32}$$

Finally (3.30)–(3.32) imply (3.24). In turn, (3.23), (3.24) and (2.3) imply,

$$I_1 \leq c(N, q) W_F(0). \tag{3.33}$$

Estimate of I_2. Let $\sigma > 0$ and $\{a_n\}$ be a sequence of positive numbers. Then,

$$\sum_{n=k}^{\infty} a_n \leq 2^{-\sigma k} \Big(\frac{1}{1 - 2^{-\sigma q'}}\Big)^{\frac{1}{q'}} \Big(\sum_{n=k}^{\infty} 2^{\sigma n q} a^q_n\Big)^{\frac{1}{q}}.$$

Applying this inequality with $a_n = \mathbb{G}_D[\mu_n](\xi)$ we obtain

$$\begin{aligned}I_2 &\leq c(N, q, \sigma) \sum_{k=-1}^{\infty} \int_{T_k} g_D(0, \xi) 2^{-\sigma q k} \sum_{n=k+2}^{\infty} 2^{\sigma n q} \mathbb{G}_D[\mu_n](\xi)^q\, d\xi \\ &\leq c \sum_{n\geq 1} 2^{\sigma n q} \sum_{1\leq k<n-2} \int_{T_k} 2^{-\sigma k q} g_D(0, \xi) \mathbb{G}_D[\mu_n](\xi)^q d\xi \\ &\leq c \sum_{n\geq 1} 2^{\sigma n q} \sum_{1\leq k<n-2} \int_{T_k} 2^{-\sigma k q} 2^{k(N-2)} \mathbb{G}_D[\mu_n](\xi)^q d\xi,\end{aligned} \tag{3.34}$$

where, in the last inequality, we used the fact that

$$g_D(0, \xi) \leq c_N 2^{k(N-2)} \quad \forall \xi \in T_k.$$

Choosing $\sigma = (N-1)/q$ we obtain,

$$I_2 \leq c(N,q)\sum_{n\geq 1} 2^{n(N-1)} \sum_{1\leq k<n-2} \int_{T_k} 2^{-k}\mathbb{G}_D[\mu_n](\xi)^q d\xi. \tag{3.35}$$

Next we estimate the term

$$J_{k,n} := \int_{T_k} \mathbb{G}_D[\mu_n](\xi)^q d\xi, \tag{3.36}$$

in the case $1 \leq k < n-2$. In view of (3.13) we have

$$\mathbb{G}_D[\mu_n](\xi) = \int_{F_n} g_D(\xi, z)d\mu_n(z) = 2^{-n(N-2q')}\int_{2^nF_n} \tilde{g}(\xi', z')d\nu_n(z')$$

where

$$\xi' = 2^n\xi, \quad \tilde{g}(\xi', z') = g_D(2^{-n}\xi', 2^{-n}z').$$

Observe that, if $\xi \in T_k$ then $\xi' \in T_{k-n}$. Thus

$$J_{k,n} = 2^{-nN}\int_{T_{k-n}} \Big(2^{-n(N-2q')}\int_{2^nF_n} \tilde{g}(\xi', z')d\nu_n(z')\Big)^q d\xi'.$$

Since

$$\tilde{g}(\xi', z') \leq c_N 2^{-n(2-N)}\left|\xi' - z'\right|^{2-N}$$

we obtain

$$J_{k,n} \leq c(N,q)2^{-n(N-2q')}\int_{T_{k-n}} \Big(\int_{2^nF_n} \left|\xi' - z'\right|^{2-N} d\nu_n(z')\Big)^q d\xi'. \tag{3.37}$$

Since $z' \in 2^nF_n \subset B_1(0)$ while $\xi' \in T_{k-n}$, $k-n < -2$ it follows that $|\xi'| \geq 2$ and consequently

$$|\xi' - z'| \geq \frac{1}{2}|\xi'|.$$

Therefore

$$\int_{T_{k-n}} \Big(\int_{2^nF_n} \left|\xi' - z'\right|^{2-N} d\nu_n(z')\Big)^q d\xi' \leq c\nu_n(2^nF_n)^q \int_{T_{k-n}} |\xi'|^{(2-N)q}\, d\xi'$$

$$\leq c(N,q)C_{2,q'}(2^nF_n)^q \int_{2^{n-k-1}}^{2^{n-k}} r^{(2-N)q+N-1}dr \leq c(N,q)C_{2,q'}(2^nF_n)A(q,N)$$

where we used the fact that $C_{2,q'}(2^nF_n) \leq C_{2,q'}(B_1)$ and

$$A(q,N) = \begin{cases} 2^{(2-N)q+N} & \text{if } q > q_c \\ \ln 2 & \text{if } q = q_c. \end{cases}$$

Thus, for $k \geq n-2 \geq -2$,

$$\begin{aligned} J_{k,n} &\leq c(N,q)2^{-n(N-2q')}\left\|\nu_n\right\|^q_{W^{-2,q}(\mathbb{R}^N)} \\ &= c(N,q)2^{-n(N-2q')}C_{2,q'}(2^nF_n). \end{aligned} \tag{3.38}$$

By (3.35) and (3.38),

$$
\begin{aligned}
I_2 &\le c(N,q)\sum_{n\ge 0}2^{n(N-1)}\sum_{k\ge n-2}2^{-k}2^{-n(N-2q')}C_{2,q'}(2^nF_n)\\
&\le c(N,q)\sum_{n\ge 0}2^{n(-2+2q')}C_{2,q'}(2^nF_n) \qquad (3.39)\\
&= c(N,q)\sum_{n\ge 0}2^{\frac{2n}{q-1}}C_{2,q'}(2^nF_n)=c(N,q)W_F(0).
\end{aligned}
$$

Estimate of I_3 By (3.21) and the notation (3.36) we have

$$
\begin{aligned}
I_3 &\le 5^q\sum_{k=-1}^{\infty}\int_{T_k}g_D(0,\xi)\sum_{n=(k-2)_+}^{k+2}\mathbb{G}_D[\mu_n](\xi)^q\,d\xi \qquad (3.40)\\
&\le \sum_{k=-1}^{\infty}2^{k(N-2)}\sum_{n=(k-2)_+}^{k+2}J_{k,n}.
\end{aligned}
$$

By (3.37)

$$J_{k,n}\le c(N,q)2^{-n(N-2q')}\int_{T_{k-n}}\Big(\int_{2^nF_n}\left|\xi'-z'\right|^{2-N}d\nu_n(z')\Big)$$

and, in the present case $-2\le n-k\le 2$. Therefore $T_{k-n}\subset B_4(0)$ and consequently, for (ξ',z') in the domain of integration of the integral above,

$$\left|\xi'-z'\right|^{2-N}\approx\mathcal{B}_2(\xi',z')$$

where $\mathcal{B}_2$ denotes the Bessel kernel with index 2. Hence,

$$
\begin{aligned}
&\int_{T_{k-n}}\Big(\int_{2^nF_n}\left|\xi'-z'\right|^{2-N}d\nu_n(z')\Big)^q d\xi'\le\\
&c(N,q)\left\|\nu_n\right\|^q_{W^{-2,q}(\mathbb{R}^N)}=c(N,q)C_{2,q'}(2^nF_n).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
I_3 &\le c(N,q)\sum_{k=-1}^{\infty}2^{k(N-2)}\sum_{n=(k-2)_+}^{k+2}2^{-n(N-2q')}C_{2,q'}(2^nF_n)\\
&\le c(N,q)\sum_{k=-1}^{\infty}2^{k(N-2)}2^{-k(N-2q')}C_{2,q'}(2^kF_k) \qquad (3.41)\\
&= c(N,q)\sum_{k=-1}^{\infty}2^{2k/(q-1)}C_{2,q'}(2^kF_k)\le c(N,q)W_F(0)
\end{aligned}
$$

Combining (3.22) with the inequalities (3.33), (3.39) and (3.41) we obtain

$$\mathbb{G}_D[(\mathbb{G}_D[\mu])^q](0)\le c(N,q)W_F(0). \qquad (3.42)$$

Finally, we combine (3.9) with (3.17) and (3.42) and replace μ by $\epsilon\mu$, $\epsilon > 0$, to obtain

$$u^{\epsilon\mu}(0) \geq (c_1(N,q)\epsilon - c_2(N,q)\epsilon^q)W_F(0). \tag{3.43}$$

Choosing $\epsilon := (c_1(N,q)/2c_2(N,q))^{1/(q-1}$ we obtain (3.11) with $c(N,q) = c_1(N,q)\epsilon/2$. □

4. Properties of U_F for F compact

As before we assume that F is a compact set. Combining the capacitary estimates contained in Theorems 2.1, 3.2 and (2.3)we have

$$U_F \sim W_F \sim W_F^* \quad \text{in } D = \mathbb{R}^N \setminus F \tag{4.1}$$

In the present section we use this result in order to establish several properties of the maximal solution.

4.1. *The maximal solution is σ-moderate.*

Theorem 4.1. *$U_F = V_F$; consequently U_F is σ-moderate.*

Proof. By (4.1) there exists a constant $c = c(N,q)$ such that

$$U_F \leq cV_F\,. \tag{4.2}$$

If the two solutions are not identical we have

$$V_F(x) < U_F(x) \quad \forall x \in D. \tag{4.3}$$

Let $\alpha = \frac{1}{2c}$ and put $v = (1+\alpha)V_F(x) - \alpha U_F$. Then $\alpha V_F(x) < v < U_F$ and (as $0 < \alpha < 1$) $\alpha V_F(x)$ is a subsolution of (1.1) in D. As in [24] we find that v is a supersolution. It follows that there exists a solution w such that $\alpha V_F(x) \leq w \leq v < V_F(x)$. But, by the definition of V_F (see (3.4)), it is easy to see that the smallest solution of (1.1) dominating $\alpha V_F(x)$ is $V_F(x)$. Therefore $w = V_F(x)$. This contradicts (4.3).

By a standard argument, the definition of $V_F(x)$ implies that it is σ-moderate. □

4.2. *A continuity property of U_F relative to capacity.*

Lemma 4.2. *There exists a positive constant c depending only on N, q such that, for every compact set $K \subset B_1(0)$, there exists an open neighborhood N_K of K such that*

$$C_{2,q'}(N_K) \leq 4C_{2,q'}(K) \quad \text{and} \quad \int_{B_1(0)\setminus N_K} U_K\,dx \leq cC_{2,q'}(K). \tag{4.4}$$

Note. In general $\int_{B_1(0)\setminus K} U_K\,dx$ may be infinite. Of course, (4.4) is meaningful only if $4C_{2,q'}(K) < C_{2,q'}(B_1(0))$.

Proof. Let $\bar{c}$ be the constant in (2.7) with $R = 2$. Assume that

$$C_{2,q'}(K) \leq a := C_{2,q'}(B_1)/8 \tag{4.5}$$

and pick γ_1 so that

$$0 < \gamma_1 \leq C_{2,q'}(K). \tag{4.6}$$

By Lemma 2.2 and (2.21) there exists $\eta \in X_K(\mathbb{R}^N)$ such that

$$\begin{aligned} \|\eta\|^{q'}_{W^{2,q'}(\mathbb{R}^N)} &\leq C_{2,q'}(K) + \gamma_1, \\ \int_{B_2(0)\setminus K} U_K (1-\eta)^{2q'}\,dx &\leq \bar{c}(C_{2,q'}(K) + \gamma_1). \end{aligned} \tag{4.7}$$

Fix η and denote,

$$K_{(\alpha)} = \{x \in B_1(0) : (1-\alpha) \leq \eta\} \quad \forall \alpha \in (0,1).$$

Then $K \subset K_{(\alpha)}$ and

$$\begin{aligned} C_{2,q'}(K_{(\alpha)}) &\leq (1-\alpha)^{-q'} \|\eta\|^{q'}_{W^{2,q'}(\mathbb{R}^N)} \\ &\leq (1-\alpha)^{-q'}(C_{2,q'}(K) + \gamma_1) \leq \frac{2C_{2,q'}(K)}{(1-\alpha)^{q'}}. \end{aligned}$$

Therefore, using (4.5), we obtain

$$(1-\alpha)^{-q'} = 2 \Longrightarrow C_{2,q'}(K_{(\alpha)}) \leq 4C_{2,q'}(K) \leq C_{2,q'}(B_1)/2.$$

Hence, by (4.7),

$$\int_{B_2(0)\setminus K_{(\alpha)}} U_K\,dx \leq \bar{c}\alpha^{-2q'}(C_{2,q'}(K) + \gamma_1) \leq (4\bar{c})C_{2,q'}(K) \tag{4.8}$$

where $\alpha = 1 - 2^{-1/q'}$. □

4.3. *Wiener criterion for blow up of U_F.*

Theorem 4.3. *For every point $y \in F$,*

$$\lim_{F^c \ni x \to y} U_F(x) = \infty \iff W_F(y) = \infty. \tag{4.9}$$

Proof. Without loss of generality we may assume that $y = 0$ and that $F \subset B_1(0)$. In order to justify the second part of this remark we observe that, for every $m \in \mathbb{N}$,

$$\begin{aligned} 2^{-2m/(q-1)} U_F(2^{-m}x) &= U_{2^m F}(x) \quad \forall x \in (2^m F)^c, \\ W_F(0) &= 2^{2m/(q-1)} W_{2^m F}(0). \end{aligned} \tag{4.10}$$

Denote

$$a_m(x) = C_{2,q'}\left(2^m F_m(x)\right), \quad a^*_m(x) = C_{2,q'}\left(2^m F^*_m(x)\right) \tag{4.11}$$

There exists a constant $c = c(N,q)$ such that for every Borel set $A \subset B_1(0)$,

$$C_{2,q'}(2A) \leq cC_{2,q'}(A). \tag{4.12}$$

If $x, \xi \in \mathbb{R}^N$, $|x-\xi| \le r_m = 2^{-m}$ and $0 \le k \le m$ then

$$2^k F_k^*(\xi) = 2^k(F \cap B_{r_k}(\xi)) \subset 2^k(F \cap B_{2r_k}(x)) = 2(2^{k-1}(F \cap B_{r_{k-1}}(x)).$$

Hence

$$a_k^*(\xi) \le c a_{k-1}^*(x) \text{ for } 0 \le k \le m, \quad \sum_{k=0}^{m} 2^{2k/(q-1)} a_k^*(\xi) \le c W_F^*(x). \tag{4.13}$$

As $x \to \xi$, $m \to \infty$ and we obtain

$$W_F^*(\xi) \le c(N,q) \liminf_{x \to \xi} W_F^*(x). \tag{4.14}$$

By (4.1), (4.14) implies that (4.9) holds in the direction $\Longleftarrow$.

In order to prove (4.9) in the opposite direction we derive the inequality,

$$\liminf_{F^c \ni x \to 0} W_F^*(x) \le c(N,q) W_F^*(0). \tag{4.15}$$

If $W_F^*(0) = \infty$ there is nothing to prove. Therefore we assume that

$$W_F^*(0) = \sum_{-\infty}^{M(0)} 2^{\frac{2m}{q-1}} C_{2,q'}\left(2^m F_m^*(0)\right) < \infty.$$

By Lemma 4.2, with $K_m = 2^m F_m^*(0)$, there exists an open neighborhood G_m of K_m such that

$$C_{2,q'}(G_m) \le 4C_{2,q'}(K_m) \quad \text{and} \quad \int_{B_1(0)\setminus G_m} U_{K_m} dx \le c(N,q) C_{2,q'}(K_m).$$

Put $T' := [5/8 \le |x| \le 7/8]$ and let E_m be a compact subset of $T' \setminus G_m$ such that

$$\begin{aligned} C_{2,q'}(E_m) &> \frac{1}{2} C_{2,q'}(T' \setminus G_m) \ge \frac{1}{2} C_{2,q'}(T') - 2C_{2,q'}(K_m) \\ &\ge \frac{1}{2} C_{2,q'}(T') - 2^{1-\frac{2m}{q-1}} W_F^*(0). \end{aligned}$$

Therefore, there exists an integer m_0 such that, for $m \ge m_0$,

$$\begin{aligned} \inf_{E_m} U_{K_m} &\le |E_m|^{-1} \int_{E_m} U_{K_m} dx \\ &\le |E_m|^{-1} c(N,q) C_{2,q'}(K_m) \le c(N,q) C_{2,q'}(K_m)/C_{2,q'}(E_m) \\ &< 4c(N,q) C_{2,q'}(B_1(0))^{-1} C_{2,q'}(K_m) = c(N,q) C_{2,q'}(K_m). \end{aligned} \tag{4.16}$$

Hence, by (4.10),

$$\inf_{2^{-m}E_m} U_{F_m^*(0)} = 2^{2m/(q-1)} \inf_{E_m} U_{K_m} < c(N,q) 2^{2m/(q-1)} C_{2,q'}(K_m),$$

which implies, for $m \ge m_0$,

$$\inf_{(2^{-m}T')\setminus F} U_{F_m^*(0)} \le c(N,q) 2^{2m/(q-1)} C_{2,q'}(K_m) \le c(N,q) W_F^*(0). \tag{4.17}$$

Fix $j > m_0$ and let $\xi \in (2^{-j}T') \setminus F$. Denote

$$F^j := F \setminus B_{2^{-j+1}}(0), \quad E_{k,j} := F^j \cap \{x \in B_1(0) : |x-\xi| \leq 2^{-k}\}, \quad \bar{j} := \left[\frac{j}{8}\right].$$

Since dist $(\xi, F^j) \geq 2^{-j/8}$,

$$W_{F^j}(\xi) = \sum_{-\infty}^{\bar{j}} 2^{2k/(q-1)} C_{2,q'}(2^k E_{k,j}).$$

For $k \leq \bar{j}$

$$x \in E_{k,j} \Longrightarrow |x| \leq 2^{-k} + 2^{-j} \leq 2^{-k} + 2^{-8(k-1)} \leq 2^9 2^{-k}.$$

Thus $E_{k,j} \subset F^*_{k-9}$ and

$$W_{F^j}(\xi) = \sum_{-\infty}^{\bar{j}} 2^{2k/(q-1)} C_{2,q'}(2^k F^*_{k-9}(0)) \leq c(N,q) W^*_F(0) \tag{4.18}$$

for every $\xi \in (2^{-j}T') \setminus F$. By (4.17), we can choose $\xi^j \in (2^{-j}T') \setminus F$ such that

$$U_{F^*_j(0)}(\xi^j) \leq c(N,q) W^*_F(0).$$

Hence, by (4.18), bearing in mind that $U_K \sim W_K \sim W^*_K$ for every compact K we obtain

$$U_F(\xi^j) \leq U_{F^*_j(0)}(\xi^j) + U_{F^j}(\xi^j) \leq c(N,q) W^*_F(0) \quad \forall j \geq m_0. \tag{4.19}$$

This implies (4.15) and completes the proof. □

Corollary 4.4. *Define*

$$\widetilde{W}_F(x) = \liminf_{y \to x} W_F(y) \quad \forall x \in \mathbb{R}^N.$$

Then, $\tilde{W}_F$ is l.s.c. in $\mathbb{R}^N$ and $\widetilde{W}_F \sim W_F$. In addition, $\tilde{W}_F$ satisfies Harnack's inequality in compact subsets of $\mathbb{R}^N \setminus F$.

Proof. The lower semi-continuity of $\widetilde{W}_F$ follows from its definition. The equivalence $\widetilde{W}_F \sim W_F$ follows from (4.14) and (4.15). The last statement follows from the fact that U_F satisfies Harnack's inequality and $\tilde{W}_F \sim U_F$. □

4.4. *U_F is an almost large solution.*

Theorem 4.5. *For every compact set $F \subset \mathbb{R}^N$, U_F is an almost large solution.*

Proof. In view of Theorem 4.3 it is enough to show that there exists a set $A \subset F$ such that

$$C_{2,q'}(A) = 0 \quad \text{and} \quad W_F(y) = \infty \quad \forall y \in F \setminus A. \tag{4.20}$$

It is known (see [1, Ch. 6]) that every point in F, with the possible exception of a set A_1 of $C_{2,q'}$ capacity zero, is a $C_{2,q'}$-thick point of F, i.e.,

$$\Lambda_F^{2,q'}(y) := \sum_0^\infty \left(2^{2m/(q-1)} C_{2,q'}(F_m^*(y))\right)^{q-1} = \infty \quad \forall y \in F \setminus A_1. \tag{4.21}$$

We show that,

$$\Lambda_F^{2,q'}(y) \le c(N,q)(W_F(y))^{\tilde{q}}, \quad \tilde{q} = \min(1, q-1) \quad \forall q > 1. \tag{4.22}$$

Recall that,

$$C_{2,q'}(F_m^*(y)) \le c(N,q) 2^{-2m/(q-1)} C_{2,q'}(2^m F_m^*(y))$$

so that

$$\Lambda_F^{2,q'}(y) \le c(N,q) \sum_0^\infty \left(C_{2,q'}(2^m F_m^*(y))\right)^{q-1}. \tag{4.23}$$

In view of the fact that $C_{2,q'}(2^m F_m^*(y)) \le C_{2,q'}(B_1)$, if $q \ge 2$, (4.23) implies (4.22). If $1 < q < 2$,

$$\sum_0^\infty \left(C_{2,q'}(2^m F_m^*(y))\right)^{q-1} \le$$
$$\Big(\sum_0^\infty 2^{-\frac{2m(2-q)}{q-1}}\Big)^{2-q} \Big(\sum_0^\infty 2^{2m/(q-1)} C_{2,q'}(2^m F_m^*(y))\Big)^{q-1},$$

which again implies (4.22). Clearly (4.21) and (4.22) imply (4.20). □

5. 'Maximal solutions' on arbitrary sets and uniqueness I

For any Borel set E put

$$\mathcal{T}(E) := \{\mu \in W_+^{-2,q}(\mathbb{R}^N) : \ \mu(E^c) = 0\}, \tag{5.1}$$

$$V_E := \sup\{u_\mu : \mu \in \mathcal{T}(E)\}, \tag{5.2}$$

where u_μ denotes the solution of (3.3). If $C_{2,q'}(E) = 0$ the only measure $\mu \in W_+^{-2,q}(\mathbb{R}^N)$ that is concentrated on E is the measure zero. Therefore in this case $V_E = 0$.

By Theorem 4.1,

$$E \text{ compact} \implies V_E = U_E. \tag{5.3}$$

Therefore the definition of V_E can be seen as an extension, to general sets, of the notion of 'maximal solution', previously defined for compact sets. However, by its definition, V_E *dominates only σ-moderate solutions in E^c*, i.e., solutions of the form $\lim u_{\mu_n}$ where $\{\mu_n\}$ is an increasing sequence of measures in $W_+^{-2,q}(\mathbb{R}^N)$ concentrated in E.

At this stage, it is not clear in which sense V_E is a solution of (1.1) in E^c, which, in general, is not an open set. This question will be discussed in the following sections.

The $C_{2,q'}$ fine topology (see [1] for definition and details) plays a central role in the remaining part of the paper. If A is a set in $\mathbb{R}^N$ we denote by $\widetilde{A}$ the closure of A in the $C_{2,q'}$ fine topology and by $\mathrm{int}_q A$ the interior of A relative to this topology.

Recall that a set $A \subset \mathbb{R}^N$ is $C_{2,q'}$*-quasi open* if, for every $\epsilon > 0$, there exists an open set G_ϵ such that

$$A \subset G_\epsilon, \qquad C_{2,q'}(G_\epsilon \setminus A) < \epsilon.$$

A set is $C_{2,q'}$*-quasi closed* if its complement is quasi open.

Every $C_{2,q'}$-finely open set is $C_{2,q'}$-quasi-open. On the other hand, if E is $C_{2,q'}$-quasi open then (see [1, Section 6.4])

$$C_{2,q'}(E \setminus \mathrm{int}_q E) = 0.$$

This implies that every $C_{2,q'}$-quasi closed set F can be written in the form

$$F = \bigcup_n^\infty K_n \bigcup Z,$$

where $\{K_n\}$ is an increasing sequence of compact sets and

$$C_{2,q'}(F \setminus K_n) \to 0, \qquad C_{2,q'}(Z) = 0.$$

Furthermore, if E is $C_{2,q'}$-quasi closed then

$$C_{2,q'}(\widetilde{E} \setminus E) = 0.$$

In the first two theorems below we describe some basic properties of V_E. These results are then used in order to establish a rather general uniqueness result for almost large solutions.

Theorem 5.1. *Let F be a $C_{2,q'}$-quasi closed set. Then*

$$\lim_{F^c \ni x \to y} V_F(x) = \infty \quad \text{for } C_{2,q'}\text{-a.e. } y \in F \tag{5.4}$$

and V_F satisfies

$$\frac{1}{c} W_F \le V_F \le c W_F, \tag{5.5}$$

where c depends only on N, q. Finally, for every $x \in F^c$,

$$W_F(x) < \infty \implies \lim_{\substack{C_{2,q'}(E) \to 0 \\ E \subset F}} V_E(x) = 0. \tag{5.6}$$

Proof. There exists an increasing sequence of compact sets $\{K_n\}$ such that $K_n \subset F$ and $C_{2,q'}(F \setminus K_n) \to 0$. By Theorem 4.1 $U_{K_n} = V_{K_n}$ and, obviously, $V_{K_n} \le V_F$. By Theorem 4.5, (5.4) holds if F is replaced by K_n. Therefore, by taking the limit as $n \to \infty$, we obtain (5.4) in the general case.

If $\mu \in \mathcal{T}_F$ then $u_\mu = \lim u_{\mu_n}$ where $\mu_n = \mu \chi_{K_n}$. Therefore

$$V_F = \lim U_{K_n}. \tag{5.7}$$

Since U_{K_n} satisfies estimates (2.5) and (3.6) for every n, it follows that V_F satisfies (5.5).

We turn to the proof of the last assertion. Let $\{E_j\}$ be a sequence of subsets of F such that $C_{2,q'}(E_j) \to 0$. We must show that

$$\xi \in F^c, \quad \limsup V_{E_j}(\xi) > 0 \implies W_F(\xi) = \infty. \tag{5.8}$$

By taking a subsequence we may assume that there exists $a > 0$ such that $V_{E_j}(\xi) > a$ for all j. Since $V_{\widetilde{E}_j}(\xi) = V_{E_j}(\xi)$ and $C_{2,q'}(E_j) \to 0$ implies $C_{2,q'}(\widetilde{E}_j) \to 0$ we may assume that the sets E_j are $C_{2,q'}$-finely closed. By (5.7) it follows that, for every j, there exists a compact set $K_j \subset E_j$ such that

$$U_{K_j}(\xi) > a. \tag{5.9}$$

By negation, suppose that $W_F(\xi) < \infty$. Then

$$\lim_{J\to\infty} \sum_J^\infty 2^{2j/(q-1)} C_{2,q'}(2^j F_j(\xi)) \to 0,$$

F_j being defined as in (2.1). Pick a positive integer J such that

$$\sum_J^\infty 2^{2j/(q-1)} C_{2,q'}(2^j F_j(\xi)) < a/4C, \tag{5.10}$$

where C is the constant in (2.5).

Pick a subsequence of $\{K_j\}$, say $\{K_{j_n}\}$, such that $C_{2,q'}(K_{j_n}) < \epsilon/2^n$, with ϵ to be determined. The set $A := \bigcup_1^\infty K_{j_n}$ is $C_{2,q'}$-quasi closed and $C_{2,q'}(A) \le \sum_1^\infty C_{2,q'}(K_{j_n}) < \epsilon$. Further,

$$\begin{aligned} W_A(\xi) = \sum_{-\infty}^\infty 2^{2j/(q-1)} C_{2,q'}(2^j A_j(\xi)) &\le \sum_{-\infty}^{-1} 2^{2j/(q-1)} C_{2,q'}(2^j A) \\ &+ \sum_0^{J-1} 2^{2j/(q-1)} C_{2,q'}(2^j A) + \sum_J^\infty 2^{2j/(q-1)} C_{2,q'}(2^j F_j(\xi)), \end{aligned}$$

where $A_j(\xi)$ is defined as in (2.1) with F replaced by A. (We used the fact that $A_j(\xi) \subset A \subset F$.) By (2.28),

$$\sum_{-\infty}^{-1} 2^{2j/(q-1)} C_{2,q'}(2^j A) \le c_1(N,q) \sum_{-\infty}^{-1} 2^{jN} C_{2,q'}(A) \le c_2(N,q)\epsilon.$$

By (2.29),

$$\sum_0^{J-1} 2^{2j/(q-1)} C_{2,q'}(2^j A) \le c_1(N,q,J) \sum_0^{J-1} 2^{2jN} C_{2,q'}(A) \le c_2(N,q,J)\epsilon.$$

Therefore, choosing $\epsilon = (a/4C)(c_2(N,q) + c_2(N,q,J))^{-1}$ and using (5.10) we obtain,

$$W_A(\xi) < a/2C.$$

Since V_A satisfies (2.5) we conclude that $V_A(\xi) < a/2$. As

$$U_{K_{j_n}} = V_{K_{j_n}} \le V_A,$$

this contradicts (5.9). □

Theorem 5.2. *Let F be a $C_{2,q'}$-quasi closed set and let $\{F_n\}$ be an increasing sequence of compact subsets of F such that $C_{2,q'}(F\setminus F_n)\to 0$. Then*

$$V_F=\lim U_{F_n}. \tag{5.11}$$

Furthermore, there exists an increasing sequence of non-negative measures $\{\mu_n\}\subset W^{-2,q}(\mathbb{R}^N)$ such that $\mu_n(F_n^c)=0$ and $u_{\mu_n}\to V_F$ in F^c.

Finally, for every $y\in F$,

$$W_F(y)<\infty \iff \liminf_{F^c\ni x\to y} V_F(x)<\infty. \tag{5.12}$$

Proof. From the definition of V_F it follows that $V_F=\lim V_{F_n}$. By Theorem 4.1, $V_{F_n}=U_{F_n}$.

Let $\xi\in D_n=F_n^c$ and let $\{\tau_k^n\}_{k=1}^\infty$ be a sequence in $W^{-2,q}(\mathbb{R}^N)$ such that $\tau_k^n(D_n)=0$ and $u_{\tau_k^n}(\xi)\to U_{F_n}(\xi)$. Note that $w_m^n=\max(u_{\tau_1^n},\dots,u_{\tau_m^n})$ is a subsolution of the equation

$$-\Delta w+w^q=\mu_m^n:=\max(\tau_1^n,\dots,\tau_m^n)\quad\text{in}\quad D_n.$$

Therefore $v_m^n=u_{\mu_m^n}$ is the smallest solution in D_n dominating w_m^n. The sequence $\{v_m^n\}_{m=1}^\infty$ is increasing, bounded by U_{F_n} and $v^n:=\lim_{m\to\infty}v_m^n$ is a solution of (1.1) in D_n such that $v^n(\xi)=U_{F_n}(\xi)$. The fact that $v^n\leq U_{F_n}$ and equals it at a point $\xi\in D_n$ implies that $v^n=U_{F_n}$.

Put

$$\tau^{(n)}:=\sum_m a_m^n\mu_m^n,\quad a_m^n:=2^{-m}\left\|\mu_m^n\right\|_{W_+^{-2,q}(\mathbb{R}^N)}. \tag{5.13}$$

Then

$$U_{F_n}=\lim_{k\to\infty}u_{k\tau^{(n)}}. \tag{5.14}$$

Finally, if $\mu_n:=\sum_1^n\tau^{(j)}$ then $\{\mu_n\}$ is increasing and $u_{\mu_n}\to V_F$.

The last statement of the theorem is proved exactly as in the case that F is compact (see Theorem 4.3). □

Theorem 5.3. *Let E be a Borel set such that $C_{2,q'}(E)>0$. Then*

$$V_E=V_{\widetilde{E}} \tag{5.15}$$

and, if $\mu\in W_+^{-2,q}(\mathbb{R}^N)$,

$$u_\mu<V_E \iff \mu(\mathbb{R}^N\setminus\tilde{E})=0. \tag{5.16}$$

Furthermore, if E is $C_{2,q'}$-quasi closed, there exists $\tau\in W_+^{-2,q}(\mathbb{R}^N)$ such that $\tau(E^c)=0$ and

$$V_E=\lim_{k\to\infty}u_{k\tau}. \tag{5.17}$$

Proof. We prove (5.15) under the assumption that E is bounded, say, $\overline{E} \subset B_R$. For the general case we observe that

$$\lim_{R\to\infty} V_{E\cap B_R} = V_E.$$

<u>Assertion 1</u>: *Let $\overline{\mathcal{T}}(E)$ denote the closure of $\mathcal{T}(E)$ in $W^{-2,q}_{+}(B_R)$. If $\mu \in \overline{\mathcal{T}}(E)$ then $u_\mu \le V_E$.*

Let $\{\nu_n\}$ be a sequence in $\mathcal{T}(E)$ such that $\nu_n \to \nu$ in $W^{-2,q}_{+}(B_R)$. Let u_n be the solution of

$$-\Delta u_n + u_n^q = \nu_n \ \text{ in } \ B_R, \quad u_n = 0 \ \text{ on } \ \Sigma_R.$$

Then $\{u_n\}$ converges in $L^q(B_R)$ and the limit u is a weak solution of

$$-\Delta u + u^q = \nu \ \text{ in } \ B_R, \quad u = 0 \ \text{ on } \ \Sigma_R.$$

Since $u_n \le V_E$ it follows that $u = u_\nu \le V_E$.

<u>Assertion 2</u>:

$$\nu \in \mathcal{T}(\widetilde{E}) \Longrightarrow \nu \in \overline{\mathcal{T}}(E). \tag{5.18}$$

Suppose that $\nu \in \mathcal{T}(\widetilde{E})$ but $\nu \notin \overline{\mathcal{T}}_E$. Then there exists $\phi \in W^{2,q'}(\mathbb{R}^N)$ such that

$$\|\phi\|_{W^{2,q'}(\mathbb{R}^N)} = 1, \quad \langle \phi, \nu\rangle > 0, \quad \langle \phi, \mu\rangle = 0 \quad \forall \mu \in \mathcal{T}_E. \tag{5.19}$$

We choose ϕ to be a $C_{2,q'}$-finely continuous representative of its equivalence class (see [1, Proposition 6.1.2]). Thus the inverse image (by ϕ) of every open interval is quasi-open (see [1, Proposition6.4.10]). It follows that

$$A_0 := \{\sigma : \ \phi(\sigma) = 0\} \ \text{ is } C_{2,q'}\text{-finely closed.}$$

We show that

$$C_{2,q'}(\widetilde{E} \setminus A_0) = 0. \tag{5.20}$$

Put $A_1 := \widetilde{E} \setminus A_0$ and

$$A_1^+ = \{x \in A_1 : \ \phi(x) > 0\}, \quad A_1^- = \{x \in A_1 : \ \phi(x) < 0\}.$$

If (5.20) does *not* hold then

$$\text{either} \quad C_{2,q'}(A_1^+) > 0, \ \text{ or } \ C_{2,q'}(A_1^-) > 0.$$

Each of these sets is $C_{2,q'}$-finely open relative to $\widetilde{E}$, i.e., there exist $C_{2,q'}$-finely open sets Q_1, Q_2 such that $Q_1 \cap \widetilde{E} = A_1^+$ and $Q_2 \cap \widetilde{E} = A_1^-$. If, say, $C_{2,q'}(Q_1 \cap \widetilde{E}) > 0$ then $C_{2,q'}(Q_1 \cap E) > 0$ (because $C_{2,q'}(G) \sim C_{2,q'}(\widetilde{G})$ for any Borel set $G \subset B_R$). Let $\mu \in W^{-2,q}_{+}(\mathbb{R}^N)$ be a non-trivial measure , supported in a compact subset of $Q_1 \cap E$. Then

$$\langle \phi, \mu\rangle > 0.$$

This contradicts (5.19) and proves (5.20).

Further (5.20) implies that $\phi = 0$ $C_{2,q'}$-a.e. on $\widetilde{E}$ which implies $\langle \phi, \nu\rangle = 0$ in contradiction to (5.19). This proves Assertion 2.

Combining these assertions we conclude:

$$\nu \in \mathcal{T}(\widetilde{E}) \Longrightarrow u_\nu \leq V_E \Longrightarrow V_{\widetilde{E}} = sup\{u_\nu : \nu \in \mathcal{T}(\tilde{E})\} \leq V_E. \tag{5.21}$$

Since, trivially, $V_E \leq V_{\widetilde{E}}$ we obtain (5.15).

If E is $C_{2,q'}$-quasi closed then, by Theorem 5.2, there exists an increasing sequence $\{\mu_n\}$ in $W_+^{-2,q}(\mathbb{R}^N)$ such that $\mu_n(E^c) = 0$ and $u_{\mu_n} \to V_E$. Put

$$\tau := \sum a_n\mu_n, \quad a_n := 2^{-n} \|\mu_n\|_{W_+^{-2,q}(\mathbb{R}^N)}. \tag{5.22}$$

Then $\tau \in W_+^{-2,q}(\mathbb{R}^N)$, $\tau(E^c) = 0$ and (5.17) holds.

We turn to the proof of (5.16). The implication

$$\mu(\mathbb{R}^N \setminus \tilde{E}) = 0 \Longrightarrow u_\mu < V_E$$

is a consequence of (5.15). To prove the implication in the opposite direction we may assume that E is compact. (This follows from Theorem 5.2.) By negation, suppose there exists $\mu \in W^{-2,q}(\mathbb{R}^N)$ such that $u_\mu < V_E$ but $\mu(\mathbb{R}^N \setminus \tilde{E}) > 0$. It follows that there exists a compact set $K \subset \mathbb{R}^N \setminus \tilde{E}$ such that $\mu(K) > 0$. Let $v_n := u_{n\mu\chi_K}$. Then $v_n \leq nu_\mu$ because nu_μ is a supersolution of the equation $-\Delta w + w^q = n\mu\chi_K$. On the other hand, V_E is the largest solution dominated by nV_E, for every n. Therefore

$$v = \lim v_n \leq V_E. \tag{5.23}$$

If A is an open neighborhood of K such that $\operatorname{dist}(A, E) > 0$ then $V_E \in L^q(A)$. On the other hand

$$\int_{A\setminus K} v^q = \infty.$$

Therefore $(v - V_E)_+$ is positive in an open subset of $A \setminus K$. This contradicts (5.23). □

Theorem 5.4. *Let Ω be an open bounded set in $\mathbb{R}^N$ such that $\Omega = \cup\Omega_n$, where $\{\Omega_n\}$ is an increasing family of open sets satisfying*

$$C_{2,q'}(\Omega \setminus \Omega_n) \to 0. \tag{5.24}$$

Put

$$\begin{aligned} &F_n := \partial\Omega_n, \quad D_n = \mathbb{R}^N \setminus \bar{\Omega}_n, \quad \Omega^n = \Omega \setminus \Omega_n \\ &F := \partial_q\Omega = \widetilde{\Omega} \setminus \Omega, \quad D := \mathbb{R}^N \setminus \widetilde{\Omega} \end{aligned} \tag{5.25}$$

and assume that

$$C_{2,q'}(F_n \setminus \widetilde{D}_n) \to 0. \tag{5.26}$$

Under these assumptions, $V_{\widetilde{D}}$ is the unique ∂_q-large solution in Ω.

The proof is based on several lemmas.

Lemma 5.5. *Let Ω be a bounded open set such that, with the notation $F := \partial\Omega$, $D := \mathbb{R}^N \setminus \bar{\Omega}$,*

$$C_{2,q'}(F \setminus \widetilde{D}) = 0. \tag{5.27}$$

Then $V_{\widetilde{D}}$ is the unique ∂_q-large solution in Ω.

Proof. Let v be a ∂_q-large solution in Ω. First we show that

$$V_D \le v \quad \text{in } \Omega. \tag{5.28}$$

If $\mu \in \mathcal{T}_D$ then $\mu = \sup\{\mu\chi_K : K \subset D, \ K \text{ compact}\}$ and $u_\mu = \sup u_{\mu\chi_K}$ over compact sets K as above. Therefore it is sufficient to show that

$$u_\mu \le v \tag{5.29}$$

for every $\mu \in W_+^{-2,q}(\mathbb{R}^N)$ supported in a compact set $K \subset D$. Since $K \cap \bar{\Omega} = \emptyset$, u_μ is uniformly bounded in $\bar{\Omega}$.

Let

$$A_v := \{y \in F : \liminf_{\Omega \ni x \to y} v(x) < \infty\}.$$

Note that

$$\partial_q D \subset \partial D \subset \partial\Omega = F,$$

$$\partial_q D \subset \partial_q(\mathbb{R}^N \setminus \widetilde{\Omega}) = \partial_q \Omega.$$

By (5.27) $C_{2,q'}(F \setminus \partial_q D) = 0$; therefore $C_{2,q'}(F \setminus \partial_q \Omega) = 0$. Therefore any ∂_q-large solution in Ω is an almost large solution in Ω. Hence $C_{2,q'}(A_v) = 0$.

Let G_ϵ be an open neighborhood of $A_v(F)$ such that $C_{2,q'}(G_\epsilon) < \epsilon$. Put

$$\Omega_\delta = \{x \in \Omega : \text{dist}\,(x, F) < \delta\}, \quad \Omega'_\delta = \{x \in \Omega : \text{dist}\,(x, F) > \delta\}.$$

Let Ω^*_δ be a smooth domain such that $\Omega'_\delta \subset \Omega^*_\delta \Subset \Omega'_{\delta/2}$. Put

$$G_{\epsilon,\delta} := G_\epsilon \cap (\Omega \setminus \overline{\Omega^*_\delta}).$$

Then $v + V_{G_{\epsilon,\delta}}$ is a supersolution of (1.1) in Ω^*_δ and, if δ is sufficiently small,

$$u_\mu \le v + V_{G_{\epsilon,\delta}} \quad \text{on } \partial\Omega^*_\delta.$$

Thus

$$u_\mu \le v + V_{G_{\epsilon,\delta}} \quad \text{in } \Omega^*_\delta.$$

Since $C_{2,q'}(G_\epsilon) \to 0$ as $\epsilon \to 0$, Theorem 5.1 implies that, for fixed $\delta > 0$,

$$\lim_{\epsilon \to 0} V_{G_{\epsilon,\delta}} = 0 \quad \text{in } \Omega^*_\delta.$$

Letting $\delta \to 0$ we obtain (5.29) and hence (5.28). Further, by Theorem 5.3,

$$V_{\widetilde{D}} = V_D \le v \quad \text{in } \Omega. \tag{5.30}$$

Next we show that the opposite inequality,

$$v \le V_{\widetilde{D}}, \tag{5.31}$$

is also valid. (A-priori this is not obvious because we do not assume that v is σ-moderate.)

By (5.27) $C_{2,q'}(\overline{D} \setminus \widetilde{D}) = 0$; hence $V_{\overline{D}} = V_{\widetilde{D}}$.

Let R be sufficiently large so that $\overline{\Omega}\subset B_R(0)$. Then

$$(i)\quad V_{\overline{D}}\leq V_{\overline{D\cap B_r}}+V_{B^c_R},\quad (ii)\quad U_{\overline{D}}\leq U_{\overline{D\cap B_R}}+U_{B^c_R}.$$

and $V_{\overline{D}}$ (resp. $U_{\bar{D}}$) is the largest solution in Ω, dominated by the right hand side of inequality (i) (resp. (ii)). Since $\overline{D\cap B_R}$ is compact, $V_{\overline{D\cap B_R}}=U_{\overline{D\cap B_R}}$. The uniqueness of large solutions in smooth domains implies that

$$U_{B^c_R}=U_{\partial B_R}=V_{\partial B_R}=V_{B^c_R}.$$

Combining these facts we conclude that

$$U_{\overline{D}}=V_{\overline{D}}=V_{\widetilde{D}}.$$

By definition, $v\leq U_{\overline{D}}$ in Ω; hence $v\leq V_{\widetilde{D}}$. □

Lemma 5.6. *Let v be a solution of* (1.1) *in a bounded open set Ω. Suppose that A is a $C_{2,q'}$-finely closed subset of $\partial\Omega$ such that*

$$\lim_{x\to y}v(x)=\infty\quad\forall y\in\partial\Omega\setminus A. \tag{5.32}$$

If $D:=\mathbb{R}^N\setminus\widetilde{\Omega}$ then,

$$V_{\widetilde{D}}=V_D\leq v+V_A\ \ in\ \ \Omega. \tag{5.33}$$

Proof. Let μ be a measure in $W^{-2,q}_+(\mathbb{R}^N)$ concentrated on a compact set $K\subset D$. Let $\{O_n\}$ be a decreasing sequence of open sets such that

$$A\subset O_n,\quad C_{2,q'}(O_n\setminus A)\to 0,\qquad \lim_{\Omega\ni x\to\partial\Omega\setminus O_n}v(x)=\infty.$$

Let $\Omega^*_{n,\delta}$ be as in the proof of Lemma 5.5 and let $\{\delta_n\}$ be a sequence of positive numbers decreasing to zero. Denote

$$G^n:=O_n\cap(\Omega_n\setminus\overline{\Omega^*_{n,\delta_n}}).$$

As in the proof of Lemma 5.5, we obtain

$$u_\mu\leq V_{G^n}+v\ \text{ in }\ \Omega^*_{n,\delta_n}.$$

Since $A\subset G^n$ and $C_{2,q'}(G^n\setminus A)\to 0$ it follows that $V_{G^n}\downarrow V_A$. Letting $n\to\infty$ we obtain

$$u_\mu\leq V_A+v$$

which in turn implies (5.33). □

Lemma 5.7. *Put*

$$S_{n,1}:=\overline{D}_n\setminus\widetilde{D}_n,\quad S_{n,2}=(\partial_qD_n)\Delta\partial_q(\mathbb{R}^N\setminus\widetilde{\Omega}_n),\quad E_n:=F_n\Delta F.$$

Then, under the assumptions of the theorem,

$$(a)\ \ C_{2,q'}(S_{n,1})\to 0,\quad (b)\ \ C_{2,q'}(S_{n,2})\to 0,\quad (c)\ \ C_{2,q'}(E_n)\to 0. \tag{5.34}$$

Proof. Since $S_{n,1} \subset F_n \setminus \widetilde{D}_n$, (a) follows from (5.26).

Since $D_n \subset \mathbb{R}^N \setminus \widetilde{\Omega}_n$ it follows that

$$\partial_q D_n \subset \partial_q(\mathbb{R}^N \setminus \widetilde{\Omega}_n) \cup \partial_q(\bar{D}_n \setminus \widetilde{D}_n), \quad \partial_q(\mathbb{R}^N \setminus \widetilde{\Omega}_n) \subset \partial_q D_n \cup \partial_q(\bar{D}_n \setminus \widetilde{D}_n).$$

But (5.34) (a) implies that $C_{2,q'}(\partial_q(\bar{D}_n \setminus \widetilde{D}_n)) \to 0$. Therefore, the previous relations imply (5.34) (b).

In order to establish (c) we observe that,

$$F \subset \partial_q \Omega_n \cup \partial_q \Omega^n, \quad \partial_q \Omega_n \subset \partial_q \Omega^n \cup F.$$

It is known that (see [1]) there exists a constant $c(N,q)$ such that, for every Borel set A,

$$C_{2,q'}(\widetilde{A}) \leq c C_{2,q'}(A). \tag{5.35}$$

Therefore (5.24) implies that $C_{2,q'}(\widetilde{\Omega}^n) \to 0$, which in turn implies that $C_{2,q'}(\partial_q \Omega^n) \to 0$. We conclude that

$$C_{2,q'}(F \Delta \partial_q \Omega_n) \to 0. \tag{5.36}$$

Hence, as $\partial_q \Omega_n \subset F_n$,

$$C_{2,q'}(F \setminus F_n) \leq C_{2,q'}(F \setminus \partial_q \Omega_n) \to 0. \tag{5.37}$$

On the other hand,

$$F_n \setminus F \subset (F_n \setminus \partial_q \Omega_n) \cup (\partial_q \Omega_n \setminus F). \tag{5.38}$$

Since

$$\partial_q \Omega_n \supseteq \partial_q \widetilde{\Omega}_n = \partial_q(\mathbb{R}^N \setminus \widetilde{\Omega}_n).$$

(5.34) (b) implies

$$C_{2,q'}(\partial_q D_n \setminus \partial_q \Omega_n) \to 0.$$

This fact and assumption (5.26) imply

$$C_{2,q'}(F_n \setminus \partial_q \Omega_n) \to 0. \tag{5.39}$$

Finally, (5.36), (5.38) and (5.39) imply

$$C_{2,q'}(F_n \setminus F) \to 0. \tag{5.40}$$

This together with (5.37) yields (5.34) (c). □

Proof of Theorem 5.4. Let $A_n = F_n \setminus F$. By (5.34) (b), $C_{2,q'}(A_n) \to 0$. If v is a ∂_q-large solution in Ω then v blows up $C_{2,q'}$ a.e. on F and consequently it blows up $C_{2,q'}$ a.e. on $F_n \setminus A_n$. Applying Lemma 5.6 to v in Ω_n we obtain

$$V_{\widetilde{D}_n} = V_{D_n} \leq v + V_{A_n} \text{ in } \Omega_n.$$

Note that

$$D_n \setminus D = \widetilde{\Omega} \setminus \overline{\Omega}_n = (\Omega \setminus \overline{\Omega}_n) \cup (F \setminus \overline{\Omega}_n) \subset (\Omega \setminus \overline{\Omega}_n) \cup (F \setminus F_n)$$

and

$$D \setminus D_n = \overline{\Omega}_n \setminus \widetilde{\Omega} = (\Omega_n \setminus \widetilde{\Omega}) \cup (F_n \setminus \widetilde{\Omega}) \subset (F_n \setminus F).$$

Therefore, (5.24) and (5.34) (c) imply that

$$C_{2,q'}(D_n \Delta D) \to 0. \tag{5.41}$$

The definition of V_E (see (5.2)) implies

$$V_{D_n} \leq V_D + V_{D_n \setminus D}$$

and, by (5.41) and Theorem 5.1, $V_{D_n \setminus D} \to 0$. Hence

$$V_{\widetilde{D}_n} \to V_{\widetilde{D}}. \tag{5.42}$$

By Theorem 5.1, $V_{A_n} \to 0$ in Ω. Therefore, letting $n \to \infty$, we obtain

$$V_{\widetilde{D}} \leq v \text{ in } \Omega.$$

It remains to show that $v \leq V_{\widetilde{D}}$. As $U_{\overline{D}_n}$ is the maximal solution in Ω_n,

$$v \leq V_{\overline{D}_n} = U_{\overline{D}_n}.$$

Lemma 5.7, implies that

$$V_{\overline{D}_n} - V_{\widetilde{D}_n} \to 0 \text{ in } \Omega.$$

Indeed, as an immediate consequence of the definition of V_E (see (5.2)),

$$V_{\overline{D}_n} \leq V_{\widetilde{D}_n} + V_{\overline{D}_n \setminus \widetilde{D}_n}.$$

By (5.34) (a), $C_{2,q'}(\overline{D}_n \setminus \widetilde{D}_n) \to 0$. Hence, by Theorem 5.1, $V_{\overline{D}_n \setminus \widetilde{D}_n} \to 0$ in Ω_n. It follows that

$$\lim V_{\overline{D}_n} \leq \lim V_{\widetilde{D}_n}.$$

The limits exist because of monotonicity. Since $V_{\widetilde{D}_n} \leq V_{\overline{D}_n}$ we obtain,

$$\lim V_{\overline{D}_n} = \lim V_{\widetilde{D}_n}.$$

Therefore

$$v \leq \lim V_{\widetilde{D}_n} = V_{\widetilde{D}}.$$

□

Corollary 5.8. *Suppose that $\Omega = \cup_1^\infty Q_n$ where $\{Q_n\}$ is a sequence of open sets such that*

$$\sum_1^\infty C_{2,q'}(Q_n) < \infty. \tag{5.43}$$

For every $n \in \mathbb{N}$, put

$$S_n = \cup_1^n \overline{Q}_k, \quad D_n = \mathbb{R}^N \setminus \overline{S}_n$$

and assume that

$$C_{2,q'}(\partial S_n \setminus \widetilde{D}_n) \to 0. \tag{5.44}$$

Then there exists a unique almost large solution in Ω.

Remark. If $y \in \partial S_n$ and there exists an open cone C_y, with vertex y, such that $C_y \subset \mathbb{R}^N \setminus S_n$ then $y \in \partial_q S_n$. Hence if, for every $n \in \mathbb{N}$, this condition is satisfied $C_{2,q'}$ a.e. on ∂S_n then (5.44) holds. In particular, if $\{Q_n\}$ is a sequence of balls, (5.44) is satisfied.

Proof. Let $\Omega_n = S_n^0 := S_n \setminus \partial S_n$. Then $\{\Omega_n\}$ is an increasing sequence of open sets, (5.44) implies (5.26) and (5.43) implies (5.24). Therefore the corollary is an immediate consequence of Theorem 5.4. □

Example. Let $\{x^m\}$ be a sequence of distinct points in $B_1(0)$. Let $\{r_n\}$ be a decreasing sequence of positive numbers such that $\{B_{r_n}(x^n)\}$ is a sequence of balls contained in $B_1(0)$ and

$$\begin{aligned} &\sum r_n^{N-2q'} < \infty && \text{if } N > 2q', \\ &\sum (1-\log r_n)^{1-q'} < \infty && \text{if } N = 2q'. \end{aligned} \tag{5.45}$$

Then there exists a unique large solution in $\Omega := \cup_1^\infty B_{r_n}(x^n)$.

Indeed $C_{2,q'}(B_r) \sim r^{N-2q'}$ if $N > 2q'$ and $C_{2,q'}(B_r) \sim \log(1-\log r)$ if $N = 2q'$ and $0 < r < 1$. Therefore the conditions of Corollary 5.8 are satisfied.

Note that $\widetilde{\Omega} = \cup \overline{B}_{r_n}(x^n)$, but, in general $\overline{\Omega}$ is much larger. For instance, if $\{x^m\}$ is a dense sequence in $B_1(0)$ then $\overline{\Omega} = \overline{B}_1(0)$. Therefore it is important that our conditions in Corollary 5.8 require $C_{2,q'}(\partial_q\Omega \setminus \widetilde{D}) = 0$ and not $C_{2,q'}(\partial\Omega \setminus \widetilde{D}) = 0$.

6. Very weak subsolutions

In this section F is a $C_{2,q'}$-finely closed set contained in $B_1(0)$ and $D = B_2(0) \setminus F$. Note that D is a $C_{2,q'}$-finely open set, but not necessarily open in the Euclidean topology.

We denote by $W^{2,q'}(D)$ the set $\{h\big|_D : h \in W^{2,q'}(\mathbb{R}^N)\}$. If $f \in W^{2,q'}(\mathbb{R}^N)$ we denote by $\mathrm{supp}_{(2,q')} f$ (= the $C_{2,q'}$-fine support of f) the intersection of all $C_{2,q'}$-finely closed sets E such that $f = 0$ a.e. in $\mathbb{R}^N \setminus E$.

The following subspace of $W^{2,q'}(D)$ serves as a space of test functions in our study:

$$W^{2,q'}_{0,\infty}(D) := \{h\big|_D : h \in W^{2,q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N),\ \mathrm{supp}_{(2,q')} h \Subset D\}. \tag{6.1}$$

The notation $E \Subset D$ means: E is 'strongly contained' in D, i.e., $\overline{E}$ is a compact subset of D. Some features of this space are discussed in Appendix A.

The following statement was established in [29] (see Lemma 2.6). (The framework in [29] is somewhat different, but the proof, with obvious modifications, applies to the present case as well.)

Lemma 6.1. *Let D be a bounded $C_{2,q'}$-finely open set. Then there exists an increasing sequence of compact sets $\{E_n\}$ such that*

$$\begin{aligned} &E_n \subset \mathrm{int}_q E_{n+1}, && \cup_1^\infty E_n \subset D, \\ &C_{2,q'}(D \setminus \cup_1^\infty E_n) = 0, && C_{2,q'}(E_n) \to C_{2,q'}(D). \end{aligned} \tag{6.2}$$

A sequence of sets $\{E_n\}$ as above is called a *q-exhaustion of D*.

We denote by $L^q_{\ell(2,q')}(D)$ the space of measurable functions f in D such that, for every positive $\phi \in W^{2,q'}_{0,\infty}(D)$, $f \in L^q(D;\phi)$, i.e., $|f|^q\phi \in L^1(D)$. We endow this space with the topology determined by the family of semi-norms

$$\{\|\cdot\|_{L^q(D;\phi)} : \phi \in W^{2,q'}_{0,\infty}(D),\ \phi \geq 0\}. \tag{6.3}$$

This topology will be denoted by $\tau_{\ell(2,q')}(D)$.

Further we denote by $\mathfrak{M}_{\ell(2,q')}(D)$ the space of positive Borel measures μ in D such that

$$\begin{array}{lll} (a) & K \subset D,\ K \text{ compact} & \Longrightarrow \mu(K) < \infty, \\ (b) & E \subset D,\ E \text{ Borel},\ C_{2,q'}(E) = 0 & \Longrightarrow \mu(E) = 0. \end{array} \tag{6.4}$$

We observe that,

Lemma 6.2. *If $\mu \in \mathfrak{M}_{\ell(2,q')}(D)$ then:*
(i) There exists an increasing sequence $\{\mu_n\}$ of positive, bounded measures in $W^{-2,q}(\mathbb{R}^N)$ such that $\mu_n(D^c) = 0$ and $\mu_n \uparrow \mu$.
(ii) $W^{2,q'}_{0,\infty}(D) \subset L^1(\mu)$.

Proof. (i) This is well known in the case that μ is a positive, bounded measure [6] and it follows from Lemma 6.1 in the case that μ is a positive measure in $\mathfrak{M}_{\ell(2,q')}(D)$.

(ii) If $\varphi \in W^{2,q'}_{0,\infty}(D)$, it vanishes outside a compact set $K_\varphi \subset D$. By definition, $\mu(K_\varphi) < \infty$. Furthermore φ is the limit $C_{2,q'}$ a.e. of smooth functions; consequently it is μ-measurable. Since φ is bounded, it is integrable relative to μ. □

Notation. A sequence $\{\mu_n\}$ as in Lemma 6.2 (i) will be called a *determining sequence* for μ.

We introduce below a very weak type of subsolution of (1.2) defined as follows.

Definition 6.3. Assume that the measure μ in (1.2) belongs to $\mathfrak{M}_{\ell(2,q')}(D)$. A non-negative measurable function u is a *very weak subsolution* of (1.2) in D if, for every non-negative $\phi \in W^{2,q'}_{0,\infty}(D)$,

$$u \in L^q(D;\zeta) \quad \textit{where} \quad \zeta := \phi^{2q'}, \tag{6.5}$$

$$-\int_D u\Delta\zeta dx + \int_D u^q \zeta dx \leq \int_D \zeta d\mu. \tag{6.6}$$

Remarks. (a) If (6.5) holds for every non-negative $\phi \in W^{2,q'}_{0,\infty}(D)$ then

$$u\Delta\zeta \in L^1(D). \tag{6.7}$$

This is proved in the next lemma.

(b) Let $\phi \in W^{2,\gamma}_{0,\infty}(D)$, $\gamma \geq 1$. By interpolation, $|\nabla\phi|^2 \in L^\gamma(D)$ and

$$\left\||\nabla\phi|^2\right\|_{L^\gamma(D)} \leq c(q,N)L\left\|D^2\phi\right\|_{L^\gamma(D)}, \quad L := \|\phi\|_{L^\infty(D)}. \tag{6.8}$$

where $|D^2\phi| := \sum_{|\alpha|=2} |D^\alpha \phi|$.

(c) If $\phi \in W^{2,\gamma}_{0,\infty}(D)$, $\gamma \geq 1$ then

$$
\begin{aligned}
&|\phi|^{2\gamma} = (\phi^2)^\gamma \in W^{2,q'}_{0,\infty}(D),\\
&\nabla(|\phi|^{2\gamma}) = 2\gamma(\phi^2)^{\gamma-1/2}\nabla\phi,\\
&\Delta(\phi^{2\gamma}) = 2\gamma(2\gamma-1)|\phi|^{2\gamma-2}|\nabla\phi|^2 + 2\gamma|\phi|^{2\gamma-1}\Delta\phi.
\end{aligned}
\tag{6.9}
$$

The last two formulas are easily verified for $\phi \in C_c^\infty(\mathbb{R}^N)$; in the general case they are obtained by the usual density argument. The fact that $|\phi|^{2\gamma} \in W^{2,q'}_{0,\infty}(D)$ is a consequence of these formulas and (6.8). (6.9) imply,

$$
\left\| |\phi|^{2\gamma} \right\|_{W^{2,\gamma}(D)} \leq AL^{2\gamma-1}\max(1,L)\, \|\phi\|_{W^{2,\gamma}(D)}\,. \tag{6.10}
$$

Theorem 6.4. (i) *If u is a non-negative measurable function satisfying* (6.5) *then $u\Delta\zeta \in L^1(D)$.*

(ii) *If u is a very weak subsolution of* (1.1) *in D (i.e. $\mu = 0$) then, for every non-negative $\phi \in W^{2,q'}_{0,\infty}(D)$,*

$$
\int_D u|\Delta\zeta|dx + \int_D u^q\zeta \leq c\left(L\left\|D^2\phi\right\|_{L^\gamma(D)}\right)^\gamma, \tag{6.11}
$$

where $\zeta := \phi^{2q'}$, $c = c(N,q)$ and $L := \|\phi\|_{L^\infty(D)}$.

(iii) *Let $\mu \in W^{-2,q}(\mathbb{R}^N)$ be a positive bounded measure vanishing outside D. If u is a non-negative very weak subsolution of* (1.2) *then*

$$
\begin{aligned}
\|u\|_{L^q(D,\zeta)} \leq & cL^{1/(q-1)}\Big(\big(\left\|D^2\phi\right\|_{L^{q'}(D)}\big)^{1/(q-1)} +\\
&\big(L^{1/(q-1)}\,\|\mu\|_{W^{-2,q'}}\left\|D^2\phi\right\|_{L^{q'}(D)}\big)^{1/q}\Big)
\end{aligned}
\tag{6.12}
$$

Finally, if $L \leq \overline{L}$, u satisfies

$$
\begin{aligned}
&\|u\|_{L^q(D,\phi)} \leq\\
&c(N,q,\overline{L})L^{\frac{1}{q-1}}\Big(\left\|D^2\phi\right\|^{\frac{1}{q-1}}_{L^{q'}(D)} + \|\mu\|^{\frac{1}{q}}_{W^{-2,q'}}\left\|D^2\phi\right\|^{\frac{1}{q}}_{L^{q'}(D)}\Big).
\end{aligned}
\tag{6.13}
$$

Proof. Let ϕ be a non-negative function in $W^{2,q'}_{0,\infty}(D)$. By (6.9), with $\gamma = q'$, we obtain

$$
|\Delta\zeta| \leq c(q)\zeta^{1/q}M(\phi), \quad M(\phi) := \big(|\nabla\phi|^2 + \phi\,|\Delta\phi|\big)
$$

and hence, using (6.8),

$$
\int_D u\,|\Delta\zeta|\,dx \leq c(q)\left(\int_D u^q\zeta\,dx\right)^{1/q}\left(\int_D M(\phi)^{q'}dx\right)^{1/q'}, \tag{6.14}
$$

$$
\int_D M(\phi)^{q'}dx \leq c(q,N)L\left\|D^2\phi\right\|^{q'}_{L^{q'}(D)}. \tag{6.15}
$$

Assuming that $u \in L^q(D,\zeta)$ we obtain $u\Delta\zeta \in L^1(D)$.

We turn to the proof of (ii) and (iii). We assume that u is a non-negative very weak subsolution as in Definition 6.3. Put

$$A := \left(\int_D u^q \zeta dx\right)^{1/q}, \; B := \left(\int_D M(\phi)^{q'} dx\right)^{1/q'}, \; C := \|\mu\|_{W^{-2,q'}} \|\zeta\|_{W^{2,q'}(D)}$$

By (6.6) and (6.14)

$$A^q = \int_D u^q \zeta dx \leq \int_D u\Delta\zeta dx + \int_D \zeta \, d\mu \leq c(q,N)AB + C. \tag{6.16}$$

This implies

$$A^q \leq \frac{1}{q}A^q + \frac{1}{q'}(cB)^{q'} + C \Longrightarrow A^q \leq (cB)^{q'} + q'C \leq c'(N,q)\max(B^{q'}, C).$$

Thus

$$A \leq c(q,N)\left(B^{1/(q-1)} + C^{1/q}\right). \tag{6.17}$$

By Poincaré's inequality

$$\|\zeta\|_{W^{2,q'}(D)} \leq c(q,N)\left\|D^2\zeta\right\|_{L^{q'}}$$

and therefore, by the same computation as in (6.9),

$$\|\zeta\|_{W^{2,q'}(D)} \leq c(q,N)\Big(\left\|\phi^{2/(q-1)}(\nabla\phi)^2\right\|_{L^{q'}} + \left\|\phi^{(1+q)/(q-1)}D^2\phi\right\|_{L^{q'}}\Big).$$

Therefore by (6.8) and (6.15),

$$\|\zeta\|_{W^{2,q'}(D)} \leq L^{\frac{1}{q}}(L + L^{\frac{1}{q}})\left\|D^2\phi\right\|_{L^{q'}}. \tag{6.18}$$

This estimate and (6.17) imply (6.12). Further, if $\mu = 0$, (6.12), (6.14) and (6.15) imply (6.11).

Now let ψ be a non-negative function in $W^{2,q'}_{0,\infty}(D)$ and put

$$\phi := (1+\psi)^{\frac{1}{2q'}} - 1, \quad \zeta := \phi^{2q'}.$$

Then $\phi \in W^{2,q'}_{0,\infty}(D)$, $\zeta \sim \psi$ and

$$\left\|D^2\phi\right\|_{L^{q'}(D)} \leq c(N,q)\left\|D^2\psi\right\|_{L^{q'}(D)}(1 + \|\psi\|_{L^\infty(D)}).$$

This inequality and (6.12) imply (6.13).

□

Lemma 6.5. *If F is a Borel set such that $C_{2,q'}(F) = 0$ then the only non-negative very weak subsolution of* (1.1) *in $D = F^c$ is the trivial solution.*

Remark. A set of capacity zero is $C_{2,q'}$-finely closed by defintion. Therefore the notion of very weak subsolution in F^c is well defined in the present case.

Proof. Since $C_{2,q'}(F)=0$, there exists a sequence $\{\eta_n\}$ in $W^{2,q'}(\mathbb{R}^N)$ such that $0\leq\eta_n\leq 1$, $\|\eta_n\|_{W^{2,q'}}\to 0$ and $\eta_n=1$ on a neighborhood of F (depending on n). Applying (6.11) to u and $\phi_n=1-\eta_n$ yields:

$$\int_D u^q\phi_n^{2q'}\,dx\leq c\left\|D^2\eta_n\right\|^{q'}_{L^{q'}(D)}\to 0.$$

Since $\|\eta_n\|_{L^{q'}}\to 0$ it follows that there exists a subsequence converging to zero a.e.. Therefore

$$\liminf\int_D u^q\phi_n^{2q'}\,dx\geq\int_D u^q\,dx.$$

This implies that $u=0$ a.e. □

7. $C_{2,q'}$-strong solutions in finely open sets and uniqueness II

We start with the definition of '$C_{2,q'}$-strong' solutions of (1.2) in a $C_{2,q'}$-finely open set or more generally in a $C_{2,q'}$-quasi open set. We recall that a set E is $C_{2,q'}$-quasi open if, for every $\epsilon>0$ there exists an open set O such that $E\subset O$ and $C_{2,q'}(O\setminus E)<\epsilon$. Every $C_{2,q'}$-finely open set is $C_{2,q'}$-quasi open; if E is $C_{2,q'}$-quasi open then $E\overset{q}{\sim}\mathrm{int}_qE$, (see [1, Chapter 6]).

Definition 7.1. Let D be a $C_{2,q'}$-quasi open set, let $\mu\in\mathfrak{M}_{\ell(2,q')}(D)$ be a non-negative measure and let $\{\mu_n\}$ be a determining sequence for μ (see Lemma 6.2).

(i) A positive function $u\in L^q_{\ell(2,q')}(D)$ is a $C_{2,q'}$*-strong solution* of (1.2) in D if there exists a decreasing sequence of open sets $\{\Omega_n\}$, such that $D\subset\Omega_n$ and, for each n, there exists a positive solution $u_n\in L^q_{\mathrm{loc}}(\Omega_n)$ of the equation

$$-\Delta u_n+u_n^q=\mu_n \tag{7.1}$$

such that

$$u_n\to u\ \ in\ \ L^q_{\ell(2,q')}(D). \tag{7.2}$$

We say that $\{(u_n,\Omega_n)\}$ is *a determining sequence for u in D.*

(ii) A $C_{2,q'}$-strong subsolution is defined in the same way as above except that u_n is only required to be a subsolution of (7.1) in Ω_n.

(iii) A positive $C_{2,q'}$-strong solution of (1.2) in D is σ*-moderate* if, in addition, the sequence $\{u_n\}$ is non-decreasing and there exists a sequence $\{v_n\}$ such that $v_n\in L^1(\Omega_n)$ and

$$-\Delta v_n=\mu_n,\quad u_n\leq v_n\ \text{ in }\ \Omega_n,\quad n=1,2,\dots. \tag{7.3}$$

(iv) If μ is bounded and $\{\|v_n\|_{L^1(\Omega_n)}\}$ is bounded we say that u is a *moderate* solution.

Remark. If D is an open set we may choose $\Omega_n=D$ for every n. Therefore any non-negative solution of (1.2) in D is a $C_{2,q'}$-strong solution in D. Furthermore, if u is a σ-moderate solution of (1.1) in D in the standard sense (i.e. the limit of an increasing sequence of moderate solutions) then it is a σ-moderate $C_{2,q'}$-strong solution in the sense of part (iii) of the above definition.

Definition 7.2. (a) A $C_{2,q'}$-strong solution v of (1.1) in D is called a ∂_q-*large solution* if

$$\lim_{x\to\partial_q D}^{q} v(x)=\infty \quad C_{2,q'} \text{ a.e. at } \partial_q D. \tag{7.4}$$

This condition is understood as follows. There exists a determining sequence $\{(v_n,\Omega_n)\}$ for v in D such that

$$\sum_1^\infty C_{2,q'}(\Omega_n\setminus D)<\infty, \tag{7.5}$$

and, for every $M>0$, $k\in\mathbb{N}$, there exists an open set $Q_{k,M}$ such that

$$\begin{aligned}&\cup_{n=k}^\infty \widetilde{\Omega_n\setminus D}\subset Q_{k,M}, \quad \lim_{k\to\infty} C_{2,q'}(Q_{k,M})=0,\\ &\liminf_{\substack{x\to\partial_q D\setminus Q_{k,M}\\ x\in\Omega_n}} v_n(x)\geq M \quad \forall n\geq k.\end{aligned} \tag{7.6}$$

Note that $\partial_q D\setminus Q_{k,M}\subset\partial\Omega_n$ for all $n\geq k$.

If F is a quasi closed subset of ∂D, the condition

$$\lim_{x\to F}^{q} v(x)=\infty \quad C_{2,q'} \text{ a.e. at } F \tag{7.7}$$

is defined in the same way except that the second line in (7.6) reads

$$\liminf_{x\to F\setminus Q_{k,M}} v_n(x)\geq M \quad \forall n\geq k.$$

(b) Let v be a non-negative $C_{2,q'}$-strong subsolution of (1.1) in D. The condition

$$\lim_{x\to\partial D}^{q} v(x)=0 \quad C_{2,q'} \text{ a.e. at } \partial_q D \tag{7.8}$$

is understood as follows. There exists a determining sequence $\{(v_n,\Omega_n)\}$ for v in D satisfying (7.5) and a family of open sets

$$\{Q_{k,\epsilon}:\ \epsilon>0,\ k\in\mathbb{N}\}$$

such that,

$$\cup_{n=k}^\infty \widetilde{\Omega_n\setminus D}\subset Q_{k,\epsilon}, \quad \lim_{k\to\infty} C_{2,q'}(Q_{k,\epsilon})=0, \tag{7.9}$$

$$\limsup_{\substack{x\to\partial_q D\setminus Q_{k,\epsilon}\\ x\in\Omega_n}} v_n(x)\leq\epsilon \quad \forall n\geq k. \tag{7.10}$$

If F is a quasi closed subset of ∂D, the condition

$$\lim_{x\to F}^{q} v(x)=0 \quad C_{2,q'} \text{ a.e. at } F \tag{7.11}$$

is defined in the same way except that (7.10) is replaced by

$$\limsup_{x\to F\setminus Q_{k,\epsilon}} v_n(x) \le \epsilon \quad \forall n \ge k. \tag{7.12}$$

(c) Let v be a non-negative $C_{2,q'}$-strong solution of (1.1) in D and let u be a non-negative classical solution in a domain $G \supseteq D$. We say that $u \overset{q}{\le} v$ at $\partial_q D$ if there exists a determining sequence $\{(v_n, \Omega_n)\}$ for v in D satisfying (7.5) and a family of open sets $\{Q_{k,\epsilon} : \epsilon > 0,\ k \in \mathbb{N}\}$ satisfying (7.9) such that

$$\limsup_{\substack{x\to \partial_q D\setminus Q_{k,\epsilon}\\ x\in\Omega_n}} (u - v_n)(x) \le \epsilon \quad \forall n \ge k. \tag{7.13}$$

If F is a quasi closed subset of ∂D, the condition $u \overset{q}{\le} w$ at F is defined in the same way except that (7.13) is replaced by

$$\limsup_{x\to F\setminus Q_{k,\epsilon}} (u - v_n)(x) \le \epsilon \quad \forall n \ge k. \tag{7.14}$$

We present several results concerning $C_{2,q'}$-strong solutions. The main ingredients in these proofs are: Theorem 6.4, the results of Section 5 concerning V_F and the results of Appendix A.

Theorem 7.3. *For every $\overline{L} > 0$, there exist constants $c = c(N, q)$ and $c = c(\overline{L})$ such that, for every non-negative measure $\mu \in \mathfrak{M}_{\ell(2,q')}(D)$ and every non-negative $C_{2,q'}$-strong solution u of* (1.2) *in D the following holds:*

$$\begin{aligned}&\|u\|^q_{L^q(D,\phi)} \le \\ &c(N,q)\Big((\|\phi\|_{L^\infty(D)} \left\|D^2\phi\right\|)^{q'}_{L^{q'}(D)} + c(\overline{L})\,\|\mu\|_{W^{-2,q'}} \left\|D^2\phi\right\|_{L^{q'}(D)}\Big)\end{aligned} \tag{7.15}$$

for every $\phi \in W^{2,q'}_{0,\infty}(D)$ such that $0 \le \phi \le \overline{L}$.

Every $C_{2,q'}$-strong solution u as above satisfies,

$$u^q\phi \in L^1(D), \quad u\Delta(\phi\psi) \in L^1(D) \tag{7.16}$$

$$-\int_D u\Delta(\phi\psi)dx + \int_D u^q(\phi\psi)dx = \int_D \phi\psi\, d\mu \tag{7.17}$$

for any non-negative $\phi, \psi \in W^{2,q'}_{0,\infty}(D)$. Finally u satisfies the estimate

$$u \le c(N,q)W_F \quad a.e.\ in\ D. \tag{7.18}$$

Proof. We use the notation of Definition 7.1. If u_n is a solution of (1.2) in Ω_n and $\phi \in W^{2,q'}_{0,\infty}(D)$ then

$$\int_D u_n\Delta\phi dx + \int_D u_n^q\phi dx = \int_D \phi\, d\mu_n \tag{7.19}$$

Evidently, u_n is, in particular, a very weak subsolution in D; consequently it satisfies inequality (6.13). By assumption, $u_n \to u$ in $L^q_{\ell(2,q')}$; hence u satisfies (6.13).

Assume that $\phi,\psi\in W^{2,q'}_{0,\infty}(D)$ and $0\le\phi\le\overline{L}$ and the same for ψ. Clearly, (7.17) holds for u_n. In addition,

$$\int_D u_n^q(\phi\psi)dx\to\int_D u^q(\phi\psi)dx,$$
$$\int_D(\phi\psi)\,d\mu_n\to\int_D(\phi\psi)\,d\mu.$$

Further,

$$\Delta(\phi\psi)=\phi\Delta\psi+\psi\Delta\phi+2\nabla\phi\cdot\nabla\psi,$$

so that

$$\int_D u_n|\Delta(\phi\psi)|dx\le\int_D u_n(\phi|\Delta\psi|+\psi|\Delta\phi|)\,dx+2\int_D u_n|\nabla\phi\cdot\nabla\psi|dx.$$

Using again the fact that $u_n\to u$ in $L^q_{\ell(2,q')}$

$$\int_D u_n(\phi\Delta\psi+\psi\Delta\phi)\,dx\to\int_D u(\phi\Delta\psi+\psi\Delta\phi)\,dx.$$

In addition,

$$\int_D u_n|\nabla\phi\cdot\nabla\psi|dx\le(\int_D u_n(\nabla\phi)^2dx)^{1/2}(\int_D u_n(\nabla\psi)^2dx)^{1/2}$$

By (7.19)

$$\int_D u_n\Delta\phi^2dx+\int_D u_n^q\phi^2dx=\int_D\phi^2\,d\mu_n$$

so that

$$\int_D u_n(\nabla\phi)^2dx\le\frac12\int_D\phi^2\,d\mu_n+\int_D u_n\phi|\Delta\phi|\,dx.$$

By Fatou, this implies,

$$\int_D u(\nabla\phi)^2dx\le\frac12\int_D\phi^2\,d\mu+\int_D u\phi|\Delta\phi|\,dx.$$

Now assume temporarily that $\{u_n\}$ is non-decreasing so that $u_n\uparrow u$. Then, by the dominated convergence theorem,

$$\int_D u_n(\nabla\phi\cdot\nabla\psi)dx\to\int_D u(\nabla\phi\cdot\nabla\psi)dx. \tag{7.20}$$

The convergence results obtained above and (7.19) imply (7.16) and (7.17). This in turn implies, by Theorem 6.4, the estimate (7.15).

Discarding the assumption of monotonicity, put $v_n:=\max(u_1,\dots,u_n)$. Then v_n is a subsolution of the equation

$$-\Delta v+v^q=\mu_n\ \text{ in }\ \Omega_n$$

and there exists a solution $\bar v_n$ of this equation which is the smallest among those dominating v_n. Then $\{\bar v_n\}$ is non-decreasing and, by Theorem 6.4,

$$\sup_n\int_D\bar v_n^q\phi\,dx<\infty$$

for any non-negative $\phi \in W^{2,q'}_{0,\infty}(D)$. Therefore $w = \lim \bar{v}_n \in L^q_{\ell(2,q')}(D)$ and by the previous part of the proof w is a $C_{2,q'}$-strong solution in D. In particular

$$w|\nabla\phi||\nabla\psi| \in L^1(D) \quad \forall \phi, \psi \in W^{2,q'}_{0,\infty}(D).$$

Clearly,

$$u_n|\nabla\phi \cdot \nabla\psi| \leq w|\nabla\phi||\nabla\psi|.$$

Therefore, once again by the dominated convergence theorem, we obtain (7.20) which together with the previous convergence results imply (7.15), (7.16) and (7.17).

Put $F_n = B_R \setminus \widetilde{\Omega}_n$, $F = B_R \setminus \widetilde{D}$. In order to prove the last assertion, we observe that, in Ω_n, $u_n \leq c(N,q)W_{F_n}$ with constant independent of n. As $\{F_n\}$ increases, $W_{F_n} \uparrow W_F$ everywhere in D. By (7.2) and Lemma A.4, we can extract a subsequence of $\{u_n\}$ which converges to u a.e. in D. Hence $u \leq cW_F$. □

Theorem 7.4. (i) *If F is a Borel set such that $C_{2,q'}(F) = 0$ then the only non-negative $C_{2,q'}$-strong subsolution of* (1.1) *in F^c is the trivial solution.*

(ii) *If F is a $C_{2,q'}$-finely closed set then V_F is a σ-moderate $C_{2,q'}$-strong solution in F^c.*

(iii) *Let F be a $C_{2,q'}$-finely closed set. If v is a $C_{2,q'}$-strong solution in $D := \mathbb{R}^N \setminus F$ then $v \leq V_F$.*

Proof. (i) By definition, a $C_{2,q'}$-strong solution u in $D = \mathbb{R}^N \setminus F$ is the limit of classical solutions in open sets containing D. In the case that $C_{2,q'}(F) = 0$, any such classical solution is the zero solution. Hence $u = 0$.

(ii) This is a consequence of Theorem 5.2.

(iii) By Theorem 7.3 :

$$v \leq c(N,q)W_F \leq c'(N,q)V_F \text{ a.e. in } \mathbb{R}^N \setminus F.$$

In addition, for every $\alpha \geq 1$,

$$\sup\{u : u \ C_{2,q'}\text{-strong solution in } F^c,\ u \leq \alpha V_F\} = V_F.$$

Hence $v \leq V_F$. □

Theorem 7.5. *Let D be a $C_{2,q'}$-finely open set and let $\{v_k\}$ be a sequence of non-negative $C_{2,q'}$-strong solutions of* (1.1) *in D converging a.e. in D. Then $v := \lim v_k$ is a $C_{2,q'}$-strong solution in D.*

Proof. By Lemma A.4 there exists an increasing sequence of compact sets $\{E'_n\}$ such that $\cup E'_n \subset D$ and $C_{2,q'}(D \setminus \cup E'_n) = 0$ and $\{v_k\}$ is uniformly bounded in $L^q(E'_n)$ for every n. Since $\{v_k\}$ converges a.e. it follows that it converges in $L^1(E'_n)$ for every n. By Theorem 7.4 (iii) V_{D^c} dominates $\{v_k\}$. By the dominated convergence theorem, $v_k \to v$ in the topology $\tau^q_{\ell(2,q')}(D)$.

By assumption, for each k, v_k is a $C_{2,q'}$-strong solution. This means that there exists a decreasing sequence of open sets $\{\Omega_{m,k}\}_{m=1}^{\infty}$, such that

$$D \subset \Omega_{m,k}, \quad \lim_{m\to\infty} C_{2,q'}(\Omega_{m,k} \setminus D) = 0 \tag{7.21}$$

and, for each m, there exists a positive solution $u_{m,k} \in L^q_{\mathrm{loc}}(\Omega_{m,k})$ of the equation $-\Delta u + u^q = 0$ in $\Omega_{m,k}$ such that

$$u_{m,k} \to v_k \text{ in } L^q_{\ell(2,q')}(D).$$

By Lemma A.3 the space $L^q_{\ell(2,q')}(D)$ with the topology $\tau^q_{\ell(2,q')}(D)$ is a metric space. We denote a metric for this topology by $d_{\ell(2,q')}$. For each k let m_k be sufficiently large so that

$$d_{\ell(2,q')}(v_k, u_{m_k,k}) < 2^{-k} \text{ and } C_{2,q'}(\Omega_{m_k,k} \setminus D) < 2^{-k}.$$

Denote $v'_k = u_{m_k,k}$ and $\Omega'_k = \cap_{j=1}^k \Omega_{m_j,j}$. Then $\{(v'_k, \Omega'_k)\}$ is a determining sequence for v in D. □

Theorem 7.6. *Let F be a $C_{2,q'}$-finely closed set and let $\{A_n\}$ be a sequence of $C_{2,q'}$-finely closed subsets of F. For each n, let v_n be a $C_{2,q'}$-strong solution in $D_n := \mathbb{R}^N \setminus A_n$.*

If $C_{2,q'}(A_n) \to 0$ then $v_n \to 0$ a.e. in $\mathbb{R}^N \setminus F$.

In particular, if $\sum C_{2,q'}(A_n) < \infty$ and v_n^* denotes the extension of v_n to $\mathbb{R}^N$ such that $v_n^* = \infty$ in A_n then,

$$v_n^* \to 0 \text{ a.e. in } \mathbb{R}^N. \tag{7.22}$$

Proof. By Theorem 7.4(iii)

$$v_n \le c(N,q) W_{A_n} \le c'(N,q) V_{A_n} \text{ a.e. in } \mathbb{R}^N \setminus A_n.$$

By Theorem 5.1 $V_{A_n} \to 0$ a.e. in $\mathbb{R}^N \setminus F$. This proves the first assertion. To verify the second assertion we apply the first to the sequence $\{A_n\}_{n=k}^{\infty}$ with F replaced by $F^k = \cup_{n=k}^{\infty} A_n$. Note that F^k is $C_{2,q'}$-finely closed up to a set of capacity zero. □

Theorem 7.7. *Suppose that z is a non-negative $C_{2,q'}$-strong subsolution of (1.1) in a $C_{2,q'}$-quasi open set D. Then there exists a $C_{2,q'}$-strong solution dominating it.*

Proof. Let $\{(z_n, \Omega_n)\}$ be a determining sequence for z. Since $(z_n)_+$ is also a subsolution we may assume that $z_n \ge 0$. Let Z_n be the smallest solution in Ω_n which dominates $\max(z_1, \cdots, z_n)$. Then $Z_n \le Z_{n+1}$ in Ω_{n+1}. Furthermore, by Theorem 7.4(iii) $Z_n \le V_{D^c}$ in D. Therefore, by Theorem 7.5, $Z = \lim Z_n$ is a $C_{2,q'}$-strong solution in D. □

Theorem 7.8. *Let Ω be a $C_{2,q'}$-quasi open set. Suppose that there exists a sequence of open sets $\{G_n\}$ such that*

$$\begin{aligned} &\text{(a)} \quad C_{2,q'}(G_n \Delta \Omega) \to 0, \\ &\text{(b)} \quad C_{2,q'}(\partial G_n \setminus \partial_q \widetilde{G}_n) \to 0. \end{aligned} \tag{7.23}$$

If v is a ∂_q-large solution of (1.1) *in Ω then $v = V_{\widetilde{D}}$ in Ω, $D := \mathbb{R}^N \setminus \widetilde{\Omega}$. Thus $V_{\widetilde{D}}$ is the unique ∂_q-large solution in Ω.*

Remark. Every $C_{2,q'}$-quasi open set Ω is $C_{2,q'}$-equivalent to the intersection of a sequence of open sets $\{O_n\}$ such that $C_{2,q'}(O_n \setminus \Omega) \to 0$. However, in the statement of the theorem, we do not require that G_n contain Ω. Instead we require (7.23) (b).

The proof of the theorem is based on several lemmas. The first collects several useful formulas:

Lemma 7.9. *Let A, E_1, E_2 be sets in $\mathbb{R}^N$. Then the following relations hold:*

$$
\begin{aligned}
&\text{(i)} && \partial_q A^c = \partial_q A,\\
&\text{(ii)} && \partial_q(E_1 \cup E_2) \subset \partial_q E_1 \bigcup \partial_q E_2,\\
&\text{(iii)} && \partial_q(E_1 \cap E_2) \subset \partial_q E_1 \bigcup \partial_q E_2,\\
&\text{(iv)} && \partial_q E_1 \subset \partial_q E_2 \bigcup \partial_q(E_2 \setminus E_1) \bigcup \partial_q(E_1 \setminus E_2),\\
&\text{(v)} && \partial_q E_1 \Delta \partial_q E_2 \subset \partial_q(E_2 \setminus E_1) \bigcup \partial_q(E_1 \setminus E_2),\\
&\text{(vi)} && \partial_q A \subset \partial A, \quad \partial_q \widetilde{A} \subset \partial_q A.
\end{aligned}
\tag{7.24}
$$

Proof. (i),(ii) and (vi) follow immediately from the definition of boundary. (iii) follows from (i), (ii) and the relation

$$(E_1 \cap E_2)^c = (E_1^c \cup E_2^c).$$

By (ii),

$$\partial_q E_1 \subset \partial_q(E_1 \cap E_2) \bigcup \partial_q(E_1 \setminus E_2).$$

By (i) and (iii), the relation,

$$E_1 \cap E_2 = E_2 \cap (E_2 \setminus E_1)^c,$$

implies that

$$\partial_q(E_1 \cap E_2) \subset \partial_q E_2 \bigcup \partial_q(E_2 \setminus E_1).$$

These relations imply (iv) which in turn implies (v). □

Notation. Let $\{A_n\}$ and $\{B_n\}$ be two sequences of sets.

(a) The notation $A_n \overset{lim}{\subset} B_n$ means that $C_{2,q'}(A_n \setminus B_n) \to 0$.

(b) The notation $A_n \overset{lim}{\sim} B_n$ means that $C_{2,q'}(A_n \Delta B_n) \to 0$.

Lemma 7.10. *Under the assumptions of the theorem,*

$$\partial_q G_n \overset{lim}{\sim} \partial_q \widetilde{G}_n \overset{lim}{\sim} \partial G_n, \quad \widetilde{G}_n \overset{lim}{\sim} \overline{G}_n, \tag{7.25}$$

$$\partial_q G_n \overset{lim}{\sim} \partial_q \Omega \tag{7.26}$$

and

$$C_{2,q'}(\overline{G}_n \Delta \widetilde{\Omega}) \to 0. \tag{7.27}$$

In addition

$$\partial_q\Omega \overset{q}{\sim} \partial_q\widetilde{\Omega}. \tag{7.28}$$

Proof. By (7.23)(b) and Lemma 7.9 (vi) we have

$$\partial_q G_n \subset \partial G_n \overset{lim}{\subset} \partial_q \widetilde{G}_n \subset \partial_q G_n.$$

This proves (7.25).

Condition (7.23) (a) implies that

$$C_{2,q'}(\partial_q(G_n \setminus \Omega)) \to 0, \quad C_{2,q'}(\partial_q(\Omega \setminus G_n)) \to 0. \tag{7.29}$$

This fact and Lemma 7.9 (v) imply (7.26).

Next observe that,

$$\widetilde{G}_n \setminus \widetilde{\Omega} \subset (G_n \setminus \Omega) \cup (\partial_q G_n \setminus \partial_q \Omega), \quad \widetilde{\Omega} \setminus \widetilde{G}_n \subset (\Omega \setminus G_n) \cup (\partial_q\Omega \setminus \partial_q G_n).$$

Therefore (7.23)(a) and (7.26) imply

$$C_{2,q'}(\widetilde{G}_n \Delta \widetilde{\Omega}) \to 0. \tag{7.30}$$

This fact and (7.25) imply (7.27).

By (7.24) and (7.30),

$$C_{2,q'}(\partial_q \widetilde{G}_n \Delta \partial_q \widetilde{\Omega}) \to 0. \tag{7.31}$$

This fact together with (7.25) and (7.26) imply (7.28).

□

Lemma 7.11. *Let G be an open set and Q be a $C_{2,q'}$-quasi open set. Assume that $C_{2,q'}(\partial G \setminus \partial_q G) = 0$. Let v be a $C_{2,q'}$-strong solution in $G' = G \setminus \widetilde{Q}$ and let u be a (classical) solution of* (1.1) *in a domain G_0 such that $\bar{G} \subset G_0$. Suppose that u, v are non-negative and*

$$u \overset{q}{\leq} v \quad \text{at } F := \partial_q G \setminus Q. \tag{7.32}$$

Then

$$u \leq v + V_{\widetilde{Q}} \quad \text{in } G'. \tag{7.33}$$

Proof. Let ϵ be a positive number. Condition (7.32) means that there exists a determining sequence $\{(v_n, \Omega_n)\}$ for the $C_{2,q'}$-strong solution v in G' and a family of open sets $\{Q_{k,\epsilon}\}$ satisfying (7.5), (7.9) and (7.14) (with D replaced by G'). We may and shall assume that $\Omega_n \subset G$, that $\{\Omega_n\}$ is decreasing and that, for every $\epsilon > 0$, $\{Q_{k,\epsilon}\}_{k=1}^{\infty}$ is decreasing.

In the next part of the proof we keep ϵ fixed. If K is a compact subset of $F \setminus Q_{n,\epsilon}$ then (7.14) implies that there exists an open neighborhood of K, say O_K, such that

$$u - v_n \leq \epsilon \text{ in } O_K \cap \Omega_n.$$

Therefore there exists an increasing sequence of compact sets $\{K_{n,\epsilon}\}$ and a sequence of open sets $\{O_{n,\epsilon}\}$ such that

$$K_{n,\epsilon} \subset F \setminus Q_{n,\epsilon}, \quad C_{2,q'}(F \setminus K_{n,\epsilon}) \to 0, \tag{7.34}$$

$$K_{n,\epsilon} \subset O_{n,\epsilon}, \quad u - v_n \leq \epsilon \text{ in } O_{n,\epsilon} \cap \Omega_n. \tag{7.35}$$

Let $\{O'_{n,\epsilon}\}$ be a decreasing family of open sets such that

$$\begin{aligned} &(\partial G \setminus \partial_q G) \cup (F \setminus K_{n,\epsilon}) \cup \widetilde{Q} \subset O'_{n,\epsilon}, \\ &C_{2,q'}(O'_{n,\epsilon}) \to C_{2,q'}(Q), \quad \cap_{n=1}^{\infty} O'_{n,\epsilon} \overset{q}{\sim} \widetilde{Q}. \end{aligned} \tag{7.36}$$

Then $E_{n,\epsilon} := O_{n,\epsilon} \cup O'_{n,\epsilon}$ is an open neighborhood of ∂G and

$$G \setminus E_{n,\epsilon} \subset G' \subset \Omega_n.$$

Consequently there exist smooth domains $\Omega_{n,\epsilon}$ such that

$$\{x \in \Omega_n : \operatorname{dist}(x, \partial\Omega_n) \geq 2^{-n}\} \subset \Omega_{n,\epsilon} \subset \overline{\Omega}_{n,\epsilon} \subset \Omega_n, \quad \partial\Omega_{n,\epsilon} \subset E_{n,\epsilon}.$$

The function $w_{n,\epsilon} := (u - v_n - \epsilon)_+$ is a classical subsolution in Ω_n and it vanishes in $\Omega_n \cap O_{n,\epsilon}$. Put $S_{n,\epsilon} = \partial\Omega_{n,\epsilon} \setminus O_{n,\epsilon}$ and

$$z_{n,\epsilon} := \begin{cases} w_{n,\epsilon} & \text{in } \bar{\Omega}_{n,\epsilon} \setminus S_{n,\epsilon} \\ 0 & \text{in } \mathbb{R}^N \setminus \bar{\Omega}_{n,\epsilon}. \end{cases}$$

Then $z_{n,\epsilon}$ is a (classical) subsolution in $\mathbb{R}^N \setminus S_{n,\epsilon}$. Since $v_n \to v$ in $L^q_{\ell(2,q')}(G')$, it follows that there exists a subsequence (still denoted $\{v_n\}$) such that $v_n \to v$ a.e. in G'. Therefore $\{z_{n,\epsilon}\}$ converges a.e. in $D := \mathbb{R}^N \setminus \widetilde{Q}$ to the function

$$z_\epsilon := \begin{cases} (u - v - \epsilon)_+ & \text{in } G' \\ 0 & \text{in } \mathbb{R}^N \setminus G. \end{cases}$$

In addition

$$\sup_{\mathbb{R}^N \setminus S_{n,\epsilon}} z_{n,\epsilon} \leq \sup_{\bar{G}} u < \infty.$$

Note that $D \subset \mathbb{R}^N \setminus S_{n,\epsilon}$ for all n. Therefore, by the dominated convergence theorem, $z_{n,\epsilon} \to z_\epsilon$ in $L^q_{\ell(2,q')}(D)$; consequently z_ϵ is a $C_{2,q'}$-strong subsolution in D. In fact $\{(z_{n,\epsilon}, \mathbb{R}^N \setminus S_{n,\epsilon})\}$ is a determining sequence for z_ϵ in D.

By Theorem 7.7, there exists a $C_{2,q'}$-strong solution Z_ϵ in D such that $z_\epsilon \leq Z_\epsilon$. By Theorem 7.4 (iii), $Z_\epsilon \leq V_{\widetilde{Q}}$ in D. Thus $z \leq V_{\widetilde{Q}}$ and so $u - v - \epsilon \leq V_{\widetilde{Q}}$ in G'. Letting $\epsilon \to 0$ we obtain (7.33). □

Corollary 7.12. *Let G, Q, G' and u, v be as in the statement of the lemma. If*

$$\overset{q}{\lim_{x \to \partial_q G'}} v(x) = \infty \quad C_{2,q'} \text{ a.e. at } \partial_q G' \tag{7.37}$$

then (7.33) *holds.*

Proof. Since u is bounded in G, (7.37) implies (7.32). Therefore the previous lemma implies (7.33). □

Proof of Theorem 7.8. Let K be a compact subset of $D=\mathbb{R}^N\setminus\widetilde{\Omega}$ and let $\mu\in W^{-2,q}(\mathbb{R}^N)$ be a non-negative measure supported in K. We prove that

$$u_\mu\leq v \quad\text{in}\quad \Omega \tag{7.38}$$

As $\mu(\widetilde{\Omega})=0$, (7.27) implies

$$\mu'_n:=\mu\chi_{\overline{G}_n}=\mu\chi_{\overline{G}_n\setminus\tilde{\Omega}}\to 0. \tag{7.39}$$

If ν is a bounded measure such that $\nu(\overline{G}_n)=0$ and $O_{n,k}$ is a sequence of open neighborhoods of $\overline{G}_n$ such that $\cap_k O_{n,k}=\overline{G}_n$ then

$$\nu(O_{n,k}\setminus\overline{G}_n)\to 0 \quad\text{as } k\to\infty.$$

Applying this observation to $\nu=\mu-\mu'_n$ and using (7.39) we conclude that, for every $n\in\mathbb{N}$, there exists a non-negative measure μ_n such that

$$\mu_n\leq\mu,\quad \operatorname{supp}\mu_n\cap\overline{G}_n=\emptyset,\quad (\mu-\mu_n)(\mathbb{R}^N)\to 0. \tag{7.40}$$

As $K_n:=\operatorname{supp}\mu_n$ is a compact set disjoint from $\overline{G}_n$ it follows that u_{μ_n} is a bounded solution of (1.1) in a neighborhood of $\overline{G}_n$.

Let $Q_n:=G_n\setminus\widetilde{\Omega}$. By (7.27) $C_{2,q'}(Q_n)\to 0$ and therefore

$$C_{2,q'}(\widetilde{Q}_n)\to 0. \tag{7.41}$$

Applying Corollary 7.12 to G_n, Q_n with $u=u_{\mu_n}$ and $w=v$ we obtain

$$u_{\mu_n}\leq v+V_{\widetilde{Q}_n} \quad\text{in}\quad G_n\setminus\widetilde{Q}_n. \tag{7.42}$$

By (7.40) $u_{\mu_n}\to u_\mu$ and, by (7.41), $V_{\widetilde{Q}_n}\to 0$. Therefore, in view of (7.23) (a),

$$u_\mu\leq v \qquad C_{2,q'}\text{ a.e. in }\Omega. \tag{7.43}$$

This holds for every non-negative measure $\mu\in W^{-2,q}(\mathbb{R}^N)$ supported in a compact subset of D. Therefore

$$V_{\widetilde{D}}=V_D\leq v \qquad C_{2,q'}\text{ a.e. in }\Omega. \tag{7.44}$$

On the other hand, by Theorem 7.4 (iii), $v\leq V_{\mathbb{R}^N\setminus\Omega}$. But (7.28) implies that

$$\mathbb{R}^N\setminus\Omega=D\cup\partial_q\Omega\overset{q}{\sim}D\cup\partial_q\widetilde{\Omega}.$$

As $\partial_q D=\partial_q\widetilde{\Omega}$ it follows that $\mathbb{R}^N\setminus\Omega\overset{q}{\sim}\widetilde{D}$. Thus $V_{\widetilde{D}}=V_{\mathbb{R}^N\setminus\Omega}$ and finally $v=V_{\widetilde{D}}$.

□

Example. Let $\{x^m\}$ be a sequence of distinct points in $B_1(0)$. Let $\{r_n\}$ be a decreasing sequence of positive numbers such that (5.45) holds and $\overline{B}_{r_n}(x^n)\subset B_1(0)$. Put

$$\Omega_n=B_1(0)\setminus\cup_1^n\overline{B}_{r_k}(x^k).$$

Then there exists a unique large solution in

$$\Omega:=\cap_1^\infty\Omega_n=B_1(0)\setminus\cup_1^\infty\overline{B}_{r_k}(x^k).$$

Appendix A. On the space $W^{2,q'}_{0,\infty}$

We establish some features of the space $W^{2,q'}_{0,\infty}$ which show that it is sufficiently rich in order to serve as a space of test functions in $C_{2,q'}$-finely open sets. These are used mainly in Section 7.

Lemma A.1. *Suppose that D a $C_{2,q'}$ finely open set and K is a bounded $C_{2,q'}$-finely closed subset of D. Then, for every $a > 0$, there exists $\phi_a \in W^{2,q'}(\mathbb{R}^N)$ such that :*

$$\begin{aligned}&(i)\quad 0 \le \phi_a \le 1, \quad (ii)\quad \operatorname{supp}_{(2,q')}\phi_a \Subset D,\\ &(iii)\quad C_{2,q'}(\{x \in K : \phi_a(x) < 1\}) < a.\end{aligned} \tag{A.1}$$

Proof. Let $0 < \epsilon(1 + 2^{q'}) < a$. Let K' be a compact set and D' be an open set such that,

$$K' \subset K, \quad D \subset D', \quad C_{2,q'}(K \setminus K') < \epsilon, \quad C_{2,q'}(\widetilde{D' \setminus D}) < \epsilon.$$

Let ϕ be a smooth function with compact support in D' such that $0 \le \phi \le 1$ and $\phi = 1$ on a neighborhood of K'. Let $\{A_n\}$ be a decreasing sequence of open neighborhoods of $\widetilde{D' \setminus D}$ such that

$$C_{2,q'}(\tilde{A}_n) \to C_{2,q'}(\widetilde{D' \setminus D}).$$

Further, let $\{\eta_n\}$ be a sequence of functions in $W^{2,q'}(\mathbb{R}^N)$ such that

$$0 \le \eta_n, \quad \eta_n \ge 1\ C_{2,q'} \text{ a.e. in } \tilde{A}_n, \quad \|\eta_n\|^{q'}_{W^{2,q'}(\mathbb{R}^N)} = C_{2,q'}(\tilde{A}_n).$$

(See [1, Thm.2.3.10] for the existence of such functions.)

Let $\alpha \in (0, 1)$ and put $E_n = \{x \in D : \eta_n(x) \ge 1 - \alpha\}$. Then

$$(1-\alpha))^{-q'} \|\eta_n\|_{W^{2,q'}(\mathbb{R}^N)} \ge C_{2,q'}(E_n)$$

so that

$$\limsup C_{2,q'}(E_n) \le C_{2,q'}(\tilde{A}_n)/(1-\alpha)^{q'} < \epsilon/(1-\alpha)^{q'}.$$

Let h be a monotone, smooth cutoff function such that

$$h(t) = \begin{cases} 0 & \text{if } t < \alpha/4 \\ h(t) = t & \text{if } t > \alpha/2. \end{cases}$$

Then $\phi_n := h \circ (\phi - \eta_n) \in W^{2,q'}(\mathbb{R}^N)$ and

$$\phi_n \ge \alpha \text{ on } K'_n := K' \setminus E_n, \quad \phi_n = 0\ C_{2,q'} \text{ a.e. in } A_n.$$

Thus, choosing $\alpha = 1/2$,

$$\begin{cases} \phi_n/\alpha \ge 1 \text{ on } K'_n, \quad \operatorname{supp}_{(2,q')}\phi_n \subset (\operatorname{supp}\phi) \setminus A_n \Subset D, \\ \limsup C_{2,q'}(K \setminus K'_n) < \epsilon(1 + (1-\alpha)^{-q'}). \end{cases} \tag{A.2}$$

By applying (to ϕ_n/α) another smooth cutoff function which approximates $\min(\cdot, 1)$, we obtain a sequence of functions which, for n sufficiently large, satisfy the statement of the lemma. □

Corollary A.2. *Let D be a bounded $C_{2,q'}$-finely open set and let $\{E_n\}$ be a q-exhaustion of D (see Lemma 6.1). Then there exists a sequence $\{\varphi_n\}$ in $W^{2,q'}(\mathbb{R}^N)$ such that :*

$$\text{(A.3)}\qquad \begin{array}{ll} (i)\ 0\leq\varphi_n\leq 1, & (ii)\ \operatorname{supp}_{(2,q')}\varphi_n\Subset E_{n+1},\\ (iii)\ \displaystyle\sum_{n=1}^{\infty}C_{2,q'}(E_n\setminus[\varphi_n=1])<\infty, & (iv)\ \{\varphi_n\}\ \textit{is non-decreasing.}\end{array}$$

In particular $\varphi_n\uparrow 1$ $C_{2,q'}$ a.e. in D.

Proof. We construct ϕ_n as in Lemma A.1 with K and D replaced by E_n and $\mathrm{int}_q E_{n+1}$, $a=2^{-n}$ and $\alpha=1/2$. Then we put $\tilde\varphi_n:=\sum_1^n\varphi_m$ and finally apply to $2\tilde\varphi_n$ a smooth cutoff function which approximates $\min(\cdot,1)$. □

Lemma A.3. *Let D be a $C_{2,q'}$-finely open set and let $\tau_{\ell(2,q')}(D)$ be the topology in $L^q_{\ell(2,q')}(D)$ defined by the family of seminorms (6.3). Then $\tau_{\ell(2,q')}(D)$ is a metric topology.*

Proof. It is sufficient to show that the space is separable. For each fixed $\phi\in W^{2,q'}_{0,\infty}(D)$, the space $L^q(D;\phi)$ is separable. Let $\{\varphi_n\}$ be as in Corollary A.2. Then, for every $f\in L^q_{\ell(2,q')}(D)$,

$$\int_D f\psi(1-\varphi_m)dx\to 0\quad\forall\psi\in W^{2,q'}_{0,\infty}(D).$$

Therefore, if $\{h_{k,m}\}_{k=1}^{\infty}$ is a dense set in $L^q(D;\varphi_m)$ then

$$\{h_{k,m}:\,k,m\in\mathbb{N}\}$$

is a dense set in $L^q_{\ell(2,q')}(D)$. □

Lemma A.4. *Assume that F is a $C_{2,q'}$-finely closed set and $F\subset B_{R/2}(0)$. Put $D=B_R(0)\setminus F$. Let $\{E_n\}$ be a q-exhaustion of D. Then there exists a q-exhaustion $\{E'_n\}$ such that*

$$\text{(A.4)}\qquad E'_n\subset E_n,\quad C_{2,q'}(E_n\setminus E'_n)\to 0,$$

for which the following statement holds:

The set of non-negative very weak subsolutions of (1.1) in D is uniformly bounded in $L^q(E'_n)$ for every $n\in\mathbb{N}$.

Proof. Let $\{\varphi_n\}$ be as in Corollary A.2 and let $A_{n,k}$ be an open neighborhood of $E_n\setminus[\varphi_n=1]$ such that

$$C_{2,q'}(A_{n,k})\leq(1+2^{-k})C_{2,q'}(E_n\setminus[\varphi_n=1]),\quad \tilde A_{n,k+1}\subset A_{n,k}\quad\forall k\geq n,\ n\in\mathbb{N}.$$

Put $E'_n=E_n\setminus\cup_{k=n}^{\infty}A_{n,k}$. Then $\{E'_n\}$ is a q-exhaustion of D and (A.4) holds. Furthermore, $\varphi_n=1$ on E'_n. Hence, by Theorem 6.4, every non-negative very weak subsolution u of (1.1) in D satisfies

$$\int_{E'_n}u^q\,dx\leq\int_D u^q\varphi_n^{2q'}dx\leq c(q,N)\left\|D^2\varphi_n\right\|^{q'}_{L^{q'}(D)}.$$

□

Lemma A.5. *Let K be a bounded $C_{2,q'}$-finely closed subset of D. Then, for every $\epsilon > 0$, there exists a compact set $K_\epsilon \subset K$ such that*

$$C_{2,q'}(K \setminus K_\epsilon) < \epsilon, \quad f \in L^q(K_\epsilon) \quad \forall f \in L^q_{\ell(2,q')}(D)$$

Proof. This is an immediate consequence of Lemma A.1. □

Appendix B. Open problems

There are many interesting problems related to possible *extensions* of the theory of solutions in finely open sets, presented in Section 7. We do not describe here problems of this nature, but only problems directly related to results presented in the present paper.

In order to formulate the first problem, it is convenient to introduce an additional definition.

Definition B.1. Let u be a non-negative measurable function in a $C_{2,q'}$-finely open set D and let $\mu \in \mathfrak{M}_{\ell(2,q')}(D)$ be a non-negative measure. We say that u is a $C_{2,q'}$*-weak solution* of (1.2) in Ω if u satisfies (7.16) and (7.17).

Problem I. We know that if u is a $C_{2,q'}$-strong solution then it is also a $C_{2,q'}$-weak solution, (see Theorem 7.3). Does the opposite implication hold: is it true that every $C_{2,q'}$-weak solution of (1.2) is a $C_{2,q'}$-strong solution?

Problem II. This problem is related to Theorem 7.5. The question is if the following related assertion is valid:

Let D be a $C_{2,q'}$-finely open set and let $\{v_k\}$ be a sequence of non-negative $C_{2,q'}$-strong solutions of (1.1) *in D. Then there exists a subsequence $\{v_{k_j}\}$, converging in $L^q_{\ell(2,q')}(D)$.*

We observe that if such a subsequence exists then one can extract a further subsequence which converges a.e. in D and, by Theorem 7.5, its limit is a $C_{2,q'}$-strong solution in D.

The next problem is related to the uniqueness result Lemma 5.5. It is known that in the subcritical case condition (5.27) is necessary in order to guarantee uniqueness of large solutions. (In the subcritical case the notion of 'large solution' and '∂_q-large solution' coincide.) The situation is essentially different with respect to ∂_q-large solutions in the supercritical case. In fact it is likely that condition (5.27) is not necessary in this case.

Problem III. *Let Ω be a bounded open set and put $F := \mathbb{R}^N \setminus \Omega$. We know that V_F is an almost large solution and, a-fortiori, a ∂_q-large solution in Ω. Question: Is V_F the unique ∂_q-large solution in Ω?*

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Department of Mathematics, Technion Haifa, ISRAEL
E-mail address: marcusm@math.technion.ac.il

Laboratoire de Mathématiques, Faculté des Sciences Parc de Grandmont, 37200 Tours, FRANCE
E-mail address: veronl@lmpt.univ-tours.fr