

# QUASI-COMPLETE HOMOGENEOUS CONTACT MANIFOLD ASSOCIATED TO A CUBIC FORM

JUN-MUK HWANG<sup>1</sup>, LAURENT MANIVEL

*Dedicated to S. Ramanan*

## 1. INTRODUCTION

This note is at the crossroad of two different lines of study.

On the one hand, we propose a general construction of a homogeneous quasi-projective manifold  $X_c$  associated to a cubic form with a mild genericity property. These manifolds are rationally chain connected (Proposition 2), a property which relates our study to that of certain types of homogeneous spaces considered in [2, 3, 4].

On the other hand, we show that our manifolds  $X_c$  are endowed with natural contact structures (Proposition 3). Our construction thus appears as part of the general study of contact projective and quasi-projective manifolds. Of course the projective case is the most interesting one, the main open problem in this area being the Lebrun-Salamon conjecture: the only Fano contact manifolds should be the projectivizations of the minimal nilpotent orbits in the simple Lie algebras. As explained in section 4, our construction is in fact modeled on these homogeneous contact manifolds, which are known to be associated to very special cubic forms: the determinants of the simple cubic Jordan algebras.

Under both points of view, one of the most interesting questions one may ask about the quasi-projective contact manifolds  $X_c$  is about their compactifications. Even the existence of a small compactification (that is, with a boundary of codimension at least two) is not clear. Also, it is extremely tempting to try to construct new projective contact manifolds by compactifying some  $X_c$  in such a way that the contact structure extends. We show that this is possible if and only if the cubic  $c$  is the determinant of a simple cubic Jordan algebra (Proposition 6). This can be interpreted as an evidence for the Lebrun-Salamon conjecture.

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<sup>1</sup> supported by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD)(KRF-2006-341-C00004).

## 2. HOMOGENEOUS SPACES DEFINED FROM CUBICS

Let  $V$  be a complex vector space of dimension  $p$ . Let  $c \in S^3V^*$  be a cubic form on  $V$ . Let  $B : S^2V \rightarrow V^*$  be the system of quadrics defined by

$$B(v_1, v_2) = c(v_1, v_2, \cdot).$$

In all the sequel we make the following

ASSUMPTION ON  $c$ . *The homomorphism  $B$  is surjective.*

Let  $W$  be a complex vector space of dimension 2. Fix a choice of a non-zero 2-form  $\omega \in \wedge^2W^*$ .

Let  $\mathfrak{n} := \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathfrak{n}_3$  where

$$\mathfrak{n}_1 := V \otimes W, \quad \mathfrak{n}_2 := V^*, \quad \mathfrak{n}_3 := W.$$

Define a graded Lie algebra structure on  $\mathfrak{n}$ , by

$$\begin{aligned} [v_1 \otimes w_1, v_2 \otimes w_2] &= \omega(w_1, w_2)B(v_1, v_2), \\ [v_1^*, v_2 \otimes w_2] &= v_1^*(v_2) w_2. \end{aligned}$$

The Jacobi identity holds because  $\dim W = 2$ .

Let  $N$  be the nilpotent Lie group with Lie algebra  $\mathfrak{n}$ . For a point  $\ell \in \mathbb{P}W$ , denote by  $\hat{\ell} \subset W$  the corresponding 1-dimensional subspace. Let

$$\mathfrak{a}_\ell := V \otimes \hat{\ell} \subset V \otimes W = \mathfrak{n}_1$$

be the abelian subalgebra of  $\mathfrak{n}$  and  $A_\ell \subset N$  be the corresponding additive abelian subgroup. We have the smooth subvariety  $A \subset N \times \mathbb{P}W$  defined by

$$A := \{(g, \ell), g \in A_\ell\}.$$

This variety  $A$  can be viewed as a family of abelian subgroups parametrized by  $\mathbb{P}W$ . Let  $\psi : X_c \rightarrow \mathbb{P}W$  be the family of relative quotients

$$X_c := \{N/A_\ell, \ell \in \mathbb{P}W\}$$

with the quotient map  $\xi : N \times \mathbb{P}W \rightarrow X_c$ . Then

$$\dim X_c = \dim N + 1 - p = 2p + 3.$$

Observe that  $\psi$  is a locally trivial fibration whose fibers are isomorphic to affine spaces. But it is not a vector nor an affine bundle. In fact the transition functions are quadratic, because the nilpotence index of  $N$  is three.

The variety  $X_c$  is homogeneous under the action of the group

$$G := N \triangleleft SL(W) \quad (\text{semi-direct product}).$$

Let  $o \in N$  be the identity and  $\ell \in \mathbb{P}W$  be a fixed base point. Then  $x_\ell := \xi(o \times \ell)$  will be our base point for  $X_c$ . Its stabilizer is  $H = A_\ell \triangleleft B_\ell$ , if  $B_\ell$  denotes the stabilizer of  $\ell$  in  $SL(W)$ . Moreover a Borel subgroup of  $G$  is  $B = N \triangleleft B_\ell$ , and we have a sequence of quotients

$$G \rightarrow G/B_\ell \xrightarrow{\xi} G/H = X_c \xrightarrow{\psi} G/B = \mathbb{P}W.$$

**Proposition 1.** (1)  $X_c$  is simply connected.  
 (2) Let  $L := \psi^* \mathcal{O}_{\mathbb{P}W}(1)$ . Then  $\text{Pic}(X_c) = \mathbb{Z}L$ .

*Proof.* As a variety, each  $A_\ell$  is nothing but an affine space. So the variety  $X_c$  being fibered in simply connected manifolds over the projective line, is simply connected. This proves (1).

The character group  $X(G)$  of  $G$  being trivial, the forgetful map

$$\alpha : \text{Pic}^G(X_c) \rightarrow \text{Pic}(X_c)$$

is injective ([9] Proposition 1.4). Moreover the Picard group of  $G$  is trivial, so  $\alpha$  is in fact an isomorphism (see the proof of Proposition 1.5 in [9]). But  $\text{Pic}^G(X_c) \simeq X(H)$  and an easy computation shows that  $X(H) = X(B_\ell)$ . This implies (2).  $\square$

Note that  $G$  is generated by  $H$  and  $SL(W)$  such that  $H \cap SL(W)$  is a Borel subgroup of  $SL(W)$ . Thus we can apply Proposition 4.1 in [4] to deduce:

**Proposition 2.** *The variety  $X_c$  is rationally chain connected. In particular,  $X_c$  is quasi-complete, i.e., there is no non-constant regular function on  $X_c$ .*

In fact, it is easy to show that for any  $n \in \mathbb{N}$ , the image of  $\{n\} \times \mathbb{P}W$  under  $\xi$  is a smooth rational curve on  $X_c$  with normal bundle of the form  $\mathcal{O}(1)^p \oplus \mathcal{O}^{p+2}$ .

### 3. CONTACT STRUCTURES

Consider the tangent spaces

$$T_{o \times \ell}(N \times \mathbb{P}W) = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathfrak{n}_3 \oplus T_\ell(\mathbb{P}W)$$

$$T_{x_\ell}(X_c) = \mathfrak{n}_1/\mathfrak{a}_\ell \oplus \mathfrak{n}_2 \oplus \mathfrak{n}_3 \oplus T_\ell(\mathbb{P}W).$$

Using the subspace  $\hat{\ell} \subset W = \mathfrak{n}_3$ , we define the hyperplane

$$D_{x_\ell} := \mathfrak{n}_1/\mathfrak{a}_\ell \oplus \mathfrak{n}_2 \oplus \hat{\ell} \oplus T_\ell(\mathbb{P}W)$$

inside  $T_{x_\ell}(X_c)$ . This hyperplane is invariant under the action of the stabilizer  $H$  of  $x_\ell$  in  $G = N \triangleleft SL(W)$ , so we get a well-defined hyperplane distribution  $D \subset T(X_c)$  with  $T(X_c)/D \cong L$ .

**Proposition 3.** *The distribution  $D \subset T(X_c)$  defines a contact structure.*

*Remark.* Observe that since

$$D_{x_\ell} = (V \otimes W/\hat{\ell}) \oplus V^* \oplus \hat{\ell} \oplus \text{Hom}(\hat{\ell}, W/\hat{\ell}),$$

there is a natural  $W/\hat{\ell}$ -valued symplectic pairing on  $D_{x_\ell}$ . Note that  $W/\hat{\ell}$  is the fiber of  $L$  at  $x_\ell$ . This shows that the bundle  $D$  has an  $L$ -valued symplectic form, and indeed this symplectic form comes from the contact structure

$$0 \rightarrow D \rightarrow T(X_c) \rightarrow L \rightarrow 0.$$

*Proof.* Let  $Y_c$  be the variety defined as the complement  $L^\times$  of the zero section in the total space of the line bundle dual to  $L$ . Let  $\theta$  be the  $L$ -valued 1-form on  $X_c$  defining  $D$ . To check that  $\theta$  is a contact form, it suffices to show that the 2-form  $d\tilde{\theta}$  where  $\tilde{\theta}$  is the pull-back of  $\theta$  to  $Y_c$ , is symplectic (see [1], Lemma 1.4).

To check this we make a local computation. Let  $m \in \mathbb{P}W$  be some point distinct from  $\ell$  and  $\hat{m} \subset W$  be the corresponding line. Then

$$\mathfrak{n}_m := (V \otimes \hat{m}) \oplus V^* \otimes W \subset \mathfrak{n}$$

defines a complement to  $\mathfrak{a}_\ell \subset \mathfrak{n}$ . We can define a local analytic chart on  $X_c$  around  $x_\ell$  by

$$x(X, p) = \xi(\exp(X) \times p),$$

where  $X \in \mathfrak{n}_m$  and  $p \in \mathbb{P}W - m$ . Let us write down  $\theta$  in that local chart. Since the chart preserves the fibration over  $\mathbb{P}W$  we just need to compute over  $\ell$ . The differential  $e_X$  of the exponential map at  $X$ , seen as an endomorphism of  $\mathfrak{n}_m$ , is defined by the relation

$$\exp(X + te_X(Y) + O(t^2))x_\ell = \exp(X)\exp(tY)x_\ell.$$

Now we can use the fact that  $N$  being 3-nilpotent, the Campbell-Hausdorff formula in  $N$  is quite simple: we have  $\exp(X)\exp(Y) = \exp(H(X, Y))$  for  $X, Y \in \mathfrak{n}$ , with

$$H(X, Y) = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [Y, X]].$$

We easily deduce that  $e_X(Y) = Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]]$ . Now we can decompose this formula with respect to the three-step grading of  $\mathfrak{n}$ . If  $Z = e_X(Y) = Z_1 + Z_2 + Z_3$ , we find that

$$\begin{aligned} Z_1 &= Y_1, \\ Z_2 &= Y_2 + \frac{1}{2}[X_1, Y_1], \\ Z_3 &= Y_3 + \frac{1}{2}[X_1, Y_2] + \frac{1}{2}[X_2, Y_1] + \frac{1}{12}[X_1, [X_1, Y_1]], \end{aligned}$$

which can be inverted as

$$\begin{aligned} Y_1 &= Z_1, \\ Y_2 &= Z_2 - \frac{1}{2}[X_1, Z_1], \\ Y_3 &= Z_3 - \frac{1}{2}[X_1, Z_2] - \frac{1}{2}[X_2, Z_1] + \frac{5}{12}[X_1, [X_1, Z_1]]. \end{aligned}$$

Since the hyperplane  $D_{x_\ell}$  is defined by the condition that  $Y_3$  belongs to  $\hat{\ell}$ , we deduce that the contact form is given at  $x(X, \ell)$ , in our specific chart, by the formula

$$\theta_{x(X, \ell)}(Z) = Z_3 - \frac{1}{2}[X_1, Z_2] - \frac{1}{2}[X_2, Z_1] + \frac{5}{12}[X_1, [X_1, Z_1]] \quad \text{mod } \hat{\ell}.$$

Even more explicitly, if we write  $Z_1 = z_1 \otimes m$  and  $X_1 = x_1 \otimes m$ , we have  $[X_1, Z_1] = 0$ ,  $[X_1, Z_2] = Z_2(x_1)m$  and  $[X_2, Z_1] = -X_2(z_1)m$ , so

$$\theta_{x(X,\ell)}(Z) = Z_3 + \frac{1}{2}(X_2(z_1) - Z_2(x_1))m.$$

Now we pull-back  $\theta$  to  $Y_c = L^\times$ . A local section of  $L^\times$  around  $\ell$  is given by  $m^* - z\ell^*$  over the point  $p = \ell + zm$  of  $\mathbb{P}W$ . Over  $\phi = y(m^* - z\ell^*)$ , we get the 1-form on  $L^\times$  given in our local chart by

$$\tilde{\theta}_{x(X,p),\phi}(Z, Y) = y(m^* - z\ell^*)(Z_3) + \frac{y}{2}(X_2(z_1) - Z_2(x_1)).$$

If  $Z_3 = Z_3^1 m + Z_3^2 \ell$ , this can also be written as:

$$\tilde{\theta}_{x(X,p),\phi} = y(dX_3^1 - z dX_3^2) + \frac{y}{2}(X_2 dX_1 - X_1 dX_2).$$

We can easily differentiate this expression and evaluate it at  $x_\ell$ . We obtain

$$d\tilde{\theta}_{x_\ell, ym^*} = dy \wedge dX_3^1 - y dz \wedge dX_3^2 + y dX_2 \wedge dX_1.$$

Since  $y$  is a non zero scalar this 2-form is everywhere non-degenerate. By homogeneity this remains true over the whole of  $L^\times$ , and the proof is complete.  $\square$

#### 4. PROJECTIVE HOMOGENEOUS CONTACT VARIETIES

Consider a complex simple Lie algebra  $\mathfrak{g}$  and the adjoint variety

$$Y_{\mathfrak{g}} = \mathbb{P}\mathcal{O}_{min} \subset \mathbb{P}\mathfrak{g},$$

the projectivization of the minimal nilpotent orbit  $\mathcal{O}_{min}$ . Then  $Y_{\mathfrak{g}}$  is homogeneous under the action of the adjoint group  $\mathbf{G} = \text{Aut}(\mathfrak{g})$ . Suppose that  $\text{Pic}(Y_{\mathfrak{g}}) \simeq \mathbb{Z}$  (this is the case if and only if  $\mathfrak{g}$  is not of type A). Then the variety  $F$  of lines on  $Y_{\mathfrak{g}}$  is  $\mathbf{G}$ -homogeneous and we can describe a line as follows. Choose  $T \subset B \subset \mathbf{G}$  a maximal torus and a Borel subgroup. Let  $\mathfrak{g}_\psi$  denote the root space in  $\mathfrak{g}$  associated to the highest root  $\psi$ . Then  $Y_{\mathfrak{g}} = \mathbf{G}\mathfrak{g}_\psi$  and the stabilizer of  $\mathfrak{g}_\psi$  is the maximal parabolic subgroup  $P_\alpha$  of  $\mathbf{G}$  defined by the unique simple root  $\alpha$  such that  $\psi - \alpha$  is a root. Moreover the line  $\ell = \langle \mathfrak{g}_\psi, \mathfrak{g}_{\psi-\alpha} \rangle$  is contained in the adjoint variety  $Y_{\mathfrak{g}}$ , and  $F = \mathbf{G}.\ell$ .

There is a five-step grading on  $\mathfrak{g}$  defined by the highest root  $\psi$ , as follows. Define  $H_\psi \in [\mathfrak{g}_\psi, \mathfrak{g}_{-\psi}]$  by the condition that  $\psi(H_\psi) = 2$ . Then the eigenvalues of  $\text{ad}(H_\psi)$  are  $0, \pm 1, \pm 2$  and the eigenspace decomposition yields the five-step grading

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$

We have  $\mathfrak{g}_2 = \mathfrak{g}_\psi$ , while  $\mathfrak{g}_\alpha$  and  $\mathfrak{g}_{\psi-\alpha}$  are respectively lines of lowest and highest weights in  $\mathfrak{g}_1$ .

Since  $\mathfrak{g}_{\psi-\alpha}$  defines, exactly as  $\mathfrak{g}_\psi$ , a point of the adjoint variety, we can use the root  $\psi - \alpha$  to define another five-step grading. Since  $\text{ad}(H_\psi)$  and  $\text{ad}(H_{\psi-\alpha})$  commute, we get a double grading on  $\mathfrak{g}$ . Moreover, the stabilizer

$\mathfrak{s} \subset \mathfrak{g}$  of the line  $\ell \subset \mathfrak{g}$  decomposes as follows (where the grading defined by  $ad(H_\psi)$  can be read horizontally):

$$\begin{array}{ccccccc} & & & & & & \mathfrak{g}_{-\alpha} \\ & & & & & & \mathfrak{g}_{00} & \mathfrak{g}_{10} \\ & & & & & & \mathfrak{g}_\alpha & \mathfrak{g}_{11} & \mathfrak{g}_{21} & \mathfrak{g}_{\psi-\alpha} \\ & & & & & & & & & \mathfrak{g}_\psi \end{array}$$

Let  $W = \mathfrak{g}_{\psi-\alpha} \oplus \mathfrak{g}_\psi \simeq \mathbb{C}^2$  and  $V = \mathfrak{g}_{21}^*$ . The map  $\mathfrak{g}_{10} \otimes \mathfrak{g}_{21} \rightarrow \mathfrak{g}_{31} = \mathfrak{g}_{\psi-\alpha}$  defined by the Lie bracket is a perfect pairing, as well as  $\mathfrak{g}_{11} \otimes \mathfrak{g}_{21} \rightarrow \mathfrak{g}_{32} = \mathfrak{g}_\psi$ , giving a natural identification

$$\mathfrak{g}_{10} \oplus \mathfrak{g}_{11} \simeq V \otimes W$$

and isomorphisms

$$\mathfrak{g}_{10} \xrightarrow{\phi} \mathfrak{g}_{11} \cong V.$$

The positive part of the vertical grading of  $\mathfrak{s}$  thus reads

$$(V \otimes W) \oplus V^* \oplus W = \mathfrak{n}.$$

Note that the degree zero part of this grading reads  $\mathfrak{sl}(W) \times \mathfrak{h}_{00}$ , where  $\mathfrak{g}_{00} = [\mathfrak{g}_{-\alpha}, \mathfrak{g}_\alpha] \oplus \mathfrak{h}_{00}$  is an orthogonal decomposition with respect to the Killing form.

The cubic form  $c$  on  $V$  is defined (up to scalar) once we identify  $\mathfrak{g}_{10}$  with  $\mathfrak{g}_{11}$ , through the map  $\phi$ . We also need to choose a generator  $X_\psi$  of  $\mathfrak{g}_\psi$ . Then we can define  $c$  by the formula

$$[\phi(X), [\phi(X), X]] = c(X)X_\psi \quad \forall X \in \mathfrak{g}_{10} \simeq V.$$

*Remark.* This construction is closely related to the *ternary models* for simple Lie algebras considered in [7], section 2. These models are of the form

$$\mathfrak{g} = \mathfrak{h} \times \mathfrak{sl}(U) \oplus (U \otimes V) \oplus (U^* \otimes V^*),$$

where  $U$  is three dimensional, and  $V$  is an  $\mathfrak{h}$ -module. To define a Lie bracket on  $\mathfrak{g}$ , one needs a cubic form  $c$  on  $V$ , a cubic form  $c^*$  on  $V^*$ , and a map  $\theta : V \otimes V^* \rightarrow \mathfrak{h}$ . Then the Jacobi identity implies a series of conditions on these data, including that

$$\mathfrak{h} \subset \text{Aut}(c) \cap \text{Aut}(c^*).$$

These conditions should ultimately lead to a cubic Jordan algebra structure on  $V$ . If we choose a maximal torus in  $\mathfrak{sl}(U)$  and use the associated grading on  $U, U^*$ , we get an hexagonal model as in Figure 2 of [8] :

$$\begin{array}{cccc}
 & & \mathfrak{g}_{-\alpha} & \\
 \mathfrak{g}_{-\psi} & & \mathfrak{g}_{-1-1} & \mathfrak{g}_{10} & \mathfrak{g}_{\psi-\alpha} \\
 & \mathfrak{g}_{-2-1} & & \mathfrak{g}_{00} & \mathfrak{g}_{21} \\
 \mathfrak{g}_{\alpha-\psi} & & \mathfrak{g}_{-10} & \mathfrak{g}_{11} & \mathfrak{g}_{\psi} \\
 & & & & \mathfrak{g}_{\alpha}
 \end{array}$$

The subalgebra we denoted  $\mathfrak{n}$  is the sum of the factors in the last three columns. and we can add the factor  $\mathfrak{sl}(W)$  from the middle column in order to get  $\mathfrak{s}$ .

Once we have defined the cubic  $c$  associated to the simple Lie algebra  $\mathfrak{g}$ , we have the associated homogeneous space  $X_c$  with its natural contact structure. A direct verification gives:

**Proposition 4.** *The homogeneous space  $X_c$  is an open subset of  $Y_{\mathfrak{g}}$ , with a codimension two boundary. Its contact structure is the restriction of the natural contact structure on  $Y_{\mathfrak{g}}$ .*

It is tempting, but illusory, as we shall see, to try to construct new projective contact manifolds as suitable compactifications of our homogeneous spaces  $X_c$  for other types of cubics.

### 5. COMPACTIFICATIONS

Since all regular functions on  $X_c$  are constant, we can expect that  $X_c$  admits a *small compactification*, that is, a projective variety  $\bar{X}_c$  containing  $X_c$  as an open subset in which the boundary of  $X_c$  has codimension two or more. By Theorem 1 in [3] and Lemma 1, it is enough to check that the algebra of sections

$$R(X_c, L) = \bigoplus_{k=0}^{\infty} \Gamma(X_c, L^k)$$

is of finite type, as well as all the  $R(X_c, L^m)$ , for  $m \geq 1$ . We have not been able to prove this but we can make the following observations.

Since the Lie algebra  $\mathfrak{g}$  of  $G = N \triangleleft SL(W)$  preserves the contact structure we have defined on  $X_c$ , there must be a morphism  $\varphi$  from  $X_c$  into  $\mathbb{P}\mathfrak{g}^*$  (see [1], Section 1). In fact, any contact vector field on a contact manifold defines a holomorphic section of the contact line bundle  $L$ . Thus  $\mathfrak{g}$  defines a linear subsystem in  $|L|$ . The morphism is always etale over its image, and since our  $X_c$  is simply connected we conclude that  $\varphi$  embeds  $X_c$  as a coadjoint orbit in  $\mathfrak{g}^*$ . We thus have a natural projective compactification of  $X_c$  in  $\mathbb{P}\mathfrak{g}^*$ .

Note that the inclusion of  $X_c$  in  $\mathbb{P}\mathfrak{g}^*$  is just the projectivization of the moment map of the symplectic variety  $Y_c = L^\times$ . We have a commutative diagram

$$\begin{array}{ccc} Y_c & \xrightarrow{\mu} & \mathfrak{g}^* \\ \downarrow & & \downarrow \\ X_c & \xrightarrow{\nu} & \mathbb{P}\mathfrak{g}^* \end{array}$$

Here  $\mu$  denotes the ( $G$ -equivariant) moment map and  $\nu$  is its quotient by the  $\mathbb{C}^*$ -action. We have  $\mathfrak{g}^* = \mathfrak{n}^* \oplus \mathfrak{sl}(W)^*$  and the component  $\mu'$  of  $\mu$  on  $\mathfrak{n}^*$  is not injective, since the  $N$  action on  $X_c$  preserve the  $\mathbb{P}^1$ -fibration. Consider  $\mu'(Y_c) \subset \mathfrak{n}^*$ .

**Proposition 5.** *Suppose that the cubic hypersurface  $Z_c \subset \mathbb{P}V$  be smooth. Then the boundary of  $\mu'(Y_c)$  has codimension at least two.*

*Proof.* We can describe explicitly the closure of  $\mu'(Y_c)$  as the set of triples  $(\phi_1, \phi_2, \phi_3) \in \mathfrak{n}^*$  such that

$$\omega(\phi_1, \phi_3) = c(\phi_2, \phi_2, \cdot),$$

where  $\omega : (V^* \otimes W) \times W \rightarrow V^*$  is the natural bilinear map.

If  $\phi_3 \neq 0$ , we are in  $\mu'(Y_c)$ . Thus on the boundary, we must have  $\phi_3 = 0$ , and then  $c(\phi_2, \phi_2, \cdot) = 0$ . But under our smoothness assumption on  $Z_c$ , this implies that  $\phi_2 = 0$ . So the boundary of  $\mu'(Y_c)$  has dimension at most  $2p = \dim \mu'(Y_c) - 2$ , the number of parameters for  $\phi_1$ .  $\square$

This seems to be a first step towards proving that  $X_c$  has a small compactification. But we have not been able even to find conditions on  $c$  that would ensure that the compactification  $\bar{X}_c \subset \mathbb{P}\mathfrak{g}^*$  is small.

What is rather surprising is that the cubics whose associated variety  $X_c$  has a smooth contact compactification can be completely classified. By this, we mean a smooth projective variety  $\bar{X}_c$  compactifying  $X_c$ , with a contact structure extending that of  $X_c$ .

**Proposition 6.** *There exists a smooth contact compactification  $\bar{X}_c$  of  $X_c$  if and only if  $c$  is the cubic norm of a semi-simple Jordan algebra.*

*Proof.* We will deduce this statement from a study the variety of minimal rational tangents  $\mathcal{C}_{x_\ell} \subset \mathbb{P}D_{x_\ell}$ . Note that the space of lines on  $X_c$  through  $x_\ell$  is just  $\xi^{-1}(x_\ell) \cong A_\ell \cong V$ .

We claim that the tangent map sending a line through  $x_\ell$  to its tangent direction in  $\mathbb{P}D_{x_\ell}$  is equal, up to scalars, to the rational map

$$\begin{aligned} \tau : V &\rightarrow \mathbb{P}D_{x_\ell} = \mathbb{P}(V \oplus V^* \oplus \mathbb{C} \oplus \mathbb{C}) \\ \tau(v) &:= [v : B(v, v) : c(v, v, v) : 1]. \end{aligned}$$

Indeed, a line through  $x_\ell$  in  $X_c$  is of the form  $\ell_g = \xi(g \times \mathbb{P}W)$  for  $g \in A_\ell$ . To write this line in the local chart we used in the proof of Proposition 3 (we use the same notations), we must write

$$\xi(g \times p) = \exp(Z)\xi(o \times p).$$

If  $g = \exp(X)$  with  $X \in V \otimes \ell$ , this amounts to solving the equation  $\exp(X) = \exp(Z)\exp(W)$ , with  $Z \in \mathfrak{n}_m$  and  $W \in V \otimes p$ . So  $X = H(Z, W)$ , and if we write  $X = v \otimes \ell$  for some  $v \in V$ , we must have  $W = v \otimes p$  and then we get, up to term of order at least two in  $z$ ,

$$\begin{aligned} Z_1 &= -zv \otimes m, \\ Z_2 &= -\frac{1}{2}[Z_1, W] = \frac{z}{2}B(v, v), \\ Z_3 &= -\frac{1}{6}[Z_1, [Z_1, W]] = -\frac{z}{6}c(v)\ell. \end{aligned}$$

This proves the claim.

We can now conclude the proof as follows. By the results of [5], the closure of the image of this map must be smooth if there exists a smooth contact compactification of  $X_c$ . So the closure of the image of the map  $\tau$  must be smooth. But then we can apply Corollary 26 in [6].  $\square$

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JUN-MUK HWANG: KOREA INSTITUTE FOR ADVANCED STUDY, HOEGIRO 87, SEOUL, 130-722, KOREA

*E-mail address:* `jmhwang@kias.re.kr`

LAURENT MANIVEL: INSTITUT FOURIER, UMR 5582 CNRS/UJF, UNIVERSITÉ JOSEPH FOURIER, 38402 SAINT MARTIN D'HÈRES, FRANCE

*E-mail address:* `Laurent.Manivel@ujf-grenoble.fr`