Existence of semilinear relaxation shocks

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Abstract

We establish existence with sharp rates of decay and distance from the Chapman–Enskog approximation of small-amplitude shock profiles of a class of semilinear relaxation systems including discrete velocity models obtained from Boltzmann and other kinetic equations. Our method of analysis is based on the macro–micro decomposition introduced by Liu and Yu for the study of Boltzmann profiles, but applied to the stationary rather than the time-evolutionary equations. This yields a simple proof by contraction mapping in weighted H^s spaces.

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1 Introduction

We consider the problem of existence of relaxation profiles

(1.1) $$U(x,t) = \bar{U}(x - st), \quad \lim_{z \to \pm\infty} \bar{U}(z) = U_\pm$$

of a semilinear relaxation system

(1.2) $$U_t + F(U)_x = Q(U),$$

in one spatial dimension, with the following structure:

Assumption 1.1. *(H1) The flux F is linear in U, so that $F(U) = AU$ for some constant matrix A.*

(H2) There are linear coordinates such that

(1.3) $$U = \begin{pmatrix} u \\ v \end{pmatrix}, \quad Q = \begin{pmatrix} 0 \\ q \end{pmatrix},$$

$u \in \mathbb{R}^n$, $v \in \mathbb{R}^r$.

(H3) q has nondegenerate equilibria parametrized by u; more precisely there are a smooth function v_* from $\mathbb{R}^n$ to $\mathbb{R}^r$ and $\theta > 0$ such that for all u:

(1.4) $$q(u, v_*(u)) = 0, \qquad \mathrm{Re}\ \sigma(\partial_v q(u, v_*(u))) \leq -\theta,$$

$\sigma(\cdot)$ *denoting spectrum and* Re *the real part.*

Common examples are discrete kinetic models obtained by discrete velocity or other approximation from continuous kinetic models such as Boltzmann or Vlasov–Poisson equations; for example, Broadwell and other lattice gas models [PI]. Other examples are the semilinear relaxation schemes introduced by Jin–Xin [JX] and Natalini [N] for the purpose of numerical approximation of hyperbolic systems. Here, we are thinking particularly of the case n bounded and $r >> 1$ arising through discretization of the Boltzmann equation, or the case $r \to \infty$ arising in Boltzmann itself; that is, *we seek estimates and proof independent of the dimension of* v.

For fixed n, r, the existence problem has been treated in [YZ, MaZ1] under the additional assumption

$$\det(dF - sI) \neq 0 \tag{1.5}$$

corresponding to nondegeneracy of the traveling-wave ODE. However, as pointed out in [MaZ2, MaZ3], this assumption is unrealistic for large models, and in particular is not satisfied for the Boltzmann equations, for which the eigenvalues of dF are constant particle speeds of all values. Our goal here, therefore, is to revisit the existence problem *without the assumption* (1.5), with the eventual aim being to establish a simple proof of existence of small-amplitude Boltzmann profiles. Of course, existence of such was established some time ago in [CN]; however, the proof is rather complicated, involving detailed resolvent estimates in weighted L^∞ spaces in spatial and velocity variables, and so it seems of use to seek a simpler approach based on weighted L^2 spaces and standard energy estimates.

Our method of analysis is motivated by the "macro-micro decomposition" technique introduced by Liu and Yu [LY], in which fluid (macroscopic, or equilibrium) and transient (microscopic) effects are separated and estimated by different techniques. This was used in [LY] to show by a study of the time-evolutionary equations that the Boltzmann profile constructed in [CN] has nonnegative probability density, that is, to show positivity of Boltzmann profiles *assuming that such a profile exists*.

Our approach here is very much in the spirit of that of [LY], based on approximate Chapman–Enskog expansion combined with Kawashima type estimates (the macro–micro decomposition of the reference), but carried out for the *stationary* (traveling-wave) rather than the time-evolutionary equations, and estimating the finite-dimensional fluid part using sharp ODE estimates in place of the energy estimates of [LY]. In this latter part, we are much aided by the more favorable properties of the stationary fluid equations, a rather standard boundary value ODE system, as compared to the time-evolutionary equations, a hyperbolic–parabolic system of PDE.

Our main result is to show existence with sharp rates of decay and distance from the Chapman–Enskog approximation of small-amplitude quasilinear relaxation shocks in the general case that the profile ODE may become degenerate. See Sections 2 and 3 for model assumptions and description of the Chapman–Enskog approximation, and Section 4 for a statement of the main theorem. In the present, semilinear case, a simple contraction-mapping argument suffices; the quasilinear case is treated by Nash–Moser iteration in [MeZ1]. In [MeZ2], we show that the argument of this paper carries over with minor modifications to the infinite-dimensional Boltzmann equation with hard potential to yield existence of small-amplitude Boltzmann shock profiles, recovering and slightly sharpening the results of [CN]. This in a sense completes the analysis of [LY], providing by a common set of techniques both existence (through the present argument) and (through the argument of [LY]) positivity. At the same time it gives a truly elementary proof of existence of Boltzmann profiles.

Finally, we note that spectral stability has been shown for general small-amplitude quasilinear relaxation profiles in [MaZ3], without the assumption (1.5), under the assumption that the profile exist and satisfy exponential bounds like those of the viscous case. The results obtained here verify that assumption, completing the analysis of [MaZ3]. It would be very interesting to continue along the same lines to obtain a complete nonlinear stability result as in [MaZ1], in particular for Boltzmann shocks.

Existence results in the absence of condition (1.5) have been obtained in special cases in [MaZ5, DY] by quite different methods (for example, center-manifold expansion near an assumed single degenerate point [DY]). However, the decay bounds as stated, though exponential, are not sufficiently sharp with respect to ε for the needs of [MaZ3]. More important, the techniques used in these analyses do not appear to generalize to the infinite-dimensional (e.g., Boltzmann) case.

2 Model, assumptions, and the reduced system

Taking without loss of generality $s = 0$, we study the traveling-wave ODE

$$AU' = Q(U), \tag{2.1}$$

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \quad A \equiv \text{constant}, \quad Q = \begin{pmatrix} 0 \\ q(u,v) \end{pmatrix} \tag{2.2}$$

governing solutions of (1.1), where q satisfies (1.4).

We use the notations

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{2.3}$$

$$f(u,v) := A_{11}u + A_{12}v. \tag{2.4}$$

We make the standard assumption of *simultaneous symmetrizability* [Y]:

Assumption 2.1. *(SS) There exists a smooth, symmetric and uniformly positive definite matrix $S(U)$ such that*

i) for all U, $S(U)A$ is symmetric,

ii) for all equilibria $U_* = (u, v_*(u))$, $S\,dQ(U_*)$ is symmetric nonpositive with

$$\dim\ker SdQ = \dim\ker dQ \equiv n. \tag{2.5}$$

We also make the Kawashima assumption of *genuine coupling* [K]:

Assumption 2.2. *(GC) For all equilibria $U_* = (u, v_*(u))$, there exists no eigenvector of A in the kernel of $dQ(U_*)$. Equivalently, given Assumption 2.1 (see [K]), there exists in a neighborhood $\mathcal{N}$ of the equilibrium manifold a skew symmetric $K = K(U)$ such that*

$$\text{Re } (KA - SdQ)(U) \geq \theta > 0 \tag{2.6}$$

for all $U \in \mathcal{N}$.

Recall from [Y] (see aslo Section 3), that the reduced, Navier–Stokes type equations obtained by Chapman–Enskog expansions are

$$f_*(u)' = (b_*(u)u')', \tag{2.7}$$

where

$$f_*(u) := f(u, v_*(u)) = A_{11}u + A_{12}v_*(u), \tag{2.8}$$
$$b_*(u) := -A_{12}c_*(u) \tag{2.9}$$

with

$$c_*(u) := \partial_v q^{-1}(u, v_*(u)) \Big(A_{21} + A_{22}dv_*(u) - dv_*(u)(A_{11} + A_{12}dv_*(u)\Big). \tag{2.10}$$

Note also, by the Implicit Function Theorem, that

$$dv_*(u) = -\partial_v q^{-1}\partial_u q(u, v_*(u)).$$

For the reduced system (2.7), simultaneous symmetrizability becomes:

(ss) There exists $s(u)$ symmetric positive definite such that $s\,df_*$ is symmetric and sb_* is symmetric positive semidefinite.

We have likewise a notion of genuine coupling [K]:

(gc) There is no eigenvector of df_* in $\ker b_*$.

We note first the following important observation of [Y].

Proposition 2.3 ([Y])**.** *Let* (2.1) *as described above be a symmetrizable system satisfying the genuine coupling condition (GC). Then, the reduced system* (2.7) *is a symmetrizable system satisfying genuine coupling condition (gc).*

Proof. We only give here a sketch, mentioning the key points and refereeing e.g. to [MaZ3] for details. Fixing U, one is reduced to constant matrices and linear algebra, with matrices

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 \\ Q_{21} & Q_{22} \end{pmatrix},$$

and symmetrizer S. With

$$P = \begin{pmatrix} \mathrm{Id} & 0 \\ V & \mathrm{Id} \end{pmatrix}, \qquad V = -Q_{22}^{-1}Q_{21},$$

the change of unknowns $U = P\widetilde{U}$ transforms the problem to an equivalent one with matrices $\widetilde{A} = P^{-1}AP$, $\widetilde{Q} = P^{-1}QP$ and symmetrizer $\widetilde{S} = P^*SP$, with

$$\widetilde{Q} = \begin{pmatrix} 0 & 0 \\ 0 & Q_{22} \end{pmatrix}.$$

Therefore $\widetilde{S}$ is block diagonal ($\widetilde{S}_{21} = 0$ and $\widetilde{S}_{12} = 0$), $\widetilde{S}_{11}\widetilde{A}_{11}$ is symmetric, $\widetilde{S}_{11}\widetilde{A}_{12} = (\widetilde{S}_{22}\widetilde{A}_{21})^*$ and $\widetilde{S}_{22}\widetilde{Q}_{22}$ is definite negative. Next, the associated matrix $\widetilde{b}$ is :

$$\widetilde{b} = -\widetilde{A}_{12}\widetilde{Q}_{22}^{-1}\widetilde{A}_{21}.$$

Thus $\widetilde{S}_{11}$ is definite positive, symmetrizes $\widetilde{A}_{11}$ and

$$\widetilde{S}_{11}\widetilde{b} = -\widetilde{S}_{11}\widetilde{A}_{12}(\widetilde{S}_{22}\widetilde{Q}_{22}^{-1})^{-1}S_{22}\widetilde{A}_{21} = -(\widetilde{S}_{22}\widetilde{A}_{21})^*(\widetilde{S}_{22}\widetilde{Q}_{22}^{-1})^{-1}\widetilde{S}_{22}\widetilde{A}_{21}$$

is symmetric and nonnegative. Noticing that

$$\tilde{A}_{11} = A_{11} + A_{12}V, \tag{2.11}$$

$$\widetilde{b} = -A_{12}Q_{22}^{-1}\big(A_{21} + A_{22}V - V(A_{11} + A_{12}V)\big) = b_* \tag{2.12}$$

this implies that the property (ss) is satisfied with symmetrizer $s = \widetilde{S}_{11}$ (In terms of the original matrices, $s = S_{11} + S_{12}V = S_{11} + S_{12}V + V^*(S_{21} + S_{22}V)$, since $S_{21} + S_{22}V = 0$ as a consequence of the block diagonal structure of $\widetilde{S}$).

Similarly, the property (GC) is transported to the system $(\widetilde{A}, \widetilde{Q})$, meaning that

$$\widetilde{A}_{11}u = \lambda u, \quad \widetilde{A}_{21}u = 0 \qquad \Rightarrow \qquad u = 0.$$

The symmetry property of $\widetilde{S}_{11}\widetilde{b}$ implies that

$$\ker \widetilde{b} = \ker \widetilde{A}_{21}$$

and the property (gc) immediately follows. □

Besides the basic properties guaranteed by Proposition 2.3, we assume that the reduced system satisfy the following important additional conditions.

Assumption 2.4. *(i) The matrix $b_*(u)$ has constant left kernel.*
(ii) For all values of u, $\ker df_*(u) \cap \ker b_*(u) = \{0\}$.

The importance of Assumption 2.4 in the present situation is that it ensures that the zero-speed profile problem for the reduced system,

$$f_*(u)' = (b_*(u)u')', \quad \lim_{z\to\pm\infty} u(z) = u_\pm \tag{2.13}$$

or, after integration from $-\infty$ to x,

$$b_*(u)u' = f_*(u) - f_*(u_\pm), \tag{2.14}$$

may be expressed as a nondegenerate ODE in u_2, coordinatizing $u = (u_1, u_2)$ with $u_1 = \pi_* u$ and $u_2 = (I - \pi_*)u$.

Condition (i) was pointed out in [Ze], condition (ii) in [MaZ3, Z1, GMWZ].

Remark 2.5. Assumption 2.4 is also central to the linearized stability analysis of general Navier–Stokes type equations in [Z2, GMWZ]. It appears to be independent from the genuine coupling conditions (GC), (gc), except in the special case that u is scalar, for which (GC), (gc) reduce to $\ker b_* = \emptyset$. It is satisfied for the important example of Boltzmann equations, which is our main motivation.

Next, we assume that the classical theory of weak shocks can be applied to (2.13), assuming that the flux f_* has a genuinely nonlinear eigenvalue near 0:

Assumption 2.6. *In a neighborhood $\mathcal{U}_*$ of a given base state u_0, df_* has a simple eigenvalue α near zero, with $\alpha(u_0) = 0$, and such that the associated hyperbolic characteristic field is genuinely nonlinear, i.e., after a choice of orientation, $\nabla\alpha \cdot r(u_0) < 0$, where r denotes the eigendirection associated with α.*

Remark 2.7. Assumption 2.6 is standard, and is satisfied in particular for the compressible Navier–Stokes equations resulting from Chapman–Enskog approximation of the Boltzmann equation.

3 Chapman–Enskog approximation

Integrating the first equation of (2.1) and noticing that the end states $(u_\pm, v_\pm)$ must be equilibria and thus satisfy $v_\pm = v_*(u_\pm)$, we obtain

$$\begin{aligned} A_{11}u + A_{12}v &= f_*(u_\pm), \\ A_{21}u' + A_{22}v' &= q(u,v). \end{aligned} \tag{3.1}$$

Because f is linear, the first equation reads

$$f_*(u) + A_{12}(v - v_*(u)) = f_*(u_\pm). \tag{3.2}$$

The idea of Chapman–Enskog approximation is that $v - v_*(u)$ is small (compared to the fluctuations $u - u_\pm$). Taylor expanding the second equation, we obtain

$$\begin{aligned} (A_{21} + A_{22}dv_*(u))u' + A_{22}(v - v_*(u))' = \partial_v q(u, v_*(u))(v - v_*(u)) \\ + O(|v - v_*(u)|^2), \end{aligned}$$

or inverting $\partial_v q$

$$\begin{aligned} v - v_*(u) = \partial_v q^{-1}(u, v_*(u))\big(A_{21} + A_{22}dv_*(u)\big)u' \\ + O(|v - v_*(u)|^2) + O(|(v - v_*(u))'|). \end{aligned} \tag{3.3}$$

The derivative of (3.2) implies that

$$\big(A_{11}u + A_{12}dv_*(u)\big)u' = O(|(v - v_*(u))'|).$$

Therefore, (3.3) can be replaced by

$$v - v_*(u) = c_*(u)u' + O(|v - v_*(u)|^2) + O(|(v - v_*(u))'|), \tag{3.4}$$

where c_* is defined at (2.10). Substituting in (3.2), we thus obtain the approximate viscous profile ODE

$$b_*(u)u' = f_*(u) - f_*(u_\pm) + O(|v - v_*(u)|^2) + O(|(v - v_*(u))'|), \tag{3.5}$$

where b_* is as defined in (2.9).

Remark 3.1. The above calculation leaves a great deal of flexibility in the choice of b_* satisfying (3.5), namely it is only specified modulo multiples of

$$A_{11} + A_{12} dv_*(u),$$

as we used when passing from (3.3) to (3.4). However, we have chosen to use the standard definition (2.8) of b_* because it is the natural choice which is invariant by change of variables (see (2.12)) and it is known by Proposition 2.3 and by the explicit example of Boltzmann to have good properties. But, it might be, for example, that a different representative could be strictly parabolic, so slightly easier to handle. This seems to be just a curiosity, as the analysis is already sufficient to treat the standard case. But, it is interesting from the viewpoint of the Chapman–Enskog expansion and possible alternative representations.

Motivated by (3.3)–(3.5), we define an approximate solution $(\bar u_{NS}, \bar v_{NS})$ of (3.1) by choosing $\bar u_{NS}$ as a solution of

$$b_*(\bar u_{NS})\bar u'_{NS} = f_*(\bar u_{NS}) - f_*(u_\pm), \tag{3.6}$$

and $\bar v_{NS}$ as the first approximation given by (3.3)

$$\bar v_{NS} - v_*(\bar u_{NS}) = c_*(\bar u_{NS})\bar u'_{NS}. \tag{3.7}$$

Small amplitude shock profiles solutions of (3.6) are constructed using the center manifold analysis of [Pe] under conditions (i)-(ii) of Assumption 2.4; see discussion in [MaZ5].

Proposition 3.2. *Under Assumptions 2.6 and 2.4, in a neighborhood of (u_0, u_0) in $\mathbb{R}^n \times \mathbb{R}^n$, there is a smooth manifold $\mathcal{S}$ of dimension n passing through (u_0, u_0), such that for $(u_-, u_+) \in \mathcal{S}$ with amplitude $\varepsilon := |u_+ - u_-| > 0$ sufficiently small, and direction $(u_+ - u_-)/\varepsilon$ sufficiently close to $r(u_0)$, the zero speed shock profile equation (3.6) has a unique (up to translation) solution $\bar u_{NS}$ in $\mathcal{U}_*$. The shock profile is necessarily of* Lax type*: i.e., with dimensions of the unstable subspace of $df_*(u_-)$ and the stable subspace of $df_*(u_+)$ summing to one plus the dimension of u, that is $n+1$.*

Moreover, there is $\theta > 0$ and for all k there is C_k independent of (u_-, u_+) and ε, such that

$$|\partial_x^k(\bar u_{NS} - u_\pm)| \le C_k \varepsilon^{k+1} e^{-\theta\varepsilon|x|}, \quad x \gtrless 0. \tag{3.8}$$

We denote by $\mathcal{S}_+$ the set of $(u_-, u_+) \in \mathcal{S}$ with amplitude $\varepsilon := |u_+ - u_-| > 0$ sufficiently small and direction $(u_+ - u_-)/\varepsilon$ sufficiently close to $r(u_0)$ such that the profile $\bar{u}_{NS}$ exists.

Given $(u_-, u_+) \in \mathcal{S}_+$ with associated profile $\bar{u}_{NS}$, we define $\bar{v}_{NS}$ by (3.7) and

$$\bar{U}_{NS} := (\bar{u}_{NS}, \bar{v}_{NS}). \tag{3.9}$$

It is an approximate solution of (3.1) in the following sense:

Corollary 3.3. *For* $(u_-, u_+) \in \mathcal{S}_+$,

$$A_{11}\bar{u}_{NS} + A_{12}\bar{v}_{NS} - f_*(u\pm) = 0 \tag{3.10}$$

and

$$\mathcal{R}_v := A_{21}\bar{u}'_{NS} + A_{22}\bar{v}'_{NS} - q(\bar{u}_{NS}, \bar{v}_{NS})$$

satisfies

$$|\partial_x^k \mathcal{R}_v(x)| \le C_k \varepsilon^{k+3} e^{-\theta\varepsilon|x|}, \quad x \gtrless 0 \tag{3.11}$$

where C_k *is independent of* (u_-, u_+) *and* $\varepsilon = |u_+ - u_-|$.

Proof. Given the choice of $\bar{v}_{NS}$, the first equation is a rewriting of the profile equation (3.6).

Next, note that

$$\bar{v}_{NS} - v_*(\bar{u}_{NS}) = O(|\bar{u}'_{NS}|), \quad \big(\bar{v}_{NS} - v_*(\bar{u}_{NS})\big)' = O(|\bar{u}''_{NS}|) + O(|\bar{u}'_{NS}|^2),$$

where here $O(\cdot)$ denote smooth functions of $\bar{u}_{NS}$ and its derivatives, which vanish as indicated. With similar notations, the Taylor expansion of q and the definition of $\bar{v}_{NS}$ show that

$$\begin{aligned}\mathcal{R}_v =& O(|\bar{v}_{NS} - v_*(\bar{u}_{NS})|^2) + O(|(\bar{v}_{NS} - v_*(\bar{u}_{NS}))'|) \\ &+ dv_*(\bar{u}_{NS}\big(A_{11} + A_{12} dv_*(\bar{u}_{NS})\big)\bar{u}'_{NS}.\end{aligned}$$

Moreover,

$$\begin{aligned}\big(A_{11} + A_{12} dv_*(\bar{u}_{NS})\big)\bar{u}'_{NS} &= \big(f_*(\bar{u}_{NS})\big)' = \big(b_*(\bar{u}_{NS})\bar{u}'_{NS})\big)' \\ &= O(|\bar{u}'_{NS}|^2) + O(|\bar{u}''_{NS}|).\end{aligned}$$

This implies that

$$\mathcal{R}_v = O(|\bar{u}'_{NS}|^2) + O(|\bar{u}''_{NS}|).$$

satisfies the estimates stated in (3.11). □

Remark 3.4. One may check that if we did not include the correction from equilibrium on the righthand side of (3.7), taking instead the simpler prescription $\bar{v}_{NS} = v_*(\bar{u}_{NS})$ as in [LY], then the residual error that would result in (3.10) would be too large for our later iteration scheme to close. This is a crucial difference between our analysis and the analysis of [LY].

4 Statement of the main theorem

We are now ready to state the main result. Define a base state $U_0 = (u_0, v_*(u_0))$ and a neighborhood $\mathcal{U} = \mathcal{U}_* \times \mathcal{V}$.

Theorem 4.1. *Let Assumptions (SS), (GC), and 2.4 hold on the neighborhood $\mathcal{U}$ of U_0, with $Q \in C^\infty$. Then, there are $\varepsilon_0 > 0$ and $\delta > 0$ such that for $(u_-, u_+) \in \mathcal{S}+$ with amplitude $\varepsilon := |u_+ - u_-| \le \varepsilon_0$, the standing-wave equation* (2.1) *has a solution $\bar U$ in $\mathcal{U}$, with associated Lax-type equilibrium shock (u_-, u_+), satisfying for all k:*

$$\begin{aligned} \left|\partial_x^k(\bar U - \bar U_{NS})\right| &\le C_k \varepsilon^{k+2} e^{-\delta\varepsilon|x|}, \\ |\partial_x^k(\bar u - u_\pm)| &\le C_k \varepsilon^{k+1} e^{-\delta\varepsilon|x|}, \quad x \gtrless 0, \\ \left|\partial_x^k(\bar v - v_*(\bar u)\right| &\le C_k \varepsilon^{k+2} e^{-\delta\varepsilon|x|}, \end{aligned} \tag{4.1}$$

where $\bar U_{NS} = (\bar u_{NS}, \bar v_{NS})$ is the approximating Chapman–Enskog profile defined in (3.9), *and C_k is independent of ε. Moreover, up to translation, this solution is unique within a ball of radius $c\varepsilon$ about $\bar U_{NS}$ in norm $\|\cdot\|_{L^2} + \varepsilon^{-1}\|\partial_x \cdot\|_{L^2} + \varepsilon^{-2}\|\partial_x^2 \cdot\|_{L^2}$, for $c > 0$ sufficiently small. (For comparison, $\bar U_{NS} - U_\pm$ is order $\varepsilon^{1/2}$ in this norm, by* (4.1)*(ii)–(iii).)*

Bounds (4.1) show that (i) the behavior of profiles is indeed well-described by the Navier–Stokes approximation, and (ii) profiles indeed satisfy the exponential decay rates required for the proof of spectral stability in [MaZ3]. From the second observation, we obtain immediately from the results of [MaZ3] the following stability result, partially generalizing that of [LY] for the Boltzmann equations.[1]

Corollary 4.2 ([MaZ3])**.** *Under the assumptions of Theorem 4.1, the resulting profiles $\bar U$ are spectrally stable for amplitude ε sufficiently small, in the sense that the linearized operator $L := \partial_x A(\bar U) - dQ(\bar U)$ about $\bar U$ has no L^2 eigenvalues λ with* $\operatorname{Re} \lambda \ge 0$ *and* $\lambda \ne 0$.

The remainder of the paper is devoted to the proof of Theorem 4.1.

Remark 4.3. Theorem 4.1 yields uniqueness only among solutions close to the Chapman–Enskog approximant $\bar U_{NS}$. The stability result of Liu–Yu [LY] should give uniqueness among solutions in a ball of small but $O(1)$ radius, assuming that they have zero relative mass compared to $\bar U_{NS}$. Indeed, it should be possible to upgrade this to general-mass perturbations to obtain ultimately a full $O(1)$ uniqueness result. Stability with respect to general-mass perturbations is an important open problem.

[1] Liu and Yu prove the stronger result of linearized stability with respect to zero-mass perturbations that are sufficiently small in an appropriate norm.

5 Outline of the proof

5.1 Nonlinear perturbation equations

Defining the perturbation variable $U := \bar U - \bar U_{NS}$, and expanding about $\bar U_{NS}$, we obtain from (3.1) the nonlinear perturbation equations

$$\begin{aligned} A_{11}u + A_{12}v &= 0 && (5.1)\\ A_{21}u' + A_{22}v' - dq(\bar U_{NS})U &= -\mathcal{R}_v + N(U) && (5.2) \end{aligned}$$

where the remainder $N(U)$ is a smooth function of U_{NS} and U, vanishing at second order at $U = 0$:

$$N(U) = \mathcal{N}(\bar U_{NS}, U) = O(|U|^2). \tag{5.3}$$

We push the reduction a little further, using that

$$M := dq(\bar u_{NS}, \bar v_{NS}) - dq(\bar u_{NS}, v_*(u_{NS})) = O(|\bar v_{NS} - v_*(\bar u_{NS})|). \tag{5.4}$$

Therefore the equation reads

$$\begin{aligned} \mathcal{L}_*^\varepsilon U &:= \begin{pmatrix} 0 & 0 \\ A_{21} & A_{22} \end{pmatrix} U' + \begin{pmatrix} A_{11} & A_{12} \\ -Q_{21} & -Q_{22} \end{pmatrix} U \\ &= \begin{pmatrix} 0 \\ -\mathcal{R}_v + MU + N(U) \end{pmatrix} \end{aligned} \tag{5.5}$$

where

$$Q_{21} = \partial_u q(\bar u_{NS}, v_*(\bar u_{NS})), \quad Q_{22} = \partial_v q(\bar u_{NS}, v_*(\bar u_{NS})). \tag{5.6}$$

Differentiating the first line, it implies that

$$L_*^\varepsilon U := AU' - dQ(\bar u_{NS}, v_*(\bar u_{NS}))U = \begin{pmatrix} 0 \\ -\mathcal{R}_v + MU + N(U) \end{pmatrix}. \tag{5.7}$$

The linearized operator $A\partial_x - dQ(\bar U)$ about an exact solution $\bar U$ of the profile equations has kernel $\bar U'$, by translation invariance, so is not invertible. Thus, the linear operators L_*^ε and $\mathcal{L}_*^\varepsilon$ are not expected to be invertible, and we shall see later that they are not. Nonetheless, one can check that $\mathcal{L}_*^\varepsilon$ is surjective in Sobolev spaces and define a right inverse $\mathcal{L}_*^\varepsilon(\mathcal{L}_*^\varepsilon)^\dagger \equiv I$, or solution operator $(\mathcal{L}_*^\varepsilon)^\dagger$ of the equation

$$\mathcal{L}_*^\varepsilon U = F := \begin{pmatrix} f \\ g \end{pmatrix}, \tag{5.8}$$

as recorded by Proposition 5.2 below. Note that L_*^ε is not surjective because the first equation requires a zero mass condition on the source term. This is why we solve the integrated equation (5.5) and not (5.7).

To define the partial inverse $(\mathcal{L}_*^\varepsilon)^\dagger$, we specify one solution of (5.8) by adding the co-dimension one internal condition:

$$\ell_\varepsilon \cdot u(0) = 0 \tag{5.9}$$

where ℓ_ε is a certain unit vector to be specified below.

Remark 5.1. There is a large flexibility in the choice of ℓ_ε. Conditions like (5.9) are known to fix the indeterminacy in the resolution of the linearized profile equation from (3.6) and it remains well adapted in the present context, see section 7 below. A possible choice, would be to choose ℓ_ε independent of ε and parallel to the left eigenvector of $df_*(u_0)$ for the eigenvalue 0 (see Assumption 2.6).

5.2 Fixed-point iteration scheme

The coefficients and the error term $\mathcal{R}_v$ are smooth functions of $\bar{u}_{NS}$ and its derivative, thus behave like smooth functions of εx. Thus, it is natural to solve the equations in spaces which reflect this scaling. We do not introduce explicitly the change of variables $\tilde{x} = \varepsilon x$, but introduce norms which correspond to the usual H^s norms in the $\tilde{x}$ variable :

$$\|f\|_{H^s_\varepsilon} = \varepsilon^{\frac{1}{2}}\|f\|_{L^2} + \varepsilon^{-\frac{1}{2}}\|\partial_x f\|_{L^2} + \cdots + \varepsilon^{\frac{1}{2}-s}\|\partial_x^s f\|_{L^2}. \tag{5.10}$$

We also introduce weighted spaces and norms, which encounter for the exponential decay of the source and solution: introduce the notations.

$$< x > := (x^2+1)^{1/2} \tag{5.11}$$

For $\delta \geq 0$ (sufficiently small), we denote by $H^s_{\varepsilon,\delta}$ the space of functions f such that $e^{\delta\varepsilon<x>} f \in H^s$ equipped with the norm

$$\|f\|_{H^s_{\varepsilon,\delta}} = \varepsilon^{\frac{1}{2}} \sum_{k\leq s} \varepsilon^{-k} \|e^{\delta\varepsilon<x>}\partial_x^k f\|_{L^2}. \tag{5.12}$$

Note that for $\delta \leq 1$, this norm is equivalent, with constants independent of ε and δ, to the norm

$$\|e^{\delta\varepsilon<x>} f\|_{H^s_\varepsilon}.$$

Proposition 5.2. *Under the assumptions of Theorem 4.1, there are constants C, $\varepsilon_0 > 0$ and $\delta_0 > 0$ and for all $\varepsilon \in]0,\varepsilon_0]$, there is a unit vector ℓ_ε such that for $\varepsilon \in]0,\varepsilon_0]$, $\delta \in [0,\delta_0]$, $f \in H^3_{\varepsilon,\delta}$, $g \in H^2_{\varepsilon,\delta}$ the operator equations (5.8) (5.9) has a unique solution $U \in H^2_{\varepsilon,\delta}$, denoted by $U = (\mathcal{L}_*^\varepsilon)^\dagger F$, which satisfies*

$$\big\|(\mathcal{L}_*^\varepsilon)^\dagger F\big\|_{H^2_{\varepsilon,\delta}} \leq C\varepsilon^{-1}\big(\|f\|_{H^3_{\varepsilon,\delta}} + \|g\|_{H^2_{\varepsilon,\delta}}\big). \tag{5.13}$$

Moreover, for $s \geq 3$, there is a constant C_s such that for $\varepsilon \in]0,\varepsilon_0]$ and $f \in H^{s+1}_{\varepsilon,\delta}$, $g \in H^s_{\varepsilon,\delta}$ the solution $U = (\mathcal{L}_*^\varepsilon)^\dagger F \in H^s_{\varepsilon,\delta}$ and

$$\big\|(\mathcal{L}_*^\varepsilon)^\dagger F\big\|_{H^s_{\varepsilon,\delta}} \leq C\varepsilon^{-1}\big(\|f\|_{H^{s+1}_{\varepsilon,\delta}} + \|g\|_{H^s_{\varepsilon,\delta}}\big) + C_s\big\|(\mathcal{L}_*^\varepsilon)^\dagger F\big\|_{H^{s-1}_{\varepsilon,\delta}}. \tag{5.14}$$

The proof of this proposition comprises most of the work of the paper. Once it is established, existence follows by a straightforward application of the Contraction-Mapping Theorem. Defining

$$\mathcal{T} := (\mathcal{L}_*^\varepsilon)^\dagger \begin{pmatrix} 0 \\ -\mathcal{R}_v + MU + N(U)) \end{pmatrix}, \tag{5.15}$$

we reduce (5.7) to the fixed-point equation

$$\mathcal{T}U := U. \tag{5.16}$$

5.3 Proof of the main theorem

Proof of Theorem 4.1. The profile $\bar{u}_{NS}$ exists if ε is small enough. The estimates (3.8) imply that

$$\|\bar{u}_{NS} - u_\pm\|_{H^s_{\varepsilon,\delta}} \le C_s\varepsilon \tag{5.17}$$

with C_s independent of ε and δ, provided that $\delta \le \theta/2$. Similarly, (3.11) implies that

$$\|\mathcal{R}_v\|_{H^s_{\varepsilon,\delta}} \le C_s\varepsilon^3, \tag{5.18}$$

and (5.4) implies that

$$\|M\|_{H^s_{\varepsilon,\delta}} \le C_s\varepsilon^2. \tag{5.19}$$

Moreover, with the choice of norms (5.10), the Sobolev inequality reads

$$\|u\|_{L^\infty} \le C\|u\|_{H^1_\varepsilon} \le C\|u\|_{H^1_{\varepsilon,\delta}} \tag{5.20}$$

with C independent of ε. Moreover, for smooth functions Φ, there are nonlinear estimates

$$\|\Phi(u)\|_{H^s_\varepsilon} \le C\big(\|u\|_{L^\infty}\big)\ \|u\|_{H^s_\varepsilon}. \tag{5.21}$$

which also extend to weighted spaces, for $\delta \le 1$:

$$\|\Phi(u)\|_{H^s_{\varepsilon,\delta}} \le C\big(\|u\|_{L^\infty}\big)\ \|u\|_{H^s_{\varepsilon,\delta}}. \tag{5.22}$$

In particular, this implies that for $s \ge 1$, $\delta \le \min\{1, \theta/2\}$ and ε small enough:

$$\begin{aligned} \|MU\|_{H^s_{\varepsilon,\delta}} &\le C\big(\|M\|_{H^1_{\varepsilon,\delta}}\|U\|_{H^s_{\varepsilon,\delta}} + \|M\|_{H^s_{\varepsilon,\delta}}\|U\|_{H^1_{\varepsilon,\delta}}\big) \\ &\le \varepsilon^2\big(C\|U\|_{H^s_\varepsilon} + C_s\|U\|_{H^1_\varepsilon}\big) \end{aligned} \tag{5.23}$$

where the first constant C is independent of s. Similarly,

$$\|N(U)\|_{H^s_{\varepsilon,\delta}} \le C\big(\|U\|_{L^\infty}\big)\|U\|_{H^1_{\varepsilon,\delta}}\|U\|_{H^s_{\varepsilon,\delta}}. \tag{5.24}$$

Combining these estimates, we find that

$$\|\mathcal{T}U\|_{H^s_{\varepsilon,\delta}} \le \varepsilon^{-1}\big(C_s\varepsilon^3 + C\varepsilon^2\|U\|_{H^s_{\varepsilon,\delta}} + C_s\varepsilon^2\|U\|_{H^1_{\varepsilon,\delta}} + C\varepsilon^{-1}\|U\|_{H^1_{\varepsilon,\delta}}\|U\|_{H^s_{\varepsilon,\delta}}\big),$$

that is

$$\|\mathcal{T}U\|_{H^s_{\varepsilon,\delta}} \le C_s\varepsilon^2 + C(\varepsilon + \|U\|_{H^1_{\varepsilon,\delta}})\|U\|_{H^s_{\varepsilon,\delta}} + C_s\varepsilon\|U\|_{H^1_{\varepsilon,\delta}} \tag{5.25}$$

provided that $\varepsilon \le \varepsilon_0$, $\delta \le \min\{1, \theta/2\}$ and $\|U\|_{L^\infty} \le 1$.

Consider first the case $s = 2$. Then, $\mathcal{T}$ maps the ball $\mathcal{B}_{\varepsilon,\delta} = \{\|U\|_{H^2_{\varepsilon,\delta}} \le \varepsilon^{1+\frac{1}{2}}\}$ to itself, if $\varepsilon \le \varepsilon_1$ where $\varepsilon_1 > 0$ is small enough. Similarly,

$$\|\mathcal{T}U - \mathcal{T}V\|_{H^2_{\varepsilon,\delta}} \le C\varepsilon^{-1}\big(\varepsilon^2 + \|U\|_{H^2_\varepsilon} + \|V\|_{H^2_\varepsilon}\big)\|U - V\|_{H^2_{\varepsilon,\delta}}, \tag{5.26}$$

provided that $\|U\|_{L^\infty} \le 1$ and $\|V\|_{L^\infty} \le 1$, from which we readily find that, for $\varepsilon > 0$ sufficiently small, $\mathcal{T}$ is contractive on $\mathcal{B}_{\varepsilon,\delta}$, whence, by the Contraction-Mapping Theorem, there exists a unique solution U^ε of (5.16) in $\mathcal{B}_{\varepsilon,\delta}$ for ε sufficiently small.

Moreover, from the contraction property

$$\|\bar{U}^\varepsilon - \mathcal{T}(0)\|_{H^2_\varepsilon} = \|\mathcal{T}(\bar{U}^\varepsilon) - \mathcal{T}(0)\|_{H^2_\varepsilon} \le c\|\bar{U}^\varepsilon\|_{H^2_\varepsilon},$$

with $c < 1$, we obtain as usual that $\|U^{\varepsilon,\delta}\|_{H^2_{\varepsilon,\delta}} \le C\|\mathcal{T}(0)\|_{H^2_{\varepsilon,\delta}}$, whence

$$\|U^\varepsilon\|_{H^2_{\varepsilon,\delta}} \le C\varepsilon^2. \tag{5.27}$$

by (5.25). In particular, $e^{\varepsilon\delta\langle x\rangle}U^\varepsilon = O(\varepsilon^2)$ in H^2_ε and by the Sobolev embedding

$$\|e^{\varepsilon\delta\langle x\rangle}U^\varepsilon\|_{L^\infty} = O(\varepsilon^2), \quad \|e^{\varepsilon\delta\langle x\rangle}\partial_x U^\varepsilon\|_{L^\infty} = O(\varepsilon^3). \tag{5.28}$$

For $s \ge 3$, the estimates (5.25) show that for $\varepsilon \le \varepsilon_1$ independent of s, the iterates $\mathcal{T}^n(0)$ are bounded in $H^s_{\varepsilon,\delta}$, and similarly that $\mathcal{T}^n(0) - \mathcal{T}(0) = O(\varepsilon^2)$ in $H^s_{\varepsilon,\delta}$, implying that the limit U belongs to $H^s_{\varepsilon,\delta}$ with norm $O(\varepsilon^2)$. Together with the Sobolev inequality (5.20), this implies the pointwise estimates (4.1).

Finally, the assertion about uniqueness follows by uniqueness in $\mathcal{B}_{c\varepsilon,\delta}$ for the choice $\delta = 0$ (noting by our argument that also $\mathcal{B}_{c\varepsilon,\delta}$ is mapped to itself for ε sufficienty small, for any $c > 0$), together with the observation that phase condition (5.9) may be achieved for any solution $\bar{U} = \bar{U}_{NS} + U$ with

$$\|U\|_{L^\infty} \le c\varepsilon^2 << \bar{U}'_{NS}(0) \sim \varepsilon^2$$

by translation in x, yielding $\bar{U}_a(x) := \bar{U}(x + a) = \bar{U}_{NS}(x) + U_a(x)$ with

$$U_a(x) := \bar{U}_{NS}(x + a) - \bar{U}_{NS}(x) + U(x + a)$$

so that $U_a(0) \sim (a + o(1))\bar{U}'_{NS}(0)$ and $\ell_\varepsilon \cdot u_a(0) \sim \ell_\varepsilon \cdot \bar{u}'_{NS}(0)$, which may be set to zero by appropriate choice of a, by the property $\ell_\varepsilon \cdot \bar{u}'_{NS}(0) \neq 0$ following from our choice of ℓ_ε (see Remark 5.1). □

It remains to prove existence of the linearized solution operator and the linearized bounds (5.14), which tasks will be the work of the rest of the paper. We concentrate first on estimates, and prove the existence next, using a viscosity method.

6 Internal and high frequency estimates

We begin by establishing a priori estimates on solutions of the equation (5.8) This will be done in two stages. In the first stage, carried out in this section, we establish energy estimates showing that "microscopic", or "internal", variables consisting of v and derivatives of (u,v) are controlled by and small with respect to the "macroscopic", or "fluid" variable, u. In the second stage, carried out in Section 7, we estimate the macroscopic variable u by Chapman–Enskog approximation combined with finite-dimensional ODE techniques such as have been used in the study of fluid-dynamical shocks; see, for example, [MaZ4, MaZ5, Z1, Z2, GMWZ].

6.1 The basic H^1 estimate

We consider the equation

$$\mathcal{L}_*^\varepsilon U := \begin{pmatrix} A_{11}u + A_{12}v \\ A_{21}u' + A_{22}v' - dq(\bar u_{NS}, v_*(\bar u_{NS}))U \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \tag{6.1}$$

and its differentiated form:

$$AU' - dQ(\bar u_{NS}, v_*(\bar u_{NS}))U = \begin{pmatrix} f' \\ g \end{pmatrix}. \tag{6.2}$$

The internal variables are $U' = (u', v')$ and $\tilde v$ where

$$\tilde v := v + pu, \qquad p = \partial_v q^{-1} \partial_u q(\bar u_{NS}, v_*(\bar u_{NS})) = -dv_*(\bar u_{NS}) \tag{6.3}$$

is the linearized version of $\bar v - v_*(\bar u)$.

Proposition 6.1. *Under the assumptions of Theorem 4.1, for there are constants C, $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that for $0 < \varepsilon \le \varepsilon_0$ and $0 \le \delta \le \delta_0$, $f \in H^2_{\varepsilon,\delta}$, $g \in H^1_{\varepsilon,\delta}$ and $U = (u,v) \in H^1_{\varepsilon,\delta}$ of (6.1) satisfies*

$$\|U'\|_{L^2_{\varepsilon,\delta}} + \|\tilde v\|_{L^2_{\varepsilon,\delta}} \le C\big(\|(f, f', f'', g, g')\|_{L^2_{\varepsilon,\delta}} + \varepsilon\|u\|_{L^2_{\varepsilon,\delta}}\big). \tag{6.4}$$

Making the change of variables $v \mapsto \tilde v$, or $U \mapsto \widetilde U = (u, \tilde v)$ and denoting $U = P(\bar u_{NS})\widetilde U$, we obtain an ODE

$$\widetilde A \widetilde U' - \widetilde Q\, \widetilde U = \widetilde F + \widetilde C \widetilde U, \tag{6.5}$$

where

$$\widetilde A = P^{-1}AP, \quad \widetilde Q = P^{-1}dQP = \begin{pmatrix} 0 & 0 \\ 0 & \widetilde Q_{22} \end{pmatrix}, \tag{6.6}$$

with $\widetilde Q_{22} = \partial_v q(\bar u_{NS}, v_*(\bar u_{NS}))$,

$$\widetilde F = \begin{pmatrix} f' \\ g + \widetilde R_{21} f' \end{pmatrix}, \tag{6.7}$$

with $\widetilde{R}_{21} = \partial_v q^{-1}\partial_u q(\bar{u}_{NS}, v_*(\bar{u}_{NS})$, and

$$\widetilde{C} = -P^{-1}AP' = \begin{pmatrix} \widetilde{C}_{11} & 0 \\ \widetilde{C}_{21} & 0 \end{pmatrix} = O(\bar{u}'_{NS}) = \varepsilon^2 \widehat{C}. \tag{6.8}$$

The equation (6.5) reads

$$\widetilde{A}\widetilde{U}' - \widetilde{Q}\widetilde{U} = \widehat{F} = \begin{pmatrix} f' + \varepsilon h \\ \tilde{g} \end{pmatrix} \tag{6.9}$$

with

$$h = \varepsilon \widehat{C}_{11} u, \qquad \tilde{g} = g + \widetilde{R}_{21} f' + \varepsilon^2 \widehat{C}_{21} u. \tag{6.10}$$

We first prove the estimate (6.4) for $\delta = 0$. Dropping hats and tildes, the ODE reads

$$AU' - QU = F, \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & Q_{22} \end{pmatrix}, \quad F = \begin{pmatrix} f' + \varepsilon h \\ g \end{pmatrix}. \tag{6.11}$$

The matrix $A = A(x)$ has end points values $A_\pm$ at $\pm\infty$ and satisfies estimates

$$|\partial_x^k (A - A_\pm)| \le C_k \varepsilon^k \tag{6.12}$$

with C_k independent of ε. There are similar estimates for Q_{22}. Moreover, A and Q are simultaneously symmetrizable by some $S = \widetilde{S}(\bar{u}_{NS})$, since this property is unaffected by coordinate changes. S is necessarily block-diagonal, SA and

$$SQ = \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}$$

are symmetric with q negative definite. Likewise, the genuine coupling condition still holds, which, by the results of [K], is equivalent to the *Kawashima condition*, and there is a smooth $K = \widetilde{K}(\bar{u}_{NS}) = -\widetilde{K}^*$ such that Re $(KA - SQ)$ is definite positive. Therefore, there is $c > 0$ such that for all $\varepsilon \le \varepsilon_0$ and $x \in \mathbb{R}$:

$$\tilde{q} \le -c\mathrm{Id}, \qquad \mathrm{Re}\,(KA - SQ) \ge c\mathrm{Id}. \tag{6.13}$$

Lemma 6.2. *There is a constant C such that for ε sufficiently small, $f \in H^2$, $g \in H^1$, $h \in H^1$ and $U \in H^1$ satisfying* (6.11), *one has*

$$\|U'\|_{L^2} + \|v\|_{L^2} \le C\big(\|f\|_{H^2} + \|h\|_{H^1} + \|g\|_{H^1} + \varepsilon\|u\|_{L^2}\big). \tag{6.14}$$

Proof. Introduce the symmetrizer

$$\mathcal{S} = \partial_x^2 \circ S + \partial_x \circ K - \lambda S. \tag{6.15}$$

One has

$$\begin{aligned}
&\text{Re } \partial_x^2 \circ S \circ (A\partial_x - Q) = \frac{1}{2}\partial_x \circ (SA)' \circ \partial_x - \partial_x \circ SQ \circ \partial_x - \text{Re } \partial_x \circ (SQ)' \\
&\text{Re } \partial_x \circ K(A\partial_x - Q) = \partial_x \circ \text{Re } KA \circ \partial_x - \text{Re}\, \partial_x \circ KQ \\
&\text{Re } S(A\partial_x - Q) = \frac{1}{2}(SA)' - SQ.
\end{aligned}$$

Thus

$$\begin{aligned}
\text{Re } \mathcal{S} \circ (A\partial_x - Q) =& \partial_x \circ (\text{Re } AK - SQ) \circ \partial_x + \lambda SQ \\
&+ \frac{1}{2}\partial_x \circ (SA)' \circ \partial_x - \frac{1}{2}\lambda (SA)' - \text{Re } \partial_x \circ (SQ)' - \text{Re } \partial_x \circ KQ.
\end{aligned}$$

Therefore, for $U \in H^2(\mathbb{R})$, (6.13) implies that

$$\begin{aligned}
\text{Re } (\mathcal{S}F, U)_{L^2} \geq\ & c\|\partial_x U\|_{L^2}^2 + \lambda c \|v\|_{L^2}^2 \\
& - \frac{1}{2}\|(SA)'\|_{L^\infty}\big(\|\partial_x U\|_{L^2}^2 + \lambda \|U\|_{L^2}^2\big) \\
& - \|(SQ)'\|_{L^\infty}\|U\|_{L^2}\|\partial_x U\|_{L^2} - \|K\|_{L^\infty}\|\partial_x U\|_{L^2}\|qv\|_{L^2}.
\end{aligned}$$

Taking

$$\lambda = \frac{2}{c}\|K\|_{L^\infty}^2 \|q\|_{L^\infty},$$

and using that

$$\|(SA)'\|_{L^\infty} + \|(SQ)'\|_{L^\infty} = O(\varepsilon^2) \tag{6.16}$$

yields

$$\|U'\|_{L^2}^2 + \|v\|_{L^2}^2 \lesssim \text{Re } (\mathcal{S}F, U)_{L^2} + \varepsilon^2\big(\|U\|_{L^2}^2 + \|U'\|_{L^2}^2\big).$$

In the opposite direction, using the block structure of S,

$$\begin{aligned}
\text{Re } (\mathcal{S}F, U)_{L^2} \leq & \|\partial_x U\|_{L^2}\big(\|\partial_x(SF)\|_{L^2} + \|K\|_{L^\infty}\|F\|_{L^2}\big) \\
& + \lambda\big(\varepsilon \|S_{11}\|_{L^\infty}\|u\|_{L^2}\|h\|_{L^2} + \|(S_{11}u)'\|_{L^2}\|f\|_{L^2} \\
& \qquad + \|S_{22}\|_{L^\infty}\|v\|_{L^2}\|g\|_{L^2}\big).
\end{aligned}$$

Using again that the derivatives of the coefficients are $O(\varepsilon^2)$, this implies that

$$\begin{aligned}
\text{Re } (\mathcal{S}F, U)_{L^2} \lesssim & \big(\|f\|_{H^2} + \|h\|_{H^1} + \|g\|_{H^1}\big)\|U'\|_{L^2} \\
& + \varepsilon \|h\|_{L^2}\|u\|_{L^2} + \varepsilon^2 \|f\|_{L^2}\|u\|_{L^2} + \|g\|_{L^2}\|v\|_{L^2},
\end{aligned}$$

The estimate (6.14) follows provided that ε is small enough.

This proves the lemma under the additional assumption that $U \in H^2$. When $U \in H^1$, the estimates follows using Friedrichs mollifiers. □

Proof of Proposition 6.1. Consider the system (6.9) Because the coefficients are functions of $\bar u_{NS}$ and its derivatives, there holds

$$\begin{aligned}\|h, h'\|_{L^2} &\le C\varepsilon\|u'\|_{L^2} + \varepsilon\|u\|_{L^2}\\ \|\tilde g, \tilde g'\|_{L^2} &\le C\big(\|f', f'', g, g'\|_{L^2} + \varepsilon^2\|u'\|_{L^2}\big)\end{aligned}$$

and

$$\|v'\|_{L^2} \le \big\|\tilde v'\big\|_{L^2} + C\big(\big\|u'\big\|_{L^2} + \varepsilon^2\|u\|_{L^2}\big).$$

Therefore, the bounds (6.14) for $(\widetilde U', \tilde v)$ imply that

$$\big\|U'\big\|_{L^2} + \big\|\tilde v\big\|_{L^2} \le C\big(\big\|(f, f', f'', h, h', \hat g, \hat g')\|_{L^2} + \varepsilon\big\|u\big\|_{L^2}\big). \tag{6.17}$$

Multiplying by $\varepsilon^{\frac12}$ and using the estimates of h and g above, yields (6.4) for $\delta = 0$.

For $\delta > 0$ small, consider $U^w = e^{\varepsilon\delta\langle x\rangle}U$. Then, U^w satisfies

$$\mathcal{L}_*^\varepsilon U^w = \begin{pmatrix} f^w \\ g^w\end{pmatrix}, \tag{6.18}$$

with $f^w = e^{\varepsilon\delta\langle x\rangle} f$ and $g^w = e^{\varepsilon\delta\langle x\rangle} g + \varepsilon\delta\langle x\rangle'(A_{21}u^w + A_{22}v^w)$. We note that,

$$\|U'\|_{L^2_{\varepsilon,\delta}} \le \|(U^w)'\|_{L^2_\varepsilon} + \varepsilon\|U^w\|_{L^2_\varepsilon},, \quad \|\tilde v\|_{L^2_{\varepsilon,\delta}} \lesssim \|\tilde v^w\|_{L^2_\varepsilon},$$

$$\|f^w, (f^w)', (f^w)''\|_{L^2_\varepsilon} \lesssim \|(f, f', f'')\|_{L^2_{\varepsilon,\delta}},$$

$$\|g^w, (g^w)'\|_{L^2_\varepsilon} \lesssim \|(g, g')\|_{L^2_{\varepsilon,\delta}} + \varepsilon\delta\|(U, U')\|_{L^2_{\varepsilon,\delta}}.$$

We use the estimate (6.4) with $\delta = 0$ for U^w, and the Proposition follows provided that δ is small enough. □

6.2 Higher order estimates

Proposition 6.3. *There are constants C, $\varepsilon_0 > 0$, $\delta_0 > 0$ and for all $k \ge 2$, there is C_k, such that $0 < \varepsilon \le \varepsilon_0$, $\delta \le \delta_0$, $U \in H^s_{\varepsilon,\delta}$, $f \in H^{s+1}_{\varepsilon,\delta}$ and $g \in H^s_{\varepsilon,\delta}$ satisfying* (6.11) *satisfies:*

$$\begin{aligned}\|\partial_x^k U'\|_{L^2_{\varepsilon,\delta}} + \|\partial_x^k \tilde v\|_{L^2_{\varepsilon,\delta}} \le{}& C\|\partial_x^k(f, f', f'', g, g')\|_{L^2_{\varepsilon,\delta}}\\ &+ \varepsilon^k C_k\big(\|U'\|_{H^{k-1}_{\varepsilon,\delta}} + \varepsilon\|\tilde v\|_{H^{k-1}_{\varepsilon,\delta}} + \varepsilon\|u\|_{L^2_{\varepsilon,\delta}}\big)\end{aligned} \tag{6.19}$$

Proof. Differentiating (6.1) k times, yields

$$A\partial_x U^k - dQ(\bar u_{NS}, v_*(\bar u_{NS}))\partial_x U^k = \begin{pmatrix}\partial_x^k f' \\ \partial_x^k g + r_k\end{pmatrix}, \tag{6.20}$$

where

$$r_k = -\sum_{l=0}^{k-1} \partial_x^{k-l} Q_{22}\, \partial_x^l \tilde v.$$

Here we have used that $dq(\bar u_{NS}, v_*(\bar u_{NS})U = Q_{22}\tilde v$. The H^1 estimate yields

$$\begin{aligned}\|\partial_x^k U'\|_{L^2_{\varepsilon,\delta}} + \|\partial_x^k v + p\partial_x^k u\|_{L^2_{\varepsilon,\delta}} \le C\big(\|\partial_x^k(f,f',f'',g,g')\|_{L^2_{\varepsilon,\delta}} \\ + \varepsilon\|\partial_x^k u\|_{L^2_{\varepsilon,\delta}} + \|\partial_x r_k\|_{L^2_{\varepsilon,\delta}} + \|r_k\|_{L^2_{\varepsilon,\delta}}\big),\end{aligned}$$

for $0 \le k \le s$, with $r_0 = 0$ when $k = 0$. Since Q is a function of $\bar u_{NS}$, its $k-l$-th derivative is $O(\varepsilon^{k-l+1})$ when $k - l > 0$. Therefore:

$$\|\partial_x r_k\|_{L^2_{\varepsilon,\delta}} + \|r_k\|_{L^2_{\varepsilon,\delta}} \le C_k \varepsilon^k \big(\|\tilde v'\|_{H^{k-1}_{\varepsilon,\delta}} + \varepsilon\|\tilde v\|_{L^2_{\varepsilon,\delta}}\big).$$

Similarly, for $k = 1$

$$\|\partial_x \tilde v_k\|_{L^2_{\varepsilon,\delta}} \le \|\partial_x v + p\partial_x u\|_{L^2_{\varepsilon,\delta}} + C\varepsilon^2\|u\|_{L^2_{\varepsilon,\delta}}$$

and for $k \ge 2$:

$$\|\partial_x^k \tilde v_k\|_{L^2_{\varepsilon,\delta}} \le \|\partial_x^k v + p\partial_x^k u\|_{L^2_{\varepsilon,\delta}} + C_k(\varepsilon^k\|u'\|_{H^{k-2}_{\varepsilon,\delta}} + \varepsilon^{k+1}\|\tilde u\|_{L^2_{\varepsilon,\delta}}).$$

□

7 Linearized Chapman–Enskog estimate

7.1 The approximate equations

It remains only to estimate $\|u\|_{L^2_{\varepsilon,\delta}}$ in order to close the estimates and establish (6.4). To this end, we work with the first equation in (6.1) and estimate it by comparison with the Chapman-Enskog approximation (see the computations Section 3).

From the second equation

$$A_{21}u' + A_{22}v' - g = \partial_u q u + \partial_v q v = \partial_v q \tilde v,$$

where we use the notations $\tilde v$ of Proposition 6.1, we find

$$\tilde v = \partial_v q^{-1}\Big((A_{21} + A_{22}\partial_v dv_*(\bar u_{NS}))u' + A_{22}\tilde v' - g\Big). \tag{7.1}$$

Introducing $\tilde v$ in the first equation, yields

$$(A_{11} + A_{12}dv_*(\bar u_{NS}))u + A_{12}\tilde v = f,$$

thus

$$(A_{11} + A_{12}dv_*(\bar u_{NS}))u' = f' - A_{12}\tilde v' - d^2v_*(\bar u_{NS})(\bar u'_{NS}, u).$$

Therefore, (7.1) can be modified to

$$\tilde v = c_*(\bar u_{NS})u' + r \tag{7.2}$$

with

$$r = d_v^{-1} q(\bar u_{NS}, v_*(\bar u_{NS}))\Big(A_{22}(\tilde v)' - g + dv_*(\bar u_{NS})\big(f' - A_{12}\tilde v' - d^2 v_*(\bar u_{NS})(\bar u'_{NS}, u)\big)\Big).$$

This implies that u satisfies the linearized profile equation

$$\bar b_* u' - \bar{df}_* u = A_{12} r - f \tag{7.3}$$

where $\bar b_* = b_*(\bar u_{NS})$ and $\bar{df}_* := df_*(\bar u_{NS}) = A_{11} + A_{12} dv_*(\bar u_{NS})$.

7.2 L^2 estimates and proof of the main estimates

Proposition 7.1. *The operator $\bar b_* \partial_x - \bar{df}_*$ has a right inverse $(b_*\partial_x - df^*)^\dagger$ satisfying*

$$\|(\bar b_* \partial_x - \bar{df}_*)^\dagger h\|_{L^2_{\varepsilon,\delta}} \le C\varepsilon^{-1} \|h\|_{L^2_{\varepsilon,\delta}}, \tag{7.4}$$

uniquely specified by the property that the solution $u = (b_*\partial_x - df^*)^\dagger h$ satisfies

$$\ell_\varepsilon \cdot u(0) = 0. \tag{7.5}$$

for certain unit vector ℓ_ε.

Taking this proposition for granted, we finish the proof of the main estimates in Proposition 5.2.

Proposition 7.2. *There are constants C, $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that for $\varepsilon \in]0, \varepsilon_0]$, $\delta \in [0, \delta_0]$, $f \in H^3_{\varepsilon,\delta}$, $g \in H^2_{\varepsilon,\delta}$ and $U \in H^2_{\varepsilon,\delta}$ satisfying* (5.8) *and* (5.9)

$$\|U\|_{H^2_{\varepsilon,\delta}} \le C\varepsilon^{-1}\big(\|f\|_{H^3_{\varepsilon,\delta}} + \|g\|_{H^2_{\varepsilon,\delta}}\big). \tag{7.6}$$

Proof. Going back now to (7.3), u satisfies

$$\bar b_* u' - \bar{df}_* u = O(|\tilde v'| + |g| + |f'| + \varepsilon^2 |u|) - f,$$

If in addition u satisfies the condition (7.5) then

$$\|u\|_{L^2_{\varepsilon,\delta}} \le C\varepsilon^{-1}(\|\tilde v'\|_{L^2_{\varepsilon,\delta}} + \|(f, f', g)\|_{L^2_{\varepsilon,\delta}} + \varepsilon^2 \|u\|_{L^2_{\varepsilon,\delta}}). \tag{7.7}$$

By Proposition 6.1 and Proposition 6.3 for $k = 1$, we have

$$\|U'\|_{L^2_{\varepsilon,\delta}} + \|\tilde v\|_{L^2_{\varepsilon,\delta}} \le C\big(\|(f, f', f'', g, g')\|_{L^2_{\varepsilon,\delta}} + \varepsilon \|u\|_{L^2_{\varepsilon,\delta}}\big). \tag{7.8}$$

$$\|U''\|_{L^2_{\varepsilon,\delta}} + \|\tilde v'\|_{L^2_{\varepsilon,\delta}} \le C\big(\|(f', f'', f''', g', g'')\|_{L^2_{\varepsilon,\delta}} + \varepsilon \|U'\|_{L^2_{\varepsilon,\delta}} + \varepsilon^2 \|u\|_{L^2_{\varepsilon,\delta}}\big). \tag{7.9}$$

Combining these estimates, this implies

$$\begin{aligned}\|\tilde v'\|_{L^2_{\varepsilon,\delta}} &\leq C\big(\|(f',f'',f''',g',g'')\|_{L^2_{\varepsilon,\delta}} + \varepsilon\|(f,f',f'',g,g')\|_{L^2_{\varepsilon,\delta}} + \varepsilon^2\|u\|_{L^2_{\varepsilon,\delta}}\big)\\ &\leq C\big(\varepsilon\|(f,f',f'',g,g')\|_{H^1_{\varepsilon,\delta}} + \varepsilon^2\|u\|_{L^2_{\varepsilon,\delta}}\big).\end{aligned}$$

Substituting in (7.7), yields

$$\varepsilon\|u\|_{L^2_{\varepsilon,\delta}} \leq C\big(\|(f,f',g)\|_{L^2_{\varepsilon,\delta}} + \varepsilon\|(f,f',f'',g,g')\|_{H^1_{\varepsilon,\delta}} + \varepsilon^2\|u\|_{L^2_{\varepsilon,\delta}}\big).$$

Hence for ε small,

$$\varepsilon\|u\|_{L^2_{\varepsilon,\delta}} \leq C\big(\|(f,f',g)\|_{L^2_{\varepsilon,\delta}} + \varepsilon\|(f,f',f'',g,g')\|_{H^1_{\varepsilon,\delta}}\big). \tag{7.10}$$

Plugging this estimate in (7.8)

$$\|U'\|_{L^2_{\varepsilon,\delta}} + \|\tilde v\|_{L^2_{\varepsilon,\delta}} + \varepsilon\|u\|_{L^2_{\varepsilon,\delta}} \leq C\|(f,f',f'',g,g')\|_{H^1_{\varepsilon,\delta}} + \big). \tag{7.11}$$

Hence, with (7.9), one has

$$\begin{aligned}\|U''\|_{L^2_{\varepsilon,\delta}} + \|\tilde v'\|_{L^2_{\varepsilon,\delta}} \leq\\ C\big(\|(f',f'',f''',g',g'')\|_{L^2_{\varepsilon,\delta}} + \varepsilon\|(f,f',f'',g,g')\|_{H^1_{\varepsilon,\delta}}\big).\end{aligned} \tag{7.12}$$

Therefore,

$$\|U'\|_{H^1_{\varepsilon,\delta}} + \|\tilde v\|_{L^2_{\varepsilon,\delta}} + \varepsilon\|u\|_{L^2_{\varepsilon,\delta}} \leq C\|f,f',f'',g,g'\|_{H^1_{\varepsilon,\delta}} \tag{7.13}$$

The left hand side dominates

$$\|U'\|_{H^1_{\varepsilon,\delta}} + \varepsilon\|U'\|_{L^2_{\varepsilon,\delta}} = \varepsilon\|U'\|_{H^2_{\varepsilon,\delta}}$$

and the right hand side is smaller than or equal to $\|f\|_{H^2_{\varepsilon,\delta}} + \|g\|_{H^1_{\varepsilon,\delta}}$. The estimate (7.6) follows. □

Knowing a bound for $\|u\|_{L^2_{\varepsilon,\delta}}$, Proposition 6.3 immediately implies

Proposition 7.3. *There are constants C, $\varepsilon_0 > 0$ and $\delta_0 > 0$ and for $s \geq 3$ there is a constant C_s such that for $\varepsilon \in]0,\varepsilon_0]$, $\delta \in [0,\delta_0]$, $f \in H^{s+1}_{\varepsilon,\delta}$, $g \in H^s_{\varepsilon,\delta}$ and $U \in H^s_{\varepsilon,\delta}$ satisfying* (5.8) *and* (5.9)*, one has*

$$\|U\|_{H^s_{\varepsilon,\delta}} \leq C\varepsilon^{-1}\big(\|f\|_{H^{s+1}_{\varepsilon,\delta}} + \|g\|_{H^s_{\varepsilon,\delta}}\big) + C_s\|U\|_{H^{s-1}_{\varepsilon,\delta}}. \tag{7.14}$$

7.3 Proof of Proposition 7.1

By Assumption 2.4(i), we may assume that there are linear coordinates $u = (u_1, u_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and $h = (h_1, h_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, with $n_2 = \text{rank } b_*(\bar{u})$ such that

$$b_*(\bar{u}) = \begin{pmatrix} 0 & 0 \\ b_{21}(\bar{u}) & b_{22}(\bar{u}) \end{pmatrix} \tag{7.15}$$

and $b_{22}(\bar{u})$ is uniformly invertible on $\mathcal{U}_*$. Introducing the new variable

$$\tilde{u}_2 = u_2 + \bar{V} u_1, \quad \bar{V} = (b^{22})^{-1} b_{21}(\bar{u}_{NS}), \tag{7.16}$$

the equation $\bar{b}_* u' - \bar{df}_* u = h$ has the form:

$$\begin{aligned} \bar{a}^{11} u_1 + \bar{a}^{12} \tilde{u}_2 &= h_1, \\ \bar{b}^{22} \tilde{u}_2' - \bar{a}^{21} u_1 - \bar{a}^{22} \tilde{u}_2 &= h_2 \end{aligned} \tag{7.17}$$

where

$$\bar{a} := \bar{df}_* \begin{pmatrix} \text{Id} & 0 \\ -\bar{V} & \text{Id} \end{pmatrix} + \bar{b} * \begin{pmatrix} 0 & 0 \\ \bar{V}' & 0 \end{pmatrix}.$$

Assumption 2.4(ii) implies that the left upper corner block $\bar{a}^{11}$ is uniformly invertible. Solving the first equation for u_1, we obtain the reduced nondegenerate ordinary differential equation

$$\bar{b}_*^{22} \tilde{u}_2' + \bar{a}^{21} (\bar{a}^{11})^{-1} \bar{a}^{12} \tilde{u}_2 - \bar{a}^{22} \tilde{u}_2 = h_2 + \bar{a}^{21} (\bar{a}^{11})^{-1} h_1$$

or

$$\check{b} u_2' - \check{a} u_2 = \check{h} = O(|h_1| + |h_2|). \tag{7.18}$$

Note that $\det \bar{d}f_* = \det \bar{a}^{11} \det \check{a}$ by standard block determinant identities, so that $\det \check{a} \sim \det \bar{d}f_*$ by Assumption 2.4(ii). Moreover, as established in [MaZ4], by Assumption 2.6 and the construction of the profile $\bar{u}_{NS}$ we find that $m := (\check{b})^{-1} \check{a}$ has the following properties:

i) with $m_\pm$ denoting the end points values of m, there is $\theta > 0$ such that for all k :

$$|\partial_x^k (m(x) - m_\pm)| \lesssim \varepsilon^{k+1} e^{-\varepsilon \theta |x|}; \tag{7.19}$$

ii) $m(x)$ has a single simple eigenvalue of order ε, dented by $\varepsilon \mu(x)$, and there is $c > 0$ such that for all x and ε the other eigenvalues λ satisfy $|\text{Re } \lambda| \geq c$;

iii) the end point values $\mu_\pm$ of μ satisfy

$$\mu_- \geq \alpha \qquad \mu_+ \leq -\alpha \tag{7.20}$$

for some $\alpha > 0$ independent of ε.

In the strictly parabolic case $\det b_* \neq 0$, this follows by a lemma of Majda and Pego [MP].

At this point, we have reduced to the case

$$u_2' - m(x)u_2 = O(|h_1| + |h_2|), \tag{7.21}$$

with m having the properties listed above. The important feature is that $m' = O(\varepsilon^2) << \varepsilon$, the spectral gap between stable, unstable, and ε-order subspaces of m. The conditions above imply that there is a matrix ω such that

$$p := \omega^{-1} m \omega = \text{blockdiag}\{p^+, \varepsilon\mu, p-\},$$

where the spectrum of $p_\pm$ lies in $\pm\text{Re}\ \lambda \geq c$. Moreover, ω and p satisfies estimates similar to (7.19). The change of variables $u_2 = \omega z$ reduces (7.21) to

$$z' - pz = \omega^{-1}\omega' z + O(|h_1| + |h_2|). \tag{7.22}$$

The equations $(z^+)' - p^+ z^+ = h^+$ and $(z^-)' - p^- z^- = h^-$ either by standard linear theory [He] or by symmetrizer estimates as in [GMWZ], admit unique solutions in weighted L^2 spaces, satisfying

$$\|e^{\delta|x|} z^\pm\|_{L^2} \leq C\|e^{\delta|x|} h^\pm\|_{L^2},$$

provided that δ remains small, typically $\delta < |\text{Re}\ p^\pm|$.

The equation $z_0' - \varepsilon\mu z_0 = h_0$ may be converted by the change of coordinates $x \to \tilde{x} := \varepsilon x$ to

$$\partial_{\tilde{x}} \tilde{z}_0 - \tilde{\mu}(\tilde{x}) z_0 = \tilde{h}_0(\tilde{x}) = \varepsilon^{-1} h_0(\tilde{x}/\varepsilon), \tag{7.23}$$

where $\tilde{z}_0(\tilde{x}) = z_0(\tilde{x}/\varepsilon)$ and $\tilde{\mu}(\tilde{x}) := \mu(\tilde{x}/\varepsilon)$. By (7.19)

$$|\tilde{\mu}(\tilde{x}) - \mu_\pm| \leq Ce^{-\theta|\tilde{x}|}$$

with $\mu_\pm$ satisfying (7.20). This equation is underdetermined with index one, reflecting the translation-invariance of the underlying equations. However, the operator $\partial_{\tilde{x}} - \tilde{\mu}$ has a bounded L^2 right inverse $(\partial_{\tilde{x}} - \tilde{\mu})^{-1}$, as may be seen by adjoining an additional artificial constraint

$$\tilde{z}_0(0) = 0 \tag{7.24}$$

fixing the phase. This can be seen by solving explicitly the equation or applying the gap lemma of [MeZ3] to reduce the problem to two constant-coefficient equations on $\tilde{x} \gtrless 0$, with boundary conditions at $z = 0$. We obtain as a result that

$$\|e^{\delta|\tilde{x}|}\ \tilde{z}_0\|_{L^2} \leq C\|e^{\delta|\tilde{x}|} \tilde{h}_0\|_{L^2}$$

if $\delta < \min\{\alpha, \theta\}$, which yields by rescaling the estimate

$$\|e^{\varepsilon\delta|x|}\ z_0\|_{L^2} \leq C\varepsilon^{-1}\|e^{\varepsilon\delta|x|} h_0\|_{L^2}$$

Together with the (better) previous estimates, this gives existence and uniqueness for the equation

$$z' - pz = h, \qquad z_0(0) = 0$$

with the estimate $\|e^{\varepsilon\delta|x|}z\|_{L^2} \le C\varepsilon^{-1}\|e^{\varepsilon\delta|x|}h\|_{L^2}$. Because $\omega^{-1}\omega' = O(\varepsilon^2)$, this implies that for ε small enough, the equation (7.22) with $z_0(0) = 0$ has a unique solution. Tracing back to the original variables u, the condition $z_0(0) = 0$ translates into a condition of the form $\ell_\varepsilon \cdot u(0) = 0$. Therefore, the equation $\bar b_* u' - \bar d f_* u = h$ has a unique solution such u that $\ell_\varepsilon \cdot u(0) = 0$, which satisfies

$$\|e^{\varepsilon\delta|x|}u\|_{L^2} \le C\varepsilon^{-1}\|e^{\varepsilon\delta|x|}h\|_{L^2}$$

for δ and ε small enough, finishing the proof of Proposition 7.1.

Remark 7.4. The estimate of Proposition 7.1 may be recognized as somewhat similar to the estimates of Goodman [Go] in the time-evolutionary case. More precisely, the argument is a simplified version of the one used by Plaza and Zumbrun [PZ] to show time-evolutionary stability of general small-amplitude waves.

Remark 7.5. The argument of Proposition 7.1 indicates that the estimate may be improved by factor ε in transverse modes $z_\pm$. However, we see no way to use this to improve the overall estimates on our iteration scheme.

8 Existence for the linearized problem

The desired estimates (5.13) and (5.14) are given by Propositions 7.2 and 7.3. It remains to prove existence for the linearized problem with phase condition $u(0) \cdot r(\varepsilon) = 0$. This we carry out using a vanishing viscosity argument.

Fixing ε, consider in place of $\mathcal{L}^\varepsilon_* U = F$ the family of modified equations

$$\mathcal{L}^{\varepsilon,\eta}_* U := \mathcal{L}^\varepsilon_* U - \eta \begin{pmatrix} u' \\ v'' \end{pmatrix} = F := \begin{pmatrix} f \\ g \end{pmatrix}, \quad \ell_\varepsilon \cdot u(0) = 0. \tag{8.1}$$

Differentiating the first equation yields

$$AU' - dQ(x)U - U'' = \begin{pmatrix} f' \\ g \end{pmatrix}, \quad \ell_\varepsilon \cdot u(0) = 0. \tag{8.2}$$

where $dQ(x)$ denotes here the matrix $dQ(\bar u_{NS}, v_*(\bar u_{NS}))$.

8.1 Uniform estimates

We first prove uniform a-priori estimates. We denote by $\mathscr{S}$ the Schwartz space and for $\delta \ge 0$, by $\mathscr{S}_{\varepsilon\delta}$ the space of functions u such that $e^{\varepsilon\delta\langle x\rangle}u \in \mathscr{S}$, with $\langle x\rangle = \sqrt{1+x^2}$ as in (5.11).

Proposition 8.1. *There are constants* $\varepsilon_0 > 0$, $\delta_0 > 0$ *and* $\eta_0 > 0$, *and for all* $s \geq 2$ *a constant* C_s, *such that for* $\varepsilon \in]0, \varepsilon_0]$, $\delta \in [0, \delta_0]$, $\eta \in]0, \eta_0]$, *and* U *and* F *in* $\mathscr{S}_{\varepsilon\delta}(\mathbb{R})$, *satisfying* (8.1)

$$\|U\|_{H^s_{\varepsilon,\delta}} \leq C_s \varepsilon^{-1} \big(\|f\|_{H^{s+1}_{\varepsilon,\delta}} + \|g\|_{H^s_{\varepsilon,\delta}} \big). \tag{8.3}$$

Proof. The argument of Proposition 6.1 goes through essentially unchanged, with new η terms providing additional favorable higher-derivative terms sufficient to absorb new higher-derivative errors coming from the Kawashima part. More precisely, consider again the change of variables $v \mapsto \tilde v = v + pu$, $p = \partial_v q^{-1} \partial_u q(\bar u_{NS}, v_*(\bar u_{NS}))$. Denoting $\widetilde U = (u, \tilde v)$ and $U = P(\bar u_{NS}) \widetilde U$, (8.1) is transformed to

$$\widetilde A \widetilde U' - \widetilde Q \widetilde U - \eta \widetilde U'' = \begin{pmatrix} f' + \varepsilon h \\ \tilde g \end{pmatrix} \tag{8.4}$$

with $\widetilde A$, $\widetilde Q$ as in (6.6), h given by (6.10) and $\tilde g$ now defined by

$$\tilde g = g + \widetilde R_{21} f' + \varepsilon^2 \widehat C_{21} u + \eta(2p'u' + p''u).$$

Thus we are led to equations of the form (6.11) with the additional term $-\eta U''$ in the left hand side. Using the symmetrizer $\mathcal{S}$ (6.15), one gains $\eta \|U''\|^2_{L^2} + \lambda \|U'\|^2_{L^2}$ in the minoration of Re $(\mathcal{S}F, U)$ and looses commutator terms which are dominated by

$$\eta \|S''\|_{L^\infty} (\|U'\|^2_{L^2} + \|U\|_{L^2} \|U'\|_{L^2}) + \eta \|K\|_{L^\infty} (\|U'\|_{L^2} + \|U\|_{L^2}) \|U''\|_{L^2},$$

which can be absorbed by the left hand side yielding uniform estimates

$$\sqrt{\eta} \|\widetilde U''\|_{L^2} + \|\widetilde U'\|_{L^2} + \|\tilde v\|_{L^2} \leq C \big(\|f\|_{H^2} + \|h\|_{H^1} + \|\tilde g\|_{H^1} + \varepsilon \|u\|_{L^2} \big). \tag{8.5}$$

Going back to (8.2), this implies uniform estimates of the form

$$\sqrt{\eta} \|U''|_{L^2_{\varepsilon,\delta}} + \big\|U'\big\|_{L^2_{\varepsilon,\delta}} + \big\|\tilde v\big\|_{L^2_{\varepsilon,\delta}} \leq C \big(\big\|(f, f', f'', g, g')\|_{L^2_{\varepsilon,\delta}} + \varepsilon \big\|u\big\|_{L^2_{\varepsilon,\delta}} \big). \tag{8.6}$$

for $\delta = 0$, and next for $\delta \in [0, \delta_0]$ with $\delta_0 > 0$ small, as in the proof of Proposition 6.1.

When commuting derivatives to the equation, the additional term $\eta \partial_x^2$ brings no new term and the proof of Proposition 6.3 can be repeated without changes, yielding estimates of the form

$$\begin{aligned} \sqrt{\eta} \|D^k_x U''\|_{L^2_{\varepsilon,\delta}} &+ \|\partial^k_x U'\|_{L^2_{\varepsilon,\delta}} + \|\partial^k_x \tilde v\|_{L^2_{\varepsilon,\delta}} \\ &\leq C \|\partial^k_x (f, f', f'', g, g')\|_{L^2_{\varepsilon,\delta}} \\ &\quad + \varepsilon^k C_k \big(\|U'\|_{H^{k-1}_{\varepsilon,\delta}} + \varepsilon \|\tilde v\|_{H^{k-1}_{\varepsilon,\delta}} + \varepsilon \|u\|_{L^2_{\varepsilon,\delta}} \big). \end{aligned} \tag{8.7}$$

Next, applying the Chapman–Enskog argument of Section 7 to the viscous system, we obtain in place of (7.3) the equation

$$\bar b_* u' - \bar d f_* u = f + O(|\tilde v'| + |g| + |f'|) + \varepsilon^2 O(|u|) + \eta O(|u'| + |U''|), \tag{8.8}$$

where the final η term coming from artificial viscosity is treated as a source. One applies Proposition 7.1 to estimate $\varepsilon\|u\|_{L^2_{\varepsilon,\delta}}$ by the $L^2_{\varepsilon,\delta}$-norm of the right hand side, and continuing as in the proof of Proposition 7.2, the estimate (7.13) is now replaced by

$$\begin{aligned}\sqrt{\eta}\|U'''\|_{L^2_{\varepsilon,\delta}} + \|U'\|_{H^1_{\varepsilon,\delta}} + \|\tilde{v}\|_{L^2_{\varepsilon,\delta}} + \varepsilon\|u\|_{L^2_{\varepsilon,\delta}} \\ \leq C\big(\|f, f', f'', g, g'\|_{H^1_{\varepsilon,\delta}} + \eta(\|U'\|_{L^2_{\varepsilon,\delta}} + \|U''\|_{L^2_{\varepsilon,\delta}})\big).\end{aligned} \tag{8.9}$$

Therefore, for η small, the new $O(\eta)$ terms can be absorbed, and (8.3) for $s = 2$ follows as before. The higher order estimates follow from (8.7). □

8.2 Existence

We now prove existence and uniqueness for (8.1). First, recast the the problem as a first-order system

$$\mathcal{U}' - \mathbb{A}\mathcal{U} = \mathcal{F} \tag{8.10}$$

with

$$\mathcal{U} = \begin{pmatrix} u \\ v \\ v' \end{pmatrix}', \qquad \mathcal{F} = \begin{pmatrix} f \\ 0 \\ g \end{pmatrix},$$

and

$$\mathbb{A} := \eta^{-1}\begin{pmatrix} A_{11} & A_{12} & 0 \\ 0 & 0 & \eta I \\ \eta^{-1}A_{21}A_{11} - Q_{21} & \eta^{-1}A_{21}A_{12} - Q_{22} & A_{22} \end{pmatrix}. \tag{8.11}$$

Next, consider this as a transmission problem or a doubled boundary value problem on $x \gtrless 0$, with boundary condtitions given by the $n + 2r$ matching conditions $\mathcal{U}(0^-) = \mathcal{U}(0^+)$ at $x = 0$ together with the phase condition $\ell_\varepsilon \cdot u(0) = 0$, that is $n + 2r + 1$ conditions in all:

$$\mathcal{U}(0^-) = \mathcal{U}(0^+), \qquad \ell_\varepsilon \cdot u(0) = 0. \tag{8.12}$$

Note that the coefficient matrix $\mathbb{A}$ converges exponentially to its endstates at $\pm\infty$.

Lemma 8.2. *There is $\theta_1 > 0$ such that for ε small enough , the matrices $\mathbb{A}_\pm$ have no eigenvalue in the strip $|\mathrm{Re}\, z| \leq \varepsilon\delta_0$.*

Proof. The proof is parallel to the proof of the estimates. Dropping the $\pm$, suppose that $i\tau$ is an eigenvalue of $\mathbb{A}$, or equivalently that there is a constant vector $U \neq 0$ such that $e^{i\tau x}U$ is a solution of of equations (8.1) Thus

$$\begin{aligned} &A_{11}u + A_{12}v = i\tau\eta u, \\ &(i\tau A - Q + \tau^2\eta)U = 0. \end{aligned} \tag{8.13}$$

Introduce once again the variable $\tilde{v} = v + Q_{22}^{-1} Q_{21} u$, so that the equation is transformed to

$$
\begin{aligned}
& A_{11}^* u + A_{12} \tilde{v} = i\tau\eta u, \\
& (i\tau \tilde{A} - \tilde{Q}^{\pm} + \tau^2 \eta) U = 0.
\end{aligned} \tag{8.14}
$$

where $\widetilde{A}$ and $\widetilde{Q}$ now denote the end point values of the matrices defined at (6.6). Denoting by $\widetilde{S}$ and $\widetilde{K}$ the end point values of the symmetrizer and Kawashima's multipliers associated to $\widetilde{A}$ and $\widetilde{Q}$, consider the multiplier

$$
\Sigma = |\tau|^2 S - i\overline{\tau} K - \lambda S.
$$

Multiplying the second equation in (8.14) by Σ and taking the real part of the scalar product with U yields

$$
\begin{aligned}
|\tau|^2 \mathrm{Re}\, (\widetilde{K}\widetilde{A} - \widetilde{S}\widetilde{Q}U, U) + \lambda(\widetilde{S}\widetilde{Q}U, U) + \eta|\tau|^4 (\widetilde{S}U, U) \\
\leq C\big(|\mathrm{Im}\, \tau|(|\tau|^2 + \lambda)\big)|U|^2 + C|\tau||\widetilde{Q}U||U| \\
+ \eta\big(|\tau|^2 |\mathrm{Im}\, \tau|^2 + |\tau|^3 + \lambda|\tau|^2)\big)|U|^2.
\end{aligned}
$$

Therefore, choosing appropriately λ, for η and $|\mathrm{Im}\, \tau|$ sufficiently small, one has

$$
(\eta|\tau|^4 + |\tau|^2)|U|^2 + |\tilde{v}|^2 \leq C|\mathrm{Im}\, \tau||u|^2 \tag{8.15}
$$

In particular, $|\tau|$ must be small if $\mathrm{Im}\, \tau$ is small.

From the equation $i\tau\widetilde{A}_{21} u + \widetilde{A}_{22}\tilde{v} - Q_{22}\tilde{v} + \eta\tau^2 v = 0$ one deduces that

$$
\tilde{v} - i\tau(\widetilde{Q}_{22})^{-1}\widetilde{A}_{21} u = O(|\tau| + \eta|\tau|^2)|\tilde{v}|.
$$

Substituting in the first equation of (8.14), we obtain the Chapman-Enskog approximation

$$
(A_{11}^* - i\tau\bar{b}_*)u = O(\eta|\tau| + |\tau| + \eta|\tau|^2)|\mathrm{Im}\, \tau|^{\frac{1}{2}}))|u|
$$

where $\bar{b}_*$ denotes the end point value of the function (2.9). Therefore,

$$
|(\bar{b}_*)^{-1} A_{11}^* u - i\tau u| \leq C|\mathrm{Im}\, \tau|^{\frac{1}{2}}|\tau||u| \tag{8.16}
$$

with arbitrarily small $c > 0$. We know from Assumption 2.6 that for ε small, $(\bar{b}_*)^{-1} A_{11}^*$ has a unique small eigenvalue, of order $O(\varepsilon)$, real. Let us denote it by $\varepsilon\mu$. Then we know that $|\mu|$ is bounded from below, see (7.20). Then (8.16) implies that there is a constant C such that for $|\mathrm{Im}\, \tau|$ small enough, and thus $|\tau|$ small, $|i\tau - \varepsilon\mu| \leq C|\mathrm{Im}\, \tau|^{\frac{1}{2}}|\tau|$. Therefore, $l\mathrm{Im}\, \tau + \varepsilon\mu| \leq \frac{1}{2}\varepsilon|\mu|$ if ε is small enough.

Summing up, we have proved that if ε is small enough, $\mathbb{A}$ has at most one eigenvalue z in the strip $|\mathrm{Re}\, z \leq \varepsilon 2|\mu|$, such that $|z - \varepsilon\mu| \leq \frac{1}{2}\varepsilon|\mu|$. This implies the lemma. □

Remark 8.3. The same reasoning can be applied to prove that $\mathbb{A}$ actually has a simple eigenvalue such that $|z - \varepsilon\mu| \leq \frac{1}{2}\varepsilon|\mu|$.

Proposition 8.4. *There are constants $\varepsilon_0 > 0$, $\delta_0 > 0$ and $\eta_0 > 0$ such that for $\varepsilon \in]0, \varepsilon_0]$, $\delta \in [0, \delta_0]$, $\eta \in]0, \eta_0]$, and F in $\mathscr{S}_{\varepsilon\delta}(\mathbb{R})$, (8.1) admits a unique solution $U \in \mathscr{S}_{\varepsilon\delta}(\mathbb{R})$.*

Proof. Noting that the coefficient matrix $\mathbb{A}$ converges exponentially to $\mathbb{A}_\pm$ at $\pm\infty$, we may apply the conjugation lemma of [MeZ1] to convert the equation (8.10) by an asymptotically trivial change of coordinates $\mathcal{U} = T(x)Z$ to a constant-coefficient problems

$$Z'_- - \mathbb{A}_- Z_- = F_-, \quad Z'_+ - \mathbb{A}_+ Z_+ = F_+, \tag{8.17}$$

on $\{\pm x \geq 0\}$, with $n + 2r + 1$ modified boundary conditions determined by the value of the transformation T at $x = 0$, where $\mathbb{A}_\pm := \mathbb{A}(\pm\infty)$, and $Z_\pm(x) := Z(x)$ for $\pm x > 0$.

By standard boundary-value theory (see, e.g., [He]), to prove existence and uniqueness in the Schwartz space for the problem (8.10) on $\{x < 0\}$ and $\{x > 0\}$ with transmission conditions (8.12), it is sufficient to show that

(i) the limiting coefficient matrices $\mathbb{A}_\pm$ are hyperbolic, i.e., have no pure imaginary eigenvalues,

(ii) the number of boundary conditions is equal to the number of stable (i.e., negative real part) eigenvalues of $\mathbb{A}_+$ plus the number of unstable eigenvalues (i.e., positive real part) of $\mathbb{A}_-$, and

(iii) there exists no nontrivial solution of the homogeneous equation $f = 0$, $g = 0$.

Moreover, since the eigenvalues of $\mathbb{A}_\pm$ are located in $\{|\mathrm{Re}\, z| \geq \theta_1\varepsilon$, the conjugated form (8.17) of the equation show that if the source term f has an exponential decay $e^{-\varepsilon\delta\langle x\rangle}$ at infinity, then the bounded solution also has the same exponential decay, provided that $\delta < \theta_1$. Therefore, the three conditions above are also sufficient to prove existence and uniqueness in $\mathscr{S}_{\varepsilon\delta}$ if ε and δ are small.

Note that (i) is a consequence of Lemma 8.2, while (iii) follows from the estimate (8.3). To verify (ii), it is enough to establish the formulae

$$\begin{aligned} \dim \mathcal{S}(\mathbb{A}_\pm) &= r + \dim \mathcal{S}(A_{11}^{*\pm}), \\ \dim \mathcal{U}(\mathbb{A}_\pm) &= r + \dim \mathcal{U}(A_{11}^{*\pm}), \end{aligned} \tag{8.18}$$

where $A_{11}^{*\pm} = df_*(u_\pm) = A_{11} + A_{12} dv_*(u_\pm)$ and $\mathcal{S}(M)$ and $\mathcal{U}(M)$ denote the stable and unstable subspaces of a matrix M. We note that $A_{11}^{*\pm} = df_*(u_\pm)$ are invertible, with dimensions of the stable subspace of A_{11}^{*+} and the unstable subspace of A_{11}^{*-} summing to $n + 1$, by Proposition 3.2. Thus, (8.18) implies that

$$\dim \mathcal{S}(\mathbb{A}_+) + \dim \mathcal{U}(\mathbb{A}_-) = 2r + \dim \mathcal{S}(A_{11}^{*+}) + \dim \mathcal{U}(A_{11}^{*-}) = 2r + n + 1$$

as claimed.

To establish (8.18), introduce the variable $\tilde v = v + Q_{22}^{-1}Q_{21}u$, and the variable corresponding to $\tilde v'$ scaled by a factor $\eta^{\frac12}$, that is $\tilde w = \eta^{\frac12} w + \eta^{-\frac12} Q_{22}^{-1}Q_{21}(A_{11}u + A_{12}v)$. After this change of variables, the matrix $\mathbb{A}$ it conjugated to $\widetilde{\mathbb{A}}$ with

$$\eta^{\frac12}\widetilde{\mathbb{A}} = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & I\\ 0 & -Q_{22} & 0\end{pmatrix} + \eta^{-\frac12}\begin{pmatrix} A_{11}^* & A_{12} & 0\\ 0 & 0 & 0\\ O(\eta^{-\frac12}) & O(\eta^{-\frac12}) & A_{22}\end{pmatrix}. \tag{8.19}$$

From (i), the matrix $\eta^{\frac12}\widetilde{\mathbb{A}}$ has no eigenvelue on the imaginary axis, and the number of eigenvalues in $\{\mathrm{Re}\,\lambda > 0\}$ is independent of η, and thus can be determined taking η to infinity. The limiting matrix has r eigenvalues in $\{\mathrm{Re}\,\lambda > 0\}$, r eigenvalues in $\{\mathrm{Re}\,\lambda < 0\}$ and the eigenvalue 0 with multiplicity n, since $-Q_{22}$ has its spectrum in $\{\mathrm{Re}\,\lambda > 0\}$. The classical perturbation theory as in [MaZ1] shows that for $\eta^{-\frac12}$ small, $\eta^{\frac12}\widetilde{\mathbb{A}}$ has n eigenvalues of order $\eta^{-\frac12}$, close to the spectrum of A_{11}^* with error $O(\eta^{-1})$. Thus, for $\eta > 0$ large, $\eta^{\frac12}\widetilde{\mathbb{A}}$ has $r + \dim \mathcal{S}(A_{11}^*)$ eigenvalue in $\{\mathrm{Re}\,\lambda < 0\}$, proving (8.18).

The proof of the Proposition is now complete. □

8.3 Proof of Proposition 5.2

Let $(\mathcal{L}_*^{\varepsilon,\eta})^\dagger$ denote the inverse operator of $\mathcal{L}_*^{\varepsilon,\eta}$ defined by (8.1), for sufficiently small $\eta > 0$. The uniform bound (8.3), and weak compactness of the unit ball in H^2, for $F \in \mathscr{S}$, we obtain existence of a weak solution $U \in H^2$ of

$$\mathcal{L}_*^{\varepsilon} U = F := \begin{pmatrix} f\\ g\end{pmatrix}, \qquad \ell_\varepsilon \cdot u(0) = 0, \tag{8.20}$$

along some weakly convergent subsequence. Proposition 7.2 implies uniqueness in H^2 for this problem, therefore the full family converges, giving sense to the definition

$$(\mathcal{L}_*^{\varepsilon})^\dagger = \lim_{\eta\to 0} (\mathcal{L}_*^{\varepsilon,\eta})^\dagger \tag{8.21}$$

acting from $\mathscr{S}$ to H^2.

For $F \in \mathcal{S}_{\varepsilon\delta}$, the uniform bounds (8.3) imply that the limit $(\mathcal{L}_*^{\varepsilon})^\dagger U \in H^s_{\varepsilon,\delta}$ and satisfies same estimate. By density, the operator $(\mathcal{L}_*^{\varepsilon})^\dagger$ extends to $f \in H^{s+1}_{\varepsilon,\delta}$ and $g \in H^1_{\varepsilon,\delta}$, with $(\mathcal{L}_*^{\varepsilon})^\dagger F \in H^s_{\varepsilon,\delta}$.

The sharp bound (5.13) and (5.14) now follow immediately from Propositions 7.2 and 7.3. The proof of Proposition 5.2 is now complete.

Remark 8.5. We have used freely the finite-dimensionality of v in our proof of linearized existence. However, as promised, it plays no role in the final linearized bounds. Thus, our result may be used together with discretization (Galerkin approximation) of v to obtain results also in the case that v is infinite-dimensional, as we do for the Boltzmann equations in [MeZ2].

9 Application to spectral stability

Proof of Corollary 4.2. In [MaZ3], under the same structural conditions assumed here, it was shown that small-amplitude profiles of general quasilinear relaxation systems are spectrally stable, provided that

$$|\bar{U}'|_{L^\infty} \le C|U_+ - U_-|^2, \qquad |\bar{U}''(x)| \le C|U_+ - U_-|\,|\bar{U}'(x)|, \tag{9.1}$$

and

$$\Big|\frac{\bar{U}'}{|\bar{U}'|} + \mathrm{sgn}(\eta) R_0\Big| \le C\,|U_+ - U_-|, \tag{9.2}$$

$$R_0 := \begin{pmatrix} r(u_0) \\ dv_*(U_0) r(u_0) \end{pmatrix},$$

where $r(u_0)$ as defined in Theorem 4.1 is the eigenvector of df_* at base point U_0 in the principal direction of the shock. From the bounds of Theorem 4.1, we immediately verify these conditions, giving the result. □

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