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Modelling seat congestion in transit assignment

Abstract

Seating or standing make distinct on-board states for a transit rider, resulting in distinct discomfort costs, with potential influence on route choice on the transit network. The paper provides a model for seat congestion: from assumptions on seat capacity, travel behaviour and priority rules, a seat allocation process is described, resulting in the notion of service modes for a transit leg, making the leg cost a random variable.

At the line level, line algorithms for flow loading and leg costing are provided, which are efficient. At the network level, a hyperpath formulation is provided for supply-demand equilibrium, with path-dependent segment cost, assignment algorithm and a property of existence. It is shown that multiple equilibria may arise.

Model extensions by adding discomfort functions, taste differentiation, or discrete choice to compete or not for getting a seat, might induce uniqueness.

Keywords

Transit assignment. Route choice. Sitting behaviour. Seated capacity. Capacity restrained assignment. Priority rules. Line algorithms. Network equilibrium.

1. Introduction

1.1 Setting

Congestion amongst riders in public transport is an important issue in respect of both the line capacity and the quality of service, as is stated and illustrated in the Transit Capacity and Quality of Service Manual (TRB, 2003).

In dynamic models of transit assignment, only one congestion effect has been modelled so far, that of a First In – First Out rule binding the passenger flow at every access point to a transit line: this was addressed by Tong and Wong (1999) and Nguyen *et al* (2001). However no distinction was made between seating and standing.

Oddly enough, there have been more contributions to model congestion in the context of static transit assignment than in that of dynamic transit assignment. Cepeda *et al* (2006) classified the static congestion models in two categories that may be called segment congestion and node congestion: congestion on a line segment is modelled by associating a discomfort function with each segment of a transit line (Spiess and Florian, 1989); whereas congestion at a node is modelled by linking the line frequency to the flows boarding, alighting or going through at a given transit stop, in a decreasing way: see De Cea and Fernandez (1993) and Cominetti and Correa (2001). Alternatively, Marcotte and Nguyen (1998) suggested to limit the boarding flow to the capacity available at that stop; this line was continued by Shimamoto *et al* (2005) who considered the probability of accessing the transit service at each stop.

Yet no consideration had been paid to the issue of getting a seat, which may be called seat congestion and identified as a category intermediary between segment congestion and node congestion since at the segment level, seating or standing make distinct states

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for the rider, resulting into distinct costs; whereas the issue of seat availability pertains to the node level.

1.2 Objectives

The paper has a core objective and a companion objective. The core objective is to provide a basic model at the line level for seat congestion, in which seat capacity, comfort states and costs, sitting behaviour and priority rules are captured in an explicit way. The core model addresses a transit line in two steps: first, the problem of line leg costing yields the average cost by transit leg, i.e. by line section from access to egress station; second, the problem of line flow loading consists in assigning the access-egress trip matrix to the vehicles and the seats. The line leg costing problem amounts to a cost-flow relationship at the line level, whereas the line flow loading problem is basically an assignment sub-model.

The companion objective is to embed the line model in the framework of traffic assignment to a transit network: then passenger comfort can be taken into consideration in route choice. Static assignment is addressed in order to demonstrate that the line treatment does not interfere with the problem of common lines.

1.3 Main results

In the line model for seat congestion, the approach is to identify the residual seat capacity at any stage along the line, and to share it amongst standing riders under priority rules. Two priority rules are assumed in the basic case: first, that standing riders with same level of priority have equal chance of getting a vacant seat; and second, that standing passengers going through a transit stop obtain access to vacant seats prior to riders boarding at that stop. The two priority rules induce the probability to sit either from the previous segment or at boarding, so by successive segments along the transit leg (transit section) a probability results for each "service mode" made up of a sequence of m segments at standing followed by $n-m$ segments at seating in a leg of n segments. At the leg level, the service mode is obtained in a somewhat random way (due to rule 1), and yields a random leg cost, with distribution characterized by both the segment-state costs and the service mode probabilities. To the leg user, the leg generalized cost is a random variable that stems from the cost of the service mode which is obtained at random.

Efficient "line algorithms" are provided to compute the flow loading by segment and state, the distribution of the service mode probabilities and costs as well as, in a simpler and more efficient way, the mean and variance of the leg cost. The algorithms are associated with mathematical functions that link, respectively:

- the sitting probabilities from on-board or at boarding by station along the line: to the flows by pair of entry-exit stops.
- the line flows by segment and state, to the flows by pair of entry-exit stops.
- The service mode probabilities, to the sitting probabilities.
- The service mode costs: to both the costs by segment and state, and the sitting probabilities.
- The mean and variance of leg cost, to both the costs by segment and state, and the sitting probabilities.

These functions for loading or costing may be included into a macroscopic transit assignment model, static or dynamic.

The inclusion of the line model into a network traffic assignment model with common lines addressed on the basis of either optimal routes or optimal strategies (hyperpaths), is worked out along the following steps:

- The representation of a transit line in terms of network nodes and arcs.
- The leg costing problem is solved prior to network costing by node-destination pair: the network representation of the line enables one to perform network costing in terms of arcs and nodes, as in the optimal strategy model of Spiess and Florian (1989).
- The same applies to network loading from all origins to a given destination: the sitting probabilities do not appear at that stage, in sharp contrast to the line choice probabilities at nodes of line choice.
- The effects of passenger congestion and comfort on route choice induce an equilibrium problem: this is cast into a variational inequality problem of the hyperpath kind, as in Nguyen and Pallotino (1988). Under mild assumptions an equilibrium state is shown to exist.
- A standard method of successive averages is advised for solution. A simple method is provided to evaluate the duality gap in a simple manner, thus yielding a rigorous criterion for convergence.
- In general uniqueness does not hold; this is shown on the basis of a crafted instance.

1.4 Paper structure

The body of the paper is organized into six sections. Section 2 sets up the core model of a line, with assumptions on comfort states, travel behaviour and priority rules; then the service modes are defined and the leg cost issue is addressed. Section 3 provides efficient line algorithms to derive, respectively, the sitting probabilities at line nodes, the service mode probabilities and costs, as well as the mean and variance of the leg cost; a short example is addressed as numerical illustration.

In Section 4, the line model is integrated into a network assignment model: each line is represented by as many network arcs as there are legs along the line. The cost-flow relationship is defined first at the arc level for non-transit arcs, then at the segment level along the transit lines for the segment costs of seating and standing, lastly at the line level for the transit legs.

The mathematical analysis of traffic equilibrium in the transit model with seat comfort is addressed in Section 5. The formula for hyperpath cost is adapted to include the leg cost – in other words the cost-flow relationship at the network level. Then the traffic equilibrium, stated as the solution of a nonlinear complementarity problem, is characterized as the solution to a variational inequality problem. Under mild assumptions it is shown that an equilibrium state must exist. The simple method to evaluate a duality gap is provided before stating the method of successive averages.

In Section 6 a classroom example involving seat congestion and path choice is addressed as numerical illustration, and also to demonstrate that multiple equilibria may arise.

Lastly, Section 7 concludes and points to potential developments: distinction of more comfort states; association of a discomfort function to each comfort state; inclusion of "transaction" costs because of effort to get a seat; inclusion of costs that are nonlinear with respect to number of segments in leg or to leg travel time; taste differentiation among users; lastly, discrete choice to compete for comfort states, which would enable one to model a higher share for long legs than for short legs.

2. The line model: assumptions and basic notions

Here the focus is on a given transit line. No other network issue is considered up to Section 4.

Our modelling assumptions pertain to comfort states and their cost (subsection 2.1), and also travel behaviour and priority rules across users (subsection 2.2). They lead us to define the notion of a service mode (subsection 2.3), which is a way to use a line "leg" from access station to egress station. As the users compete for the residual seat capacity, there is randomness in getting a seat, which makes the leg cost a random variable (subsection 2.4). To sum up, the outputs of the line model consist in sitting probabilities and leg costs.

2.1 Comfort states and their cost

In a public transport service, it frequently occurs that several categories of places with distinct attributes be provided to the users. In urban transit by bus, tram, metro or train, the distinction between seating and standing is particularly significant: a seated rider is less inconvenienced by the vehicle's acceleration and deceleration and by the slopes and curves in the vehicle's trajectory; he may invest his travel time into a complementary activity such as reading, listening to music, relaxing or working; moreover, he is much less submitted to crowding.

From observation of traffic in public transit, it appears that riders favour the seating state over the standing state to a large extent: when a crowded vehicle arrives at a station, those riders that stand and stay on-board try to get a seat given away by an outgoing passenger. At the initial station of a crowded metro line, some users prefer to wait for the next vehicle to arrive, rather than to board when there is no vacant seat. Furthermore, on some origin-destination pairs serviced by alternative metro lines, many passengers prefer to use the line with less congestion, even if the travel time is higher.

Let us assume that on a transit line segment from a station to the next, there are two riding states namely seating and standing, with associated costs that reflect the users' preference to have a seat.

Let a denote a line segment, $\underline{c}_a$ its discomfort cost to a rider at seating, and $\overline{c}_a$ its discomfort cost to a rider at standing: the basic assumption is that $\underline{c}_a \leq \overline{c}_a$ whatever the traffic load.

2.2 Travel behaviour and priority rules

It is assumed that every rider is a cost-minimizing individual decision-maker, striving to reduce his travel cost. Then a rider who is standing tries to get a seat as soon as one becomes available. As there is a limited number of available seats, say κ for capacity, and also a number x of riders that would like to sit, it may be the case where $\kappa < x$, meaning that capacity is less than demand.

In this case only a proportion κ / x of riders may sit. The issue of which riders would get a seat is addressed here in a simple way, assuming that all of them have an equal probability to sit, i.e. neglecting the individual attributes of age and physical needs, eagerness-to-sit, planned egress station etc.

The time at which seats become available and competition occurs is important: standing riders that stay on board have an advantage over the incoming riders, which is modelled by assuming two successive competitions: the first one among "through" riders, the other one among incoming riders.

Thus, in our basic model of seat congestion, two priority rules are assumed: first, that standing riders with same level of priority have equal chance of getting a vacant seat; and second, that standing passengers going through a transit stop obtain access to vacant seats prior to riders boarding at that stop.

2.3 Services modes on a transit leg

On a "leg" from an access station to an egress station along a transit line, the user is provided service in a particular way, depending on which comfort states he gets on each line segment in the leg.

Let us define a "service mode" as the sequence of segment comfort states along the leg. In a leg made up of N segments, if there are two comfort states associated to each segment, there could be as much as 2^N service modes associated to the leg, but in fact the users' behaviour reduces the number of alternative service modes to $N+1$: after getting a seat a user is assumed not to release it until arrival at his egress station.

Thus a service mode is fully described by the station $i+m$ at which the user gets a seat, with index $m \in \{0,1,2..N\}$: by convention, getting a seat on exiting at $i+N$ means standing all over the leg.

The cost of service mode m from access station i to egress station $j = i+N$ is:

$$c_{ij}^{m} = [\textstyle\sum_{k=i}^{i+m-1} \overline{c}_{a\approx(k,k+1)}] + [\textstyle\sum_{k=i+m}^{j-1} \underline{c}_{a\approx(k,k+1)}] \quad (1)$$

2.4 Leg cost as a random variable

The attribution of a given service mode does not depend solely on the user, because his preference to be seating rather than to stand may get into competition with others' preferences, resulting in a collective allocation process of seats rather than in an individual choice of a service mode.

From the modelling assumptions on travel behaviour and priority rules, the seat allocation process at each station can be summarized by two sitting probabilities: the first one say p_i^o for through riders, and the other one say p_i^+ for incoming riders. Here the sitting probabilities are taken as exogenous, so as to derive the distribution of service modes and costs.

The probability of keeping standing from i to $i+N$ is

$$\pi_{i,i+N}^{N} = (1-p_i^+).\,[\textstyle\prod_{k=1}^{N-1}(1-p_{i+k}^{o})] \quad (2a)$$

The probability of sitting at station $i+m$ is:

$$\pi^0_{i,i+N} = p^+_i \text{ for } m=0 \tag{2b}$$

$$\pi^m_{i,i+N} = (1-p^+_i).\,[\prod_{k=1}^{m-1}(1-p^o_{i+k})]\;.\,p^o_{i+m} \text{ for } m\in\{1,2..N-1\} \tag{2c}$$

The probabilities $(\pi^m_{i,i+N})_{m=0..N}$ are associated with the service modes and describe the random process of getting a service mode. Service mode *m* has probability $\pi^m_{i,i+N}$ stated in (2) and cost $c^m_{i,i+N}$ stated in (1).

Thus, to the user the leg cost is a random variable $c_{i,i+N}$, depending on which service mode is obtained. The mean leg cost is:

$$\widehat{c}_{i,i+N} \equiv \mathrm{E}[c_{i,i+N}] = \sum_{m=0}^{N} \pi^m_{i,i+N}\,.\,c^m_{i,i+N} \tag{3a}$$

The variance of the leg cost is:

$$v_{i,i+N} \equiv \mathrm{var}[c_{i,i+N}] = \sum_{m=0}^{N} \pi^m_{i,i+N}\,.\,(c^m_{i,i+N} - \widehat{c}_{i,i+N})^2 \tag{3b}$$

The cost variance may be taken into account in the generalized cost of travel along the leg. However, in the sequel of the paper the focus is on mean cost only, since cost variability in hyperpath assignment still makes an open issue which deserves specific research efforts.

3. Statement of line problems and algorithms

Having provided the assumptions and derived the formulae for the major output variables in the line model, let us now provide efficient algorithms to solve the formulae – and to be included in assignment procedures. The algorithms have minimal complexity with respect to the number of distinct variables that are outputted; this efficiency is achieved through a technique of auxiliary variable.

The section roadmap is as follows: first, simple formulae are established for the sitting probabilities along the transit line (subsection 3.1). Then, a line flow loading algorithm is provided to load an access-egress trip matrix onto the line according to the priority rules and yield all sitting priorities; letting S denote the number of stops along the transit line, the algorithm has complexity $O(S^2)$ hence it is efficient (subsection 3.2). Next, recursive formulae are established for the service mode probabilities and costs: their application makes an efficient algorithm of complexity $O(S^3)$ (subsection 3.3). Lastly, a line costing algorithm is provided to compute the mean and variance of the leg costs along the line, yielding a reduced computational complexity $O(S^2)$ which makes it efficient for its reduced purpose (subsection 3.4).

3.1 On sitting probabilities

On the line segment upstream of station *i*, let us describe the line traffic by two flow vectors indexed by egress station *j*: $\underline{x}_j$ (resp. $\overline{x}_j$) is the number of seating (resp. standing) riders with egress at *j*. Let also $\underline{x}_{\geq i} = \sum_{j\geq i}\underline{x}_j$ and $\overline{x}_{\geq i} = \sum_{j\geq i}\overline{x}_j$.

The riders boarding at *i* are described by a flow vector $\mathbf{q}^\ell_i = [q^\ell_{ij}]_{j>i}$.

Let κ_{i-1}^{-} be the residual seat capacity from the previous station: if $\bar{x}_{\geq i} \neq 0$ then $\kappa_{i-1}^{-} = 0$. Every seating rider who exits at i releases their seat, which increases the residual seat capacity to

$$\kappa_i^o = \kappa_{i-1}^{-} + \underline{x}_i \quad (4a)$$

This capacity may be used by the standing riders who continue after i, in number of $y_i^o = \sum_{j>i} \bar{x}_j$. Their probability to sit amounts to:

$$p_i^o = \min\{1, \kappa_i^o / y_i^o\} \quad (4b)$$

After their eventual sitting, the residual capacity is decreased to:

$$\kappa_i^+ = \kappa_i^o - p_i^o . y_i^o \quad (4c)$$

It is available to the incoming riders in number of $y_i^+ = \sum_{j>i} q_{ij}^{\ell}$, who sit with the following probability:

$$p_i^+ = \min\{1, \kappa_i^+ / y_i^+\} \quad (4d)$$

After their eventual sitting, the residual capacity is decreased to:

$$\kappa_i^- = \kappa_i^+ - p_i^+ . y_i^+ \quad (4e)$$

Thus formulae (4a-e) enable one to derive the residual capacities and the sitting probabilities at a given station, for riders either through or incoming, as functions of the vectors of seating and standing flows with respect to egress station and the access-egress trip matrix.

3.2 Line flow loading problem and algorithm

The **line flow loading problem** is to assign a line access-egress trip matrix to the seating and standing states along the line segments. The outputs consist basically in the sitting probabilities by station node and priority status; the link flows by comfort state and exit station can also be outputted.

A solution method for the line loading problem has been provided above for a given station: the line loading algorithm consists in applying this method to every station along the line, in turn from the initial station to the final one. Aside from applying (4), the main issue is to maintain the vectors $\underline{x}$ and $\bar{x}$ of seating and standing flows at each stage along the line.

The **line loading algorithm** addresses a line ℓ with $\bar{n}_{\ell}$ stations and seat capacity κ. Input variables also include the access-egress trip matrix $[q_{ij}^{\ell}]_{i<j\in\ell}$. The output variables basically consist in the sitting probabilities $[p_i^+]_{i\in\ell}$ and $[p_i^o]_{i\in\ell}$; other information such as segment flow by comfort state and egress station may be outputted, too. Working variables also include the $\underline{x}$ and $\bar{x}$ flow vectors together with their sums $\underline{X}$ and $\bar{X}$, to be updated at each stage.

The algorithm is comprised of the following steps:

Initialisation. Let $i := 0$; let $\underline{x}_j := 0$ and $\overline{x}_j := 0$ $\forall j \in \ell$; let $\underline{X} := 0$ and $\overline{X} := 0$.

Termination Test. If $i = \overline{n}_\ell$ then terminate else let $i := i+1$ and continue.

Progression. At station i:

- let first $\kappa_i^o := \kappa - \underline{X} + \underline{x}_i$; then $y_i^o := \overline{X} - \overline{x}_i$, and $p_i^o := \min\{1, \kappa_i^o / y_i^o\}$.
- let $\underline{x}_j := \underline{x}_j + p_i^o.\overline{x}_j$ and $\overline{x}_j := (1-p_i^o).\overline{x}_j$ $\forall j > i$.
- let $\underline{X} := \underline{X} - \underline{x}_i + p_i^o.y_i^o$ and $\overline{X} := (1-p_i^o).y_i^o$.
- let $\kappa_i^+ := \kappa_i^o - p_i^o.y_i^o$, then $y_i^+ := \sum_{j>i} q_{ij}^\ell$ and $p_i^+ := \min\{1, \kappa_i^+ / y_i^+\}$.
- let $\underline{x}_j := \underline{x}_j + p_i^+.q_{ij}^\ell$ and $\overline{x}_j := \overline{x}_j + (1-p_i^+).q_{ij}^\ell$ $\forall j > i$.
- let $\underline{X} := \underline{X} + p_i^+.y_i^+$ and $\overline{X} := \overline{X} + (1-p_i^+).y_i^+$.
- Go to *Termination Test*.

This algorithm might be streamlined somewhat further. In its current form, the treatment of a current station requires a number of operations proportional to $\overline{n}_\ell - i$, which induces a computational complexity of $O(\overline{n}_\ell^2)$ - a still modest number for typical transit lines. The complexity is minimal with respect to the number of distinct variables that can be outputted, since there are $O(\overline{n}_\ell^2)$ link flows by comfort state and egress station.

Figure 1 depicts the comfort states along a transit line: state transition may occur for a rider from standing at $i-1$ to either seating at i with probability p_i^o or standing at i with probability $1-p_i^o$, or from incoming at i to either seating at i with probability p_i^+ or standing at i with probability $1-p_i^+$.

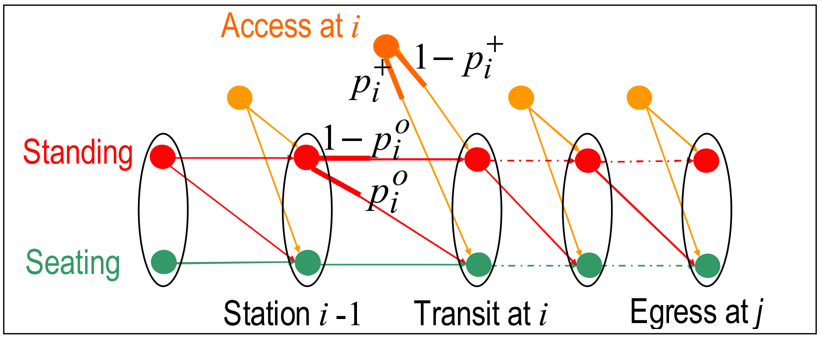

Fig. 1. Rider states along a transit line.

Instance 1. Figure 2 depicts a transit line with 4 stations and the following access-egress trip flows:

- At $i=1$ there are $y_1^+ = 120$ riders boarding, among whom $q_{12} = 50$ destined to station 2, $q_{13} = 30$ destined to station 3 and $q_{14} = 40$ destined to station 4.
- At $i=2$ there are $y_2^+ = 90$ riders boarding, among whom $q_{23} = 60$ and $q_{24} = 30$.
- At $s=3$ there are $y_3^+ = 50 = q_{34}$ boarding riders.

For a seat capacity of $\kappa = 100$, the line loading algorithm yields the following results:

- At $i=1$, $\kappa_1^o = 100 = \kappa_1^+$, $p_1^o = 1$, $y_1^+ = 120$ hence $p_1^+ = \frac{5}{6}$, $\underline{X} = 100$ and $\overline{X} = 20$, $\underline{x}_2 = 41.7$, $\underline{x}_3 = 25$, $\underline{x}_4 = 33.3$ whereas $\overline{x}_2 = 8.3$, $\overline{x}_3 = 5$, $\overline{x}_4 = 6.7$.
- At $i=2$, $\kappa_2^o = 41.7$, $p_2^o = 1$ hence all standing riders that go through get a seat, $\kappa_2^+ = 30$, $y_2^+ = 90$ hence $p_2^+ = \frac{1}{3}$, $\underline{X} = 100$ and $\overline{X} = 60$, $\underline{x}_3 = 50$, $\underline{x}_4 = 50$ whereas $\overline{x}_3 = 40$, $\overline{x}_4 = 20$.
- At $i=3$, $\kappa_3^o = 50$, $p_3^o = 1$ hence all standing riders that go through get a seat, $\kappa_3^+ = 30$, $y_3^+ = 50$ hence $p_3^+ = \frac{3}{5}$, $\underline{X} = 100$ and $\overline{X} = 20$, $\underline{x}_4 = 100$ and $\overline{x}_4 = 20$.
- At $i=4$, all riders come out.

Instance 2. Let us adapt the trip matrix by replacing its first line with $q_{1\bullet} = [150 \quad 30 \quad 140]$. The line loading algorithm yields sitting probabilities as follows:

$\mathbf{p}_i^+ = [.3125 \quad 0 \quad 0 \quad 0]$ and $\mathbf{p}_i^o = [1 \quad .4010695 \quad 0.2013423 \quad 0]$.

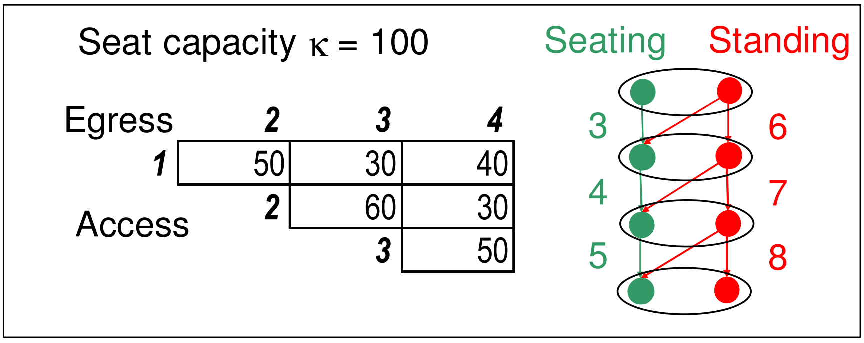

Fig. 2. Line trip matrix and segment costs by comfort state

3.3 Cost and flow share of service modes

The flow share and cost of each service mode, as stated in (1) and (2), may be readily evaluated on the basis of the sitting probabilities (taken here as inputs).

Let us first provide an algorithm to evaluate the flow share of all service modes along legs with egress at station j, by dealing with the access stations i in backward order from downstream to upstream. We shall use a sequence of auxiliary variables $[\rho_k^m : k \leq j,\, m = 0,1,...j-k]$, in order to save on computational effort.

The algorithm is made up of the following steps:

Initialisation. Let $\rho_j^0 := 1$ and $i := j$.

Progression. Let $i := i-1$. If $i = 0$ then terminate, else do:

- let $\rho_i^0 := p_i^o$.
- let $\rho_i^m := (1-p_i^o).\rho_{i+1}^{m-1}$ for $m \in \{1,2,...j-i\}$.
- Go to *Progression*.

Then $\pi_{ij}^m = p_i^+$ if $m = 0$, or $(1-p_i^+).\rho_{i+1}^{m-1}$ if $m \geq 1$.

The computation of service mode costs is still easier, on the basis of:

$$c_{i,j}^{m} = \overline{c}_{i,i+1} + c_{i+1,j}^{m-1} \quad \text{for } m \geq 1,\ i < j$$
$$c_{i,j}^{0} = \underline{c}_{i,i+1} + c_{i+1,j}^{0} \quad \text{if } \ i < j$$

The computational complexity of both algorithms is in $O(\overline{n}_\ell^3)$ since it is a product of the number of egress stations, times the number of access stations, times the number of service modes from access to egress. This complexity is minimal since there are $O(\overline{n}_\ell^3)$ output variables π_{ij}^m : thus the service mode algorithm is efficient.

Instance 2 (continued). Let us evaluate the cost and flow share of the service modes for the trips destined to station 4. By entry station i and service mode m, the costs and flow shares are, respectively:

$$[C_{i4}^m] = \begin{bmatrix} 12 & 15 & 18 & 21 \\ 9 & 12 & 15 & \\ 5 & 8 & & \end{bmatrix} \text{ and } [\pi_{i4}^m] = \begin{bmatrix} .3125 & .2757 & .0829 & .3289 \\ 0 & .2013 & .7987 & \\ 0 & 1 & & \end{bmatrix}$$

3.4 Line leg costing algorithm

To save even more on computational effort, let us now provide an algorithm to evaluate the mean and variance of leg cost. Auxiliary variables are still useful to average the cost of the downstream sub-path conditional on the upstream state of comfort.

To obtain the mean leg cost $\widehat{c}_{ij}$ from access station *i* to egress station *j*, let us associate two costs to that leg, namely:

- the seating cost $c_{ij}^0 = \sum_{k=i}^{j-1} \underline{c}_{a\approx(k,k+1)}$,
- an auxiliary mean cost γ_{ij} which is a mean cost from i to j conditional on standing on segment $(i,i+1)$.

These costs satisfy the following recursive equations, due to the law of total probability:

$$c_{i,j}^0 = \underline{c}_{a\approx(i,i+1)} + c_{i+1,j}^0 \tag{5a}$$

$$\gamma_{i,j} = \overline{c}_{a\approx(i,i+1)} + p_{i+1}^o . c_{i+1,j}^0 + (1-p_{i+1}^o) . \gamma_{i+1,j} \tag{5b}$$

$$\widehat{c}_{i,j} = p_i^+ . c_{i,j}^0 + (1-p_i^+) . \gamma_{i,j} \tag{5c}$$

The line costing algorithm addresses a given egress station j, by dealing with access stations $i \leq j$ in backward order from downstream to upstream. The initial conditions are $c_{i,j}^0 = 0$ and $\gamma_{i,j} = 0$ at $i = j$. As the treatment of i as access station amounts to computing six additions and four products, the mean costing of all legs with egress at *j* has a computational complexity of $O(j)$, making the mean costing of all legs along the line a $O(\overline{n}_\ell^2)$ burden. Thus the mean leg costing algorithm is of optimal complexity, since there are $O(\overline{n}_\ell^2)$ output variables $\widehat{c}_{ij}$.

The variance of the leg cost can be obtained in a similar way, on the basis of an auxiliary variance $\varpi_{i,j}$ defined as the variance of leg cost from i to j conditional on standing on segment $(i, i+1)$. The conditional variance $\varpi_{i,j}$ and the variance v_{ij} satisfy the following recursive formulae:

$$\varpi_{i,j} = (1 - p^o_{i+1}).[\varpi_{i+1,j} + p^o_{i+1}.(\gamma_{i+1,j} - c^0_{i+1,j})^2] \quad (6a)$$

$$v_{ij} = (1 - p^+_i).[\varpi_{i,j} + p^+_i.(\gamma_{i,j} - c^0_{i,j})^2] \quad (6b)$$

Eqn (6b) stems from the theorem of total variance: a leg from i to j is either a leg beginning at standing or an all-seating leg, with respective probabilities $1-p$ and p. The interclass variance of the leg cost amounts to

$$p.(1-p).(\Delta C)^2$$

with $\Delta C = \gamma - c^0$ the difference in class costs. Within the class of all-seating trips the cost variance is zero. Within the class of legs that begin at standing, the cost variance is $\varpi_{i,j}$ by definition. To sum up, the intraclass variance amounts to $(1-p).\varpi_{i,j}$.

A similar proof applies to (6a) by considering that a leg from i to j that begins at standing has its second segment either at standing or at seating.

Instance 1 (continued). Let us apply the line costing algorithm to station 4 as egress, on assuming the following segment costs: $\underline{c}_{12} = 3$, $\overline{c}_{12} = 6$, $\underline{c}_{23} = 4$, $\overline{c}_{23} = 7$, $\underline{c}_{34} = 5$, $\overline{c}_{34} = 8$:

At $s = 4$, $c^0_{44} = \gamma_{44} = 0 = \hat{c}_{44}$ and $v_{44} = \varpi_{44} = 0$.

At $s = 3$, $c^0_{34} = 5$, $\gamma_{34} = 8$, $\hat{c}_{34} = 6.2$, $\varpi_{34} = 0$, $v_{34} = 2.16$.

At $s = 2$, $c^0_{24} = 9$, $\gamma_{24} = 12$, $\hat{c}_{24} = 11$, $\varpi_{24} = 0$ (all riders sit at 3), $v_{24} = 2$.

At $s = 1$, $c^0_{14} = 12$, $\gamma_{14} = 15$, $\hat{c}_{14} = 12.5$, $\varpi_{14} = 0$ (all riders sit at 2), $v_{14} = 1.25$.

In this example, the cost variance takes on moderate values; it increases with respect to the leg number of segments in a less than proportional way since the more segments in a leg, the more opportunities to get a seat.

Instance 2 (continued). Let us apply the line costing algorithm to the modified trip matrix, hence to the modified sitting probabilities. By entry station i from 1 to 3 and exit station j from 2 to 4, the mean costs and cost variances are, respectively:

$$[\hat{C}_{ij}] = \begin{bmatrix} 5.0625 & 10.298 & 16.284 \\ & 7 & 14.396 \\ & & 8 \end{bmatrix} \text{ and } [v_{ij}] = \begin{bmatrix} 1.0336 & 6.4297 & 13.7479 \\ & 0 & 1.4472 \\ & & 0 \end{bmatrix}$$

Under increased entry-exit flows it turns out that the cost variance may take much larger values.

4. Network representation and cost-flow relationship

Our objective here is to embed the line model of seat congestion in the framework of traffic assignment to a transit network: thus passenger comfort can be taken into consideration in route choice. Static assignment is addressed in order to demonstrate that the line treatment does not interfere with the problem of common lines.

There are basically two static models for transit assignment, route-based vs. hyperpath-based. In a route-based model (Lamb and Havers, 1970, De Cea and Fernandez, 1989), a route consists in a sequence of access nodes from origin to destination of a trip, each access node being linked to the next one by a private path or a bundle of parallel transit legs. In a hyperpath-based model (Spiess, 1985; Nguyen and Pallotino, 1988; Spiess and Florian, 1989), a hyperpath is a directed, acyclic sub-graph that connects origin nodes to a given destination, together with routing proportions at each node that indicate the local share of flow between the arcs that come out of the node and belong to the hyperpath. Hereafter we shall focus on the hyperpath model, for the following two reasons: first, we may consider the set of routes as a subset of hyperpaths, so that the mathematical formulation for hyperpaths could be readily restricted to routes. Second, in a route-based model the main issues are to compute the leg costs, to combine the legs into leg bundles, and to add the bundle costs along a given route: as our line algorithms provide the leg costs, it is also straightforward to include seat congestion in the route formulation given by De Cea and Fernandez (1993).

The integration of the seat congestion model in a transit network assignment model is a twofold issue. On one hand, the leg costs that stem from the seat congestion model must be taken into account within the users' routing choices to their destination. On the other hand, the network flows that stem from the assignment model will induce the line trip matrices in the seat congestion model.

To tackle this issue, we shall first provide a network representation of the transit line on the basis of one arc per leg. Then we shall define the model variables and the cost-flow relationship at the network level.

4.1 Network representation

In a traffic assignment model, the network representation is purported to describe the routes that are available to the network users in terms of paths and of physical and economic attributes including travel time and money cost. It is also used to model the users' route choices and to aggregate the resulting chosen paths into path flows and link flows, hence the traffic loads.

The obvious way to address seat congestion is to associate one network arc to each leg of a transit line, tailed at the access station and headed to the egress station. For an oriented transit line ℓ with $\bar{n}_\ell$ stations i numbered in forward order from 1 to $\bar{n}_\ell$, there are $\bar{a}_\ell = \bar{n}_\ell - 1$ line segments $(i, i+1)$. The set of network nodes required to model the line is $N_\ell = N_\ell^+ \cup N_\ell^-$ with $N_\ell^+ = \{\mathrm{n}_{\ell,i}^+ : i = 1..\bar{n}_\ell\text{-}1\}$ the subset of nodes for access at station i and $N_\ell^- = \{\mathrm{n}_{\ell,i}^- : i = 2..\bar{n}_\ell\}$ for egress nodes. The line segments give rise to set $A_\ell = \{a \approx (\mathrm{n}_{\ell,i}^+, \mathrm{n}_{\ell,j}^-) : 1 \le i < j \le \bar{n}_\ell\}$ of line leg arcs, in number of $|A_\ell| = \frac{1}{2}(\bar{n}_\ell - 1)(\bar{n}_\ell - 2)$.

A leg arc has a travel cost which depends on the line access-egress trip matrix, which is associated to the set of leg arcs for that line.

In large networks this representation of the legs yields a very large assignment network; however no other network representation would save on the computer cost, since it is required to identify the line trip matrix which is analogous to the line subvector of a network flow.

The leg-as-arc representation involves an important assumption about route choice and user behaviour: that each rider who chooses a leg will travel along it up to its exit station, whatever the service mode he gets. Yet, in the basic transit assignment model of optimal strategy, the rider is assumed to be cost-minimizing under dynamic information limited to his perception. Indeed a rider who did not get a seat perceives that he is standing on board; eventually he would consider whether to keep on board or to exit at the next stop. Such leg re-routing behaviour is prohibited under the leg-as-arc representation.

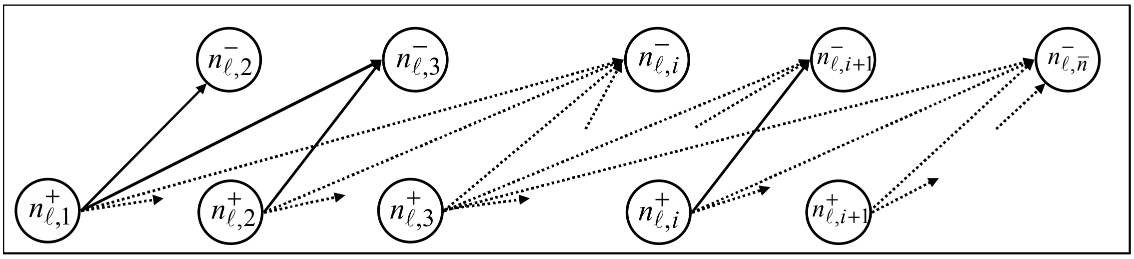

Fig. 3. Representation of transit line within assignment network

4.2 Model variables

The network $G=[N,A]$ is comprised of a set N of nodes n and a set A of arcs a with tail node n^+_a and head node n^-_a in N. Any node or arc belongs to at most one transit line $\ell\in L$ set of lines. Arcs that do not belong to a transit line are called "non-transit" arcs and correspond to other transportation modes such as by foot. Among non-transit arcs, those headed to an access node $\mathrm{n}^+_{\ell,i}$ along a transit line ℓ are called access arcs to that line.

Arc a has average traversal cost $\hat{c}_a$, and a service frequency f_a which is set to either line frequency f_ℓ when a is an access arc to line ℓ, or infinity otherwise.

The set of destinations is denoted by S – for sink nodes. To a destination s is associated a set W^s of origin-destination (OD) pairs $w=(o,s)$ with OD flow of q_{os}. To the set $W=\bigcup_{s\in S} W^s$ is associated the vector of OD flows $\mathbf{q}=[q_w]_{w\in W}$.

At the arc level, let x_{as} denote a flow of users on arc a destined to s.

A vector of arc-destination flows $\mathbf{x}_{AS}=[x_{as}:a\in A, s\in S]$ is called a **flow state**. Its restriction to the legs of a transit line ℓ is denoted as $\mathbf{x}_{\ell S}=[x_{as}:a\in A_\ell, s\in S]$.

4.3 Cost-flow relationship

The arc costs are assumed to depend on the network flow state, $\mathbf{x}_{AS}$, on the basis of the following cost-flow function:

$$\mathbf{x}_{AS} \mapsto \mathbf{c}_A(\mathbf{x}_{AS}) = [\mathrm{c}_a(\mathbf{x}_{AS}) : a \in A] \tag{7}$$

This framework encompasses a range of congestion effects:

- Local congestion induced by the local arc flow $x_a = \sum_{s \in S} x_{as}$, by restricting $\mathrm{c}_a(\mathbf{x}_{AS}) = \mathrm{c}_a(x_a)$.
- Seat congestion along a transit leg $a \in A_\ell$, by restricting $\mathrm{c}_a(\mathbf{x}_{AS}) = \mathrm{c}_a(\mathbf{x}_\ell)$ in which $\mathbf{x}_\ell = [x_a : a \in A_\ell]$.

More precisely, the effect of the seat congestion model on the leg cost can be stated in three steps. First, the line loading algorithm amounts to a pair of functions:

$$\mathbf{p}_\ell^o = \mathbf{P}_\ell^o(\mathbf{x}_\ell) \tag{8a}$$

$$\mathbf{p}_\ell^+ = \mathbf{P}_\ell^+(\mathbf{x}_\ell) \tag{8b}$$

for the sitting probabilities along the line stations, either from on-board or at entry, with respect to the line trip matrix $\mathbf{x}_\ell$.

Second, the line segment costs $\underline{c}_{i,i+1}$ at seating and $\overline{c}_{i,i+1}$ at standing can be modelled as functions of the line trip matrix:

$$\underline{c}_{i,i+1} = \underline{\mathrm{C}}_{i,i+1}(\mathbf{x}_\ell) \tag{9a}$$

$$\overline{c}_{i,i+1} = \overline{\mathrm{C}}_{i,i+1}(\mathbf{x}_\ell) \tag{9b}$$

Third, the line leg costing algorithm derives the mean leg costs from the sitting probabilities $p_{\ell,i}^o$ and $p_{\ell,i}^+$ together with the segment costs $\underline{c}_{i,i+1}$ and $\overline{c}_{i,i+1}$:

$$\hat{c}_{i,j} = \hat{\mathrm{C}}_{i,j}(\mathbf{p}_\ell^o, \mathbf{p}_\ell^+, [\underline{c}_{i,i+1}]_{i \in \ell}, [\overline{c}_{i,i+1}]_{i \in \ell}) \tag{10}$$

Thus, by the composition of functions, leg $a \approx (i, j)$ along line ℓ is associated with

$$c_a = \hat{\mathrm{C}}_{i,j}(\mathbf{P}_\ell^o(\mathbf{x}_\ell), \mathbf{P}_\ell^+(\mathbf{x}_\ell), [\underline{\mathrm{C}}_{i,i+1}(\mathbf{x}_\ell)]_{i \in \ell}, [\overline{\mathrm{C}}_{i,i+1}(\mathbf{x}_\ell)]_{i \in \ell}), \tag{11}$$

which is denoted hereafter by $c_a = \mathrm{c}_a(\mathbf{x}_{AS})$, knowing that a is a leg arc.

4.4 Hyperpath issues

A hyperpath $h=(\hbar,\hat{h})$ with destination node s is a pair of an arc set $\hbar$ and a routing field $\hat{h}$. The hyperpath arc set $\hbar$ is such that each arc $a\in\hbar$ belongs to a positive path within $\hbar$ towards s, and that it contains no oriented cycle. The routing field $\hat{h}$ is a mapping of A onto [0,1] such that $\hat{h}_a=0$ if $a\notin\hbar$, $\sum_{a\in A_m^+}\hat{h}_a=1$ for all nodes m lying between n and s along $\hbar$ (except for s), A_m^+ being the set of arcs that go out of m.

From a given node n to a destination node s, let H_{ns} denote the set of hyperpath arc sets $\hbar$ from n to s via the network, eventually submitted to some more restrictions (e.g. to obtain a route-based model in the sense of De Cea *et al*, 1988).

In transit assignment, the routing field $\hat{h}$ associated to an hyperpath arc set $\hbar$ is further constrained by $\hat{h}_a=1_{\{a\in\hbar\}}$ if arc a is not a transit access arc or, if a is a transit access arc with tail node m, by $\hat{h}_a=f_a.1_{\{a\in\hbar\}}/F_m^{\hbar}$ where $F_m^{\hbar}=\sum_{a\in A_m^+}f_a.1_{\{a\in\hbar\}}$ denotes the combined frequency of the active lines that can be accessed from this node. Thus, in the sequel a hyperpath $h=(\hbar,\hat{h})$ is assimilated to its arc set $\hbar$, from which the routing field is derived straightforwardly.

Given flow state $\mathbf{x}_{AS}$ and hyperpath arc set $h\in H_{ns}$, there is a hyperpath routing field $\hat{h}$ that provides the minimum hyperpath cost along *h*, stated as

$$\mathrm{C}_{ns}(h,\mathbf{x}_{AS})=\sum_{r\in R_{ns}(h)}\hat{h}_r\,.\,[(\sum_{m\in r}\mathrm{w}_m^h(\mathbf{x}_{AS}))+\sum_{a\in r}\mathrm{c}_a(\mathbf{x}_{AS})\,], \tag{12}$$

in which:

- $R_{ns}(h)$ is the set of elementary paths along *h* from *n* to *s*,
- $\hat{h}_r=\prod_{a\in r}\hat{h}_a$ is the path flow proportion,
- $c_a(\mathbf{x}_{AS})$ is the travel cost of arc *a* conditional on flow state $\mathbf{x}_{AS}$,
- $\mathrm{w}_m^h=\alpha_m/F_m^h$ stands for the delay at node *m*.

The dependency of mean arc cost onto the path *r* allows for relating a transit arc cost to its line leg within the path.

The waiting weighting factor α_m is equal to one if the transit lines accessible from *m* are serviced by vehicles with interarrival times that obey to a negative exponential distribution; under other service assumptions it may take a different value, so that w_m^h would in all way stand for the average waiting time at *m* until the arrival of the next vehicle from among the transit lines that are accessible (i.e. their access arc $a\in h$).

5. Equilibrium analysis

On the basis of the previous definitions, we are now ready to define a traffic equilibrium and to carry out the relevant mathematical analysis. We shall first set up the feasible set of flow states and their hyperpath representation. Then a traffic equilibrium will be defined in the form of a Nonlinear Complementarity Problem (NCP). After providing an equivalent characterization in the form of a Variational Inequality Problem (VIP), the existence of an equilibrium state will be demonstrated. Lastly, a method of successive averages will be put forward to compute an equilibrium: its convergence can be assessed on the basis of a duality gap which makes up a rigorous criterion.

5.1 On feasible flow states and hyperpath representation

Definition 1, Feasible network flow state. *A network flow state* $\mathbf{x}_{AS} = [x_{as}]_{a\in A, s\in S}$ *is feasible if it is non negative and if it satisfies the node conservation of flow by destination:*

$$x_{as} \geq 0 \quad \forall a \in A, s \in S \tag{13a}$$

$$\sum_{a\in A_m^+} x_{as} = q_{ms} + \sum_{a\in A_m^-} x_{as} \quad \forall s \in S,\, m \in N,\, m \neq s\,, \tag{13b}$$

in which A_m^+ *[resp.* A_m^-*] denotes the subset of arcs that go out [resp. come in] node* m *and* q_{ms} *is a given origin-destination flow from node* m *to destination node* s.

Let $\mathbf{E}_\mathbf{x}$ be the set of feasible network flow states.

Recalling the sets H_{ns} of hyperpaths from n to s, a **hyperpath flow state** $\mathbf{X}_{NS}$ is defined as a linear combination of elementary flows along hyperpaths h with coefficients q_{ns}^h:

$$\mathbf{X}_{NS} = [q_{ns}^h : s \in S,\, n \in N \setminus \{s\},\, h \in H_{ns}] \tag{14}$$

Definition 2, Feasible hyperpath flow state. *Given the origin-destination trip matrix* $\mathbf{q}_{NS} = [q_{ns}]_{n\in N,\ s\in S}$, *a hyperpath flow state* $\mathbf{X}_{NS}$ *is feasible if it is non negative and it satisfies the conservation of flow by origin-destination pair:*

$$q_{ns}^h \geq 0 \quad \forall s \in S,\, n \in N \setminus \{s\},\, h \in H_{ns} \tag{15a}$$

$$\sum_{h\in H_{ns}} q_{ns}^h = q_{ns} \quad \forall s \in S, n \in N, n \neq s\,. \tag{15b}$$

A hyperpath flow state $\mathbf{X}_{NS}$ induces a network flow state $\mathbf{x}_{AS} = \mathbf{A}(\mathbf{X}_{NS})$ in the following way:

$$x_{as} = \sum_{n\in N} \sum_{h\in H_{ns}} q_{ns}^h \sum_{r\in R_{ns}(h)} \hat{h}_r \, 1_{\{a\in r\}}\,, \tag{16}$$

in which $R_{ns}(h)$ denotes the set of elementary paths within h from n to s (positive paths with no node repetition), $1_{\{a\in r\}}$ is equal to 1 if $a \in r$ or 0 otherwise, and $\hat{h}_r = \prod_{a\in r} \hat{h}_a$ is the proportion of flow carried out from n to s via h by route r.

From the basic properties of hyperpaths (Nguyen and Pallotino, 1988), a feasible hyperpath flow state induces a feasible network flow sate. Non-negativity (13a) comes from (15a), (16) and the non-negativity of $\hat{h}_a$ hence of $\hat{h}_r$. The conservation of the destination flow at node m in (13b) is derived as follows:

$$\sum_{a\in A_m^+} x_{as} - \sum_{a\in A_m^-} x_{as} = \sum_{n\in N}\sum_{h\in H_{ns}} q_{ns}^h \sum_{r\in R_{ns}(h)} \hat{h}_r \,\Delta_{rm},$$

in which $\Delta_{rm} = \sum_{a\in A_m^+} 1_{\{a\in r\}} - \sum_{a\in A_m^-} 1_{\{a\in r\}}$. Now, if m is not incident to r then all the terms in the sum are zero, yielding $\Delta_{rm} = 0$. Then, if m is incident to r but not an endpoint node n_r^+ or s, then $\Delta_{rm} = 0$ since one and only one arc in A_m^+ has $1_{\{a\in r\}} = 1$ and the same applies to A_m^-. Lastly, if $m = \mathrm{n}_r^+$ then $\Delta_{rm} = 1$ since $1_{\{a\in r\}} = 0$ for all a in A_m^-. Thus

$$\begin{aligned}\sum_{a\in A_m^+} x_{as} - \sum_{a\in A_m^-} x_{as} &= \sum_{h\in H_{ms}} q_{ms}^h \sum_{r\in R_{ms}(h)} \hat{h}_r \\ &= \sum_{h\in H_{ms}} q_{ms}^h \\ &= q_{ms}\end{aligned}$$

from the assumption on $\mathbf{X}_{NS}$.

5.2 Definition of traffic equilibrium using an NCP

Definition 3, Traffic Equilibrium. *Given the OD trip matrix* $\mathbf{q}_{NS} = [q_{ns}]_{n\in N,\, s\in S}$, *a hyperpath flow state* $\mathbf{X}_{NS} = [q_{ns}^h : s\in S, n\in N\setminus\{s\}, h\in H_{ns}]$ *with* $\mathbf{x}_{AS} = \mathbf{A}(\mathbf{X}_{NS})$ *is a traffic equilibrium if there exists a matrix* $\boldsymbol{\mu}_{NS} = [\mu_{ns} : s\in S, n\in N\setminus\{s\}]$ *such that, for all* $s\in S, n\in N\setminus\{s\}$*:*

$$q_{ns}^h \geq 0 \quad \forall h\in H_{ns} \tag{17a}$$

$$\sum_{h\in H_{ns}} q_{ns}^h = q_{ns} \tag{17b}$$

$$\mathrm{C}_{ns}(h,\mathbf{x}_{AS}) - \mu_{ns} \geq 0 \quad \forall h\in H_{ns} \tag{17c}$$

$$q_{ns}^h . [\mathrm{C}_{ns}(h,\mathbf{x}_{AS}) - \mu_{ns}] = 0 \quad \forall h\in H_{ns} \tag{17d}$$

The interpretation is as follows: to each destination s, from each node n, all hyperpath costs $\mathrm{C}_{ns}(h,\mathbf{x}_{AS})$ are no less than μ_{ns} and only the hyperpaths with cost $\mathrm{C}_{ns}(h,\mathbf{x}_{AS}) = \mu_{ns}$ (hence minimal) may carry a positive flow q_{ns}^h. This coincides with the definition of user equilibrium in traffic assignment since Wardrop in 1952, according to which each user makes his routing choice so as to minimize his own travel cost.

The set of conditions (17) is a non-linear complementarity problem (NCP) in the variable $(\mathbf{X}_{NS}, \boldsymbol{\mu}_{NS})$, with associated cost function as follows:

$$(\mathbf{X}_{NS}, \boldsymbol{\mu}_{NS}) \mapsto [\mathrm{C}_{ns}(h, \mathbf{A}(\mathbf{X}_{NS})) - \mu_{ns} : s\in S, n\in N\setminus\{s\}, h\in H_{ns}].$$

5.3 Equilibrium characterization using a VIP

Theorem 1, Characterization of Traffic Equilibrium. *A hyperpath flow state* $\mathbf{X}_{NS} = [q_{ns}^h : s \in S, n \in N \setminus \{s\}, h \in H_{ns}]$ *that is feasible for the OD trip matrix* $\mathbf{q}_{NS} = [q_{ns}]_{n \in N,\, s \in S}$ *is a traffic equilibrium if and only if, for any feasible hyperpath flow state* $\mathbf{Y}_{NS} = [\eta_{ns}^h : s \in S, n \in N \setminus \{s\}, h \in H_{ns}]$*, it holds that:*

$$\chi(\mathbf{X}_{NS}).(\mathbf{Y}_{NS} - \mathbf{X}_{NS}) \geq 0\,, \tag{18}$$

in which $\chi(\mathbf{X}_{NS}) = [\mathrm{C}_{ns}(h, \mathbf{A}(\mathbf{X}_{NS})) : s \in S, n \in N \setminus \{s\}, h \in H_{ns}]$.

Proof. Assume first that $\mathbf{X}_{NS}$ is an equilibrium state. Letting $\boldsymbol{\mu}_{NS}$ be the associated matrix of dual variables as of (17), for any $\mathbf{Y}_{NS} = [\eta_{ns}^h]$ it stems from (17c) that

$$\eta_{ns}^h . [\mathrm{C}_{ns}(h, \mathbf{x}_{AS}) - \mu_{ns}] \geq 0 \quad \forall h \in H_{ns}$$

Summation over $h \in H_{ns}$ yields that $\sum_{h \in H_{ns}} \eta_{ns}^h \mathrm{C}_{ns}^h \geq \sum_{h \in H_{ns}} \eta_{ns}^h \mu_{ns} = q_{ns} \mu_{ns}$.

From (17d) we get that $\sum_{h \in H_{ns}} q_{ns}^h \mathrm{C}_{ns}^h = \sum_{h \in H_{ns}} \eta_{ns}^h \mu_{ns} = q_{ns} \mu_{ns}$, hence

$$\sum_{h \in H_{ns}} \mathrm{C}_{ns}^h (\eta_{ns}^h - q_{ns}^h) \geq 0\,.$$

This yields (18) after summing over $s \in S$ and $n \in N$.

Conversely, assume that (18) holds at $\mathbf{X}_{NS}$ and take $\mathbf{Y}_{NS} = [\eta_{ns}^h]$ equal to $\mathbf{X}_{NS}$ except perhaps on node-destination pair (n,s): if there is only one hyperpath h in H_{ns} then letting $\mu_{ns} = \mathrm{C}_{ns}(h, \mathbf{x}_{AS})$ obviously satisfies (17c,d). If there are two or more hyperpaths in H_{ns}, for any h with $q_{ns}^h > 0$ let us define $\eta_{ns}^h = q_{ns}^h - \varepsilon$ and $\eta_{ns}^{h'} = q_{ns}^{h'} + \varepsilon$: then (18) yields that

$$(\mathrm{C}_{ns}^{h'} - \mathrm{C}_{ns}^h)\varepsilon \geq 0\,,$$

which implies that the cost of any hyperpath h with positive flow q_{ns}^h in $\mathbf{X}_{NS}$ is minimal on H_{ns}: on defining $\mu_{ns} = \min_{h \in H_{ns}} \mathrm{C}_{ns}(h, \mathbf{x}_{AS})$, (17c,d) follows.

5.4 Equilibrium properties

Theorem 2, Existence of traffic equilibrium. *Assuming that the cost functions* C_a *are continuous, there exists an equilibrium state for the model of transit assignment with seat congestion.*

Proof. Using the VIP formulation, the assumption ensures that the cost functions $\mathbf{x}_{AS} \mapsto \mathrm{C}_{ns}(h, \mathbf{x}_{AS})$ are continuous with respect to the network flow state $\mathbf{x}_{AS}$. When this state is a continuous function $\mathbf{A}$ of variable $\mathbf{X}_{NS}$ (a combination of hyperpath flows), the composed function $\mathbf{X}_{NS} \mapsto \mathrm{C}_{ns}(h, \mathbf{A}(\mathbf{X}_{NS}))$ is continuous, as is function $\mathbf{X}_{NS} \mapsto \chi(\mathbf{X}_{NS})$. As the domain set $\mathbf{E}_{\mathbf{X}}$ of feasible hyperpath flows is convex and compact, this ensures that the variational inequality admits at least one solution.

The uniqueness of an equilibrium does not hold in general, as is shown in the next section.

5.5 A criterion of duality gap for convergence to equilibrium

The VIP formulation is useful to establish the existence of an equilibrium state and also to design an equilibration algorithm. As our model satisfies the assumptions in Wu *et al* (1994) about the mapping of arc costs, the two methods that these authors showed to be globally convergent are applicable to search for equilibrium: namely the linearized Jacobi method and the projection method. However, for the sake of simplicity we prefer to use the well-known Method of Successive Averages (MSA): although we have not demonstrated this method to be globally convergent on theoretical grounds, it is possible to assess its convergence in any application by evaluating the duality gap at the current flow state. Indeed, when the duality gap comes close to zero it is guaranteed that the current flow state is close to a traffic equilibrium, since the duality gap is continuous when the VIP function χ is continuous.

Then the remaining issue is to evaluate the duality gap at each current flow state along the iterations in the MSA. It would be most cumbersome to compute the duality gap in a straightforward way, since this would require not only to store all the used hyperpaths in the computer memory, but also to redo a hyperpath costing along each of them to evaluate their cost under the current traffic conditions. Hereafter a simple method is provided, which only requires the updating of one real variable along the iterations in the MSA. It is applicable to any hyperpath-based transit assignment model in which the hyperpath cost depends on the flow state only through the arc costs, not the node costs (thus the line frequency cannot be related to the traffic flows).

Let us consider hyperpath flow state $\mathbf{X}_{NS}$ [resp. $\mathbf{Z}_{NS}$] with associated network flow state $\mathbf{x}_{AS}$ [resp. $\mathbf{z}_{AS}$] and coordinate q_{ns}^{h} [resp. η_{ns}^{h}] on hyperpath h from node n to destination s. Our aim is to evaluate the cost to transport the flow $\mathbf{Z}_{NS}$ under the travel conditions associated to $\mathbf{X}_{NS}$, defined as

$$\mathbf{Z}_{NS}.\chi(\mathbf{X}_{NS}) = \sum_{s\in S}\sum_{n\in N}\sum_{h\in H_{ns}} \eta_{ns}^{h}\, \mathrm{C}_{ns}(h,\mathbf{x}_{AS}) \,. \tag{19}$$

Let us split this cost into an arc-based part and a node-based part. The arc-based part is

$$\begin{aligned}\chi_A(\mathbf{Z}_{NS}:\mathbf{X}_{NS}) &= \sum_{s\in S}\sum_{n\in N}\sum_{h\in H_{ns}} \eta_{ns}^{h} \sum_{r\in R_{ns}(h)} \hat{h}_r \sum_{a\in r} \mathrm{c}_a(\mathbf{x}_{AS}) \\ &= \sum_{a\in A} \mathrm{c}_a(\mathbf{x}_{AS}) \sum_{s\in S}\sum_{n\in N}\sum_{h\in H_{ns}} \eta_{ns}^{h} \sum_{r\in R_{ns}(h)} \hat{h}_r 1_{\{a\in r\}}\end{aligned}$$

hence

$$\chi_A(\mathbf{Z}_{NS}:\mathbf{X}_{NS}) = \sum_{a\in A} \mathrm{c}_a(\mathbf{x}_{AS})\, z_a \,, \tag{20}$$

which is easy to evaluate whatever the $\mathbf{Z}_{NS}$ state.

About the node-based part,

$$\chi_N(\mathbf{Z}_{NS}:\mathbf{X}_{NS}) = \sum_{s\in S}\sum_{n\in N}\sum_{h\in H_{ns}} \eta_{ns}^{h} \sum_{r\in R_{ns}(h)} \hat{h}_r \sum_{m\in r} \mathrm{w}_m^{h}(\mathbf{x}_{AS})$$

let us notice that w_m^h does not depend on $\mathbf{x}_{AS}$ (when no congestion effect is involved in line frequency) and that any term $\sum_{r\in R_{ns}(h)} \hat{h}_r \sum_{m\in r} \mathrm{w}_m^h$ is a function of m, h and $R_{ns}(h)$ only, from here denoted as ρ_{ns}^h. Thus

$$\chi_N(\mathbf{Z}_{NS}) = \sum_{s\in S}\sum_{n\in N}\sum_{h\in H_{ns}} \eta_{ns}^h \rho_{ns}^h , \tag{21}$$

which is a linear function of $\mathbf{Z}_{NS}$ through the η_{ns}^h.

Consider now an auxiliary hyperpath flow state $\mathbf{Y}_{NS}^{(k)}$ that arises at the k-th iteration in an MSA application with sequence of step sizes $(\zeta_k)_{k\geq 0}$ such that $\zeta_0 = 1$. The transport cost $\mathbf{Y}_{NS}^{(k)}.\chi(\mathbf{X}_{NS}^{(k)})$ of that auxiliary state under the traffic conditions induced by the current flow state $\mathbf{X}_{NS}^{(k)}$ is merely

$$\mathbf{Y}_{NS}^{(k)}.\chi(\mathbf{X}_{NS}^{(k)}) = \sum_{s\in S}\sum_{n\in N} q_{ns} u_{ns}^{(k)} ,$$

in which $u_{ns}^{(k)}$ is the unit travel cost from node n to destination s under flow state $\mathbf{X}_{NS}^{(k)}$: this stems from the definition of the auxiliary state by assignment of all the OD flow to the hyperpath of minimal cost under the current traffic conditions. On the basis of (20), it is straightforward to recover

$$\begin{aligned}\chi_N(\mathbf{Y}_{NS}^{(k)}) &= \mathbf{Y}_{NS}^{(k)}.\chi(\mathbf{X}_{NS}^{(k)}) - \chi_A(\mathbf{Y}_{NS}^{(k)} : \mathbf{X}_{NS}^{(k)}) \\ &= [\sum_{s\in S}\sum_{n\in N} q_{ns} u_{ns}^{(k)}] - \sum_{a\in A} \mathrm{c}_a(\mathbf{x}_{AS})\, z_a \end{aligned} \tag{22}$$

Let us turn our attention to the current state $\mathbf{X}_{NS}^{(k)}$, which is constructed as the weighted average of the previous auxiliary states $\mathbf{Y}_{NS}^{(j)}$, $j<k$, with weight coefficients $\zeta_j^{(k)} = \zeta_j / \Gamma_k$ where $\Gamma_k = \sum_{j=0}^{k-1} \zeta_j$. The arc-based transport cost is evaluated straightforwardly. Being a linear function, the node-based transport cost can be derived from the linear decomposition of $\mathbf{X}_{NS}^{(k)}$:

$$\chi_N(\mathbf{X}_{NS}^{(k)}) = \chi_N(\sum_{j=0}^{k-1} \zeta_j^{(k)} \mathbf{Y}_{NS}^{(j)}) = \sum_{j=0}^{k-1} \zeta_j^{(k)} \chi_N(\mathbf{Y}_{NS}^{(j)}) . \tag{23}$$

Lastly, let us define a sequence β_k in the following way: $\beta_0 = \zeta_0 \chi_N(\mathbf{Y}_{NS}^{(0)})$ and

$$\beta_{k+1} = \beta_k + \zeta_k \chi_N(\mathbf{Y}_{NS}^{(k)}) . \tag{24}$$

Then the ratio β_k / Γ_k amounts to the transport cost of the current flow state under its own traffic conditions:

$$\chi_N(\mathbf{X}_{NS}^{(k)}) = \beta_k / \Gamma_k . \tag{25}$$

To sum up, at iteration k in the MSA algorithm the duality gap is

$$\begin{aligned}\mathrm{DG}_k &= (\mathbf{X}_{NS}^{(k)} - \mathbf{Y}_{NS}^{(k)}).\boldsymbol{\chi}(\mathbf{X}_{NS}^{(k)}) \\ &= \frac{\beta_k}{\Gamma_k} + [\sum_{a\in A} \mathrm{c}_a(\mathbf{x}_{AS}^{(k)})\, x_a^{(k)}] - [\sum_{s\in S}\sum_{n\in N} q_{ns} u_{ns}^{(k)}]\end{aligned} \tag{26}$$

5.6 Assignment algorithm: an MSA

Traffic assignment to a transit network with seat congestion can be performed by means of the following equilibration algorithm, which makes use of two network flow states $\mathbf{x}_{AS}$ for a current state and $\mathbf{y}_{AS}$ for an auxiliary state, two related overall arc flow vectors $\mathbf{x}_A$ and $\mathbf{y}_A$, one set of node potentials $\mathbf{u} = [u_n]_{n\in N}$, an iteration counter k, variables β, Γ, U, W and Z. Input variables consist in G, L, arc costs $\underline{\mathbf{c}} = [\underline{c}_a]_{a\in A}$ and $\bar{\mathbf{c}} = [\bar{c}_a]_{a\in A}$, $\mathbf{f} = [f_\ell]_{\ell\in L}$, $\boldsymbol{\kappa} = [\kappa_\ell]_{\ell\in L}$, the OD trip matrix $\mathbf{q} = [q_{os}]_{o\in O_s, s\in S}$, a tolerance ε on the convergence level, and a sequence of decreasing positive numbers $(\zeta_k)_{k\geq 0}$ with $\zeta_0 = 1$.

The equilibration algorithm is made up of five steps:

Initialization. Set $\mathbf{x}_{AS} := 0$ and $\mathbf{x}_A := 0$. Let $k := 0$, $\beta := 0$ and $\Gamma := 0$.

Cost-Flow Relationship. Evaluate the arc costs $c_a = \mathrm{C}_a(\mathbf{x}_{AS})$ for all $a \in A$ as in subsection 4.3.

Network Costing and Flow Loading. Let $U := 0$ and $\mathbf{y}_A := 0$. For every destination node $s \in S$:

- Find the optimal hyperpath destined to s under the current arc costs, yielding node potentials u_n.
- Load the OD flows $[q_{ns} : n \in N]$ on the currently optimal hyperpaths to s, yielding arc flows y_{as}.
- Let $U := U + \sum_{n\in N} q_{ns} u_{ns}$. Let $y_a := y_a + y_{as}$ for all $a \in A$.

Flow Update. Let $W := U - \sum_{a\in A} c_a y_a$. Let $\Gamma := \Gamma + \zeta_k$ and $\beta := \beta + \zeta_k W$. Let $Z := \frac{\beta}{\Gamma} - U + \sum_{a\in A} c_a x_a$. Then let $\mathbf{x}_{AS} := \mathbf{x}_{AS} + \zeta_k(\mathbf{y}_{AS} - \mathbf{x}_{AS})$ and $\mathbf{x}_A := \mathbf{x}_A + \zeta_k(\mathbf{y}_A - \mathbf{x}_A)$.

Convergence Test. If $Z \leq \varepsilon$ then terminate, else let $k := k+1$ and go to step *Cost Flow Relationship*.

This is a mere method of successive averages, in which the auxiliary flow state $\mathbf{y}_{AS}$ is a user-optimized assignment of all the OD flows on the basis of the costs induced by the current flow state $\mathbf{x}_{AS}$. The convergence criterion Z is the duality gap of the previous subsection.

6. A simple case of hyperpath choice

To illustrate the seat congestion effect in conjunction with hyperpath choice in the context of traffic assignment, let us consider a simple case inspired from the transit network in the Paris area. At the morning peak period, the most crowded transit line in Paris is a heavy metro line called line A of RER (the Réseau Express Régional, in fact a regional train with very high frequency), particularly so in the westbound direction because there is a major Business Centre at station La Défense, and at the Châtelet station which allows for transfer between three RER lines and 5 metro lines. Trip-makers willing to board line A at Châtelet to La Défense have to wait for the second or third train to come, because of a queue. Thus it is highly likely that on-board they will stand within high crowding.

To avoid the congestion, some trip-makers eventually divert to an alternative route: those coming from the Belleville metro station may board line A either at Châtelet after using line 11, or at station Nation after using line 2. Boarding line A at Nation is easier as there is no queue.

6.1 Case data and test parameters

Figure 4 illustrates the network and transport data for a trip from Belleville to La Défense. On line A, the leg from Châtelet to La Défense costs either 11 min at seating or 20 min at standing, on the basis of Stated Preference data (Debrincat *et al*, 2006). The leg from Nation to Châtelet costs 6 min when seated, which is assumed here. The leg from Belleville to Châtelet via line 11 takes 15 min including transfer, as takes the leg from Belleville to Nation via line 2 also including transfer.

Service frequencies are about 20 veh/h on line 11, 30 veh/h on line 2 and 30 veh/h on line A. The waiting weighting factor α is set equal to 1.

The traffic flows and capacity are described on the basis of three parameters q, ξ and κ: q is the OD flow from Belleville to La Défense; ξ is the flow from other origins boarding line A at Châtelet; κ is the seat capacity available on line A at the Nation station.

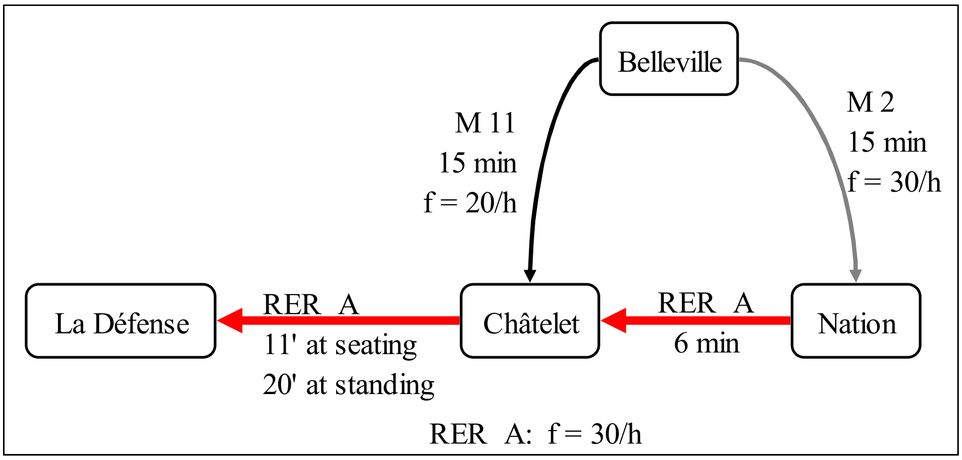

Fig 4. Transit network for Belleville – La Défense trips

6.2 On paths, hyperpaths and costs

Path 1 with access to line A at Châtelet and Path 2 with access to line A at Nation make up three hyperpaths: hyperpath 1 made up of path 1 alone; hyperpath 2 made up of path 2 alone; and hyperpath 3 made up the combination of path 1 and path 2, with line combination at the Belleville station.

The respective hyperpath costs are the following functions of the probability p to sit on boarding at the Châtelet station:

$$c_1^{\min} = 11p + 20(1-p) + \tfrac{60}{30} + 15 = 37 - 9p$$

$$\widehat{c}_1 = c_1^{\min} + \frac{\alpha}{f(\text{line }11)} = 40 - 9p$$

$$c_2^{\min} = 11 + 6 + \tfrac{60}{30} + 15 = 34$$

$$\widehat{c}_2 = c_2^{\min} + \frac{\alpha}{f(\text{line }2)} = 36$$

$$\widehat{c}_3 = \frac{\alpha + f_{11}.\, c_1^{\min} + f_2.\, c_2^{\min}}{f_{11} + f_2} = 36.4 - 3.6p$$

6.3 Equilibrium analysis and non-uniqueness

Let us first assess the domain where path 1 has cost less than path 2:

$$\widehat{c}_1 \le \widehat{c}_2 \iff p \ge \tfrac{4}{9}$$

Then let us assess the domain where the combination, hyperpath 3, is better than any single path: the conditions are that

$$\begin{cases} c_1^{\min} \le \widehat{c}_2 \\ c_2^{\min} \le \widehat{c}_1 \end{cases} \iff p \in [\tfrac{1}{9}, \tfrac{2}{3}]$$

Thus the supply-demand equilibrium takes on the following states, depending on the value of parameter p:

$p < \tfrac{1}{9}$: only path 2 is used.

$p = \tfrac{1}{9}$: both path 2 and hyperpath 3 may be used.

$p \in]\tfrac{1}{9}, \tfrac{2}{3}[$: only hyperpath 3 is efficient, not path 1 or path 2 used individually.

$p = \tfrac{2}{3}$: both path 1 and hyperpath 3 may be used.

$p > \tfrac{2}{3}$: only path 1 is used.

Knowing which hyperpaths are used, we may relate p to parameters κ, ξ and q:

When path 1 alone is used, then $p = \min\{1, \tfrac{\kappa}{\xi+q}\}$, so the requirement that $p > \tfrac{2}{3}$ yields that $q < \tfrac{3}{2}\kappa - \xi$.

When path 2 alone is used, then $p = \max\{\tfrac{\kappa-q}{\xi}, 0\}$, so the requirement that $p < \tfrac{1}{9}$ yields that $q > \kappa - \xi/9$.

If hyperpath 3 alone is used, then

$$p = \frac{\kappa - \pi_2.q}{\xi + \pi_1.q}$$

where $\pi_2 = f_2/(f_2 + f_{11})$ and $\pi_1 = 1 - \pi_2$: the requirement that $\frac{1}{9} < p < \frac{2}{3}$ yields that

$$\frac{\kappa - \xi/9}{\pi_2 + \pi_1/9} > q > \frac{\kappa - 2\xi/3}{\pi_2 + 2\pi_1/3}.$$

At states where several hyperpaths are used, the OD flow q splits into hyperpath flows x_i so that $p = \frac{\kappa - \pi_2 x_3 - x_2}{\xi + \pi_1 x_3 + x_1}$, which yields

$$px_1 + x_2 + (p\pi_1 + \pi_2)x_3 = \kappa - p\xi.$$

At $p = \frac{2}{3}$, $x_2 = 0$ so $\frac{2}{3}x_1 + (\pi_2 + \frac{2}{3}\pi_1)x_3 = \kappa - \frac{2}{3}\xi$: the smallest q complying to this requirement is $(\kappa - \frac{2}{3}\xi)/(\pi_2 + \frac{2}{3}\pi_1)$ if all OD flow is carried by x_3 and none by x_1, and the largest is $\frac{3}{2}\kappa - \xi$ all carried by x_1.

At $p = \frac{1}{9}$, $x_1 = 0$ so $x_2 + (\pi_2 + \frac{1}{9}\pi_1)x_3 = \kappa - \frac{1}{9}\xi$: the smallest q complying to this requirement is $\kappa - \frac{1}{9}\xi$ if all OD flow is carried by x_2 and none by x_3, and the largest is $(\kappa - \frac{1}{9}\xi)/(\pi_2 + \frac{1}{9}\pi_1)$ all carried by x_3.

At a given OD flow q, from one up to five traffic equilibria may exist:

- on $[0, \frac{\kappa - 2\xi/3}{\pi_2 + 2\pi_1/3}]$ there is one equilibrium state with $p > \frac{2}{3}$;
- on $[\frac{\kappa - 2\xi/3}{\pi_2 + 2\pi_1/3}, \frac{3}{2}\kappa - \xi]$ there are three equilibria, first with $p > \frac{2}{3}$, second with $p = \frac{2}{3}$ and last with $p \in]\frac{1}{9}, \frac{2}{3}[$;
- on $[\frac{3}{2}\kappa - \xi, \kappa - \frac{1}{9}\xi]$ there is one equilibrium with $p \in]\frac{1}{9}, \frac{2}{3}[$ (assuming a non-empty interval i.e. $9\kappa < 16\xi$);
- on $[\kappa - \frac{1}{9}\xi, \frac{\kappa - \xi/9}{\pi_2 + \pi_1/9}]$ there are two equilibria with either $p \in]\frac{1}{9}, \frac{2}{3}[$ or $p = \frac{1}{9}$;
- on $[\frac{\kappa - \xi/9}{\pi_2 + \pi_1/9}, \kappa]$ one equilibrium state with $p < \frac{1}{9}$ (assuming a non-empty interval);
- the case where $[\frac{\kappa - 2\xi/3}{\pi_2 + 2\pi_1/3}, \frac{3}{2}\kappa - \xi] \cap [\kappa - \frac{1}{9}\xi, \frac{\kappa - \xi/9}{\pi_2 + \pi_1/9}] \neq \emptyset$ might also arise, leading to five equilibria at any value of q within the intersection.

For $q > \kappa$ our study should be modified to take into account the seat congestion on the Nation – Châtelet segment.

Figure 5 depicts the hyperpath costs with respect to OD flow q, for parameter values of κ = 10,000 and ξ = 8,000. As $\pi_1 = .4$ and $\pi_2 = .6$, $\frac{\kappa - \xi/9}{\pi_2 + \pi_1/9} > \kappa$ so the regime with $p < \frac{1}{9}$ falls out of the range $[0, \kappa]$ for q. From 0 to $\frac{\kappa - 2\xi/3}{\pi_2 + 2\pi_1/3}$ only path 1 is used. From $\frac{\kappa - 2\xi/3}{\pi_2 + 2\pi_1/3}$ to $\frac{3}{2}\kappa - \xi$, either path 1 alone, or hyperpath 3 alone, or both can support equilibrium. From $\frac{3}{2}\kappa-\xi$ to $\kappa-\xi/9$ only hyperpath 3 is used. From $\kappa-\xi/9$ to κ, either path 2 alone, or hyperpath 3 alone, or both can support equilibrium.

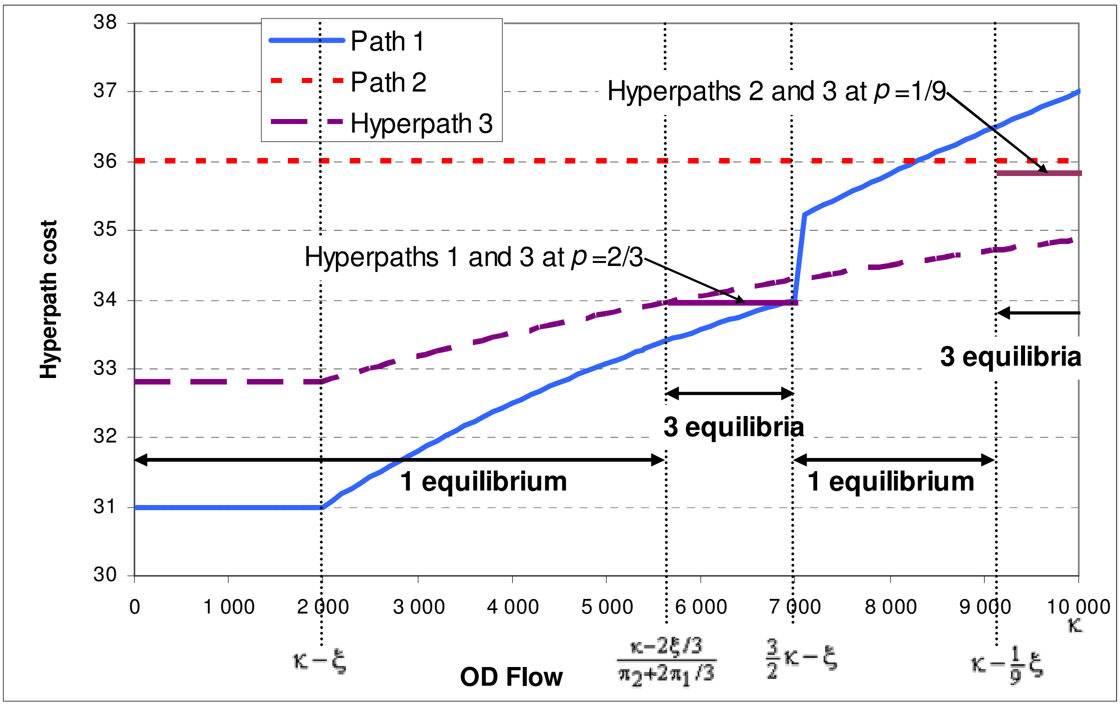

Fig. 5. Equilibrium states in a cost – flow diagram

7. Conclusion

The contribution of the paper is summarized in the abstract, from comfort state representation at the level of the leg segment, to the hyperpath formulation of supply-demand equilibrium, passing by the notions of service mode and leg cost as a random variable.

The network assignment algorithm was implemented in cooperation with the RATP – Paris metro operator, with some refinements that pertain to the issues of line sub-services and of generalized cost including mean and variance of leg cost: results are available in two related papers (Leurent *et al*, 2007, 2008). In short, the equilibrium results appear to be stable in terms of arc flows and elementary costs (including access-egress costs by service mode), though variable in terms of selected hyperpaths. The comparison between the seat-congestion model and the previous model, a standard hyperpath transit assignment model, showed that at the morning peak the transit flows could vary by as much as 30% along the metro lines that carry about 5,000 to 20,000 passengers per hour and per direction. Along the regional train lines the relative change is limited to a small percentage, because there are few alternative transit routes and their base flow is as high as 20,000 to 60,000 passengers per hour and per direction.

Further work on the seat congestion model might focus on the following issues:

- Identification of more comfort states, e.g. several types of seats or comfort classes.
- En-route choice of egress station depending on the current user's comfort state.

- A discomfort function may be associated to each comfort state. Provided that seating is still more valuable than standing, the loading algorithm is unchanged and in the costing algorithm it is required to evaluate the discomfort function only once for each segment and state.
- Inclusion of "transaction" costs because of effort to get a seat; in conjunction with modelling other congestion effects in transit such as access-egress congestion.
- Inclusion of costs which are nonlinear functions of the number of segments in leg or the leg travel time.
- Taste differentiation across the riders, so as to model diverse trade-offs between state discomfort, travel time and also the fare.
- Discrete choice model to compete or not for a seat, so that the sitting probability might be varied among candidates. This would enable to model a higher share for long legs than for short legs.
- Discrete choice model of the user's willingness to obtain a given comfort state by associating a "choice" probability to each comfort state.
- Priority rules for social or commercial purpose.

In the author's opinion, the discomfort function, taste differentiation and discrete choice extensions could yield some properties of equilibrium uniqueness.

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OUTREACH: segment congestion + infinite capacity at standing, <> frequency effect

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