Application of Mean Field Games to Growth Theory

Jean-Michel Lasry, Pierre-Louis Lions, Olivier Guéant

Abstract

This article discusses the interaction between economic growth in the sense of human capital accumulation and the dynamics of inequalities. We use a mean-field game framework in which individuals improve their human capital both to improve their wages and to avoid potential competition with less skilled individuals.
Our contribution is twofold. First, we exhibit a mechanism in which competition between a continuum of people regarding human capital accumulation lead to growth. Second, our model highlights the importance of Pareto distributions to describe inequalities since power laws appear naturally as explicit solutions of our problem.

Introduction

Recent theories of economic growth, following the Schumpeterian model developed essentially by P. Aghion and P. Howitt ([AH92]), mainly focus on research and industrial innovation as the only way to generate a non decreasing growth process.
Here, we are back to former ideas to explain growth with human capital accumulation only. However, our framework is quite new since we use the theory designed by J.-M. Lasry and P.-L. Lions on mean-field games ([LL06a, LL06b, LL07a]). This new framework allows us to model in a simple way the interaction between people and growth will be a by-product of the interaction and competition between people to improve their welfare.
We basically model a continuum of individuals whose wages depend not only on their own human capital but also on the whole distribution of human capital. This distribution dependency is important in two different ways. First, we take into account a competition effect in the labor market. Second we model the easiness or difficulty to improve human capital depending on the proximity to the technological frontier.
Like in the recent paper by Aghion et al (2001) ([AHHV01]) or as in Aghion and Howitt ([AH]), growth is fostered by an *escape competition* effect. However, in our setting, it is the threat of competition that forces people to improve their human capital and not competition by itself: because individuals less skilled than a given person represent a threat for this person, she is forced to accumulate human capital. That leads even-

tually to economic growth.
In addition to the growth process, the mean-field game framework allows us to deal with the distribution of wages across people. The distribution of wages is indeed known to be quite well described by a Pareto distribution (at least for the tail) and we show that Pareto distributions for human capital and wages are indeed stable in our setting.
Our contribution is therefore to shed light on a growth process which is a consequence of a competition effect and which is compatible with power law distributions for inequalities.

In the first section we will present the setup of our model and derive a solution using classical methods (Euler Lagrange). In a second part, we exhibit comparative statics and then, in a third part, we go deeply into an analysis of the mechanisms that are involved in our model. In the fourth section, we use the mean field game partial differential equations to solve the problem in a different way and we generalize the solution to a stochastic framework. Partial proofs are given in the appendix: a more mathematical paper to be published may contain more precise proofs.

1 The model

1.1 Introduction

We assume that there is initially a working population of size 1 with a given distribution of human capital. Human capital will be denoted q and the distribution, at time t, will be referred to as $m(t, q)$.
For a given worker, wage per hour will be a function not only depending on the individual human capital but also on the scarcity of her specific human capital[1]. In other words, the salary of a worker with human capital q is, at time t, given by:

$$w(t,q) = G(q, m(t,q)), \qquad G(\underbrace{\cdot}_{+}, \underbrace{\cdot}_{-})$$

The main interest of this equation is to model a competition effect in the labor market: unskilled people have a small salary because they are unskilled and also because most of the time they are so numerous that they can be replaced by other similar unskilled people[2].

Individuals - who live forever - can improve their human capital with a cost depending on two factors. First, the cost, in monetary terms, is a function of human capital change and, second, it is also a function of the position of initial human capital in the distribution. More precisely, we will assume that the cost (in monetary terms) at time t is given by:

[1] One can think this is a very restrictive way to model competition since it imply a very low substitution between people. Another setting that leads to similar results could be to replace the density function by the tail function.

[2] We assume here, as it will be the case in what follows, that $m(t, \cdot)$ is a decreasing function.

$$C(t,q,\frac{dq}{dt}) = H(\frac{dq}{dt},\overline{F}(t,q)), \qquad H(\underbrace{\cdot}_{+},\underbrace{\cdot}_{-})$$

where $\overline{F}(t,q) = \int_q^\infty m(t,u)du$ is the number of people in the population with a human capital greater than q. That is to say it is more costly for skilled workers to improve their human capital than for unskilled workers. This hypothesis is relevant since it is often more difficult to improve human capital for an individual in the right tail of the distribution since she is near the technological frontier.

1.2 The optimization problem

As in the classical Mincerian approch to human capital accumulation, we are going to suppose that people improve their human capital all life long. However, we do not focus on schooling choices in the sense that we consider a given working population. Human capital accumulation must therefore be seen as the consequence of on-the-job training.
Each individual chooses her effort continuously to maximize her utility. Her intertemporal utility is classically given by an expression of the form:

$$\int_0^\infty u(c_t)e^{-\rho t}dt$$

with a wealth constraint $\dot{s}_t \leq rs_t + \left(w(t,q) - C(t,q,\frac{dq}{dt})\right) - c_t$. This gives a unique intertemporal constraint that is:

$$\int_0^\infty c_t e^{-rt}dt \leq s_0 + \int_0^\infty \left(w(t,q) - C(t,q,\frac{dq}{dt})\right) e^{-rt}dt$$

Therefore, if we assume that r is exogenous the only thing the agent is going to maximize is the right hand side of the constraint which is her intertemporal wealth.
Basically, an individual with human capital q has the following program:

$$Max_{(q_s),q_0=q} \int_0^\infty \left[G(q_s,m(s,q_s)) - H(a(s,q_s),\overline{F}(s,q_s))\right] e^{-rs}ds$$

where $a(\cdot,\cdot)$ is defined by $dq_s = a(s,q_s)ds$.

1.3 Resolution

1.3.1 A specific setup

To solve the problem we need to specify the two functions G and H. Our specification is the following:

To derive the wage function we are going to start with a discrete setting. Imagine that there are n types of workers with human capital

$q_1, \dots, q_n$.
A standard production function for a representative firm would be[3]

$$Y = A \sum_{i=1}^{n} q_i^{\alpha} L_i^{1-\beta} \qquad \alpha > 0, \beta \in (0;1]$$

The wage associated to a worker of type i, w_i, is then proportional to $\frac{q_i^\alpha}{L_i^\beta}$.

Therefore, if we go from this discrete setting to a continuous one, we can assume that:

$$G(q, m(t,q)) = \begin{cases} C\frac{q^\alpha}{m(t,q)^\beta}, & \text{if } q \text{ is in the support of } m(t,\cdot) \\ 0 & \text{otherwise} \end{cases}$$

For the cost, we use the simple specification that follows:

$$H(\frac{dq}{dt}, \overline{F}(t,q)) = \frac{E}{\varphi}\frac{\left(\frac{dq}{dt}\right)^\varphi}{\overline{F}(t,q)^\delta}, \quad \forall q \text{ in the support of } m(t,\cdot)$$

where C and E are two constants and where α, β, δ and φ are four positive parameters subject to technical constraints that are: $\alpha + \beta = \varphi$, $\beta = \delta$ and we want typically φ to be strictly greater than 1 .
This specification can be considered *ad hoc* but in fact it must be regarded as quite general since we have two degrees of freedom to choose the parameters. An easier specification without any degree of freedom would have been to set $\alpha = \beta = \delta = 1$ and $\varphi = 2$ but we want to remain quite general.

Now, we assume that the initial distribution of human capital is a Pareto distribution[4]. We can use a normalization and assume that the minimal point of the initial distribution is 1. The Pareto coefficient[5] of the initial distribution is denoted k, so that:

$$m(0,q) = k\frac{1}{q^{k+1}}1_{q\geq 1}$$

This Pareto distribution is central in the study of economic inequalities and will be stable in our model in the sense that our solution involves Pareto distributions of human capital at all time.

1.3.2 Explicit resolution

To start the explicit resolution of our problem, let's begin with the Euler Lagrange equation associated to it.

[3]The limit case where $\beta = 1$ is a logarithmic case.
[4]This distribution is usually used in the literature on economic inequalities (see Piketty's papers for example, or Atkinson ([Atk05]))
[5]It is usually a measure of inequality. For example, it is related to the Gini coefficient by the formula: $G = \frac{1}{2k-1}$.

Proposition 1 (Euler-Lagrange's equation). *Let's note* $\tilde{G}(t,q) = G(q, m(t,q))$ *and* $\tilde{H}(t,q,\dot{q}) = H(\dot{q}, \overline{F}(t,q))$.
The optimal path has to satisfy the following Euler-Lagrange equation:

$$\partial_q \tilde{G}(t,q) - \partial_q \tilde{H}(t,q,\dot{q}) = -\frac{d}{dt}\left[\partial_{\dot{q}} \tilde{H}(t,q,\dot{q})\right] + r\partial_{\dot{q}} \tilde{H}(t,q,\dot{q})$$

Proof:

This is a pure application of the Euler-Lagrange's principle with a discount rate r. □

This equation can be solved easily and this is the result of the following proposition:

Proposition 2 (Growth rate). *If* $\varphi(\varphi - 1) < \beta k$ *then, there is a unique* γ *so that the solution of the preceding equation is characterized by a constant growth* γ*:*

- $q_t = q_0 \exp(\gamma t)$
- $m(t,q) = k\frac{\exp(\gamma k t)}{q^{k+1}} 1_{q \geq \exp(\gamma t)}$

Moreover, γ *is implicitly given by:*

$$\frac{\varphi(\varphi - 1) - \beta k}{\varphi}\gamma^{\varphi} = r\gamma^{\varphi - 1} - \frac{C(\varphi + \beta k)}{E k^{\beta}} \qquad (*)$$

Proof: See appendix. □

Before going into the comparative statics and the analysis of the model, we need to verify that the above solution satisfy the transversality condition. In other words we need to verify that the integral in the criterion remains finite.

Proposition 3 (Transversality condition). *For the solution exhibited in Proposition 2 to be an actual solution of the optimization problem, we need to have* $\gamma < \frac{r}{\varphi}$

Proof: See appendix. □

These propositions are central in our discussion. We have indeed proved that a constant growth rate was a possible outcome of our model if the parameters satisfied some constraints. Moreover, the distribution of human capital and hence the distribution of wealth is always of the Pareto type and this is interesting in the light of the usual theories of inequalities.

2 Comparative statics

In that part, we analyze the growth rate formula derived previously (equation $(*)$). Basically, γ can be seen as a function of three meaningful parameters: r, E and k.

- r has to be seen as a parameter linked to *impatience*. We expect the growth rate to be a decreasing function of r
- E is a measure of the cost to improve human capital. A small E indicates efficient on-the-job training in our model and we expect growth to decrease with E.
- k is a measure of the initial homogeneity in the distribution of human capital. If we consider that the initial human capital distribution is a result of the basic educational system then a high k means a very equalitarian educational system whereas a smaller k represents a more free educational system that leads to more heterogeneity. The sign of $\frac{d\gamma}{dk}$ will be interesting to evaluate the link between growth and social homogeneity.

Growth as a decreasing function of r

As expected, γ is a decreasing function of r. This is straightforward if we consider equation $(*)$ or the following graph on which we plotted the left hand side and the right side of $(*)$.

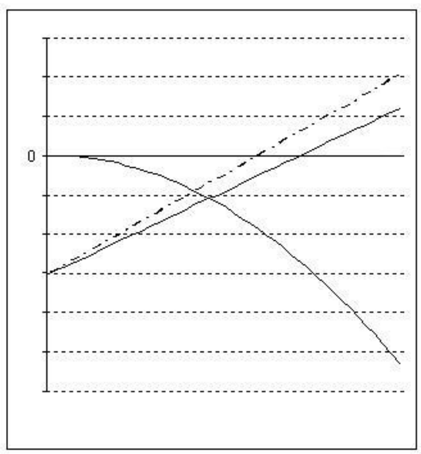

Figure 1: The impact of an increase in r

Growth as a decreasing function of E

As expected, γ is a also a decreasing function of E. The same argument as before applies.

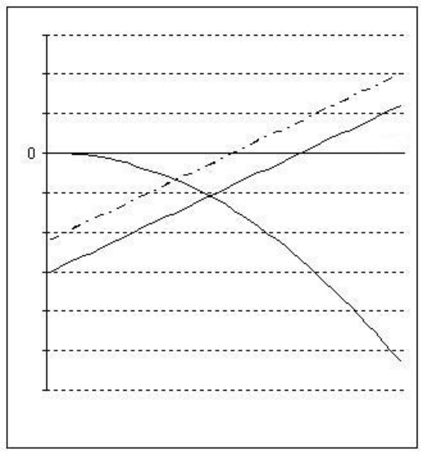

Figure 2: The impact of an increase in E

Growth can be fostered by heterogeneity

The last dependency we analyze is on k. The result is in general ambiguous but we can say the following:

Proposition 4 (Dependence on k[6]). *Suppose as before that $\varphi(\varphi - 1) < \beta k$*

- *For $\beta = 1$, the function $k \mapsto \gamma(k)$ is decreasing.*
- *For $\beta < 1$ and as k goes to infinity, $\gamma(k)$ tends to zero as $k^{-\frac{\beta}{\varphi}}$.*

Proof: see appendix.

3 Analysis of our model

Our model generates a constant growth rate for human capital, both for the entire society and for each single individual. In what follows we discuss the underlying source of growth and relate our finding to recent papers in the economic literature.

To begin with, the basic reason why people change their human capital is

[6] To relate this result to better known measures of heterogeneity, just notice for instance that the Gini coefficient in the case of a Pareto distribution is simply given by $G = \frac{1}{2k-1}$. Therefore, we basically show that γ is increasing as a function of the Gini coefficient for the human capital distribution at least if this Gini coefficient is small enough.

due to two effects. First, there is a pure wage effect since, *ceteris paribus*, wage increases with human capital. However, this effect cannot explain by itself the continuous improvement of human capital at a constant growth rate. The effect needed to ensure a convincing explanation is a competition effect, or to say it as in Aghion and Howitt ([AH]), even though the comparison is not entirely relevant, an *escape competition* effect. A given individual taken at random in the population is threaten by people who have less human capital than he has (say $\tilde{q}$). Indeed, if part of those people where to improve there human capital so that they end up with a human capital $\tilde{q}$ they will compete with our individual on the labor market, reducing her wage. This effect is the origin of continuous growth in our model. Contrary to Aghion et al., we have here a continuum of agents and therefore, for any given individual, there is always a threat. We think therefore that the Schumpeterian effect which basically assumes that people won't improve their human capital if the gains are two small is reduced to nothing because there is always a potential competitor and that's why a Darwinian effect (competition effect) dominates. Let's indeed highlight how tough is the threat effect. Each agent knows that every one is threaten by every one, and that fear will induce behaviors that will make the frightening event happen and be more important. This model shows that the growth process is not only due to those who innovate, that is to say "researchers" near the technological frontier, but is in fact a process that involves the whole population and is fostered by those who are far from the technological frontier and threaten the leaders by improving their human capital. Also, our model gives a striking example of the fact that the Darwinian competitive pressure can be much more intense between agents with rational expectations than between myopic agents. Myopic agents would fear other agents moves, while agents with rational expectations fear also the competitive moves of other agents induced by their own competitive behavior. In other words, Darwinian competition, as a general concept, when extended to competition between agents with rational expectations, leads to an extremely tough competitive scheme.

One of the characteristics of our model is also related to the structure of economic inequalities. Starting with a given Pareto distribution with parameter k, the solution exhibited above, is always a Pareto distribution of order k (with a support that depends on time obviously). Recalling that the Gini coefficient is only determined by k (it is straightforward to get the formula $G = \frac{1}{2k-1}$), we have also the interesting property that the Gini coefficient is constant and cannot therefore be modified by the kind of human capital accumulation process we model.
Is this consistent with reality? The answer depends on the country.
To deal with this issue we have to consider wages instead of human capital. Because $w(t,q) = C\frac{q^\alpha}{m(t,q)^\beta}$, the wages on the optimal path are given by:

$$w(t,q) = \frac{C}{k^\beta}\exp(-\beta k\gamma t)q^{\beta k+\varphi}$$

Distribution of wages is therefore a Pareto distribution of order $p = \frac{k}{\beta k+\varphi}$ for all t. It means that wage inequalities are very stable over time. This is difficult to test but great work has been done in Piketty ([Pik03]) and

Piketty and Saez ([PS03]) for wages, respectively in France and in the US[7]. Even if there are fluctuations in wage inequalities, these inequalities seem (perhaps surprisingly) to have been stable over the twentieth century in France. However, this is not the case for the US where wages are less and less equally distributed (*Today, the "working rich" celebrated by Forbes magazine seem to have overtaken the "coupon-clippers"* - see [PS03]).
Also, in a book printed recently ([AP07]), Atkinson and Piketty showed that this stability is true for most of non english-speaking developed countries whereas inequalities are less stable in the US, the UK, Ireland, Australia, New Zealand ...
The conclusion is that our model is quite consistent with the continental Europe experience in the long run as far as wages inequalities are concerned.

We discussed earlier the impact of initial inequalities represented either by k or by $G = \frac{1}{2k-1}$ on the growth rate (notice here that the growth rate of wages is simply given by $\varphi\gamma$: that can be derived easily from the above equation for $w(t,q)$). Our finding is that an education system which leads to a very homogeneous population could be responsible for a small growth rate. This has natural policy implications and supports liberalization of the schooling system and therefore less uniform schools. However, one must not forget that in our model, on-the-job training is always feasible and that's why the main implication of our model is certainly better expressed by: economic inequalities can be good for growth as soon as there is no segregation *i.e.* large access for everybody to the human capital accumulation process.

4 Mathematical complements and generalization to a stochastic framework

4.1 The mean field game partial differential equations

In the first part, we found a solution to our problem using a Euler-Lagrange methodology. This is not in fact the most relevant mathematical method to deal with the problem. The problem involves indeed the probability distribution function and the tail function of the human capital across the population and the mean field games partial differential equations are in a way far more relevant to solve the problem.

Let's first introduce the Bellman function of the problem:

$$J(t,q) = Max_{(q_s),q_t=q} \int_t^\infty \left[G(q_s, m(s,q_s)) - H(a(s,q_s), \bar{F}(s,q_s))\right] e^{-r(s-t)} ds$$

[7] Another article by Atkinson deals with the UK but concerns revenues and not wages

We can translate this optimization problem into the two mean field games partial differential equations:

Proposition 5 (The mean field games partial differential equations). *Our optimization problem can be represented by the two following PDEs:*

$$(HJB) \qquad G(q, m(t,q)) + \partial_t J + Max_a \left(a\partial_q J - H(a, \bar{F}(t,q))\right) - rJ = 0$$

$$(Kolmogorov) \qquad \partial_t m(t,q) + \partial_q (a(t,q)m(t,q)) = 0$$

where $a(t,q) = ArgMax_a \left(a\partial_q J - H(a, \bar{F}(t,q))\right)$ *is the optimal control function.*
In the special case we solved, the two equations can be written as:

$$C\frac{q^\alpha}{m(t,q)^\beta} + \frac{\varphi-1}{\varphi}\frac{1}{E^{\frac{1}{\varphi-1}}}\overline{F}(t,q)^{\frac{\beta}{\varphi-1}}(\partial_q J)^{\frac{\varphi}{\varphi-1}} + \partial_t J - rJ = 0$$

$$\partial_t m(t,q) + \partial_q \left(\left(\frac{\overline{F}(t,q)^\beta}{E}\partial_q J(t,q)\right)^{\frac{1}{\varphi-1}} m(t,q)\right) = 0$$

and the optimal control is given by:

$$a(t,q) = \left(\frac{\overline{F}(t,q)^\beta}{E}\partial_q J(t,q)\right)^{\frac{1}{\varphi-1}}$$

Proof: see appendix.

These PDEs can be solved easily if we add the constraint $a(t,q) = \gamma q$ that corresponds to a uniform and constant growth rate.

Proposition 6 (Resolution of the PDEs). *If* $\varphi(\varphi-1) < \beta k$, *there is a unique triple* (J, m, γ) *that satisfies both the PDEs and the additional equation on the optimal control function:* $a(t,q) = \gamma q$.

Solutions are of the following form:

$$m(t,q) = k\frac{\exp(\gamma k t)}{q^{k+1}} 1_{q \geq \exp(\gamma t)}$$

$$J(t,q) = B\exp(-\beta k \gamma t) q^{\beta k + \varphi} 1_{q \geq \exp(\gamma t)}$$

where γ *and* B *are related by* $\gamma = \left(\frac{B}{E}(\beta k + \varphi)\right)^{\frac{1}{\varphi-1}}$

Proof: see appendix.

This PDE approach allows us to generalize the model. We are indeed going to add a common noise in the model and show how this noise affects the results.

4.2 The model with common noise

So far, our model was completely deterministic whereas most applications of mean field games are in a random setting. Here, it is not natural to introduce a specific noise for each individual if we want to keep explicit solutions with m being a Pareto distribution (because of the threshold). However, it's possible to introduce randomness through a common noise on the evolution of the human capital.
More precisely, we can replace the dynamics for q by a stochastic one:

$$dq_t = a(t, q_t)dt + \sigma q_t dW_t$$

where W is a noise common to all agents. This specification seems complicated since all the functions J, m and $\overline{F}$ are now random variables. However, the intuitions developed above can be applied *mutatis mutandis* and our specification is robust and deep enough to be generalized to a complex stochastic framework.

First, consider the Bellman function J. From Proposition 6, we can see that the expression for J was in fact a function of q and of the lower bound q^m of the q's so that it's more natural in general to define here $J = J(t, q, q^m)$ as:

$$Max_{(q_s)_{s>t}, q_t=q, q_t^m=q^m} \mathbb{E}\left[\int_t^\infty \left[C\frac{q^\alpha}{m(t,q)^\beta} - \frac{E}{\varphi}\frac{a(t,q)^\varphi}{\overline{F}(t,q)^\beta}\right] e^{-r(s-t)} ds | \mathcal{F}_t\right]$$

Proposition 7 (Partial differential equations with common noise). *The Hamilton Jacobi equation corresponding to the above optimization problem can be written in the following differential form:*

$$Max_a\ C\frac{q^\alpha}{m(t,q)^\beta} - \frac{E}{\varphi}\frac{a^\varphi}{\overline{F}(t,q)^\beta} - rJ$$

$$+\partial_t J + a\partial_q J + \frac{\sigma^2}{2}q^2\partial^2_{qq}J + a'\partial_{q^m}J + \frac{\sigma^2}{2}q^{m2}\partial^2_{q^m q^m}J + \sigma^2 q q^m \partial^2_{qq^m}J = 0$$

where a' is $a(t, q_t^m)$[8].

The optimal control function is given by the same expression as in the deterministic case:

$$a(t,q) = \left(\frac{\overline{F}(t,q)^\beta}{E}\partial_q J(t,q)\right)^{\frac{1}{\varphi-1}}$$

Lemma 1. *If $a(t, q) = \gamma q$, then the probability distribution function of the q's is $m(t, q) = k\frac{(q_t^m)^k}{q^{k+1}}1_{q\geq q_t^m}$.*

Proposition 8 (Resolution of the PDEs). *If $\varphi(\varphi - 1) < \beta k$ and $r > \frac{\sigma^2}{2}\varphi(\varphi - 1)$, then, there is a unique growth rate γ compatible with the problem and J is of the form:*

$$J(q, q^m) = Bq^{\beta k+\varphi}(q^m)^{-\beta k}1_{q\geq q^m}$$

[8] This is exogenous in the optimization because individuals are atomized

where γ *and* B *are related by* $\gamma = \left(\frac{B}{E}(\beta k + \varphi)\right)^{\frac{1}{\varphi-1}}$
Moreover, γ *is given by* $(*')$*:*

$$\frac{\varphi(\varphi-1) - \beta k}{\varphi}\gamma^{\varphi} = (r - \varphi(\varphi-1)\frac{\sigma^2}{2})\gamma^{\varphi-1} - \frac{C(\varphi+\beta k)}{Ek^{\beta}} \qquad (*')$$

Proof: see appendix.

The main conclusion is that the introduction of a common noise leads to an increase in γ (as it can be seen on the following graph).

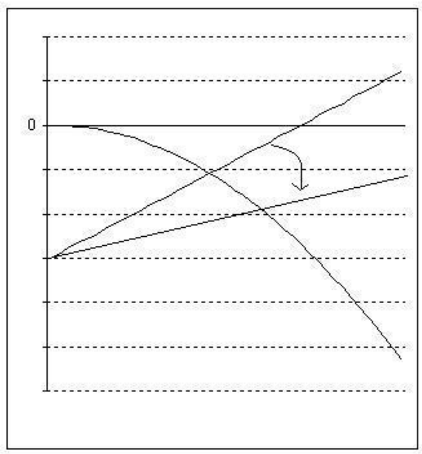

Figure 3: The introduction of a noise

To conclude this part, let's just note that the transversality condition is modified:

Proposition 9 (Transversality condition). *For the solution exhibited in the above proposition to be an actual solution of the optimization problem, we need to have* $\gamma < \frac{r}{\varphi} - (\varphi - 1)\frac{\sigma^2}{2}$.

Proof: see appendix

Conclusion

Using a mean field game framework, this paper presents a growth model where growth is fostered by the fear of individuals about the possible competition of their peers. This model can either be solved by classical Euler-Lagrange methods or using the partial differential equations of the mean field games theory. This second approach is a good way to show the robustness of the model when it comes to the introduction of randomness.

Appendix

Proof of Proposition 2:

Let's consider a solution of the form $q_t = q_0 \exp(\gamma t)$. If this is true for every single individual, then, the probability distribution function $m(t,\cdot)$ has to be of the Pareto form $m(t,q) = k\frac{\exp(\gamma k t)}{q^{k+1}}1_{q\geq \exp(\gamma t)}$. This expression for $m(t,\cdot)$ leads to $\overline{F}(t,q) = \frac{\exp(\gamma k t)}{q^k}1_{q\geq \exp(\gamma t)}$.
Therefore, for $q \geq \exp(\gamma t)$ we have:

- $\tilde{G}(t,q) = G(q, m(t,q)) = \frac{C}{k^\beta} q^{\alpha+\beta(k+1)} e^{-\gamma\beta k t}$
- $\tilde{H}(t,q,\dot{q}) = H(\dot{q}, \overline{F}(t,q)) = \frac{E}{\varphi}\dot{q}^\varphi \exp(-\gamma\delta k t) q^{\delta k}$

Hence, if we use the preceding proposition, we must have:

$$\frac{C(\alpha+\beta(k+1))}{k^\beta} q^{\alpha+\beta(k+1)-1} e^{-\gamma\beta k t} - \frac{E\delta k}{\varphi}\dot{q}^\varphi \exp(-\gamma\delta k t) q^{\delta k-1}$$

$$= -\frac{d}{dt}\left[E\dot{q}^{\varphi-1}\exp(-\gamma\delta k t)q^{\delta k}\right] + rE\dot{q}^{\varphi-1}\exp(-\gamma\delta k t)q^{\delta k}$$

Since $\dot{q} = \gamma q$ we obtain:

$$\frac{C(\alpha+\beta(k+1))}{k^\beta} q^{\alpha+\beta(k+1)-1} e^{-\gamma\beta k t} - \frac{E\delta k}{\varphi}\gamma^\varphi \exp(-\gamma\delta k t) q^{\delta k-1+\varphi}$$

$$= -\frac{d}{dt}\left[E\gamma^{\varphi-1}\exp(-\gamma\delta k t)q^{\delta k+\varphi-1}\right] + rE\gamma^{\varphi-1}\exp(-\gamma\delta k t)q^{\delta k+\varphi-1}$$

$$\Rightarrow \frac{C(\alpha+\beta(k+1))}{k^\beta} q^{\alpha+\beta(k+1)-1} e^{-\gamma\beta k t} - \frac{E\delta k}{\varphi}\gamma^\varphi \exp(-\gamma\delta k t) q^{\delta k-1+\varphi}$$

$$= \gamma\delta k E\gamma^{\varphi-1}\exp(-\gamma\delta k t)q^{\delta k+\varphi-1} - E(\delta k+\varphi-1)\gamma^{\varphi-1}\exp(-\gamma\delta k t)\gamma q^{\delta k+\varphi-1}$$
$$+ rE\gamma^{\varphi-1}\exp(-\gamma\delta k t)q^{\delta k+\varphi-1}$$

Hence, using the various assumptions on the parameters, we get:

$$\frac{C(\varphi+\beta k)}{k^\beta} - \frac{E\beta k}{\varphi}\gamma^\varphi = \beta k E\gamma^\varphi - E(\beta k+\varphi-1)\gamma^\varphi + rE\gamma^{\varphi-1}$$

$$\Rightarrow \frac{\varphi(\varphi-1)-\beta k}{\varphi}\gamma^\varphi = r\gamma^{\varphi-1} - \frac{C(\varphi+\beta k)}{Ek^\beta}$$

This equation in γ has a unique solution if, as it is supposed here, $\varphi(\varphi-1) < \beta k$.

□

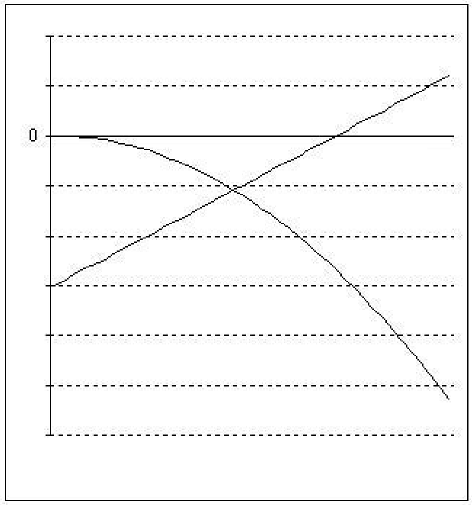

Figure 4: The solution for γ

Proof of Proposition 3:

The integrand in the criterion is:

$$\left(\frac{C}{k^{\beta}} q_0^{\beta k+\varphi} \exp((\beta k+\varphi)\gamma t - k\gamma\beta t) - \frac{E}{\varphi}\gamma^{\varphi} q_0^{\beta k+\varphi} \exp((\beta k+\varphi)\gamma t - k\gamma\beta t)\right) e^{-rt}$$

Hence, we must have:

$$(\beta k+\varphi)\gamma - k\gamma\beta - r < 0$$

$$\gamma < \frac{r}{\varphi}$$

Proof of Proposition 4:

First, let's differentiate the equation $(*)$ with respect to k:

$$(\varphi(\varphi-1)-\beta k)\frac{d\gamma}{dk}\gamma^{\varphi-1} - \frac{\beta}{\phi}\gamma^{\varphi} = r(\varphi-1)\frac{d\gamma}{dk}\gamma^{\varphi-2} + \frac{\beta C\varphi}{E}k^{-\beta-1} - \frac{\beta C(1-\beta)}{E}k^{-\beta}$$

$$\Rightarrow \frac{d\gamma}{dk}\left[(\varphi(\varphi-1)-\beta k)\gamma^{\varphi-1} - r(\varphi-1)\gamma^{\varphi-2}\right] = \frac{\beta}{\phi}\gamma^{\varphi} + \frac{\beta C\varphi}{E}k^{-\beta-1} - \frac{\beta C(1-\beta)}{E}k^{-\beta}$$

If $\beta = 1$, it's then obvious that $k \mapsto \gamma(k)$ is decreasing.
Otherwise, if $\beta < 1$, we can see from $(*)$ that the only limit point for γ is 0 and then, $\frac{\beta}{\varphi}k\gamma^{\varphi} \sim \frac{C\beta}{E}k^{1-\beta}$. This leads to the result. □

Proof of Proposition 5:

The optimal control function is given by $a(t,q) = Argmax_a a\partial_q J - \frac{E}{\varphi}\frac{a^\varphi}{\overline{F}(t,q)^\beta}$.

Hence, $a(t,q) = \left(\frac{\overline{F}(t,q)^\beta}{E}\partial_q J(t,q)\right)^{\frac{1}{\varphi-1}}$ and $Max_a a\partial_q J - \frac{E}{\varphi}\frac{a^\varphi}{\overline{F}(t,q)^\beta}$ can be replaced by $\frac{\varphi-1}{\varphi}\frac{1}{E^{\frac{1}{\varphi-1}}}\overline{F}(t,q)^{\frac{\beta}{\varphi-1}}(\partial_q J)^{\frac{\varphi}{\varphi-1}}$ in the HJB equation. □

Proof of Proposition 6:

First of all, the additional condition is equivalent to a constant growth rate for q_t and therefore, we obtain the Pareto distribution $m(t,\cdot)$ stated above.
Therefore, we have the following equation for $\partial_q J(t,q)$ if $q \geq \exp(\gamma t)$:

$$\partial_q J(t,q) = E(\gamma q)^{\varphi-1}\overline{F}(t,q)^{-\beta} = E(\gamma q)^{\varphi-1}e^{-\beta k\gamma t}q^{\beta k}$$

Hence (the constant being nought),

$$J(t,q) = \frac{E}{\beta k+\varphi}\gamma^{\varphi-1}e^{-\beta k\gamma t}q^{\beta k+\varphi}$$

If we plug this expression into the Hamilton-Jacobi equation we get:

$$\frac{C}{k^\beta}q^{\beta k+\varphi}e^{-\beta k\gamma t} + \frac{\varphi-1}{\varphi}E\gamma^\varphi q^{\beta k+\varphi}e^{-\beta k\gamma t}$$

$$-\beta k\gamma\frac{E}{\beta k+\varphi}\gamma^{\varphi-1}e^{-\beta k\gamma t}q^{\beta k+\varphi} - r\frac{E}{\beta k+\varphi}\gamma^{\varphi-1}e^{-\beta k\gamma t}q^{\beta k+\varphi} = rD$$

From this we get:

$$\frac{C}{k^\beta} + \frac{\varphi-1}{\varphi}E\gamma^\varphi - \beta k\frac{E}{\beta k+\varphi}\gamma^\varphi - r\frac{E}{\beta k+\varphi}\gamma^{\varphi-1} = 0$$

This is exactly the equation $(*)$ of Proposition 2 and therefore γ is unique. □

Proof of Proposition 8:

First, if $a(t,q) = \gamma q$ then,

$$\partial_q J(t,q,q^m) = E(\gamma q)^{\varphi-1}\overline{F}(t,q)^{-\beta} = E\gamma^{\varphi-1}q^{\beta k+\varphi-1}(q_t^m)^{-\beta k}$$

From this we deduce that the solution is of the stated form with $B = \frac{E}{\beta k+\varphi}\gamma^{\varphi-1}$.

If we want to find B or γ we need to plug the expression for J in the Hamilton Jacobi equation. This gives:

$$q^{\beta k+\varphi-1}(q^m)^{-\beta k}\left[\frac{C}{k^\beta} - \frac{E}{\varphi}\gamma^\varphi - rB + \gamma(\beta k+\varphi)B - \beta k\gamma B\right.$$

$$\left.+\frac{\sigma^2}{2}B\left((\beta k+\varphi)(\beta k+\varphi-1) + (-\beta k)(-\beta k-1) + 2(\beta k+\varphi)(-\beta k)\right)\right] = 0$$

$$\frac{C}{k^\beta} - \frac{E}{\varphi}\gamma^\varphi + \gamma\varphi B - (r - \varphi(\varphi-1)\frac{\sigma^2}{2})B = 0$$

$$\frac{C(\beta k+\varphi)}{Ek^{\beta}} - \frac{\beta k+\varphi}{\varphi}\gamma^{\varphi} + \varphi\gamma^{\varphi} - (r-\varphi(\varphi-1)\frac{\sigma^2}{2})\gamma^{\varphi-1} = 0$$
$$\frac{\varphi(\varphi-1)-\beta k}{\varphi}\gamma^{\varphi} = (r-\varphi(\varphi-1)\frac{\sigma^2}{2})\gamma^{\varphi-1} - \frac{C(\varphi+\beta k)}{Ek^{\beta}}$$

As for $(*)$, it's clear that, given our hypotheses, this equation has a unique solution. □

Proof of Proposition 9:

The expression in the integral that defines the criterion is:

$$\frac{C}{k^{\beta}} q_0^{k\beta+\varphi} (q_t^m)^{\varphi} - \frac{E}{\varphi}\gamma^{\varphi} q_0^{k\beta+\varphi}(q_t^m)^{\varphi}$$

Hence, for the solution to be well defined, the function $t \mapsto \mathbb{E}\left[(q_t^m)^{\varphi}\right] e^{-rt}$ has to be integrable.

But:

$$\mathbb{E}\left[(q_t^m)^{\varphi}\right] = \mathbb{E}\left[\exp(\varphi(\gamma - \frac{\sigma^2}{2})t + \varphi\sigma W_t)\right] = \exp(\varphi(\gamma - \frac{\sigma^2}{2})t + \varphi^2\frac{\sigma^2}{2}t)$$

We therefore need to have $r > \varphi(\gamma - \frac{\sigma^2}{2}) + \varphi^2\frac{\sigma^2}{2}$ and this is what we wanted to prove. □

References

[AAZ06] D. Acemoglu, P. Aghion, and F. Zilibotti. Distance to frontier, selection, and economic growth. *Journal of the European Economic Association*, 4(1), Mar. 2006.

[ABB+05] P. Aghion, N. Bloom, R. Blundell, R. Griffith, and P. Howitt. Competition and innovation: An inverted-u relationship. *Quarterly Journal of Economics*, 120(2), May. 2005.

[AH] P. Aghion and P. Howitt. *Forthcoming book.*

[AH92] P. Aghion and P. Howitt. A model of growth through creative destruction. *Econometrica*, 60(2), Mar. 1992.

[AHHV01] P. Aghion, C. Harris, P. Howitt, and J. Vickers. Competition, imitation and growth with step-by-step innovation. *The Review of Economic Studies*, 68(3), Jul. 2001.

[AP07] A. Atkinson and T. Piketty. *Top Incomes over the Twentieth Century: A Contrast between European and English-Speaking Countries.* Oxford University Press, 2007.

[Atk05] A. Atkinson. Top incomes in the united kingdom over the 20th century. *Journal of the Royal Statistical Society*, 168, 2005.

[Aum64] R. Aumann. Markets with a continuum of traders. *Econometrica*, 32(1/2), 1964.

[Hai00a] J.-O. Hairault. *Analyse macroéconomique, tome 1*. La Découverte, 2000.

[Hai00b] J.-O. Hairault. *Analyse macroéconomique, tome 2*. La Découverte, 2000.

[Jon95] C. Jones. R&d-based models of economic growth. *The Journal of Political Economy*, 103(4), Aug. 1995.

[LL06a] J.-M. Lasry and P.-L. Lions. Jeux à champ moyen i. le cas stationnaire. *C. R. Acad. Sci. Paris*, 343(9), 2006.

[LL06b] J.-M. Lasry and P.-L. Lions. Jeux à champ moyen ii. horizon fini et contrôle optimal. *C. R. Acad. Sci. Paris*, 343(10), 2006.

[LL07a] J.-M. Lasry and P.-L. Lions. Mean field games. *Japanese Journal of Mathematics*, 2(1), Mar. 2007.

[LL07b] J.-M. Lasry and P.-L. Lions. Mean field games. *Cahiers de la Chaire Finance et Développement Durable*, (2), 2007.

[Min58] J. Mincer. Investment in human capital and personal income distribution. *The Journal of Political Economy*, 66(4), Dec. 1958.

[Pik03] T. Piketty. Income inequality in france, 1901-1998. *Journal of Political Economy*, 111(5), 2003.

[PS03] T. Piketty and E. Saez. Income inequality in the united states, 1913-1998. *Quarterly Journal of Economics*, 118(1), Feb. 2003.

[Rom86] P. Romer. Increasing returns and long-run growth. *The Journal of Political Economy*, 94(5), Oct. 1986.

[Rom90] P. Romer. Endogenous technological change. *Journal of Political Economy*, 98(5), Oct. 1990.

[Tou02] N. Touzi. Stochastic control and application to finance. *Scuola Normale Superiore, Pisa. Special Research Semester on Financial Mathematics*, 2002.