

On the Infrared Problem for the Dressed Non-Relativistic Electron in a Magnetic Field

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ABSTRACT. We consider a non-relativistic electron interacting with a classical magnetic field pointing along the x_3 -axis and with a quantized electromagnetic field. The system is translation invariant in the x_3 -direction and we consider the reduced Hamiltonian $H(P_3)$ associated with the total momentum P_3 along the x_3 -axis. For a fixed momentum P_3 sufficiently small, we prove that $H(P_3)$ has a ground state in the Fock representation if and only if $E'(P_3) = 0$, where $P_3 \mapsto E'(P_3)$ is the derivative of the map $P_3 \mapsto E(P_3) = \inf \sigma(H(P_3))$. If $E'(P_3) \neq 0$, we obtain the existence of a ground state in a non-Fock representation. This result holds for sufficiently small values of the coupling constant.
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1. Introduction

In this paper we pursue the analysis of a model considered in [AGG1], describing a non-relativistic particle (an electron) interacting both with the quantized electromagnetic field and a classical magnetic field pointing along the x_3 -axis. An ultraviolet cutoff is imposed in order to suppress the interaction between the electron and the photons of energies bigger than a fixed, arbitrary large parameter Λ . The total system being invariant by translations in the x_3 -direction, it can be seen (see [AGG1]) that the corresponding Hamiltonian admits a decomposition of the form $H \simeq \int_{\mathbb{R}}^{\oplus} H(P_3) dP_3$ with respect to the spectrum of the total momentum along the x_3 -axis that we denote by P_3 . For any given P_3 sufficiently close to 0, the existence of a ground state for $H(P_3)$ is proven in [AGG1] provided an infrared regularization is introduced (besides a smallness assumption on the coupling parameter). Our aim is to address the question of the existence of a ground state without requiring any infrared regularization.

The model considered here is closely related to similar non-relativistic QED models of freely moving electrons, atoms or ions, that have been studied recently (see [BCFS, FGS1, Hi, CF, Ch, HH, CFP, FP] for the case of one single electron, and [AGG2, LMS, FGS2, HH, LMS2] for atoms or ions). In each of these papers, the physical systems are translation invariant, in the sense that the associated Hamiltonian H commutes with the operator of total momentum P . As a consequence, $H \simeq \int_{\mathbb{R}^3} H(P) dP$, and one is led to study the spectrum of the fiber Hamiltonian $H(P)$ for fixed P 's.

For the one-electron case, an aspect of the so-called *infrared catastrophe* lies in the fact that, for $P \neq 0$, $H(P)$ does not have a ground state in the Fock space

(see [CF, Ch, HH, CFP]). More precisely, if an infrared cutoff of parameter σ is introduced in the model in order to remove the interaction between the electron and the photons of energies less than σ , the associated Hamiltonian $H_\sigma(P)$ does have a ground state $\Phi_\sigma(P)$ in the Fock space. Nevertheless as $\sigma \rightarrow 0$, it is shown that $\Phi_\sigma(P)$ “leaves” the Fock space. Physically this can be interpreted by saying that a free moving electron in its ground state is surrounded by a cloud of infinitely many “soft” photons.

For negative ions, the absence of a ground state for $H(P)$ is established in [HH] under the assumption $\nabla E(P) \neq 0$, where $E(P) = \inf \sigma(H(P))$.

In [CF], with the help of operator-algebra methods, a representation of a *dressed 1-electron state* non-unitarily equivalent to the usual Fock representation of the canonical commutation relations is given. We shall obtain in this paper a related result, following a different approach, under the further assumption that the electron interact with a classical magnetic field and an electrostatic potential.

We shall first provide a necessary and sufficient condition for the existence of a ground state for $H(P_3)$. Namely we shall prove that the bottom of the spectrum, $E(P_3) = \inf \sigma(H(P_3))$, is an eigenvalue of $H(P_3)$ if and only if $E'(P_3) = 0$ where $E'(P_3)$ denotes the derivative of the map $P_3 \mapsto E(P_3)$. In the case $E'(P_3) \neq 0$, thanks to a (non-unitary) Bogoliubov transformation, in the same way as in [Ar, DG2], we shall define a “renormalized” Hamiltonian $H^{\text{ren}}(P_3)$ which can be seen as an expression of the physical Hamiltonian in a non-Fock representation. Then we shall prove that $H^{\text{ren}}(P_3)$ has a ground state. These results have been announced in [AFGG].

The regularity of the map $P_3 \mapsto E(P_3)$ plays a crucial role in our proof. Adapting [Pi, CFP] we shall see that $P_3 \mapsto E(P_3)$ is of class $C^{1+\gamma}$ for some strictly positive γ . Let us also mention that our method can be adapted to the case of free moving hydrogenoid ions without spin, the condition $E'(P_3) = 0$ being replaced by $\nabla E(P) = 0$ (see Subsection 1.2 for a further discussion on this point).

The remainder of the introduction is organized as follows: In Subsection 1.1, a precise definition of the model considered in this paper is given, next, in Subsection 1.2, we state our results and compare them to the literature.

1.1. The model. We consider a non-relativistic electron of charge e and mass m interacting with a classical magnetic field pointing along the x_3 -axis, an electrostatic potential, and the quantized electromagnetic field in the Coulomb gauge. The Hilbert space for the electron and the photon field is written as

$$(1.1) \quad \mathcal{H} = \mathcal{H}_{\text{el}} \otimes \mathcal{H}_{\text{ph}},$$

where $\mathcal{H}_{\text{el}} = L^2(\mathbb{R}^3; \mathbb{C}^2)$ is the Hilbert space for the electron, and \mathcal{H}_{ph} is the symmetric Fock space over $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ for the photons,

$$(1.2) \quad \mathcal{H}_{\text{ph}} = \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} S_n \left[L^2(\mathbb{R}^3 \times \mathbb{Z}_2)^{\otimes n} \right].$$

Here S_n denotes the orthogonal projection onto the subspace of symmetric functions in $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)^{\otimes n}$ in accordance with Bose-Einstein statistics. We shall use the notation $\mathbf{k} = (k, \lambda)$ for any $(k, \lambda) \in \mathbb{R}^3 \times \mathbb{Z}_2$, and

$$(1.3) \quad \int_{\mathbb{R}^3 \times \mathbb{Z}_2} d\mathbf{k} = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} dk.$$

Likewise, the scalar product in $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ is defined by

$$(1.4) \quad (h_1, h_2) = \int_{\mathbb{R}^3 \times \mathbb{Z}_2} \bar{h}_1(\mathbf{k}) h_2(\mathbf{k}) d\mathbf{k} = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \bar{h}_1(k, \lambda) h_2(k, \lambda) dk.$$

The position and the momentum of the electron are denoted respectively by $x = (x_1, x_2, x_3)$ and $p = (p_1, p_2, p_3) = -i\nabla_x$. The classical magnetic field is of the form $(0, 0, b(x'))$, where $x' = (x_1, x_2)$ and $b(x') = (\partial a_2 / \partial x_1)(x') - (\partial a_1 / \partial x_2)(x')$. Here $a_j(x')$, $j = 1, 2$, are real functions in $C^1(\mathbb{R}^2)$. The electrostatic potential is denoted by $V(x')$. The quantized electromagnetic field in the Coulomb gauge is defined by

$$(1.5) \quad \begin{aligned} A(x) &= \frac{1}{\sqrt{2\pi}} \int \frac{\epsilon^\lambda(k)}{|k|^{1/2}} \rho^\Lambda(k) \left[e^{-ik \cdot x} a^*(\mathbf{k}) + e^{ik \cdot x} a(\mathbf{k}) \right] d\mathbf{k}, \\ B(x) &= -\frac{i}{\sqrt{2\pi}} \int |k|^{1/2} \left(\frac{k}{|k|} \wedge \epsilon^\lambda(k) \right) \rho^\Lambda(k) \left[e^{-ik \cdot x} a^*(\mathbf{k}) - e^{ik \cdot x} a(\mathbf{k}) \right] d\mathbf{k}, \end{aligned}$$

where $\rho^\Lambda(k)$ denotes the characteristic function $\rho^\Lambda(k) = \mathbf{1}_{|k| \leq \Lambda}(k)$ and Λ is an arbitrary large positive real number. Note that this explicit choice of the ultraviolet cutoff function ρ^Λ is made mostly for convenience. Our results would hold without change for any ρ^Λ satisfying $\int_{|k| \leq 1} |k|^{-2} |\rho^\Lambda(k)|^2 d^3k + \int_{|k| \geq 1} |k| |\rho^\Lambda(k)|^2 d^3k < \infty$. The vectors $\epsilon^1(k)$ and $\epsilon^2(k)$ in (1.5) are real polarization vectors orthogonal to each other and to k . Besides $a^*(\mathbf{k})$ and $a(\mathbf{k})$ are the usual creation and annihilation operators obeying the canonical commutation relations

$$(1.6) \quad [a^\#(\mathbf{k}), a^\#(\mathbf{k}')] = 0 \quad , \quad [a(\mathbf{k}), a^*(\mathbf{k}')] = \delta(\mathbf{k} - \mathbf{k}') = \delta_{\lambda\lambda'} \delta(k - k').$$

The Pauli Hamiltonian H_g associated with the system we consider is formally given by

$$(1.7) \quad \begin{aligned} H_g &= \frac{1}{2m} \left(p - ea(x') - gA(x) \right)^2 - \frac{e}{2m} \sigma_3 b(x') \\ &\quad - \frac{g}{2m} \sigma \cdot B(x) + V(x') + H_{\text{ph}}, \end{aligned}$$

where the charge of the electron is replaced by a coupling parameter g in the terms containing the quantized electromagnetic field. The Hamiltonian for the photons in the Coulomb gauge is given by

$$(1.8) \quad H_{\text{ph}} = d\Gamma(|k|) = \int |k| a^*(\mathbf{k}) a(\mathbf{k}) d\mathbf{k}.$$

Finally $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is the 3-component vector of the Pauli matrices.

Noting that H_g formally commutes with the operator of total momentum in the direction x_3 , $P_3 = p_3 + d\Gamma(k_3)$, one can consider the reduced Hamiltonian associated with $P_3 \in \mathbb{R}$ that we denote by $H_g(P_3)$. For any fixed P_3 , $H_g(P_3)$ acts on $L^2(\mathbb{R}^2; \mathbb{C}^2) \otimes \mathcal{H}_{\text{ph}}$ and is formally given by

$$(1.9) \quad \begin{aligned} H_g(P_3) &= \frac{1}{2m} \sum_{j=1,2} \left(p_j - ea_j(x') - gA_j(x', 0) \right)^2 - \frac{e}{2m} \sigma_3 b(x') + V(x') \\ &\quad + \frac{1}{2m} \left(P_3 - d\Gamma(k_3) - gA_3(x', 0) \right)^2 - \frac{g}{2m} \sigma \cdot B(x', 0) + H_{\text{ph}}. \end{aligned}$$

We define the infrared cutoff Hamiltonian $H_g^\sigma(P_3)$ by replacing $A(x)$ in (1.5) with

$$(1.10) \quad A_\sigma(x) = \frac{1}{\sqrt{2\pi}} \int \frac{\epsilon^\lambda(k)}{|k|^{1/2}} \rho_\sigma^\Lambda(k) \left[e^{-ik \cdot x} a^*(\mathbf{k}) + e^{ik \cdot x} a(\mathbf{k}) \right] d\mathbf{k},$$

where $\rho_\sigma^\Lambda = \mathbf{1}_{\sigma \leq |k| \leq \Lambda}$, and similarly for $B_\sigma(x)$. We set $E_g(P_3) = \inf \sigma(H_g(P_3))$ and $E_{g\sigma}(P_3) = \inf \sigma(H_g^\sigma(P_3))$.

The electronic Hamiltonian $h(b, V)$ on $L^2(\mathbb{R}^2; \mathbb{C}^2)$ is defined by

$$(1.11) \quad h(b, V) = \sum_{j=1,2} \frac{1}{2m} (p_j - e a_j(x'))^2 - \frac{e}{2m} \sigma_3 b(x') + V(x').$$

Let $e_0 = \inf \sigma(h(b, V))$. We make the following hypothesis:

(H₀) $h(b, V)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2; \mathbb{C}^2)$ and e_0 is an isolated eigenvalue of multiplicity 1.

We refer to **[AHS, So, IT, Ra]** for possible choices of b, V satisfying Hypothesis **(H₀)**. The following proposition is established in **[AGG1, Theorem 2.3]**:

PROPOSITION 1.1. *Suppose Hypothesis **(H₀)**. For sufficiently small values of $|g|$, H_g is self-adjoint with domain $D(H_g) = D(H_0)$, and for any $\sigma \geq 0$ and $P_3 \in \mathbb{R}$, $H_g^\sigma(P_3)$ identifies with a self-adjoint operator with domain $D(H_g^\sigma(P_3)) = D(H_0(P_3))$. Moreover H_g admits the decomposition*

$$(1.12) \quad H_g = \int_{\mathbb{R}}^{\oplus} H_g(P_3) dP_3.$$

1.2. Results and comments. The key ingredient that we shall need in order to prove our main theorem (see Theorem 1.3 below) lies in the regularity of the map $P_3 \mapsto E'_{g\sigma}(P_3)$ uniformly in $\sigma \geq 0$.

THEOREM 1.2. *Assume that **(H₀)** holds. There exists $g_0 > 0$, $\sigma_0 > 0$ and $P_0 > 0$ such that for all $|g| \leq g_0$, for all $0 \leq \sigma \leq \sigma_0$, for all P_3, k_3 such that $|P_3| \leq P_0$, $|P_3 + k_3| \leq P_0$, for all $\delta > 0$,*

$$(1.13) \quad |E'_{g\sigma}(P_3 + k_3) - E'_{g\sigma}(P_3)| \leq C_\delta |k_3|^{\frac{1}{4} - \delta},$$

where C_δ is a positive constant depending only on δ .

Similar results for a free electron (that is for $b = V = 0$) interacting with the quantized electromagnetic field have been obtained recently (see **[Ch, CFP, FP]**). The model studied in the latter papers is technically simpler than the one considered here in that the fiber Hamiltonian $H(P)$ associated with a free electron does not contain the electronic part $h(b, V)$ and its (minimal) coupling to the quantized electromagnetic field. In particular the operator $H(P)$ in **[Ch, CFP, FP]** acts only on the Fock space, whereas in our case $H_{g\sigma}(P_3)$ still contains interactions between the electromagnetic field and the electronic degrees of freedom. We shall use the exponential decay of the ground states $\Phi_g^\sigma(P_3)$ in x' in order to overcome this difficulty.

It is proved in **[Ch]** (for a free electron) that $P \mapsto E(P) = \inf \sigma(H(P))$ is of class C^2 in a neighborhood of 0 thanks to a renormalization group analysis (see also **[BCFS]**). The author also shows that, still in a neighborhood of $P = 0$, the derivative $\nabla E(P)$ vanishes only at $P = 0$. In **[CFP]**, with the help of what the authors call “iterative analytic perturbation theory”, following a multiscale

analysis developed in **[Pi]**, it is proved, among other results, that $P \mapsto E(P)$ is of class $C^{5/4-\delta}$ for arbitrary small $\delta > 0$. The method has later been improved in **[FP]** leading to the C^2 property of $P \mapsto E(P)$.

In order to establish our main theorem, Theorem 1.3, the “degree of regularity” we need is reached as soon as $P_3 \mapsto E_{g\sigma}(P_3)$ is at least of order $C^{1+\gamma}$, uniformly in σ , for some $\gamma > 0$. Therefore, although one can conjecture that $P_3 \mapsto E_{g\sigma}(P_3)$ is of class C^2 uniformly in σ , Theorem 1.2 is sufficient for our purpose. In order to prove it we shall adapt **[Pi, CFP]**: First, we shall give a short proof of the existence of a spectral gap for $H_g^\sigma(P_3)$ (restricted to the space of photons of energies bigger than σ) above the non-degenerate eigenvalue $E_{g\sigma}(P_3)$. Next we shall apply “iterative analytic perturbation theory”.

We postpone the proof of Theorem 1.2 to the appendix. Since several parts are taken from **[Pi, CFP]**, we shall not give all the details, rather we shall emphasize the differences with **[Pi, CFP]**.

For $h \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$, let us define the field operator $\Phi(h)$ by

$$(1.14) \quad \Phi(h) = \frac{1}{\sqrt{2}}(a^*(h) + a(h)),$$

where the creation operator $a^*(h)$ and the annihilation operator $a(h)$ are defined respectively by

$$(1.15) \quad a^*(h) = \int_{\mathbb{R}^3 \times \mathbb{Z}_2} h(\mathbf{k}) a^*(\mathbf{k}) d\mathbf{k}, \quad a(h) = \int_{\mathbb{R}^3 \times \mathbb{Z}_2} \bar{h}(\mathbf{k}) a(\mathbf{k}) d\mathbf{k}.$$

Hence, letting $h_{j,\sigma}(x')$ and $\tilde{h}_{j,\sigma}(x')$ for $j = 1, 2, 3$ be defined respectively by

$$(1.16) \quad \begin{aligned} h_{j,\sigma}(x', \mathbf{k}) &= \pi^{-1/2} \frac{\epsilon_j^\lambda(k)}{|k|^{1/2}} \rho_\sigma^\Lambda(k) e^{ik' \cdot x'}, \\ \tilde{h}_{j,\sigma}(x', \mathbf{k}) &= -i\pi^{-1/2} |k|^{1/2} \left(\frac{k}{|k|} \wedge \epsilon^\lambda(k) \right)_j \rho_\sigma^\Lambda(k) e^{ik' \cdot x'}, \end{aligned}$$

where $k' = (k_1, k_2)$, we have $A_{j,\sigma}(x', 0) = \Phi(h_{j,\sigma}(x'))$ and $B_{j,\sigma}(x', 0) = \Phi(\tilde{h}_{j,\sigma}(x'))$. The Weyl operator associated with $h \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ is denoted by $W(h) = e^{i\Phi(h)}$. Let $f_\sigma : \mathbb{R}^3 \times \mathbb{Z}_2 \rightarrow \mathbb{C}$ be defined by

$$(1.17) \quad f_\sigma(\mathbf{k}) = -\frac{g}{\sqrt{4\pi}} \frac{\rho_\sigma^\Lambda(k) \epsilon_\lambda^3(k)}{k_3 |k|^{1/2}} \frac{E_{g\sigma}(P_3 - k_3) - E_{g\sigma}(P_3)}{E_{g\sigma}(P_3 - k_3) - E_{g\sigma}(P_3) + |k|}.$$

If $\sigma = 0$ we remove the subindex σ in the preceding notations. We recall from **[AGG1, Lemma 4.3]** that for g, σ, P_3 and $|k|$ sufficiently small,

$$(1.18) \quad E_{g\sigma}(P_3 - k_3) - E_{g\sigma}(P_3) \geq -\frac{3}{4}|k|.$$

Hence in particular for $\sigma > 0$, we have $f_\sigma \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$, whereas if $\sigma = 0$ and $P_3 \mapsto E_g(P_3)$ is of class $C^{1+\gamma}$ with $\gamma > 0$, then

$$(1.19) \quad f \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2) \iff E'_g(P_3) = 0.$$

Similarly as in **[Ar]** (see also **[DG2, Pa]**), we define the “renormalized” (Bogoliubov transformed) Hamiltonian $H_{g\sigma}^{\text{ren}}(P_3)$ by the expression

$$(1.20) \quad H_{g\sigma}^{\text{ren}}(P_3) = W(if_\sigma) H_g^\sigma(P_3) W(if_\sigma)^*.$$

Notice that the identity (1.20) might only be formal for $\sigma = 0$ since in this case, by (1.19), f might not be in L^2 . Nevertheless using usual commutation relations (see for instance [DG1]), we define for any $\sigma \geq 0$:

$$\begin{aligned} H_{g\sigma}^{\text{ren}}(P_3) &= \frac{1}{2m} \sum_{j=1,2} \left(p_j - ea_j(x') - gA_{j,\sigma}(x', 0) + g\text{Re}(h_{j,\sigma}(x'), f_\sigma) \right)^2 \\ &+ \frac{1}{2m} \left(P_3 - d\Gamma(k_3) - \Phi(k_3 f_\sigma) - \frac{1}{2}(k_3 f_\sigma, f_\sigma) - gA_{3,\sigma}(x', 0) + g\text{Re}(h_{3,\sigma}(x'), f_\sigma) \right)^2 \\ &- \frac{e}{2m} \sigma_3 b(x') - \frac{g}{2m} \sigma \cdot \left(B_\sigma(x', 0) - \text{Re}(\tilde{h}_\sigma(x'), f_\sigma) \right) + V(x') \\ &+ H_f + \Phi(|k|f_\sigma) + \frac{1}{2}(|k|f_\sigma, f_\sigma). \end{aligned}$$

In the same way as for $H_g^\sigma(P_3)$ (see Proposition 1.1), one can verify that $H_{g\sigma}^{\text{ren}}(P_3)$ is self-adjoint with domain $D(H_{g\sigma}^{\text{ren}}(P_3)) = D(H_0(P_3))$ for any $\sigma \geq 0$. Besides for $\sigma > 0$, we have that $H_{g\sigma}^{\text{ren}}(P_3)$ is unitarily equivalent to $H_g^\sigma(P_3)$, whereas for $\sigma = 0$, one can verify that $H_g^{\text{ren}}(P_3)$ is unitarily equivalent to $H_g(P_3)$ if and only if $f \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$. Our main result is:

THEOREM 1.3. *Suppose Hypothesis (\mathbf{H}_0) . There exist $g_0 > 0$ and $P_0 > 0$ such that for all $0 \leq |g| \leq g_0$ and $0 \leq |P_3| \leq P_0$,*

- (i) $H_g(P_3)$ has a ground state if and only if $E'_g(P_3) = 0$,
- (ii) $H_g^{\text{ren}}(P_3)$ has a ground state.

The proof of Theorem 1.3 can be adapted to the case of free moving hydrogenoid ions without spins¹, the condition $E'_g(P_3) = 0$ being replaced by $\nabla E_g(P) = 0$, where $E_g(P)$ denotes the bottom of the spectrum of the fiber Hamiltonian $H_g(P)$. The existence of ground states for atoms has been obtained in [AGG2] thanks to a Power-Zienau-Wooley transformation and the crucial property $Q = 0$ (here Q denotes the total charge of the atomic system). Indeed, in [HH], it is proved that for negative ions ($Q < 0$) $H_g(P)$ does not have a ground state if $\nabla E_g(P) \neq 0$. Let us also mention [LMS] where the existence of ground states for atoms is proven for any value of the coupling constant g , by adapting [GLL], under the further assumption $E_g(P) \geq E_g(0)$ which has not been proven yet. Thus in addition to these results, our method provides the existence of ground states for spinless hydrogenoid ions, both for $H_g(P)$ in the case $\nabla E_g(P) = 0$ and for $H_g^{\text{ren}}(P)$.

The two statements “ $H_g(P_3)$ has a ground state if $E'_g(P_3) = 0$ ” and “ $H_g^{\text{ren}}(P_3)$ has a ground state” shall be established following the same standard procedure: An infrared cutoff σ is introduced into the model so that the Hamiltonian $H_g^\sigma(P_3)$ (respectively $H_{g\sigma}^{\text{ren}}(P_3)$) has a ground state $\Phi_g^\sigma(P_3)$ (respectively $\Phi_{g\sigma}^{\text{ren}}(P_3)$). We then need to prove that $\Phi_g^\sigma(P_3)$ and $\Phi_{g\sigma}^{\text{ren}}(P_3)$ converge strongly as $\sigma \rightarrow 0$. To this end we control the number of photons in the states $\Phi_g^\sigma(P_3)$ and $\Phi_{g\sigma}^{\text{ren}}(P_3)$ thanks to a pull-through formula and (1.13).

We emphasize that, in the case $E'_g(P_3) \neq 0$, $H_g^{\text{ren}}(P_3)$ can be seen as an expression of the physical Hamiltonian in a representation of the canonical commutation relations non-unitarily equivalent to the Fock representation. Besides, regarding [Ch] for the case of a single freely moving electron, one can conjecture that for sufficiently small values of $|P_3|$, $E'_g(P_3) = 0$ if and only if $P_3 = 0$.

¹The hypothesis of simplicity for the electronic ground state (\mathbf{H}_0) imposes this restriction to hydrogenoid atoms or ions.

Our proof of the absence of a ground state for $H_g(P_3)$ in the case $E'_g(P_3) \neq 0$ is based on a contradiction argument and [DG2, Lemma 2.6] (see also Lemma 2.2). Again the result is achieved by deriving a suitable expression of $a(\mathbf{k})\Phi_g(P_3)$ thanks to a pull-through formula (assuming here that $H_g(P_3)$ has a ground state $\Phi_g(P_3)$). Note that the regularity property (1.13) appears again as a key property (although here only (1.13) for $\sigma = 0$ is required).

The paper is organized as follows: In Section 2, we prove Theorem 1.3. Next in the appendix we prove Theorem 1.2.

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2. Proof of Theorem 1.3

The following proposition is proven in Subsection A.1 of the appendix.

PROPOSITION 2.1. *Assume that (\mathbf{H}_0) holds. There exists $g_0 > 0$, $\sigma_0 > 0$ and $P_0 > 0$ such that for all $|g| \leq g_0$, for all $0 < \sigma \leq \sigma_0$, for all $|P_3| \leq P_0$, $H_{g\sigma}(P_3)$ has a unique normalized ground state $\Phi_g^\sigma(P_3)$, i.e.*

$$(2.1) \quad H_g^\sigma(P_3)\Phi_g^\sigma(P_3) = E_{g\sigma}(P_3)\Phi_g^\sigma(P_3), \quad \|\Phi_g^\sigma(P_3)\| = 1.$$

Notice that Proposition 2.1 is also established in [AGG1] under the weaker assumption that e_0 is an isolated eigenvalue of $h(b, V)$ of finite multiplicity. Let us recall a lemma, due to [DG2], on which is based our proof of the absence of a ground state for $H_g(P_3)$ in the case $E'_g(P_3) \neq 0$.

LEMMA 2.2. *Let $\Psi \in L^2(\mathbb{R}^2; \mathbb{C}^2) \otimes \mathcal{H}_{\text{ph}}$. Assume that*

$$(2.2) \quad \int_{\mathbb{R}^3 \times \mathbb{Z}_2} \|(a(\mathbf{k}) - h(\mathbf{k}))\Psi\|^2 d\mathbf{k} < \infty,$$

where h is a measurable function from $\mathbb{R}^3 \times \mathbb{Z}_2$ to \mathbb{C} such that $h \notin L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$. Then $\Psi = 0$.

PROOF. See [DG2, Lemma 2.6]. □

Theorem 1.3 shall follow from a suitable decomposition of $a(\mathbf{k})\Phi_g^\sigma(P_3)$ based on a pull-through formula. The latter is the purpose of the following lemma, where the equalities should be understood as identities between measurable functions from $\mathbb{R}^3 \times \mathbb{Z}_2$ to $L^2(\mathbb{R}^2; \mathbb{C}^2) \otimes \mathcal{H}_{\text{ph}}$. For a rigorous justification of the commutations used in the next proof, we refer for instance to [Ge, HH].

In order to shorten the notations, we shall write

$$(2.3) \quad \begin{aligned} H &= H_g^\sigma(P_3), & E &= E_{g\sigma}(P_3), & \Phi &= \Phi_g^\sigma(P_3), \\ \tilde{H} &= H_g^\sigma(P_3 - k_3), & \tilde{E} &= E_{g\sigma}(P_3 - k_3). \end{aligned}$$

LEMMA 2.3. *Let $\sigma \geq 0$ and let $\Phi = \Phi_g^\sigma(P_3)$ be a normalized ground state of $H = H_g^\sigma(P_3)$ (assuming it exists for $\sigma = 0$). We have:*

$$(2.4) \quad a(\mathbf{k})\Phi = L_\sigma(\mathbf{k})\Phi + R_\sigma(\mathbf{k})\Phi + \frac{1}{\sqrt{2}}f_\sigma(\mathbf{k})\Phi,$$

where L_σ and R_σ are operator-valued functions such that,

$$(2.5) \quad \int_{\mathbb{R}^3 \times \mathbb{Z}_2} \|L_\sigma(\mathbf{k})\Phi\|^2 d\mathbf{k} \leq Cg^2,$$

and

$$(2.6) \quad R_\sigma(\mathbf{k}) = -\frac{g}{2\sqrt{2\pi}} \frac{\rho_\sigma^\Lambda(k) \epsilon_3^\lambda(k) |k|^{1/2}}{k_3(\tilde{E} - E + |k|)} \frac{\tilde{H} - \tilde{E}}{\tilde{H} - E + |k|}.$$

PROOF. It follows from the canonical commutation relations (1.6) that

$$(2.7) \quad \begin{aligned} a(\mathbf{k})H &= (\tilde{H} + |k|)a(\mathbf{k}) \\ &- \frac{g}{2^{\frac{3}{2}}m} \sum_{j=1,2} \left(h_{j,\sigma}(x', \mathbf{k})(p_j - ea_j(x') - gA_{j,\sigma}(x', 0)) + \sigma_j \tilde{h}_{j,\sigma}(x', \mathbf{k}) \right) \\ &- \frac{g}{2^{\frac{3}{2}}m} \left(h_{3,\sigma}(x', \mathbf{k})(P_3 - d\Gamma(k_3) - gA_{3,\sigma}(x', 0)) + \sigma_3 \tilde{h}_{3,\sigma}(x', \mathbf{k}) \right). \end{aligned}$$

In order to control the term containing $(p_j - ea_j(x') - gA_{j,\sigma}(x', 0))$ in the right-hand-side of the previous equality, we use that (formally)

$$(2.8) \quad \frac{1}{2m}(p_j - ea_j(x') - gA_{j,\sigma}(x', 0)) = i[H, x'_j],$$

for $j = 1, 2$. Notice that an alternative would be to consider the Hamiltonian obtained through a unitary Power-Zienau-Wooley transformation (see for instance [GLL]). For a rigorous justification of (2.8), we refer to [BFP, Theorem II.10] which can easily be adapted to our case. In particular it follows that $x'_j \Phi \in D(H)$. Applying (2.7) to Φ then yields

$$(2.9) \quad \begin{aligned} a(\mathbf{k})\Phi &= \frac{ig}{2^{\frac{3}{2}}} \sum_{j=1,2} h_{j,\sigma}(x', \mathbf{k})[\tilde{H} - E + |k|]^{-1}(H - E)x'_j \Phi \\ &+ \frac{g}{2^{\frac{3}{2}}m} [\tilde{H} - E + |k|]^{-1} \sigma \cdot \tilde{h}_\sigma(x', \mathbf{k}) \Phi \\ &+ \frac{g}{2^{\frac{3}{2}}m} h_{3,\sigma}(x', \mathbf{k})[\tilde{H} - E + |k|]^{-1} (P_3 - d\Gamma(k_3) - gA_{3,\sigma}(x', 0)) \Phi. \end{aligned}$$

Note that the expressions of H and \tilde{H} imply

$$(2.10) \quad \tilde{H} - H = -\frac{k_3}{m} (P_3 - d\Gamma(k_3) - gA_{3,\sigma}(x', 0)) + \frac{k_3^2}{2m}.$$

From (1.18), we get

$$(2.11) \quad \|[\tilde{H} - E + |k|]^{-1}\| \leq C|k|^{-1}.$$

Moreover it is not difficult to show that

$$(2.12) \quad \left\| (P_3 - d\Gamma(k_3) - gA_{3,\sigma}(x', 0))[\tilde{H} - E + |k|]^{-1} \right\| \leq C|k|^{-1},$$

and consequently, by (2.10),

$$(2.13) \quad \left\| (H - E)[\tilde{H} - E + |k|]^{-1} \right\| \leq C.$$

Introducing (2.11)–(2.13) into (2.9) and recalling the definitions (1.16) of h_j and \tilde{h}_j , we thus obtain

$$(2.14) \quad \begin{aligned} a(\mathbf{k})\Phi &= L_1(\mathbf{k})\Phi \\ &+ \frac{g}{2^{\frac{3}{2}}m} h_{3,\sigma}(0, \mathbf{k})[\tilde{H} - E + |k|]^{-1} (P_3 - d\Gamma(k_3) - gA_{3,\sigma}(x', 0)) \Phi, \end{aligned}$$

where

$$(2.15) \quad \|L_1(\mathbf{k})\Phi\| \leq C|g||k|^{-1/2} (\|\Phi\| + \|x'_1 \Phi\| + \|x'_2 \Phi\|).$$

In passing from (2.9) to (2.14) we used that

$$(2.16) \quad |h_{3,\sigma}(x', \mathbf{k}) - h_{3,\sigma}(0, \mathbf{k})| \leq C|k||x'|.$$

Let us now note the following obvious identity:

$$(2.17) \quad \frac{\tilde{H} - E}{\tilde{H} - E + |k|} = \frac{\tilde{E} - E}{\tilde{E} - E + |k|} + \frac{|k|}{\tilde{E} - E + |k|} \left(\frac{\tilde{H} - \tilde{E}}{\tilde{H} - E + |k|} \right).$$

Hence, introducing (2.10) and (2.17) into (2.14) leads to

$$(2.18) \quad \begin{aligned} a(\mathbf{k})\Phi &= L_1(\mathbf{k})\Phi - \frac{gk_3}{2^{\frac{3}{2}}}h_{3,\sigma}(0, \mathbf{k})[\tilde{H} - E + |k|]^{-1}\Phi \\ &- \frac{g}{2^{\frac{3}{2}}k_3} \frac{\tilde{E} - E}{\tilde{E} - E + |k|} h_{3,\sigma}(0, \mathbf{k})\Phi \\ &- \frac{g}{2^{\frac{3}{2}}k_3} \frac{|k|}{\tilde{E} - E + |k|} h_{3,\sigma}(0, \mathbf{k}) \frac{\tilde{H} - \tilde{E}}{\tilde{H} - E + |k|} \Phi. \end{aligned}$$

We conclude the proof using again that $\|x'_j\Phi\| < \infty$. \square

The following lemma shows in particular that if the map $P_3 \mapsto E(P_3)$ is sufficiently regular, then $\mathbf{k} \mapsto \|R(\mathbf{k})\Phi\|$ is in $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$, where $R(\mathbf{k})$ denotes the operator defined in (2.6) for $\sigma = 0$.

LEMMA 2.4. *Let the parameters g, σ, P_3 be fixed. Assume that there exist $\gamma > 0$, $P_0 > 0$ and a positive constant C independent of $\sigma \geq 0$ such that for all $|k_3| \leq P_0$,*

$$(2.19) \quad |E'_{g\sigma}(P_3 + k_3) - E'_{g\sigma}(P_3)| \leq C|k_3|^\gamma.$$

Then there exists a positive constant C' , independent of σ , such that

$$(2.20) \quad \left\| (H_g^\sigma(P_3 - k_3) - E_{g\sigma}(P_3 - k_3))^{1/2}\Phi \right\| \leq C'|k_3|^{\frac{1+\gamma}{2}}.$$

PROOF. We use again the notations (2.3) and let in addition $E' = E'_{g\sigma}(P_3)$. By (2.10), we have

$$(2.21) \quad \tilde{E} - E \leq (\Phi, (\tilde{H} - H)\Phi) = -\frac{k_3}{m}(\Phi, (P_3 - d\Gamma(k_3) - A_{3,\sigma}(x', 0))\Phi) + \frac{k_3^2}{2m}.$$

Dividing by $-k_3$ and letting $k_3 \rightarrow 0$ (distinguishing the cases $k_3 > 0$ and $k_3 < 0$), we obtain the Feynman-Hellman formula:

$$(2.22) \quad E' = \frac{1}{m}(\Phi, (P_3 - d\Gamma(k_3) - A_{3,\sigma}(x', 0))\Phi).$$

Hence, by (2.10),

$$(2.23) \quad \begin{aligned} \left| (\Phi, (\tilde{H} - \tilde{E})\Phi) \right| &= \left| (\Phi, (\tilde{H} - H) - (\tilde{E} - E)\Phi) \right| \\ &\leq \left| -k_3 E' - (\tilde{E} - E) \right| + \frac{k_3^2}{2m}. \end{aligned}$$

The lemma then follows from (2.19) and the mean value theorem. \square

We are now ready to prove Theorem 1.3:

PROOF OF THEOREM 1.3. Let us begin with estimating the term $\|R_\sigma(\mathbf{k})\Phi_g^\sigma(P_3)\|$ appearing in Lemma 2.3. Recalling the notations (2.3), we write

$$(2.24) \quad \|R_\sigma(\mathbf{k})\Phi\| \leq \frac{C|g|}{|k_3||k|^{\frac{1}{2}}} \mathbf{1}_{\sigma \leq |k| \leq \Lambda}(k) \left\| \frac{(\tilde{H} - \tilde{E})^{1/2}}{\tilde{H} - E + |k|} \right\| \|(\tilde{H} - \tilde{E})^{1/2}\Phi\|.$$

It follows from the Spectral Theorem and (1.18) that

$$(2.25) \quad \left\| \frac{(\tilde{H} - \tilde{E})^{1/2}}{\tilde{H} - E + |k|} \right\| = \sup_{r \geq 0} \left| \frac{r^{\frac{1}{2}}}{r + \tilde{E} - E + |k|} \right| \leq \sup_{r \geq 0} \left| \frac{r^{\frac{1}{2}}}{r + |k|/4} \right| \leq \frac{C}{|k|^{\frac{1}{2}}}.$$

Thus, Theorem 1.2 together with Lemma 2.4 yield

$$(2.26) \quad \|R_\sigma(\mathbf{k})\Phi\| \leq \frac{C|g|}{|k_3|^{\frac{1}{2}-\frac{\gamma}{2}}|k|} \mathbf{1}_{\sigma \leq |k| \leq \Lambda}(k),$$

where $\gamma = 1/4 - \delta$, and where δ in Theorem 1.2 is chosen such that $0 < \delta < 1/4$. Hence

$$(2.27) \quad \int_{\mathbb{R}^3 \times \mathbb{Z}_2} \|R_\sigma(\mathbf{k})\Phi\|^2 d\mathbf{k} \leq Cg^2.$$

Let us now prove (i). First assume that $E'_g(P_3) = 0$. In order to get the existence of a ground state for $H_g(P_3)$ our aim is to prove that $\Phi_g^\sigma(P_3)$ converges strongly as $\sigma \rightarrow 0$. Using Lemma A.7 (see also Remark A.8), we obtain from (1.17) that

$$(2.28) \quad |f_\sigma(\mathbf{k})| \leq C \left(\frac{g^2\sigma}{|k_3||k|^{\frac{3}{2}}} + \frac{|g|(E_g(P_3 - k_3) - E_g(P_3))}{|k_3||k|^{\frac{3}{2}}} \right) \mathbf{1}_{\sigma \leq |k| \leq \Lambda}(k).$$

Hence, since $E'_g(P_3) = 0$ by assumption, (1.13) implies

$$(2.29) \quad |f_\sigma(\mathbf{k})| \leq C \left(\frac{g^2\sigma}{|k_3||k|^{3/2}} + \frac{|g|k_3^{\frac{1}{4}-\delta}}{|k|^{\frac{3}{2}}} \right) \mathbf{1}_{\sigma \leq |k| \leq \Lambda}(k).$$

Therefore

$$(2.30) \quad \|f_\sigma\|_{L^2(\mathbb{R}^3 \times \mathbb{Z}_2)} \leq C|g|.$$

Combining Lemma 2.3 with (2.30) and (2.27), we obtain

$$(2.31) \quad (\Phi_g^\sigma(P_3), \mathcal{N}\Phi_g^\sigma(P_3)) = \int_{\mathbb{R}^3 \times \mathbb{Z}_2} \|a(\mathbf{k})\Phi_g^\sigma(P_3)\|^2 d\mathbf{k} \leq Cg^2,$$

where $\mathcal{N} = d\Gamma(I)$ denotes the number operator. For a sufficiently small fixed $|g|$, the strong convergence of $\Phi_g^\sigma(P_3)$ as $\sigma \rightarrow 0$ is then obtained by following for instance [BFS], showing that $|(\Phi_g^\sigma(P_3), \Phi_{\text{el}} \otimes \Omega)| \geq C > 0$ uniformly in $\sigma \geq 0$. Here Φ_{el} denotes a normalized ground state of $h(b, V)$.

Assume next that $E'_g(P_3) \neq 0$ and let us prove that $H_g(P_3)$ does not have a ground state. By Lemmata 2.2, 2.3 and Estimate (2.27), it suffices to prove that $f \notin L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$. The latter follows from the fact that

$$(2.32) \quad \left| \frac{E_g(P_3 - k_3) - E_g(P_3)}{k_3} \right| \geq C > 0$$

uniformly for small k_3 since $E'_g(P_3) \neq 0$. Hence Theorem 1.3(i) is proven.

Let us finally prove (ii). For $\sigma > 0$, we set

$$(2.33) \quad \Phi^{\text{ren}} = W(if_\sigma)\Phi_g^\sigma(P_3).$$

Obviously Φ^{ren} is a normalized ground state of $H_{g\sigma}^{\text{ren}}(P_3)$. By Lemma 2.3 we have

$$\begin{aligned} a(\mathbf{k})\Phi^{\text{ren}} &= W(if_\sigma)a(\mathbf{k})\Phi + [a(\mathbf{k}), W(if_\sigma)]\Phi \\ &= W(if_\sigma)L_\sigma(\mathbf{k})\Phi + W(if_\sigma)R_\sigma(\mathbf{k})\Phi + \frac{1}{\sqrt{2}}f_\sigma(\mathbf{k})\Phi^{\text{ren}} + [a(\mathbf{k}), W(if_\sigma)]\Phi. \end{aligned}$$

One can compute the commutator $[a(\mathbf{k}), W(if_\sigma)] = -2^{-1/2}f_\sigma(\mathbf{k})$, so that

$$(2.34) \quad a(\mathbf{k})\Phi^{\text{ren}} = W(if_\sigma)L_\sigma(\mathbf{k})\Phi + W(if_\sigma)R_\sigma(\mathbf{k})\Phi.$$

Therefore, since $W(if_\sigma)$ is unitary, $\|a(\mathbf{k})\Phi^{\text{ren}}\|$ can be estimated in the same way as $\|a(\mathbf{k})\Phi\|$ (in the case $E'_g(P_3) = 0$), using (2.5) and (2.27). This leads to the existence of a ground state for $H_g^{\text{ren}}(P_3)$ and concludes the proof of Theorem 1.3. \square

Appendix A. Uniform regularity of the map $P_3 \mapsto E_{g\sigma}(P_3)$

In this appendix we shall prove Theorem 1.2. The structure follows [Pi] and [CFP]: First, we give a simple proof of the existence of a spectral gap for the infrared cutoff Hamiltonian $H_g^\sigma(P_3)$, considered as an operator on the space of photons of energies $\geq \sigma$. Our proof is based on the min-max principle. Then we establish (1.13) by adapting [Pi, CFP] (see also [BFP]). In comparison to [CFP], the main technical difference comes from the terms in $H_g(P_3)$ containing the interaction between the electronic variables x'_j and the quantized electromagnetic field. This shall be handled in Lemma A.11 below thanks to the exponential decay of $\Phi_g^\sigma(P_3)$ in x'_j .

In some parts of our presentation, we shall only sketch the proof, emphasizing the differences that we have to include, and referring otherwise to [Pi], [BFP], or [CFP].

Let us begin with some definitions and notations. Henceforth we remove the subindex g to simplify the notations, and for $\sigma \geq 0$, we replace $H^\sigma(P_3)$ by its Wick-ordered version $H^\sigma(P_3) - \frac{g^2}{2m}(\Lambda^2 - \sigma^2)$ (which we still denote by $H^\sigma(P_3)$). Note that this shall not affect our discussion below on the regularity of the ground state energy since the two operators only differ by a constant. We decompose

$$(A.1) \quad H^\sigma(P_3) = h_0(P_3) + H_I^\sigma(P_3),$$

where

$$(A.2) \quad h_0(P_3) = h(b, V) \otimes \mathbf{1} + \mathbf{1} \otimes \left[\frac{1}{2m} (P_3 - d\Gamma(k_3))^2 + H_f \right],$$

and

$$(A.3) \quad \begin{aligned} H_I^\sigma(P_3) &= -\frac{g}{m} \sum_{j=1,2} \left(A_{j,\sigma}(x', 0) (p_j - ea_j(x')) + \frac{g^2}{2m} A_{j,\sigma}(x', 0)^2 \right) \\ &\quad - \frac{g}{2m} A_{3,\sigma}(x', 0) (P_3 - d\Gamma(k_3)) - \frac{g}{2m} (P_3 - d\Gamma(k_3)) A_{3,\sigma}(x', 0) \\ &\quad + \frac{g^2}{2m} A_{3,\sigma}(x', 0)^2 - \frac{g}{2m} \sigma \cdot B_\sigma(x', 0) - \frac{g^2}{2m} (\Lambda^2 - \sigma^2). \end{aligned}$$

Let Φ_{el} denote a normalized ground state of $h(b, V)$. For any $|P_3| < m$, one can easily check that $\Phi_{\text{el}} \otimes \Omega$ is a ground state of $h_0(P_3)$, with ground state energy

$e_0(P_3) = e_0 + P_3^2/2m$. Note that for $\tau \leq \sigma$, we have

$$\begin{aligned}
& H^\tau(P_3) - H^\sigma(P_3) \\
&= -\frac{g}{m} \sum_{j=1,2} A_{j,\tau}^\sigma(x', 0) \left(p_j - ea_j(x') - gA_{j,\sigma}(x', 0) \right) - \frac{g^2}{2m} (\sigma^2 - \tau^2) \\
&+ \frac{g^2}{2m} A_\tau^\sigma(x', 0)^2 - \frac{g}{2m} A_{3,\tau}^\sigma(x', 0) \left(P_3 - d\Gamma(k_3) - gA_{3,\sigma}(x', 0) \right) \\
&- \frac{g}{2m} \left(P_3 - d\Gamma(k_3) - gA_{3,\sigma}(x', 0) \right) A_{3,\tau}^\sigma(x', 0) - \frac{g}{2m} \sigma \cdot B_\tau^\sigma(x', 0),
\end{aligned} \tag{A.4}$$

where

$$A_\tau^\sigma(x', 0) = \frac{1}{\sqrt{2\pi}} \int \frac{\epsilon^\lambda(k)}{|k|^{1/2}} \rho_\tau^\sigma(k) \left[e^{-ik' \cdot x'} a_\lambda^*(k) + e^{ik' \cdot x'} a_\lambda(k) \right] d\mathbf{k}, \tag{A.5}$$

and likewise for $B_\tau^\sigma(x', 0)$. Let $\mathcal{H}_\sigma = L^2(\mathbb{R}^2; \mathbb{C}^2) \otimes \mathcal{F}_\sigma$, where \mathcal{F}_σ denotes the symmetric Fock space over $L^2(\{\mathbf{k} \in \mathbb{R}^3 \times \mathbb{Z}_2, |\mathbf{k}| \geq \sigma\})$. The restriction of $H^\sigma(P_3)$ to \mathcal{H}_σ is denoted by $H_\sigma(P_3)$:

$$H_\sigma(P_3) = H^\sigma(P_3)|_{\mathcal{H}_\sigma}, \tag{A.6}$$

and, similarly,

$$h_{0,\sigma}(P_3) = h_0(P_3)|_{\mathcal{H}_\sigma}, \quad H_{I,\sigma}(P_3) = H_I^\sigma(P_3)|_{\mathcal{H}_\sigma}. \tag{A.7}$$

Let Ω_σ be the vacuum in \mathcal{F}_σ . Then for $|P_3| < m$, $\Phi_{\text{el}} \otimes \Omega_\sigma$ is a ground state of $h_{0,\sigma}(P_3)$ with ground state energy $e_0(P_3)$, and

$$\text{Gap}(h_{0,\sigma}(P_3)) \geq \left(1 - \frac{|P_3|}{m}\right)\sigma, \tag{A.8}$$

where $\text{Gap}(H) = \inf(\sigma(H) \setminus \{E(H)\}) - \inf(\sigma(H))$ for any self-adjoint and semi-bounded operator H with ground state energy $E(H)$. We also define

$$H_\tau^\sigma(P_3) = (H^\tau(P_3) - H^\sigma(P_3))|_{\mathcal{H}_\tau}. \tag{A.9}$$

The symmetric Fock space over $L^2(\{\mathbf{k} \in \mathbb{R}^3 \times \mathbb{Z}_2, \tau \leq |\mathbf{k}| \leq \sigma\})$ is denoted by \mathcal{F}_τ^σ . Note that there exists a unitary operator $\mathcal{V} : \mathcal{H}_\tau \rightarrow \mathcal{H}_\sigma \otimes \mathcal{F}_\tau^\sigma$. We shall identify \mathcal{H}_τ and $\mathcal{H}_\sigma \otimes \mathcal{F}_\tau^\sigma$ in the sequel in order to simplify the notations. We let Ω_τ^σ be the vacuum in \mathcal{F}_τ^σ .

A.1. Existence of a spectral gap.

LEMMA A.1. *There exist $g_0 > 0$, $\sigma_0 > 0$ and $P_0 > 0$ such that the following holds: Let $|g| \leq g_0$, $0 \leq \sigma \leq \sigma_0$ and $|P_3| \leq P_0$ be such that $H_\sigma(P_3)$ has a normalized ground state $\Phi_\sigma(P_3)$ and $\text{Gap}(H_\sigma(P_3)) \geq \gamma\sigma$ for some $\gamma > 0$. Then for all $0 \leq \tau \leq \sigma$, $\Phi_\sigma(P_3) \otimes \Omega_\tau^\sigma$ is a normalized ground state of $H^\sigma(P_3)|_{\mathcal{H}_\tau}$, and*

$$\text{Gap}(H^\sigma(P_3)|_{\mathcal{H}_\tau}) \geq \min(\gamma\sigma, \tau/4). \tag{A.10}$$

PROOF. To simplify the notations, let us remove the dependence on P_3 throughout the proof. First, one can readily check that $\Phi_\sigma \otimes \Omega_\tau^\sigma$ is an eigenstate of $H^\sigma|_{\mathcal{H}_\tau}$

associated with the eigenvalue E_σ . For any v we let $[v]$ and $[v]^\perp$ denote respectively the subspace spanned by v and its orthogonal complement. We write

$$\begin{aligned} & \inf_{\Phi \in [\Phi_\sigma \otimes \Omega_\tau^\sigma]^\perp, \|\Phi\|=1} (\Phi, H^\sigma |_{\mathcal{H}_\tau} \Phi) \\ & \geq \min \left(\inf_{\Phi \in [\Phi_\sigma]^\perp \otimes [\Omega_\tau^\sigma], \|\Phi\|=1} (\Phi, H^\sigma |_{\mathcal{H}_\tau} \Phi), \inf_{\Phi \in \mathcal{H}_\sigma \otimes [\Omega_\tau^\sigma]^\perp, \|\Phi\|=1} (\Phi, H^\sigma |_{\mathcal{H}_\tau} \Phi) \right). \end{aligned}$$

The assumption $\text{Gap}(H_\sigma) \geq \gamma\sigma$ implies

$$\inf_{\Phi \in [\Phi_\sigma]^\perp \otimes [\Omega_\tau^\sigma], \|\Phi\|=1} (\Phi, H^\sigma |_{\mathcal{H}_\tau} \Phi) \geq E_\sigma + \gamma\sigma.$$

On the other hand, using that the number operator $\int_{\tau \leq |k| \leq \sigma} a^*(\mathbf{k})a(\mathbf{k})d\mathbf{k}$ commutes with $H^\sigma |_{\mathcal{H}_\tau}$, one can prove as in [P1] that

$$\inf_{\Phi \in \mathcal{H}_\sigma \otimes [\Omega_\tau^\sigma]^\perp, \|\Phi\|=1} (\Phi, H^\sigma |_{\mathcal{H}_\tau} \Phi) \geq \inf_{\tau \leq |k| \leq \sigma} (E_\sigma(P_3 - k_3) - E_\sigma(P_3) + |k|).$$

We conclude the proof thanks to (1.18) \square

COROLLARY A.2. *Under the conditions of Lemma A.1, for all $0 \leq \tau \leq \sigma$,*

$$(A.11) \quad E_\tau(P_3) \leq E_\sigma(P_3) \leq e_0(P_3).$$

PROOF. It follows from Lemma A.1 that

$$(A.12) \quad \begin{aligned} E_\tau(P_3) & \leq (\Phi_\sigma(P_3) \otimes \Omega_\tau^\sigma, H_\tau(P_3) \Phi_\sigma(P_3) \otimes \Omega_\tau^\sigma) \\ & = (\Phi_\sigma(P_3) \otimes \Omega_\tau^\sigma, H^\sigma(P_3) |_{\mathcal{H}_\tau} \Phi_\sigma(P_3) \otimes \Omega_\tau^\sigma) = E_\sigma(P_3). \end{aligned}$$

Hence the first inequality in (A.11) is proven. To prove the second one, it suffices to write similarly

$$(A.13) \quad \begin{aligned} E_\sigma(P_3) & \leq (\Phi_{\text{el}} \otimes \Omega_\sigma, H_\sigma(P_3) \Phi_{\text{el}} \otimes \Omega_\sigma) \\ & = (\Phi_{\text{el}} \otimes \Omega_\sigma, h_{0,\sigma}(P_3) \Phi_{\text{el}} \otimes \Omega_\sigma) = e_0(P_3). \end{aligned}$$

\square

We shall establish the existence of a spectral gap of order $O(\sigma)$ above the bottom of the spectrum of $H_\sigma(P_3)$ by induction. More precisely, let $\mathbf{Gap}(\sigma)$ denote the assertion

$$\mathbf{Gap}(\sigma) \left\{ \begin{array}{l} \text{(i)} \quad E_\sigma(P_3) \text{ is a simple eigenvalue of } H_\sigma(P_3), \\ \text{(ii)} \quad \text{Gap}(H_\sigma(P_3)) \geq \sigma/8. \end{array} \right.$$

We shall prove

PROPOSITION A.3. *There exists $g_0 > 0$, $\sigma_0 > 0$ and $P_0 > 0$ such that, for all $|g| \leq g_0$, $0 < \sigma \leq \sigma_0$ and $|P_3| \leq P_0$, the assertion $\mathbf{Gap}(\sigma)$ above holds.*

Let us begin with two preliminary useful estimates:

LEMMA A.4. *Fix the parameters g , σ and P_3 such that $0 \leq |g| \leq g_0$, $0 \leq \sigma \leq \sigma_0$ and $0 \leq |P_3| \leq P_0$, for some sufficiently small g_0 , σ_0 and P_0 . For any $0 < \rho < 1$,*

$$(A.14) \quad \begin{aligned} & \left\| [h_{0,\sigma}(P_3) - e_0(P_3) + \rho]^{-1/2} H_{I,\sigma}(P_3) [h_{0,\sigma}(P_3) - e_0(P_3) + \rho]^{-1/2} \right\| \\ & \leq C|g|\rho^{-1/2}, \end{aligned}$$

where C is a positive constant (depending only on Λ). Likewise,

$$(A.15) \quad \left\| [H^\sigma(P_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + \rho]^{-1/2} H_\tau^\sigma(P_3) [H^\sigma(P_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + \rho]^{-1/2} \right\| \leq C|g|\sigma^{1/2}\rho^{-1/2}.$$

PROOF. Let us prove (A.15), Estimate (A.14) would follow similarly. We introduce the expression of $H_\tau^\sigma(P_3)$ given by (A.4) and (A.9) and estimate each term separately. Consider for instance

$$(A.16) \quad |g| \left\| \left[H^\sigma(P_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + \rho \right]^{-1/2} \int_{\tau \leq |k| \leq \sigma} \frac{\epsilon_\lambda^{(3)}(k)}{|k|^{1/2}} e^{ik' \cdot x'} a^*(\mathbf{k}) d\mathbf{k} \right. \\ \left. \left(P_3 - d\Gamma(k_3) + gA_{3,\sigma}(x', 0) \right) \left[H^\sigma(P_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + \rho \right]^{-1/2} \right\|.$$

Using that

$$(A.17) \quad \left\| \left(P_3 - d\Gamma(k_3) + gA_{3,\sigma}(x', 0) \right) \left[H^\sigma(P_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + \rho \right]^{-1/2} \right\| \leq C\rho^{-1/2},$$

we get

$$(A.16) \leq C|g|\rho^{-1/2} \left\| \left[H^\sigma(P_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + \rho \right]^{-1/2} \int_{\tau \leq |k| \leq \sigma} \frac{\epsilon_\lambda^{(3)}(k)}{|k|^{1/2}} e^{ik' \cdot x'} a^*(\mathbf{k}) d\mathbf{k} \right\|.$$

Moreover, for any $\Phi \in D(H^\sigma(P_3)|_{\mathcal{H}_\tau})$,

$$\left\| \left[H^\sigma(P_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + \rho \right]^{-1/2} \int_{\tau \leq |k| \leq \sigma} \frac{\epsilon_\lambda^{(3)}(k)}{|k|^{1/2}} e^{ik' \cdot x'} a^*(\mathbf{k}) d\mathbf{k} \Phi \right\|^2 \\ \leq \int_{\tau \leq |k|, |\tilde{k}| \leq \sigma} \frac{C}{|k|^{1/2} |\tilde{k}|^{1/2}} \left| \left(\Phi, a(\mathbf{k}) \left[H^\sigma(P_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + \rho \right]^{-1} a^*(\tilde{\mathbf{k}}) \Phi \right) \right| d\mathbf{k} d\tilde{\mathbf{k}}.$$

Now, for any \mathbf{k} such that $\tau \leq |k| \leq \sigma$, we have the pull-through formula

$$(A.18) \quad a(\mathbf{k}) H^\sigma(P_3)|_{\mathcal{H}_\tau} = \left[H^\sigma(P_3 - k_3)|_{\mathcal{H}_\tau} + |k| \right] a(\mathbf{k}),$$

since $a(\mathbf{k})$ commutes with $A_\sigma(x', 0)$. Hence

$$\left(\Phi, a(\mathbf{k}) \left[H^\sigma(P_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + \rho \right]^{-1} a^*(\tilde{\mathbf{k}}) \Phi \right) \\ = \delta(\mathbf{k} - \tilde{\mathbf{k}}) \left(\Phi, \left[H^\sigma(P_3 - k_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + |k| + \rho \right]^{-1} \Phi \right) \\ + \left(a(\tilde{\mathbf{k}}) \Phi, \left[H^\sigma(P_3 - k_3 - \tilde{k}_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + |k| + |\tilde{k}| + \rho \right]^{-1} a(\mathbf{k}) \Phi \right).$$

Using that $H^\sigma(P_3 - k_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + |k| \geq |k|/4$ for any k sufficiently small (see (1.18)), we get

$$\left\| \left[H^\sigma(P_3 - k_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + |k| + \rho \right]^{-1} \right\| \leq \frac{C}{|k|}.$$

Let $H_{f,\tau}^\sigma = \int_{\tau \leq |k| \leq \sigma} |k| a^*(\mathbf{k}) a(\mathbf{k}) d\mathbf{k}$. As in [Pi, Lemma 1.1], it follows from the proof of Lemma A.1 that $H_{f,\tau}^\sigma \leq C(H^\sigma(P_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3))$ for any P_3 sufficiently

small. This yields

$$\left\| \left[H_{f,\tau}^\sigma + |k| + |\tilde{k}| \left[H^\sigma(P_3 - k_3 - \tilde{k}_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + |k| + |\tilde{k}| + \rho \right]^{-1} \right\| \leq C.$$

Thus, combining the previous estimates we obtain

$$\begin{aligned} & \left\| \left[H^\sigma(P_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + \rho \right]^{-1/2} \int_{\tau \leq |k| \leq \sigma} \frac{\epsilon_\lambda^{(3)}(k)}{|k|^{1/2}} e^{ik' \cdot x'} a^*(\mathbf{k}) d\mathbf{k} \Phi \right\|^2 \\ & \leq C \int_{\tau \leq |k| \leq \sigma} \frac{d\mathbf{k}}{|k|^2} + C \left[\int_{\tau \leq |k| \leq \sigma} \frac{d\mathbf{k}}{|k|^{1/2}} \left\| [H_{f,\tau}^\sigma + |k|]^{-1/2} a(\mathbf{k}) \Phi \right\| \right]^2 \leq C\sigma. \end{aligned}$$

Since $D(H^\sigma(P_3)|_{\mathcal{H}_\tau})$ is dense in \mathcal{H}_τ , the result is proven as for the term we have chosen to consider, that is (A.16) $\leq C|g|\sigma^{1/2}\rho^{-1/2}$. Since the other terms in the expression of H_τ^σ given by (A.4) can be treated in the same way, the lemma is established. \square

The next lemma corresponds to the root in the induction procedure leading to the proof of Proposition A.3.

LEMMA A.5. *There exist $g_0 > 0$, $\sigma_0 > 0$, $P_0 > 0$ and a positive constant C_0 such that for all $|g| \leq g_0$ and $|P_3| \leq P_0$, for all σ such that $C_0 g^2 \leq \sigma \leq \sigma_0$, the assertion **Gap**(σ) holds.*

PROOF. To simplify the notations, we write H_σ for $H_\sigma(P_3)$, E_σ for $E_\sigma(P_3)$, and similarly for other quantities depending on P_3 . Let μ_σ denote the first point above E_σ in the spectrum of H_σ . By the min-max principle,

$$(A.19) \quad \mu_\sigma \geq \inf_{\psi \in [\Phi_{\text{el}} \otimes \Omega_\sigma]^\perp, \|\psi\|=1} (\psi, H_\sigma \psi),$$

where $[v]^\perp$ denotes the orthogonal complement of the vector space spanned by v . It follows from (A.14) that for any $\psi \in [\Phi_{\text{el}} \otimes \Omega_\sigma]^\perp$, $\|\psi\| = 1$, and any $\rho > 0$,

$$(A.20) \quad \begin{aligned} (\psi, H_\sigma \psi) & \geq (\psi, H_{0,\sigma} \psi) - C|g|\rho^{-1/2} (\psi, [h_{0,\sigma} - e_0(P_3) + \rho] \psi) \\ & \geq \left(1 - C|g|\rho^{-1/2}\right) (\psi, H_{0,\sigma} \psi) + C|g|\rho^{-1/2} e_0(P_3) - C|g|\rho^{1/2}. \end{aligned}$$

By (A.8), for any $\psi \in [\Phi_{\text{el}} \otimes \Omega_\sigma]^\perp$, $(\psi, h_{0,\sigma} \psi) \geq e_0(P_3) + (1 - |P_3|/m)\sigma$ provided that σ_0 is chosen sufficiently small. Hence for any ρ such that $\rho^{1/2} > C|g|$,

$$(A.21) \quad (\psi, H_\sigma \psi) \geq e_0(P_3) + \left(1 - C|g|\rho^{-1/2}\right) \left(1 - \frac{|P_3|}{m}\right) \sigma - C|g|\rho^{1/2}.$$

Choosing $\rho^{1/2} = 4C|g|$ and P_0 sufficiently small, by Corollary A.2, we obtain

$$(A.22) \quad \begin{aligned} (\psi, H_\sigma \psi) & \geq E_\sigma + \frac{3}{4} \left(1 - \frac{|P_3|}{m}\right) \sigma - 4C^2 g^2 \\ & \geq E_\sigma + \frac{1}{2} \sigma - 4C^2 g^2. \end{aligned}$$

Together with (A.19), this leads to the statement of the lemma provided that the constant C_0 is chosen such that $C_0 > 32C^2/3$. \square

The following lemma corresponds to the induction step of the induction process in the proof of Proposition A.3.

LEMMA A.6. *There exists $g_0 > 0$, $\sigma_0 > 0$ and $P_0 > 0$ such that for all $|g| \leq g_0$ and $|P_3| \leq P_0$, for all σ such that $0 < \sigma \leq \sigma_0$,*

$$\mathbf{Gap}(\sigma) \Rightarrow \mathbf{Gap}(\sigma/2).$$

PROOF. Again, throughout the proof, we drop the dependence on P_3 in all the considered quantities. Let $\mathbf{Gap}(\sigma)$ be satisfied for some $0 < \sigma$, let Φ_σ be a ground state of H_σ , and let $\tau = \sigma/2$. As in the proof of Lemma A.5, let μ_τ denote the first point above E_τ in the spectrum of H_τ . By the min-max principle,

$$(A.23) \quad \mu_\tau \geq \inf_{\psi \in [\Phi_\sigma \otimes \Omega_\tau^\sigma]^\perp, \|\psi\|=1} (\psi, H_\tau \psi),$$

where Ω_τ^σ is the vacuum in \mathcal{F}_τ^σ and where $[\Phi_\sigma \otimes \Omega_\tau^\sigma]^\perp$ denotes the orthogonal complement of the vector space spanned by $\Phi_\sigma \otimes \Omega_\tau^\sigma$ in $\mathcal{H}_\sigma \otimes \mathcal{F}_\tau^\sigma$. It follows from (A.15) that for any $\rho > 0$,

$$\begin{aligned} (\psi, H_\tau \psi) &\geq (\psi, H^\sigma|_{\mathcal{H}_\tau} \psi) + (\psi, H_\tau^\sigma \psi) \\ &\geq \left[1 - C|g|\sigma^{1/2}\rho^{-1/2}\right] (\psi, H^\sigma|_{\mathcal{H}_\tau} \psi) + C|g|\sigma^{1/2}\rho^{-1/2}E_\sigma - C|g|\sigma^{1/2}\rho^{1/2}. \end{aligned}$$

Next, from $\mathbf{Gap}(\sigma)$ and Property (A.10), since $\tau = \sigma/2$, we obtain that for any ψ in $[\Phi_\sigma \otimes \Omega_\tau^\sigma]^\perp$, $\|\psi\| = 1$,

$$(A.24) \quad (\psi, H^\sigma|_{\mathcal{H}_\tau} \psi) \geq E_\sigma + \min\left(\frac{\sigma}{8}, \frac{\tau}{4}\right) \geq E_\sigma + \sigma/8,$$

provided that $|g|$ is sufficiently small. Hence for any $\rho > 0$ such that $\rho^{1/2} > C|g|\sigma^{1/2}$,

$$(A.25) \quad (\psi, H_\tau \psi) \geq E_\sigma + \left[1 - C|g|\sigma^{1/2}\rho^{-1/2}\right] \frac{\sigma}{8} - C|g|\sigma^{1/2}\rho^{1/2}.$$

Choosing $\rho^{1/2} = 4C|g|\sigma^{1/2}$, by Corollary A.2, we get

$$(A.26) \quad (\psi, H_\tau \psi) \geq E_\sigma + \frac{3}{32}\sigma - 4C^2g^2\sigma \geq E_\tau + \frac{3}{16}\tau - 8C^2g^2\tau.$$

Hence $\mu_\tau \geq E_\tau + \tau/8$ provided that $|g| \leq (8C)^{-1}$, which proves the lemma. \square

PROOF OF PROPOSITION A.3 As mentioned above, Proposition A.3 easily follows from Lemmata A.5 and A.6, and an induction argument. \square

Let us conclude this Subsection with a bound on the difference $|E_\tau - E_\sigma|$.

LEMMA A.7. *Under the conditions of Proposition A.3, there exists a positive constant C such that for all $0 \leq \tau \leq \sigma \leq \sigma_0$,*

$$(A.27) \quad |E_\tau(P_3) - E_\sigma(P_3)| \leq C|g|\sigma.$$

PROOF. By Corollary A.2, we already have $E_\tau(P_3) \leq E_\sigma(P_3)$. The inequality $E_\sigma(P_3) \leq E_\tau(P_3) + C|g|\sigma$ follows similarly, using (A.15) and a variational argument. \square

REMARK A.8. Lemma A.7 remains true if the operators under consideration are not Wick-ordered. More precisely in this case we have

$$(A.28) \quad E_\tau(P_3) \leq E_\sigma(P_3) + Cg^2\sigma \leq E_\tau(P_3) + C|g|\sigma.$$

A.2. Proof of Theorem 1.2. The key property used in the proof of Theorem 1.2 lies in the estimate of $|E'_\tau(P_3) - E'_\sigma(P_3)|$ for $\tau \leq \sigma$.

PROPOSITION A.9. *There exists $g_0 > 0$, $\sigma_0 > 0$ and $P_0 > 0$ such that for all $0 < |g| \leq g_0$ and $|P_3| \leq P_0$, for all $\sigma, \tau > 0$ such that $\tau \leq \sigma \leq \sigma_0$, for all $\delta > 0$,*

$$|E'_\tau(P_3) - E'_\sigma(P_3)| \leq C_\delta \sigma^{1/2-\delta},$$

where C_δ is a positive constant depending only on δ .

We shall divide the main part of the proof of Proposition A.9 into two lemmata. Let us begin with some definitions and notations. For $\sigma > 0$ and $\rho \geq 0$, we define the function $g_{\sigma,\rho} \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ by

$$g_{\sigma,\rho}(\mathbf{k}) = g \mathbf{1}_{\sigma \leq |\mathbf{k}| \leq \Lambda}(k) \frac{\epsilon_\lambda^3(k)}{\sqrt{2\pi} |k|^{1/2} |k| - k_3 \rho}.$$

Depending on the context, the Weyl operator $W(ig_{\sigma,\rho})$ will represent an operator on \mathcal{H}_σ , \mathcal{H}_τ (for $\tau \leq \sigma$), or \mathcal{H} .

From now on, to simplify the notations, we drop the dependence on P_3 everywhere unless a confusion may arise. For g , σ and P_3 as in Proposition A.3, let Φ_σ denote a normalized ground state of H_σ . Define

$$H_{\sigma,\rho}^{\text{ren}} = W(ig_{\sigma,\rho}) H_\sigma W(ig_{\sigma,\rho})^*, \quad \Phi_{\sigma,\rho}^{\text{ren}} = W(ig_{\sigma,\rho}) \Phi_\sigma,$$

and let $P_{\sigma,\rho}^{\text{ren}}$ be the orthogonal projection onto the vector space spanned by $\Phi_{\sigma,\rho}^{\text{ren}}$. Note that $\Phi_{\sigma,\rho}^{\text{ren}}$ is a normalized, non-degenerate ground state of $H_{\sigma,\rho}^{\text{ren}}$, associated with the ground state energy E_σ . Recall that, by Lemma A.1, $[\Phi_\sigma \otimes \Omega_\tau^\sigma]$ is a ground state of $H^\sigma|_{\mathcal{H}_\tau}$. We set

$$H_{\sigma,\rho,\tau}^{\text{ren}} = W(ig_{\sigma,\rho}) H^\sigma|_{\mathcal{H}_\tau} W(ig_{\sigma,\rho})^*, \quad \Phi_{\sigma,\rho,\tau}^{\text{ren}} = W(ig_{\sigma,\rho}) [\Phi_\sigma \otimes \Omega_\tau^\sigma],$$

and the projection onto the vector space spanned by $\Phi_{\sigma,\rho,\tau}^{\text{ren}}$ is denoted by $P_{\sigma,\rho,\tau}^{\text{ren}}$. Since $W(ig_{\sigma,\rho}) = e^{i\Phi(i g_{\sigma,\rho}) \otimes \mathbf{1}}$, it can be seen that $\Phi_{\sigma,\rho,\tau}^{\text{ren}} = [W(ig_{\sigma,\rho}) \Phi_\sigma] \otimes \Omega_\tau^\sigma = \Phi_{\sigma,\rho}^{\text{ren}} \otimes \Omega_\tau^\sigma$.

LEMMA A.10. *There exists $g_0 > 0$, $\sigma_0 > 0$ and $P_0 > 0$ such that for all $0 < |g| \leq g_0$ and $|P_3| \leq P_0$, for all $\sigma, \tau > 0$ such that $\tau \leq \sigma \leq \sigma_0$,*

$$(A.29) \quad |E'_\sigma - E'_\tau| \leq C \left[\left\| P_{\sigma,E'_\sigma,\tau}^{\text{ren}} - P_{\tau,E'_\sigma}^{\text{ren}} \right\| + g^2 \sigma \right],$$

where C is a positive constant.

PROOF. By the Feynman-Hellman formula (see (2.22)),

$$(A.30) \quad E'_\sigma = \frac{1}{m} (\Phi_\sigma, [P_3 - d\Gamma(k_3) - gA_{3,\sigma}(x', 0)] \Phi_\sigma)_{\mathcal{H}_\sigma}.$$

It follows from (A.30) and commutation relations with $W(ig_{\sigma,E'_\sigma})$ that

$$(A.31) \quad E'_\sigma = \frac{1}{m} \left(\Phi_{\sigma,E'_\sigma}^{\text{ren}}, \left[P_3 - d\Gamma(k_3) - \Phi(k_3 g_{\sigma,E'_\sigma}) - \frac{1}{2} (k_3 g_{\sigma,E'_\sigma}, g_{\sigma,E'_\sigma}) \right. \right. \\ \left. \left. - gA_{3,\sigma}(x', 0) + g\text{Re}(h_{3,\sigma}(x'), g_{\sigma,E'_\sigma}) \right] \Phi_{\sigma,E'_\sigma}^{\text{ren}} \right)_{\mathcal{H}_\sigma},$$

Consequently, for $\tau \leq \sigma$, we can write

$$(A.32) \quad E'_\sigma = \frac{1}{m} \left(\Phi_{\sigma, E'_\sigma, \tau}^{\text{ren}}, \left[P_3 - d\Gamma(k_3) - \Phi(k_3 g_{\tau, E'_\sigma}) - \frac{1}{2}(k_3 g_{\sigma, E'_\sigma}, g_{\sigma, E'_\sigma}) \right. \right. \\ \left. \left. - gA_{3, \tau}(x', 0) + g\text{Re}(h_{3, \sigma}(x'), g_{\sigma, E'_\sigma}) \right] \Phi_{\sigma, E'_\sigma, \tau}^{\text{ren}} \right)_{\mathcal{H}_\tau},$$

whereas

$$(A.33) \quad E'_\tau = \frac{1}{m} \left(\Phi_{\tau, E'_\sigma}^{\text{ren}}, \left[P_3 - d\Gamma(k_3) - \Phi(k_3 g_{\tau, E'_\sigma}) - \frac{1}{2}(k_3 g_{\tau, E'_\sigma}, g_{\tau, E'_\sigma}) \right. \right. \\ \left. \left. - gA_{3, \tau}(x', 0) + g\text{Re}(h_{3, \tau}(x'), g_{\tau, E'_\sigma}) \right] \Phi_{\tau, E'_\sigma}^{\text{ren}} \right)_{\mathcal{H}_\tau}.$$

The expression into brackets being uniformly bounded with respect to $H_{\sigma, E'_\sigma, \tau}^{\text{ren}}$, one can prove that

$$(A.34) \quad \left\| \left[P_3 - d\Gamma(k_3) - \Phi(k_3 g_{\tau, E'_\sigma}) - \frac{1}{2}(k_3 g_{\sigma, E'_\sigma}, g_{\sigma, E'_\sigma}) \right. \right. \\ \left. \left. - gA_{3, \tau}(x', 0) + \text{Re}(h_{3, \sigma}(x'), g_{\sigma, E'_\sigma}) \right] \Phi_{\sigma, E'_\sigma, \tau}^{\text{ren}} \right\| \leq C,$$

and likewise with $\Phi_{\tau, E'_\sigma}^{\text{ren}}$ replacing $\Phi_{\sigma, E'_\sigma, \tau}^{\text{ren}}$. In addition, we have

$$(A.35) \quad |(k_3 g_{\sigma, E'_\sigma}, g_{\sigma, E'_\sigma}) - (k_3 g_{\tau, E'_\sigma}, g_{\tau, E'_\sigma})| \leq Cg^2\sigma,$$

and, similarly,

$$(A.36) \quad \left\| [\text{Re}(h_{3, \tau}(x'), g_{\tau, E'_\sigma}) - \text{Re}(h_{3, \sigma}(x'), g_{\sigma, E'_\sigma})] \Phi_{\tau, E'_\sigma}^{\text{ren}} \right\| \leq C|g|\sigma.$$

Estimating the difference of (A.32) and (A.33) then leads to

$$(A.37) \quad |E'_\sigma - E'_\tau| \leq C \left[\left\| \Phi_{\sigma, E'_\sigma, \tau}^{\text{ren}} - \Phi_{\tau, E'_\sigma}^{\text{ren}} \right\|_{\mathcal{H}_\tau} + g^2\sigma \right]$$

The statement of the lemma now follows by choosing the non-degenerate ground states $\Phi_{\sigma, E'_\sigma, \tau}^{\text{ren}}$ and $\Phi_{\tau, E'_\sigma}^{\text{ren}}$ in such a way that

$$(A.38) \quad \left\| \Phi_{\sigma, E'_\sigma, \tau}^{\text{ren}} - \Phi_{\tau, E'_\sigma}^{\text{ren}} \right\|_{\mathcal{H}_\tau} \leq C \left\| P_{\sigma, E'_\sigma, \tau}^{\text{ren}} - P_{\tau, E'_\sigma}^{\text{ren}} \right\|.$$

Note that this choice is indeed possible due to the non-degeneracy of the ground states $\Phi_{\sigma, E'_\sigma, \tau}^{\text{ren}}$ and $\Phi_{\tau, E'_\sigma}^{\text{ren}}$. \square

For g, P_3, σ, ρ as above, let us define the operator $\nabla H_{\tau, \rho}^{\text{ren}}$ by

$$\nabla H_{\sigma, \rho}^{\text{ren}} = \frac{1}{m} W(ig_{\sigma, \rho}) [P_3 - d\Gamma(k_3) - gA_{3, \sigma}(x', 0)] W(ig_{\sigma, \rho})^* \\ = \frac{1}{m} \left[P_3 - d\Gamma(k_3) - \Phi(k_3 g_{\sigma, \rho}) - \frac{1}{2}(k_3 g_{\sigma, \rho}, g_{\sigma, \rho}) \right. \\ \left. - gA_{3, \sigma}(x', 0) + g\text{Re}(h_{3, \sigma}(x'), g_{\sigma, \rho}) \right].$$

LEMMA A.11. Let $\Gamma_{\sigma,\mu}$ be the curve $\Gamma_{\sigma,\mu} = \{\mu\sigma e^{i\nu}, \nu \in [0, 2\pi[\}$. There exist $g_0 > 0$, $\sigma_0 > 0$, $\mu > 0$ and $P_0 > 0$, such that for all $0 < |g| \leq g_0$, $|P_3| \leq P_0$, for all $\sigma > 0$ and $\tau > 0$ such that $\sigma/2 \leq \tau \leq \sigma \leq \sigma_0$,

$$(A.39) \quad \left\| P_{\sigma, E'_\sigma, \tau}^{\text{ren}} - P_{\tau, E'_\sigma}^{\text{ren}} \right\| \leq C|g|^{1/2} \sigma^{1/2} \sup_{z \in \Gamma_{\sigma,\mu}} \left[1 + \left| \left(\nabla H_{\sigma, E'_\sigma}^{\text{ren}} - E'_\sigma \right) \Phi_{\sigma, E'_\sigma}^{\text{ren}}, \right. \right. \\ \left. \left. \left[H_{\sigma, E'_\sigma}^{\text{ren}} - E_\sigma - z \right]^{-1} \left(\nabla H_{\sigma, E'_\sigma}^{\text{ren}} - E'_\sigma \right) \Phi_{\sigma, E'_\sigma}^{\text{ren}} \right|^{1/2} \right],$$

where C is a positive constant.

PROOF. By [BFP, Lemma II.11],

$$(A.40) \quad \left\| P_{\sigma, E'_\sigma, \tau}^{\text{ren}} - P_{\tau, E'_\sigma}^{\text{ren}} \right\| = \left| \left(\Phi_{\sigma, E'_\sigma, \tau}^{\text{ren}} [P_{\sigma, E'_\sigma, \tau}^{\text{ren}} - P_{\tau, E'_\sigma}^{\text{ren}}] \Phi_{\sigma, E'_\sigma, \tau}^{\text{ren}} \right) \right|^{1/2}.$$

It follows from Lemma A.1 and Proposition A.3 that $\text{Gap}(H_{\sigma, E'_\sigma, \tau}^{\text{ren}}) \geq \sigma/8$ and $\text{Gap}(H_{\tau, E'_\sigma}^{\text{ren}}) \geq \tau/8 \geq \sigma/16$. Therefore, since $|E_\sigma - E_\tau| \leq C|g|\sigma$ by Lemma A.7, we can write

$$P_{\sigma, E'_\sigma, \tau}^{\text{ren}} - P_{\tau, E'_\sigma}^{\text{ren}} = \frac{i}{2\pi} \oint_{\Gamma_{\sigma,\mu}} \left(\left[H_{\sigma, E'_\sigma, \tau}^{\text{ren}} - E_\sigma - z \right]^{-1} - \left[H_{\tau, E'_\sigma}^{\text{ren}} - E_\sigma - z \right]^{-1} \right) dz,$$

provided $\mu < 1/16$ and $|g|$ is sufficiently small. Expanding $\left[H_{\tau, E'_\sigma}^{\text{ren}} - E_\sigma - z \right]^{-1}$ into a (convergent) Neumann series yields

$$P_{\sigma, E'_\sigma, \tau}^{\text{ren}} - P_{\tau, E'_\sigma}^{\text{ren}} = \frac{i}{2\pi} \sum_{n \geq 1} \oint_{\Gamma_{\sigma,\mu}} (-1)^n \left[H_{\sigma, E'_\sigma, \tau}^{\text{ren}} - E_\sigma - z \right]^{-1} \\ \left(\left[H_{\tau, E'_\sigma}^{\text{ren}} - H_{\sigma, E'_\sigma, \tau}^{\text{ren}} \right] \left[H_{\sigma, E'_\sigma, \tau}^{\text{ren}} - E_\sigma - z \right]^{-1} \right)^n dz.$$

Let us compute the difference $H_{\tau, E'_\sigma}^{\text{ren}} - H_{\sigma, E'_\sigma, \tau}^{\text{ren}}$ explicitly. We have:

$$H_{\sigma, E'_\sigma, \tau}^{\text{ren}} = \frac{1}{2m} \sum_{j=1,2} \left(p_j - ea_j(x') - gA_{j,\sigma}(x', 0) + g\text{Re}(h_{j,\sigma}(x'), g_{\sigma, E'_\sigma}) \right)^2 \\ + \frac{m}{2} (\nabla H_{\sigma, E'_\sigma}^{\text{ren}})^2 - \frac{e}{2m} \sigma_3 b(x') - \frac{g}{2m} \sigma \cdot \left(B_\sigma(x', 0) - \text{Re}(\tilde{h}_\sigma(x'), g_{\sigma, E'_\sigma}) \right) \\ + V(x') + H_f + \Phi(|k|g_{\sigma, E'_\sigma}) + \frac{1}{2} (|k|g_{\sigma, E'_\sigma}, g_{\sigma, E'_\sigma}) - \frac{g^2}{2m} (\Lambda^2 - \sigma^2),$$

and

$$H_{\tau, E'_\sigma}^{\text{ren}} = \frac{1}{2m} \sum_{j=1,2} \left(p_j - ea_j(x') - gA_{j,\tau}(x', 0) + g\text{Re}(h_{j,\tau}(x'), g_{\tau, E'_\sigma}) \right)^2 \\ + \frac{m}{2} (\nabla H_{\tau, E'_\sigma}^{\text{ren}})^2 - \frac{e}{2m} \sigma_3 b(x') - \frac{g}{2m} \sigma \cdot \left(B_\sigma(x', 0) - \text{Re}(\tilde{h}_\tau(x'), g_{\tau, E'_\sigma}) \right) \\ + V(x') + H_f + \Phi(|k|g_{\tau, E'_\sigma}) + \frac{1}{2} (|k|g_{\tau, E'_\sigma}, g_{\tau, E'_\sigma}) - \frac{g^2}{2m} (\Lambda^2 - \tau^2).$$

Let us decompose:

$$(A.41) \quad H_{\tau, E'_\sigma}^{\text{ren}} - H_{\sigma, E'_\sigma, \tau}^{\text{ren}} = [a] + [b] + [c] + [d] + [e],$$

with

$$\begin{aligned}
[a] &= \frac{1}{m} \sum_{j=1,2} \left(-gA_{j,\tau}^\sigma(0,0) + g\operatorname{Re}(h_{j,\tau}(0), g_{\tau,E'_\sigma}^\sigma) \right) \\
&\quad \times \left(p_j - ea_j(x') - gA_{j,\sigma}(x',0) + g\operatorname{Re}(h_{j,\sigma}(x'), g_{\sigma,E'_\sigma}) \right), \\
[b] &= \frac{1}{2m} \sum_{j=1,2} \left(-gA_{j,\tau}^\sigma(x',0) + g\operatorname{Re}(h_{j,\tau}(x'), g_{\tau,E'_\sigma}^\sigma) \right)^2 - \frac{g^2}{2m} (\sigma^2 - \tau^2) \\
&\quad + \frac{1}{2m} \left(-\Phi(k_3 g_{\tau,E'_\sigma}^\sigma) - \frac{1}{2}(k_3 g_{\tau,E'_\sigma}^\sigma, g_{\tau,E'_\sigma}^\sigma) - gA_{3,\tau}^\sigma(x',0) + g\operatorname{Re}(h_{3,\tau}(x'), g_{\tau,E'_\sigma}^\sigma) \right)^2, \\
&\quad + \frac{g}{2m} \sigma \cdot \operatorname{Re} \left(\tilde{h}_\tau(x') - \tilde{h}_\sigma(x'), g_{\tau,E'_\sigma} \right) \\
[c] &= \frac{1}{m} \sum_{j=1,2} \left(-g(A_{j,\tau}^\sigma(x',0) - A_{j,\tau}^\sigma(0)) + g\operatorname{Re}(h_{j,\tau}(x') - h_{j,\tau}(0), g_{\tau,E'_\sigma}^\sigma) \right) \\
&\quad \times \left(p_j - ea_j(x') - gA_{j,\sigma}(x',0) + g\operatorname{Re}(h_{j,\sigma}(x'), g_{\sigma,E'_\sigma}) \right) \\
&\quad - gE'_\sigma [A_{3,\tau}^\sigma(x',0) - A_{3,\tau}^\sigma(0,0)] + gE'_\sigma \operatorname{Re}(h_{3,\tau}(x') - h_{3,\tau}(0), g_{\tau,E'_\sigma}^\sigma), \\
[d] &= gE'_\sigma (h_{3,\tau}(0), g_{\tau,E'_\sigma}^\sigma) - \frac{1}{2} E'_\sigma (k_3 g_{\tau,E'_\sigma}^\sigma, g_{\tau,E'_\sigma}^\sigma), \\
[e] &= \frac{1}{2} \left(-\Phi(k_3 g_{\tau,E'_\sigma}^\sigma) - \frac{1}{2}(k_3 g_{\tau,E'_\sigma}^\sigma, g_{\tau,E'_\sigma}^\sigma) - gA_{3,\tau}^\sigma(x',0) + g\operatorname{Re}(h_{3,\tau}(x'), g_{\tau,E'_\sigma}^\sigma) \right) \\
&\quad \times \left(\nabla H_{\sigma,E'_\sigma}^{\operatorname{ren}} - E'_\sigma \right) + \frac{1}{2} \left(\nabla H_{\sigma,E'_\sigma}^{\operatorname{ren}} - E'_\sigma \right) \\
&\quad \times \left(-\Phi(k_3 g_{\tau,E'_\sigma}^\sigma) - \frac{1}{2}(k_3 g_{\tau,E'_\sigma}^\sigma, g_{\tau,E'_\sigma}^\sigma) - gA_{3,\tau}^\sigma(x',0) + g\operatorname{Re}(h_{3,\tau}(x'), g_{\tau,E'_\sigma}^\sigma) \right).
\end{aligned}$$

Note that we have added and subtracted E'_σ , using the identity $(E'_\sigma k_3 - |k|)g_{\sigma,E'_\sigma} = -gE'_\sigma h_{3,\sigma}(0)$ and likewise with g_{τ,E'_σ} replacing g_{σ,E'_σ} . Let us now consider, for some $n \geq 1$,

$$\begin{aligned}
(A.42) \quad & \oint_{\Gamma_{\sigma,\mu}} \left(\Phi_{\sigma,E'_\sigma,\tau}^{\operatorname{ren}} \left[H_{\sigma,E'_\sigma,\tau}^{\operatorname{ren}} - E_\sigma - z \right]^{-1} \right. \\
& \left. \left(\left[H_{\tau,E'_\sigma}^{\operatorname{ren}} - H_{\sigma,E'_\sigma,\tau}^{\operatorname{ren}} \right] \left[H_{\sigma,E'_\sigma,\tau}^{\operatorname{ren}} - E_\sigma - z \right]^{-1} \right)^n \Phi_{\sigma,E'_\sigma,\tau}^{\operatorname{ren}} \right).
\end{aligned}$$

We insert (A.41) into the right-hand side of (A.42), thus obtaining a sum of terms that we estimate separately. We claim that all the terms where at least one of the operators $[a]$, $[b]$, or $[c]$ appear, are bounded by $C\sigma(C'|g|)^n$ where C, C' are two positive constants. The latter can be proven by means of rather standard estimates involving pull-through formulas (see for instance [**BFS**, **Pi**, **BFP**, **CFP**]), so we shall not give all the details. Let us still emphasize that in order to deal with $[a]$ or $[c]$ we need to use the exponential decay of $\Phi_{\sigma,E'_\sigma,\tau}^{\operatorname{ren}}$ in x' (proven in [**AGG2**, Appendix A]). This is the main difficulty we encounter compared to the proof of [**CFP**]. In order to overcome it, we adapt a method due to [**Si**] (see also [**AFFS**, Section 5]). Let us give an example: Consider

$$(A.43) \quad \left(\Phi_{\sigma,E'_\sigma,\tau}^{\operatorname{ren}} [e] \left[H_{\sigma,E'_\sigma,\tau}^{\operatorname{ren}} - E_\sigma - z \right]^{-1} [a] \left[H_{\sigma,E'_\sigma,\tau}^{\operatorname{ren}} - E_\sigma - z \right]^{-1} [e] \Phi_{\sigma,E'_\sigma,\tau}^{\operatorname{ren}} \right).$$

We shall take advantage of the identity

$$(A.44) \quad \left(p_j - ea_j(x') - gA_{j,\sigma}(x', 0) + g\text{Re}(h_{j,\sigma}(x'), g_{\sigma,E'_\sigma}) \right) = 2i \left[H_{\sigma,E'_\sigma,\tau}^{\text{ren}}, x'_j \right]$$

which holds in the sense of quadratic forms on $D(H_{\sigma,E'_\sigma,\tau}^{\text{ren}}) \cap D(x'_j)$. The field operator $A_{j,\sigma}^\tau(0,0) = \Phi(h_{j,\sigma}^\tau)$ in $[a]$ decompose into a sum of a creation operator and an annihilation operator that are estimated separately. Take for instance the creation operator. Using a pull-through formula, we have to bound:

$$(A.45) \quad g \int h_{j,\tau}^\sigma(\mathbf{k}) \left(\Phi_{\sigma,E'_\sigma,\tau}^{\text{ren}}[e] a^*(\mathbf{k}) \left[H_{\sigma,E'_\sigma,\tau}^{\text{ren}}(P_3 - k_3) - E_\sigma + |k| - z \right]^{-1} \right. \\ \left. \left[H_{\sigma,E'_\sigma,\tau}^{\text{ren}}, x'_j \right] \left[H_{\sigma,E'_\sigma,\tau}^{\text{ren}} - E_\sigma - z \right]^{-1} [e] \Phi_{\sigma,E'_\sigma,\tau}^{\text{ren}} \right) d\mathbf{k}.$$

Let $\gamma > 0$ be such that $\|e^{\gamma\langle x' \rangle} \Phi_{\sigma,E'_\sigma,\tau}^{\text{ren}}\| < \infty$. Undoing the commutator $[H_{\sigma,E'_\sigma,\tau}^{\text{ren}}, x'_j]$ gives two terms. We write the first one under the form

$$g \int h_{j,\tau}^\sigma(\mathbf{k}) \left(\left(H_{\sigma,E'_\sigma,\tau}^{\text{ren}} - E_\sigma \right) \left[H_{\sigma,E'_\sigma,\tau}^{\text{ren}}(P_3 - k_3) - E_\sigma + |k| - \bar{z} \right]^{-1} a(\mathbf{k}) [e]^* \Phi_{\sigma,E'_\sigma,\tau}^{\text{ren}}, \right. \\ \left. x'_j e^{-\gamma\langle x' \rangle} e^{\gamma\langle x' \rangle} \left[H_{\sigma,E'_\sigma,\tau}^{\text{ren}} - E_\sigma - z \right]^{-1} e^{-\gamma\langle x' \rangle} [e] e^{\gamma\langle x' \rangle} \Phi_{\sigma,E'_\sigma,\tau}^{\text{ren}} \right) d\mathbf{k}.$$

Now we have the following estimates:

$$(A.46) \quad \left\| e^{\gamma\langle x' \rangle} \left[H_{\sigma,E'_\sigma,\tau}^{\text{ren}} - E_\sigma - z \right]^{-1} e^{-\gamma\langle x' \rangle} [e] e^{\gamma\langle x' \rangle} \Phi_{\sigma,E'_\sigma,\tau}^{\text{ren}} \right\| \leq C|g|,$$

$$(A.47) \quad \left\| x'_j e^{-\gamma\langle x' \rangle} \right\| \leq C,$$

$$(A.48) \quad \left\| \left[H_{\sigma,E'_\sigma,\tau}^{\text{ren}}(P_3 - k_3) - E_\sigma + |k| - z \right]^{-1} \left(H_{\sigma,E'_\sigma,\tau}^{\text{ren}} - E_\sigma \right) \right\| \leq C,$$

$$(A.49) \quad \left\| a(\mathbf{k}) [e]^* \Phi_{\sigma,E'_\sigma,\tau}^{\text{ren}} \right\| \leq C|g| |k|^{-1/2}.$$

Note that in (A.48) and (A.49), we used that $\tau \leq |k| \leq \sigma$, and thus in particular that $a(\mathbf{k}) \Phi_{\sigma,E'_\sigma,\tau}^{\text{ren}} = 0$. Since the other term coming from the commutator $[H_{\sigma,E'_\sigma,\tau}^{\text{ren}}, x'_j]$ can be estimated in the same way, this yields

$$(A.50) \quad |(A.45)| \leq C|g|^3 \int |h_{j,\tau}^\sigma(\mathbf{k})| |k|^{-1/2} d\mathbf{k} \leq C|g|^3 \sigma^2.$$

Taking into account the factor σ coming from the integration in (A.42) would finally lead to our claim in the case of the example (A.43). The same holds for the terms containing $[c]$ at least once (except that the use of (A.44) is then not required). Besides, since $[d]$ is constant,

$$\oint_{\Gamma_{\sigma,\mu}} \left(\Phi_{\sigma,E'_\sigma,\tau}^{\text{ren}} \left[H_{\sigma,E'_\sigma,\tau}^{\text{ren}} - E_\sigma - z \right]^{-1} \left([d] \left[H_{\sigma,E'_\sigma,\tau}^{\text{ren}} - E_\sigma - z \right]^{-1} \right)^n \Phi_{\sigma,E'_\sigma,\tau}^{\text{ren}} \right) = 0.$$

Therefore it remains to consider the terms containing only $[d]$ or $[e]$, with $[e]$ appearing at least in one factor. One can prove that this leads to

$$\begin{aligned} \left\| P_{\sigma, E'_\sigma, \tau}^{\text{ren}} - P_{\tau, E'_\sigma}^{\text{ren}} \right\| &\leq C |g|^{1/2} \sigma^{1/2} \sup_{z \in \Gamma_{\sigma, \mu}} \left[1 + \sigma^{-1} \left\| H_{\sigma, E'_\sigma, \tau}^{\text{ren}} - E_\sigma - z \right\|^{-1/2} \right. \\ &\quad \left. \left(-\Phi(k_3 g_{\tau, E'_\sigma}^\sigma) - \frac{1}{2} (k_3 g_{\tau, E'_\sigma}^\sigma, g_{\tau, E'_\sigma}^\sigma) - g A_{3, \tau}^\sigma(x', 0) + g \text{Re}(h_{3, \tau}(x'), g_{\tau, E'_\sigma}^\sigma) \right) \right. \\ &\quad \left. \left\| \left[\nabla H_{\sigma, E'_\sigma}^{\text{ren}} - E'_\sigma \right] \Phi_{\sigma, E'_\sigma, \tau}^{\text{ren}} \right\| \right]. \end{aligned}$$

Using again the exponential decay of $\Phi_{\sigma, E'_\sigma}^{\text{ren}}$ in x' , we may replace $\text{Re}(f_{3, \tau}(x'), g_{\tau, E'_\sigma}^\sigma)$ with $\text{Re}(f_{3, \tau}(0), g_{\tau, E'_\sigma}^\sigma)$ in the previous expression. Proceeding then as in [CFP, Lemma A.3], since both

$$(\Phi(k_3 g_{\tau, E'_\sigma}^\sigma) + g A_{3, \tau}^\sigma(x', 0)) (\nabla H_{\sigma, E'_\sigma}^{\text{ren}} - E'_\sigma) \Phi_{\sigma, E'_\sigma, \tau}^{\text{ren}} \quad \text{and} \quad (\nabla H_{\sigma, E'_\sigma}^{\text{ren}} - E'_\sigma) \Phi_{\sigma, E'_\sigma, \tau}^{\text{ren}}$$

are orthogonal to $\Phi_{\sigma, E'_\sigma, \tau}^{\text{ren}}$, we obtain Inequality (A.39) (notice in particular that σ_0 and μ must be fixed sufficiently small to pass from the last estimate to (A.39)). \square

PROOF OF PROPOSITION A.9 To conclude the proof of Proposition A.9, in view of Lemmata A.10 and A.11, it suffices to show that

$$\left| \left(\left(\nabla H_{\sigma, E'_\sigma}^{\text{ren}} - E'_\sigma \right) \Phi_{\sigma, E'_\sigma}^{\text{ren}}, \left[H_{\sigma, E'_\sigma}^{\text{ren}} - E_\sigma - z \right]^{-1} \left(\nabla H_{\sigma, E'_\sigma}^{\text{ren}} - E'_\sigma \right) \Phi_{\sigma, E'_\sigma}^{\text{ren}} \right) \right| \leq \frac{C_\delta}{|g| \sigma^{2\delta}},$$

for any $z \in \Gamma_{\sigma, \mu}$ and any $\delta > 0$. This corresponds to the bound (IV.68) in [CFP] and can be proven in the same way as in [CFP, Subsection IV.5, step (4)], using an induction procedure. We therefore refer the reader to [CFP] for a proof. \square

PROOF OF THEOREM 1.2 Fix P_3 and k_3 such that $|P_3| \leq P_0$, $|P_3 + k_3| \leq P_0$. One can see that there exist positive constants C_0 and C such that, for any $0 < \beta < 1$ and $\sigma \geq C_0 |k_3|^\beta$,

$$(A.51) \quad |E'_\sigma(P_3 + k_3) - E'_\sigma(P_3)| \leq C |k_3|^{\frac{1}{2}(1-\beta)}.$$

This can be proven by estimating $|E'_\sigma(P_3 + k_3) - E'_\sigma(P_3)|$ in terms of $\|\Phi_\sigma(P_3 + k_3) - \Phi_\sigma(P_3)\|$, then using the second resolvent equation to estimate $\|[H_\sigma(P_3 + k_3) - z]^{-1} - [H_\sigma(P_3) - z]^{-1}\|$. Now, for $\sigma \leq C_0 |k_3|^\beta$, we use Proposition A.9, which yields

$$\begin{aligned} &|E'_\sigma(P_3 + k_3) - E'_\sigma(P_3)| \\ &\leq \left| E'_\sigma(P_3 + k_3) - E'_{C_0 |k_3|^\beta}(P_3 + k_3) \right| + \left| E'_{C_0 |k_3|^\beta}(P_3 + k_3) - E'_{C_0 |k_3|^\beta}(P_3) \right| \\ &\quad + \left| E'_\sigma(P_3) - E'_{C_0 |k_3|^\beta}(P_3) \right| \\ &\leq C_\delta \left[|k_3|^{\frac{1}{2}(1-\beta)} + |k_3|^{\frac{1}{2}\beta(1-\delta)} \right]. \end{aligned}$$

The theorem follows by choosing $\beta = [2 - \delta]^{-1}$. \square

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