

# Games with imperfect information

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## Abstract

Games have been extensively studied, either in computer science, mathematics or even economy. Nevertheless, each discipline has its own interest in using this formalism. Computer science for instance is attached to calculability issues. These results have some direct applications in model checking or compilation. Recently, a new type of game has been introduced: games with imperfect information. They allow the modeling of more sophisticated systems, but bring also new calculability problems. In this document, we introduce a general method to prove the determinacy of any type of game. This method is used several times, and allow us to solve some open problems. This document introduces also several examples of important games stating for important properties. Then, a new type of game unifying the concepts of concurrency and imperfect information is presented. Finally, we discuss of the extension on infinite arenas.

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# 1 Notations

Let  $A$  be a finite set. We denote  $\mathcal{D}(A)$  the set of probabilities distributed over  $A$ , that is to say:

$$\mathcal{D}(A) = \{P \in A \rightarrow [0; 1] \mid \sum_{a \in A} P(a) = 1\}.$$

The support of a probability distribution  $P \in \mathcal{D}(A)$  is defined by  $support(P) = P^{-1}(]0; 1])$ .  $|A|$  denotes the cardinal of  $A$ ,  $2^A$  the set of subsets of elements in  $A$ .

By  $\omega$  we denote the first transfinite ordinal. For more simplicity, we will often use  $\omega$  instead of  $\mathbb{N}$ . Hence, if  $A$  is a finite set,  $A^\omega$  denotes the set of infinite sequences on  $A$ . If  $L$  is a language over  $A$ ,  $L^\omega$  is the  $\omega$ -iteration of language  $L$ , that is to say:  $L^\omega = \{\lambda \in A^\omega \mid \lambda = s_0 s_1 \dots \text{ such that } \forall i \in \omega, s_i \in L\}$ .

Given an infinite sequence  $\lambda$  on a possibly infinite set  $Z$ ,  $Inf(\lambda) = \{z \in Z \mid \lambda \in (Z^* z)^\omega\}$  is the subset of elements in  $Z$  that appears infinitely often in  $\lambda$ . Let  $support(\lambda) = \{z \in Z \mid \lambda \in Z^* z Z^\omega\}$  be the subset of elements in  $Z$  that appears in  $\lambda$ .

From now on, we will consider Eve and Adam to be two players.

## 2 Introduction

### 2.1 Games

The games we are considering are based on the one introduced by John Von Neumann in the middle of the twentieth century [20]. They all may be unified under the formalism of repeated games with imperfect information, which was introduced by Aumann and Maschler in 1966, and are studied in economy and mathematics ([15] and [13]). Such a game is modelised by a family of matrices, all corresponding to a state of the game. There are two players (in full generality, more than one player) with a given value, and their possible actions are the indexes of the matrices. A play in the game is performed as follows: the two players simultaneously choose their

actions, which determine both a value and the new state for the game. This process is repeated possibly infinitely often. The values given by the matrices are called “payments” and are taken from one player value to the other one, hence these games are zero-sum games. If his value satisfies a given property, then the first player wins, otherwise the second player wins. The imperfect information comes from the fact that the players do not know the exact state they are in but only an equivalence class on them. Both players may use strategies to choose their next action. A strategy is a function that maps the history of a play to a distribution on the actions of a player.

However, the interests for these domains differ from the one in computer science. In mathematics, the main question is the existence of values in such games, not their calculability. In economy, the hot spot is the research of Nash equilibrium. In a game where the winner is the player with the highest value, a Nash equilibrium is a pair of strategies such that any modification of one of them lower the value of its owner. In computer science, we are interested in computability issues. Those issues are to find the winning regions for a given player and its corresponding winning strategies. A winning region is a set of states of the game such that the player has a winning strategy from them.

In this document, we will introduce several formalisms of games and some results on them. The structure is the following: We first present a theorem on the calculability of positional strategies. Then we introduce some examples of games showing the difference between various objectives. We propose a proof that parity concurrent games may require infinite memory winning strategies. We then study the games with imperfect information introducing some new results, mainly on co-Büchi winning condition. Later, we introduce a new extended formalism of games, with a reduction to find winning strategies. Finally, we discuss on the pushdown model.

## 2.2 Almost-sure positional strategies

This section introduces a result that will be used several times in this document. This result may be applied to any repeated game. It shows that if there exist an almost-surely winning positional strategy in a game, then this strategy can be transformed to an almost-surely winning positional

strategy where all choices are uniformly distributed. Therefore, if there exists a positional almost-surely winning strategy and if one can decide for a given such strategy if it is winning, then one can be computed (the most naïve method being to test all the uniform positional strategies, which are finitely many).

We fix two strategies for Eve and Adam. Let  $S_1$  and  $S_2$  be two disjoint finite sets, and  $\mathcal{T}$  the infinite weighted tree on  $S_1 \cup S_2$ , weights being in  $]0; 1]$ , corresponding to the set of plays engendered by the two strategies. The weights represent the probability for a transition to be taken. The set of infinite paths in the tree is denoted  $\rho_{\mathcal{T}}$ . By the Carathéodory unique extension theorem (2), the weights induce an unique measure  $\mu$  on  $2^{\rho_{\mathcal{T}}}$ . Given an objective  $\Omega \subseteq (S_1 \cup S_2)^\omega$ , an almost-winning tree is a tree such that  $\mu(\rho_{\mathcal{A}} \cap \Omega) = 1$ .

A 1-positional tree is a tree such that for any state  $s$  in  $S_1$ , the successors of a node labeled  $s$  and the weights to reach them are the same. Such a tree corresponds to a positional strategy for Eve. A 1-positional uniform tree is a 1-positional tree such that for any state in  $S_1$ , the weights leaving the nodes labeled by this state are uniformly distributed. Such a tree corresponds to an uniformly distributed positional strategy for Eve.

The following result holds:

**Theorem 1.** *Given two disjoint finite sets  $S_1$  and  $S_2$ , and an objective  $\Omega \subseteq (S_1 \cup S_2)^\omega$ , if Eve has a strategy such that for all strategy of Adam, the tree engendered is an almost-winning tree, then Eve has an uniformly distributed strategy such that for all strategy of Adam, the tree engendered is an almost-winning tree.*

This result is very easy to show, as a 1-positional uniform almost-winning tree may be obtained from 1-positional almost-winning tree simply by doing this modification: all weights leaving states in  $S_1$  are made uniformly distributed. Let  $\mu$  be the initial measure and  $\mu'$  the one on the new tree. Given a path  $\rho$  in the first tree (the paths are the same in both trees), it is obvious that  $\mu(\rho) = 0$  if and only if  $\mu'(\rho) = 0$ . Hence, if the new tree is not almost-winning, then there is a subset of paths  $\rho_0$  such that  $\mu'(\rho_0) > 0$  and  $\rho_0 \cap \Omega = \emptyset$ . Hence  $\mu(\rho) > 0$  and the initial tree is not a winning one. This achieves the proof of Theorem 1.

### 3 Classical turn-based games

The turn based games were first introduced by McNaughton ([11]) 1965. Since then, they have been widely studied in literature. One could for instance see [22] which give some applications on automata theory on infinite trees or [17].

#### 3.1 Definitions

A turn-based arena  $\mathcal{A}$  is a tuple  $\langle S = S_1 \cup S_2, \Delta \subseteq S \times S \rangle$  where:

- $S$  is the set of states,
- $S_1$  and  $S_2$  are respectively the subset of states of Eve and Adam ( $S_1 \cap S_2 = \emptyset$ ),
- $\Delta$  is the transition relation on states.

We suppose the arena has no dead-end, that is to say  $\forall s \in S, \exists s' \in S$  such that  $(s, s') \in \Delta$ . A play on the arena  $\mathcal{A}$  is performed as follows: from an initial state  $s_0$ , the player who owns this state chooses a new state  $s_1$  such that  $(s_0, s_1) \in \Delta$ . Once again,  $s_1$  belongs to one player, this one chooses the new state of the play  $s_2$  such that  $(s_1, s_2) \in \Delta$ . This iteration is repeated infinitely often. Therefore, a play on  $\mathcal{A}$  is an infinite sequence of states  $\lambda = s_0 s_1 \dots$  such that  $\forall i \in \omega, (s_i, s_{i+1}) \in \Delta$ .

A game on  $\mathcal{A}$  is a tuple  $\mathbb{G} = \langle \mathcal{A}, \Omega \rangle$  where  $\Omega \subseteq S^\omega$ .  $\Omega$  is called the objective of game  $\mathbb{G}$ .

Eve wins some play  $\lambda$  on  $\mathbb{G} = \langle \mathcal{A}, \Omega \rangle$  if and only if  $\lambda \in \Omega$ .

#### 3.2 Winning conditions

One may find in the literature a lot of references studying different objectives. But mainly, the objectives considered in computer science are always the same, belonging to the  $\omega$ -regular class which will be described later (see [16] and [18]). Some of these objectives are described in this section.

The most natural objective is the reachability one. This objective is the set of infinite sequences of states where one state from a given subset of states appears at least one time. Formally, if  $Q \subseteq S$ , the reachability objective on  $Q$  is  $S^\omega Q S^\omega$ . This objective may be written as  $\{\lambda \in S^\omega \mid \text{support}(\lambda) \cap Q \neq \emptyset\}$ .

Dually is defined the safety objective. This objective is the set of infinite sequences of states where no state from a given subset of states appears. Hence, if  $Q \subseteq S$ , the safety objective on  $Q$  is  $(S - Q)^\omega$ .

The other objectives we describe beneath only depend on the set of infinitely often visited states. Let  $S_\infty$  be this subset of states.

The Büchi objective requires that  $S_\infty$  contains at least one state from a given subset of states. Formally, let  $Q \subseteq S$  be a subset of states, then the Büchi objective on  $Q$  is  $\{\lambda \in S^\omega \mid \text{Inf}(\lambda) \cap Q \neq \emptyset\}$ . This objective can be seen as an infinite version of the reachability objective.

The co-Büchi objective is the infinite version of the safety objective. That is to say, if  $Q \subseteq S$  is a subset of states, the co-Büchi objective on  $Q$  is the set of plays where no state in  $Q$  appears infinitely often. Formally, the co-Büchi objective on  $Q$  is  $\{\lambda \in S^\omega \mid \text{Inf}(\lambda) \cap Q = \emptyset\}$ .

The Muller objective is a natural generalization of the Büchi and the co-Büchi objectives. It can be described as the objective obtained with boolean conditions on Büchi objectives. Given a set of subset of states  $T$ , the Muller objective on  $T$  is the set of plays such that the set of states visited infinitely often is a subset of  $T$ . Formally, the Muller objective on  $T$  is the set  $\{\lambda \in S^\omega \mid \text{Inf}(\lambda) \in T\}$ . It is immediate that the Büchi objective on  $Q \subseteq S$  is the Muller objective on  $2^Q - \emptyset$  and the co-Büchi objective on  $Q$  is the Muller objective on  $2^{S-Q} - \emptyset$ .

Given a coloring function  $\rho : S \rightarrow \omega$ , the parity objective is the set of plays such that the lowest color appearing infinitely often is even. Formally, the parity objective on  $\rho$  is the set  $\{\lambda \in S^\omega \mid \min \rho(\text{Inf}(\lambda)) \text{ is even}\}$ . Once again, the parity objective is a generalisation of the Büchi objective and the co-Büchi objective. To see this, one should mark the states in  $Q$  by the color 0 for the Büchi objective and by 1 in the co-Büchi case, the other states are then marked by 1 and 2 respectively.

### 3.3 Strategies

Given a turn-based game  $\mathbb{G} = \langle S = S_1 \cup S_2, \Delta \rangle, \Omega \rangle$ , a strategy for Eve is a function  $\varphi_1 : S^*S_1 \rightarrow S$  and a strategy for Adam is a function  $\varphi_2 : S^*S_2 \rightarrow S$ . Although strategies may be defined on any sequence of states, the one we will consider will be defined on prefixes of plays on  $\mathbb{G}$ . The set of strategies for Eve is denoted  $\Phi_1$  and the set of strategies for Adam is denoted  $\Phi_2$ . A positional strategy is a strategy that only depends on the last state the game is in. In other words, a positional strategy is a function  $\varphi$  such that  $\forall s \in S, \forall h_1, h_2 \in S^*, \varphi(h_1s) = \varphi(h_2s)$ .

There is no need to consider randomized strategies as far as turn-based games with regular objectives are concerned. This result will be explained later.

Given a couple of strategies ( $\varphi_1 \in \Phi_1, \varphi_2 \in \Phi_2$ ) and a starting state  $s_0$ , there exists a unique play  $\lambda = s_0s_1 \dots$  such that  $\forall i \in \omega$ ,

$$s_{i+1} = \begin{cases} \varphi_1(s_0s_1 \dots s_i) & \text{if } s_i \in S_1 \\ \varphi_2(s_0s_1 \dots s_i) & \text{if } s_i \in S_2 \end{cases}$$

This play defines the value of the couple of strategies from initial state  $s_0$ , denoted  $\|\varphi_1, \varphi_2\|_{s_0}$ , by:

$$\|\varphi_1, \varphi_2\|_{s_0} = \begin{cases} 1 & \text{if } \lambda \in \Omega \\ 0 & \text{if } \lambda \notin \Omega \end{cases}$$

Then is defined the value of a strategy for Eve from  $s_0$ , denoted  $\|\varphi_1\|_{s_0}$ , by:

$$\|\varphi_1\|_{s_0} = \inf_{\varphi_2 \in \Phi_2} \|\varphi_1, \varphi_2\|_{s_0}.$$

The value of a strategy for Adam is defined dually:

$$\|\varphi_2\|_{s_0} = \sup_{\varphi_1 \in \Phi_1} \|\varphi_1, \varphi_2\|_{s_0}.$$

If the value of a strategy for Eve (respectively for Adam) is 1, the strategy is said to be winning, that is to say that no matter what do the opponent, Eve (respectively Adam) is assured to win if she always makes the choices given by her strategy. The value of the game for Eve is defined by:

$$\|\mathbb{G}\|_{s_0}^1 = \sup_{\varphi_1 \in \Phi_1} \|\varphi_1\|_{s_0}.$$

And the value of the game for Adam is:

$$\|\mathbb{G}\|_{s_0}^2 = \inf_{\varphi_2 \in \Phi_2} \|\varphi_2\|_{s_0}.$$

The Martin determinacy theorem ([10]) states that for all starting state  $s_0 \in S$  and any objective in those described before,  $\|\mathbb{G}\|_{s_0}^2 + \|\mathbb{G}\|_{s_0}^1 = 1$ , that is to say that either Eve or Adam has a winning strategy from this state. This is why there is no need to consider random strategies.

### 3.4 Important results

#### 3.4.1 $\omega$ -regular languages

The infinite language theory holds that the parity winning condition is the condition described before that induces the larger class of objectives. This class is called the  $\omega$ -regular class. Actually, any of the class defined before but Muller one may be reduced to the parity one with a polynomial cost in the number of states. Although the Muller condition induces exactly the same class as the parity one, the reduction from one to the other is not trivial. From now on, we will not consider this condition anymore. Hence, as far as complexity and calculability are concerned, the results are presented in this document from the smaller classes (reachability and safety), to the biggest one (parity) passing by the intermediate ones (Büchi and co-Büchi). Figure 1 depicts this hierarchy.

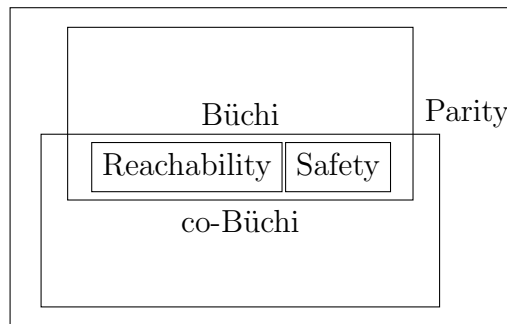


Figure 1: The hierarchy between winning conditions.

### 3.4.2 Parity games

A main result in turn-based games is the determinacy of games. This result states that any state in a parity game is a winning one either for one player or the other one ([6]). Moreover, in parity games, winning strategies can be chosen positional ([6]). Hence, as Büchi, co-Büchi, reachability and safety conditions can be directly expressed as parity ones, the corresponding games have positional winning strategies too. On the other hand, Muller condition cannot be expressed directly as a parity condition, and winning strategies may require memory. Figure 2 shows an example of a Muller game where memory is required. In this game, Adam has no state, so Eve is the only player to choose the states in a play. To win, she must visit infinitely often the three states of the game. She has a trivial winning strategy consisting of visiting states  $s_1$ ,  $s_0$ ,  $s_2$  then  $s_0$  infinitely often. Nevertheless, if she uses a positional strategy, then she has either to go every time to  $s_1$  or  $s_2$  being on state  $s_0$ . Whatever she chooses to do, she will not visit the three states infinitely often (actually, not even once if the starting state is  $s_0$ ). Nevertheless, Muller games does not require infinite memory, as finite memory is sufficient. This aspect will not be developed in this document.

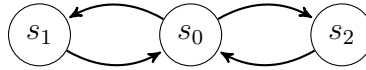


Figure 2: Example of a Muller turn-based game where memory is required to win. The Muller condition is to visit states  $s_0$ ,  $s_1$  and  $s_2$  infinitely often.

Solving parity games (that is to say finding which state is winning for which player) can be solved using a  $NP \cap coNP$  algorithm. Some improvements have been made reaching a  $UP \cap coUP$  complexity ([7]) and the actual most efficient algorithms are described in [9], [8] and [19]. It is still an open question whether this complexity can be reduced to a polynomial bound or not. Nevertheless, if the number of colors is fixed, the game can be solved with a polynomial complexity.

## 4 Concurrent games

Even if they are newer, concurrent arenas have been widely studied since a decade. A lot of results may be discussed in [3].

### 4.1 Definitions

A concurrent arena is a tuple  $\mathcal{A} = \langle S, \Sigma_1, \Sigma_2, \delta \rangle$  where:

- $S$  is the set of states,
- $\Sigma_1$  is the set of states for Eve,
- $\Sigma_2$  is the set of states for Adam,
- $\delta : S \times \Sigma_1 \times \Sigma_2 \rightarrow S$  is the transition total function.

Turn based arenas can be simulated by concurrent arenas. Indeed, let  $\mathcal{A}_t = \langle S = S_1 \cup S_2, \Delta \rangle$  be a turn based arena. We define the new transition function as follows:  $\forall s \in S_1, \forall s' \in S$ , if  $(s, s') \in \Delta$  then  $\exists \sigma_1 \in \Sigma_1, \forall \sigma_2 \in \Sigma_2, \delta(s, \sigma_1, \sigma_2) = s'$  and  $\forall \sigma_1 \in \Sigma_1$ , if  $\exists \sigma_2 \in \Sigma_2$  such that  $\delta(s, \sigma_1, \sigma_2) = s'$  then  $\forall \sigma_2 \in \Sigma_2, \delta(s, \sigma_1, \sigma_2) = s'$ . The construction is done symmetrically for states in  $S_2$ .

Given a set  $\Omega \subseteq S^\omega$ ,  $\mathbb{G} = \langle \mathcal{A}, \Omega \rangle$  is a concurrent game and  $\Omega$  is called the objective of the game  $\mathbb{G}$ .

A play in  $\mathbb{G}$  is an infinite sequence of the following iteration: being on a state, Eve and Adam choose *simultaneously* a move respectively in  $\Sigma_1$  and  $\Sigma_2$ . The new state is then given by the transition function  $\delta$ . Formally, a play in  $\mathbb{G}$  is an infinite sequence of states  $\lambda = \lambda_0 \lambda_1 \dots$  such that:

$$\forall i \in \omega, \exists \sigma_1 \in \Sigma_1, \exists \sigma_2 \in \Sigma_2 \text{ such that } \lambda_{i+1} = \delta(\lambda_i, \sigma_1, \sigma_2).$$

Given a non empty prefix  $\lambda_p$  of a play  $\lambda$ ,  $last(\lambda_p)$  denotes the last state in the sequence  $\lambda_p$ .

A strategy for Eve (respectively Adam) is a function  $\varphi_1 : S^+ \rightarrow \mathcal{D}(\Sigma_1)$  (respectively  $\varphi_2 : S^+ \rightarrow \mathcal{D}(\Sigma_2)$ ). The set of strategies for Eve (respectively Adam) is denoted by  $\Phi_1$  (respectively  $\Phi_2$ ).

A positional strategy is a strategy that only depends on the last state, formally:  $\forall s \in S, \forall h_1 \in S^*, \forall h_2 \in S^*, \varphi(h_1s) = \varphi(h_2s)$ . The set of positional strategies for Eve (respectively Adam) is denoted by  $\Phi_1^p$  (respectively  $\Phi_2^p$ ).

## 4.2 Objectives

Contrary to turn based games where for omega regular winning conditions, there exists deterministic winning strategies, concurrent games may require randomized strategies even with  $\omega$ -regular conditions. Figure 3 shows an example of an  $\omega$ -regular concurrent game where there exists an optimal randomized strategy but no deterministic one. Indeed, whatever is the deterministic strategy of Eve, Adam can choose the deterministic counter-strategy that consists of choosing the move that makes him staying on state  $s_0$ . On the other hand, if Eve plays randomly uniformly moves  $a$  and  $b$ , she is sure to visit infinitely often states  $s_0$  and  $s_1$  with probability one.

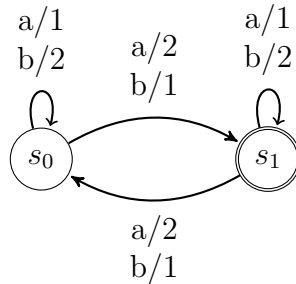


Figure 3: Concurrent game where a randomized strategy is strictly better than any deterministic one. The objective is for Eve to visit the state marked by a double circle infinitely often (Büchi objective).

Given two randomized strategies, there is no longer an unique play associated to the strategies but a possibly infinite set of such plays. This refines the notion of victory. Indeed, a trivial adaptation of the initial objective could be to force all the plays conform with the two strategies to be winning for the same player. It is easy to prove that, with this objective, winning

strategies can be chosen deterministic. Because of this loss of generality and interest, this objective will not be discussed later.

Given a starting state  $s_0$ , a pair of strategies  $(\varphi_1, \varphi_2) \in \Phi_1 \times \Phi_2$  induces a measurable set of infinite plays. This set can be defined as follow. On finite plays of length  $i \in \omega$  one inductively defines a measure  $\mu_{s_0, i}^{\varphi_1, \varphi_2}$  by:

- The only play of length 1 ( $s_0$ ) has measure 1.
- Given a play  $\lambda_{i+1} = \lambda_i s_{i+1}$ ,  $s_{i+1} \in S$  of length  $i + 1$ ,  $i \in \omega$ ,

$$\mu_{s_0, i+1}^{\varphi_1, \varphi_2}(\lambda_{i+1}) = \sum_{\sigma_1, \sigma_2 \text{ such that } \delta(\text{last}(\lambda_i), \sigma_1, \sigma_2) = s_{i+1}} P(\sigma_1 \mid \lambda_i) P(\sigma_2 \mid \lambda_i) \mu_{s_0, i}^{\varphi_1, \varphi_2}(\lambda_i)$$

where  $P(\sigma_1 \mid \lambda_i)$  denotes the probability for Eve to play  $\sigma_1$  knowing the history is  $\lambda_i$  and  $P(\sigma_2 \mid \lambda_i)$  denotes the probability for Adam to play  $\sigma_2$  knowing the history is  $\lambda_i$ .

By the Carathéodory unique extension theorem (2), these measures induce an unique measure  $\mu_{s_0}^{\varphi_1, \varphi_2}$  on infinite plays.

This measure allows us to define the value of a pair of strategies from  $s_0$  by:

$$\|\varphi_1, \varphi_2\| = \mu_{s_0}^{\varphi_1, \varphi_2}(\Omega).$$

The value of a strategy  $\varphi_1$  for Eve is defined from  $s_0$  by:

$$\|\varphi_1\|_{s_0} = \inf_{\varphi_2 \in \Phi_2} \|\varphi_1, \varphi_2\|_{s_0}.$$

The value of a strategy  $\varphi_2$  for Adam is defined dually. Finally, we define the value of the game for Eve on  $s_0$  by:

$$\|\mathbb{G}^1\|_{s_0}^{\mathbb{G}} = \sup_{\varphi_1 \in \Phi_1} \|\varphi_1\|_{s_0} = \sup_{\varphi_1 \in \Phi_1} \inf_{\varphi_2 \in \Phi_2} \|\varphi_1, \varphi_2\|_{s_0}.$$

The Blackwell optimality theorem adds that:

$$\|\mathbb{G}^1\|_{s_0} = \inf_{\varphi_2 \in \Phi_2} \sup_{\varphi_1 \in \Phi_1} \|\varphi_1, \varphi_2\|_{s_0}.$$

We may now define refined notions of objectives naturally induced by this formalism. First, we will say a strategy is almost-surely winning if the probability that a play conform with this strategy is a winning one is one. Formally, a strategy is almost winning if

$$\|\varphi_1\|_{s_0} = 1.$$

Then, we define the limit-sure objective. A sequence of strategies is limit-surely winning if the limit of the values of these strategies is one. Intuitively, a sequence of strategies is limit-surely winning if the probability for a play to be a winning one can be made arbitrary close to one. Formally, a sequence  $(\varphi_1^i)_i \in \omega$  is limit-surely winning if:

$$\lim_{i \rightarrow \infty} \|\varphi_1^i\|_{s_0} = 1.$$

Those conditions and examples of such games are presented in [5].

### 4.3 Winning conditions

As the winning condition are the same as in the turn based case, they will not be recalled here. Solving almost-sure reachability, Büchi, and co-Büchi concurrent games can be done using the method described in the next section on imperfect information games as concurrent games can be modeled by imperfect information games with a linear cost in the number of states. Those games admit positional winning strategies.

#### 4.3.1 Büchi games may have different almost-sure and limit-sure values

In this section, we prove that Büchi concurrent games may have different almost-sure and limit-sure values. To do this, let us consider the Büchi game depicted in Figure 4. The only final state is marked by a double circle. Thus, Eve wins a play iff this state is visited infinitely often. This game has different almost-sure and limit-sure values. Actually, the central state is not winning for Eve in the almost-sure case while it is in the limit-sure case.

The proof is quite easy. Let us consider first the almost-sure case where winning strategies can be chosen positional. Hence, two cases are possible:

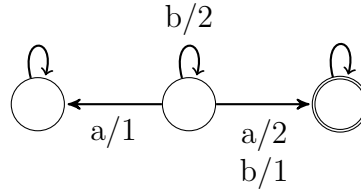


Figure 4: Example of a Büchi game with different almost-sure and limit-sure values.

- $P(a) = \eta > 0$ . In this case, Adam may choose to play always move 1. The winning probability is then  $P(b) < 1$ .
- $P(b) = 1$ . Adam may choose to play always move 2, and he is almost-sure to win.

Therefore, in both cases Eve cannot almost-surely win the game. On the other hand, in the limit-sure case Eve can choose the following strategy:

$$f(x) = \begin{cases} \varepsilon & \text{if } x = a \\ 1 - \varepsilon & \text{if } x = b \end{cases}$$

Let us fix  $x \in \{a, b\}$ . We introduce:

$$\begin{aligned} P_x &= \sum_{k=0}^{\infty} P(x/1 \mid \underbrace{b/2 \dots b/2}_{k \text{ times}}) P(\underbrace{b/2 \dots b/2}_{k \text{ times}}) \\ &= \sum_{k=0}^{\infty} P(x \mid \underbrace{b/2 \dots b/2}_{k \text{ times}}) P(1 \mid \underbrace{b/2 \dots b/2}_{k \text{ times}}) P(\underbrace{b/2 \dots b/2}_{k \text{ times}}) \\ &= P(x) \underbrace{\sum_{k=0}^{\infty} P(1 \mid \underbrace{b/2 \dots b/2}_{k \text{ times}}) P(\underbrace{b/2 \dots b/2}_{k \text{ times}})}_{=\eta} \\ &= P(x)\eta \end{aligned}$$

Actually, Adam cannot assure to stay in the central state. Therefore, the winning ( $P_{win}$ ) and losing ( $P_{loose}$ ) probabilities verify:  $P_{loose} = P_a$  and

$P_{win} \geq P_b$ . Hence:

$$\begin{aligned} \frac{P_{win}}{P_{loose}} &\geq \frac{P_b}{P_a} \\ &\geq \frac{P(b)\eta}{P(a)\eta} \\ &\geq \frac{1-\varepsilon}{\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} +\infty \end{aligned}$$

As  $P_{win} + P_{loose} = 1$ , this shows that the central state is winning in the limit-sure case.

### 4.3.2 Parity concurrent games

If concurrent Büchi and co-Büchi games admit positional strategies, this is no longer the case for parity winning conditions. This section shows an example of a parity concurrent game described in [4] where almost-surely winning the game requires infinite memory. Nevertheless, the proof of this example is not in [4] and no proof has been yet proposed. We show here a proof of this result which is not complete as we do not consider all the possible strategies for Adam but only a sub-class. Though, this sub-class should be large enough to convince anybody of the truth of the result.

We consider the game depicted in Figure 5.

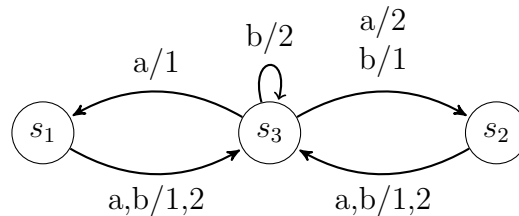


Figure 5: Example of a parity concurrent game which requires infinite memory for Eve to win. Colors are the indices of states.

In this game, Eve has no finite memory almost-surely winning strategy. Indeed, in this game, either Eve plays move  $b$  with probability one, and Adam can then choose to play deterministically move 2 and wins with probability one. Either Eve plays regularly move  $a$  with a probability always greater than some  $\eta > 0$  and Adam can play deterministically move 1 to be sure to visit  $s_1$  infinitely often.

On the other hand, Eve has a strategy to almost-surely win the game. This strategy consists in playing move  $a$  with probability  $\frac{1}{2^k}$  and move  $b$  with probability  $1 - \frac{1}{2^k}$  where  $k$  is the number of prior visits to  $\{s_1, s_2\}$ .

We first point out that there is a probability zero to stay on the central state as the probability to leave it is invariant and strictly greater than zero. Hence the play can be seen as an infinite sequence of the following events: "the next state visited that is not  $s_3$ ". We denote by  $f(h)$  the probability for Adam to play 1 knowing that the history is  $h$ . Our proof requires that the strategy used by Adam does not depend on the number of times the play has stayed in the central state since the last visit to  $\{s_1, s_2\}$  when he has to make his choice (but the strategy can depend on anything that is before the last visit to a non-central state). The probability of the event to be  $s_1$  is

$$\begin{aligned} f(h) \frac{1}{2^k} \sum_{i \in \omega} \left( \left(1 - \frac{1}{2^k}\right) (1 - f(h)) \right)^i &= \frac{f(h)}{2^k} \frac{1}{f(h) \left(\frac{1}{2^k} + 1\right) + \frac{1}{2^k}} \\ &= \frac{1}{2^k} \frac{1}{\frac{1}{2^k} + 1 + \frac{1}{2^k f(h)}} \\ &\leq \frac{1}{2^k} \end{aligned}$$

where  $k$  is the number of times the event has happened and  $h$  is the current history of the game while not considering the number of visits to the central state since the last visit to a non-central state. Moreover,

$$\sum_{k \in \omega} \frac{1}{2^k} < +\infty.$$

Hence Borel-Cantelli Lemma allows us to conclude that the number of visits to state  $s_1$  is finite, hence state  $s_2$  is visited infinitely often.

It is still an open question whether almost-surely winning strategies in parity concurrent games can be easily and finitely described.

## 5 Games with Imperfect information

Games with imperfect information have been introduced in [12]. They are strictly larger than the games with semi-perfect information introduced in [2] which are equivalent (with a polynomial blow up in complexities) to concurrent games.

### 5.1 Definitions

An *arena with imperfect information*  $\mathcal{A}_i$  is a tuple  $\langle S, \Sigma, \Delta \subseteq S \times \Sigma \times S, O, \rho : S \rightarrow O \rangle$  where:

- $S$  is the set of states,
- $\Sigma$  is the set of moves,
- $\Delta$  is the transition relation,
- $O$  is the set of observations,
- and  $\rho$  is a function which maps each state to its corresponding observation.

We force the transition relation to allow any moves in any states. Formally, we require that :  $\forall s \in S, \forall \sigma \in \Sigma, \exists s' \in S : (s, \sigma, s') \in \Delta$ . This property is not restrictive for the results we discuss later. Indeed, one could for instance add a sink state any forbidden move leads to. Then this state should be marked accordingly to the winning condition.

We define the morphism  $\rho$  on possibly infinite sequences of states by

$$\rho(s_0 s_1 \dots s_i) = \rho(s_0) \rho(s_1) \dots \rho(s_i).$$

Classical turn based arenas can be easily transported to arenas with imperfect information. Indeed, let suppose we have a classical turn based game with labeled states. For all Adam states, the outgoing transitions are directly mapped to the preceding Eve states, adding the label of the state on the new transition. The set of observations is then defined as the set of states and the function  $\rho$  is the identity function on the states. This arena with imperfect information have exactly the same behavior than the initial one.

Given a set  $\Omega \subseteq O^\omega$  of infinite observations, an imperfect information game over  $\mathcal{A}_i$  is a tuple  $\mathbb{G}_i = \langle \mathcal{A}_i, \Omega \rangle$ .  $\Omega$  is said the *objective* of the game. From now on, we will consider  $\Omega$  to be fixed.

A *play* in  $\mathbb{G}_i$  with starting state  $s_0$  is an infinite succession of the following iteration: being in state  $s \in S$ , Eve chooses a move  $\sigma \in \Sigma$ . This move defines a set of possible new states  $Post_\sigma(s) = \{s' \in S \mid (s, \sigma, s') \in \Delta\}$ . Then Adam chooses a state in  $Post_\sigma(s)$  which will correspond to the new state of the play and tells Eve the observation associated with this new state. The imperfect information property comes from the fact Eve doesn't know the exact state the play is in but only a class of possible states.

Therefore, a *strategy for Eve* is a function  $\varphi_1 : SO^* \rightarrow \mathcal{D}(\Sigma)$ .

The set of strategies for Eve is denoted  $\Phi_1$ . A positional strategy for Eve is a function that only depends on the last observation:  $\forall h_1, h_2 \in SO^*, \forall o \in O, \varphi_1(h_1o) = \varphi_1(h_2o)$ . The set of positional strategies for Eve is denoted  $\Phi_1^p$ .

A *strategy for Adam* is a function  $\varphi_2 : S^+ \times \Sigma \rightarrow \mathcal{D}(S)$  where

$$support(\varphi_2(S^*s, \sigma)) \subseteq post_\sigma(s).$$

The set of strategies for Adam is denoted  $\Phi_2$ . A positional strategy for Adam is a strategy that only depends on the last state and the last move:  $\forall h_1, h_2 \in S^*, \forall s \in S, \forall \sigma \in \Sigma, \varphi_2(h_1s\sigma) = \varphi_2(h_2s\sigma)$ . The set of positional strategies for Adam is denoted  $\Phi_2^p$ . From now on, when we will introduce a pair of strategies, this will mean the first coordinate will be a strategy for Eve and the second one a strategy for Adam.

A play associated to a pair of strategies  $(\varphi_1, \varphi_2)$  is an infinite sequence of states  $\lambda = s_0s_1 \dots$  where

$$\forall i \in \omega, s_{i+1} \in support(\varphi_2(s_0s_1 \dots s_i, support(\varphi_1(\rho(s_0)\rho(s_1) \dots \rho(s_i))))).$$

The set of plays associated to a pair of strategies  $(\varphi_1, \varphi_2)$  is denoted  $\Lambda_{s_0}^{\varphi_1, \varphi_2}$ .

An example of such a game is the game of the bar man blind with boxing gloves. This game consists of the following: on a round table are placed four glasses forming a square having the same center as the table. Some of the glasses are upside down. The bar man blind with boxing gloves can't see if a glass is to the place or not but he can reverse as much glasses as he wants. When he has finished, a client may quarter turn the table at will. The objective for the bar man is to get the four glasses on the same side. If he manage to, then the game will be stopped and the barman will be winning.

An immediate reduction consists in pointing out there are just four different classes of states in this game: either the opposite glasses are on the same side, or exactly two adjacent glasses are on the same side, or three glasses are on the same side or all glasses are on the same side. This reduction is depicted on figure 6. In this new class of states, the barman may choose three actions: to reverse two opposite glasses, to reverse two adjacent glasses, or to reverse a single glass, they will be denoted respectively  $o, a$  and  $u$ .



Figure 6: The class of states after the reduction of the game of the bar man blind with boxing gloves.

With this reduction, the game can be described as in figure 7.

## 5.2 Knowledge

Given a subset  $Q \subseteq S$  of states, a move  $\sigma \in \Sigma$  and an observation  $o \in O$ , the knowledge  $K$  is defined by  $K(Q, \sigma, o) = Post_\sigma(Q) \cap \rho^{-1}(o)$ . Intuitively, if  $Q$  is the set of states player Eve knows she can be in,  $K(Q, \sigma, o)$  represents the set of states she could be in if she plays move  $\sigma$  and if then Adam tells her the observation is  $o$ .

**Remark:** By definition, all the states in  $K(K, \sigma, o)$  maps to the same observation, namely  $o$ .

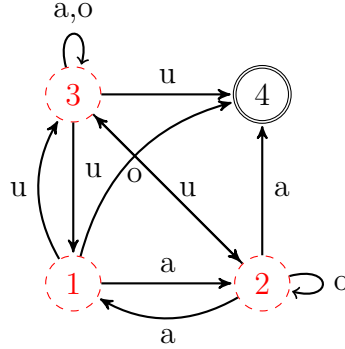


Figure 7: The game of the bar man blind with boxing gloves with reduced states. The red states map to the same observation.

We build inductively the *set*  $\mathcal{K}$  of *knowledges*:  $\mathcal{K}^0 = \{\{s\} \mid s \in S\}$ , and  $\forall i, \mathcal{K}^{i+1} = \mathcal{K}^i \cup K(\mathcal{K}^i, \Sigma, O)$ . Then

$$\mathcal{K} = \lim_{i \rightarrow \infty} \mathcal{K}^i.$$

This limit exists as  $(\mathcal{K}^i)_i$  is monotone and bounded:  $\forall i, \mathcal{K}^i \subset 2^S$ . It is easily seen that  $\mathcal{K}$  may equals  $2^S$  for a well chosen arena with states  $S$ .

The trace of a play  $\lambda = s_0 s_1 \dots$  where the sequence of choices made by Eve is  $\sigma = \sigma_0 \sigma_1 \dots$ , denoted  $(\text{trace}(\lambda)_\sigma^i)_{i \in \omega}$ , is the unique sequence of knowledges characterized by:

- $\text{trace}(\lambda)_\sigma^0 = \{s_0\}$ ,
- $\forall i \in \omega, \text{trace}(\lambda)_\sigma^{i+1} = K(\text{trace}(\lambda)_\sigma^i, \sigma_i, \rho(s_{i+1}))$ .

Intuitively, the trace of a play is the most precise information Eve may have on it. A sub play of a trace  $\beta$  is a sequence of states  $\lambda = s_0 s_1 \dots$  such as  $\exists \sigma \in \Sigma^\omega, \text{trace}(\lambda)_\sigma = \beta$ . By induction, it is easy to see that the same  $\sigma \in \Sigma^\omega$  can be chosen for all sub plays of a given trace. Moreover, the trace only depends on  $s_0, \rho(\lambda)$  and  $\sigma$ , so it can be built by Eve during the play. From now on, we will consider strategies of Eve to be defined on  $K^*$ .

### 5.3 Objectives

Let  $s_0$  be the starting state of a game. We inductively define a sequence of measures  $\mu_{s_0, i}^{\varphi_1, \varphi_2}$  on the prefix of plays:

- The only play of length 1 ( $\lambda = s_0$ ) has measure 1,
- Be a play of length  $i + 1, i \in \omega : \lambda_{i+1} = s_0 s_1 \dots s_i s_{i+1} = \lambda_i s_{i+1}$ , then

$$\mu_{s_0, i+1}^{\varphi_1, \varphi_2}(\lambda_{i+1}) = \mu_{s_0, i}^{\varphi_1, \varphi_2}(\lambda_i) \sum_{\sigma \in \Sigma} P_{\varphi_2}(s_{i+1} \mid \lambda_i \sigma) P_{\varphi_1}(\sigma \mid \rho(\lambda_i))$$

where  $(P_{\varphi_2}(s_{i+1} \mid \lambda_i \sigma))$  is the probability for Adam to choose state  $s_{i+1}$  knowing the history is  $\lambda_i \sigma$  and  $P_{\varphi_1}(\sigma \mid \rho(\lambda_i))$  is the probability for Eve to choose move  $\sigma$  knowing the observation history is  $\rho(\lambda_i)$ .

By the Carathódory unique extension theorem (2), this measure induces an unique measure  $\mu_{s_0}^{\varphi_1, \varphi_2}$  on  $\Lambda_{s_0}^{\varphi_1, \varphi_2}$ .

Given a starting state  $s_0$ , we define the value of a pair of strategies  $(\varphi_1, \varphi_2)$  by

$$\|\varphi_1, \varphi_2\|_{s_0}^{\mathbb{G}_i} = \mu_{s_0}^{\varphi_1, \varphi_2}(\Lambda_{s_0}^{\varphi_1, \varphi_2} \cap \Omega).$$

The value of a strategy  $\varphi_1$  for Eve is defined by:

$$\|\varphi_1\|_{s_0}^{\mathbb{G}_i} = \inf_{\varphi_2 \in \Phi_2} \|\varphi_1, \varphi_2\|_{s_0}^{\mathbb{G}_i}.$$

The value of a strategy  $\varphi_2$  for Adam is defined dually. Finally, we define the value of the game for Eve on  $s_0$  by:

$$|||\mathbb{G}_i^1|||_{s_0}^{\mathbb{G}_i} = \sup_{\varphi_1 \in \Phi_1} \|\varphi_1\|_{s_0}^{\mathbb{G}_i} = \sup_{\varphi_1 \in \Phi_1} \inf_{\varphi_2 \in \Phi_2} \|\varphi_1, \varphi_2\|_{s_0}^{\mathbb{G}_i}.$$

One should note that the Blackwell optimality theorem shows the following:

$$|||\mathbb{G}_i^1|||_{s_0}^{\mathbb{G}_i} = \inf_{\varphi_2 \in \Phi_2} \sup_{\varphi_1 \in \Phi_1} \|\varphi_1, \varphi_2\|_{s_0}^{\mathbb{G}_i}.$$

We will now say a strategy  $\varphi_1$  for player Eve is *almost-optimal* if  $\|\varphi_1\|_{s_0}^{\mathbb{G}_i} = |||\mathbb{G}_i^1|||_{s_0}$ . A sequence of strategies  $(\varphi_1^i)_{i \in \omega}$  is said to be *limit-optimal* if

$$\lim_{i \rightarrow \infty} \|\varphi_1^i\|_{s_0}^{\mathbb{G}_i} = |||\mathbb{G}_i^1|||_{s_0}.$$

If  $|||\mathbb{G}_i^1|||_{s_0} = 1$ , an *almost-optimal* strategy will be called *almost-sure* and a *limit-optimal* strategy will be called *limit-sure*.

By extension, the almost-sure winning region for Eve in the game  $\mathbb{G}_i$  is the subset of states  $W_1^{\mathbb{G}_i} = \{s \in S \mid \exists \varphi_1 \in \Phi_1, \|\varphi_1, \varphi_2\|_{s_0}^{\mathbb{G}_i} = 1\}$ . We symmetrically define the same values for player Adam.

## 5.4 Winning conditions

The winning conditions available in classical games have to be extended. We choose to restrain the winning condition to those on observations rather than on states. To explain this choice, we first point out it is natural for a player to know whether he wins or not. To see that it is not insured with winning conditions on states, consider Figure 8, which presents an example of a co-Büchi game with winning condition on states. In this example, Adam can't win with probability one: if he uses a deterministic strategy, it is easy to find a counter-strategy for Eve which force not to go to the final state with probability one. If he chooses randomly how to play from  $s_0$ , Eve can play accordingly to the counter-strategy corresponding to the higher probability choice, forcing the probability to win to be lower than the sum of the two lowest probabilities: Adam can't win with probability higher than  $\frac{2}{3}$ . On the other hand, the strategy  $\varphi$  which consists of playing one of the three states  $\{s_1, s_2, s_3\}$  with probability  $\frac{1}{3}$  each assures  $\|\varphi\| = \frac{2}{3}$ , so  $\|\mathbb{G}_i^2\|_{s_0} = \frac{2}{3}$ . Hence the game can't be translated on observations, as considering the observation red as accepting (or not) changes the value of the game to  $\|\mathbb{G}_i^2\|_{s_0} = 1$  (respectively  $\|\mathbb{G}_i^2\|_{s_0} = 0$ ). And as Eve only sees the same observation for all plays: she can't know whether she wins or not. Therefore, we will define winning conditions on observations.

Some classical winning conditions are *reachability*, *Büchi*, *co-Büchi* and *parity*. As we will show later, the parity condition is yet a too large condition to expect good results in games with imperfect information. It is why we don't introduce other conditions. They are defined that way:

**Reachability condition:** Let  $F \subseteq O$  be a subset of observations. A play  $\lambda = s_0s_1\dots$  is winning for Eve iff  $F \cap \rho(\lambda) \neq \emptyset$ .

**Büchi condition:** Let  $F \subseteq O$  be a subset of observations. A play  $\lambda = s_0s_1\dots$  is winning for Eve iff  $F \subseteq \text{inf}(\rho(\lambda))$ .

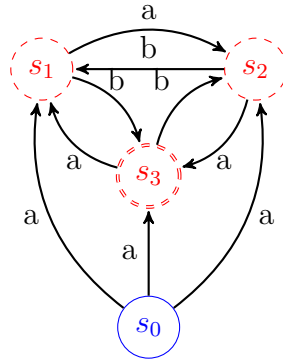


Figure 8: co-Büchi imperfect information game with winning condition on states where Eve may don't know whether she wins or not and condition can't be translated to observations. Observations are modeled by colors.

**co-Büchi condition:** Let  $F \subseteq O$  be a subset of observations. A play  $\lambda = s_0 s_1 \dots$  is winning for Eve iff  $F \cap \text{inf}(\rho(\lambda)) = \emptyset$ .

**Parity condition:** Let  $\tau : O \rightarrow \omega$  be a function. A play  $\lambda = s_0 s_1 \dots$  is winning for Eve iff  $\min(\tau(\text{inf}(\rho(\lambda)))) \in 2\omega$ .

## 5.5 Reduction

To solve imperfect information games, it is natural to use a knowledge-based structure, as this reduction contains all the information useful for Eve. However, this is not sufficient to keep the values of the game. To prove this, let us consider the Büchi game 1 described on Figure 9. In this game, Eve has an almost-sure winning strategy : this strategy consist in playing uniformly motive  $a$  and motive  $ba$  when in knowledge  $\{s_1, s_2\}$ . The game 2 presents a natural reduction of game 1 in an only knowledge-based structure. In this game, Adam has an immediate sure winning strategy. This suggest that one must keep trace of the knowledge together with the real state the play is in, and so restrain the strategies for Eve to those coherent with the initial game.

A valid knowledge-based reduction for omega-regular winning conditions has been proposed in [1]: Be  $\mathbb{G}_i = \langle \langle S, \Sigma, \Delta, O, \rho \rangle, \Omega \rangle$  an imperfect

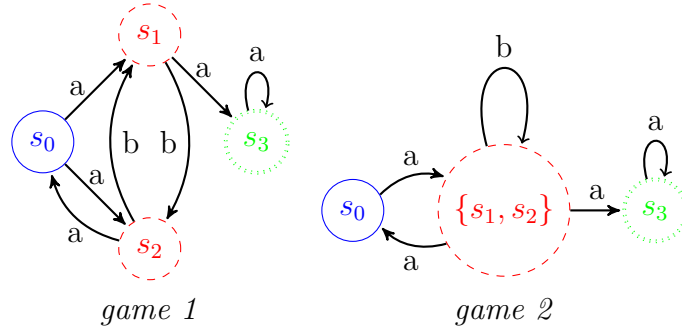


Figure 9: An example of a Büchi imperfect information game and its only knowledge-based reduction : both games have different values.

information game. The knowledge-based game associated to  $\mathbb{G}_i$  is the game  $\mathbb{H}_{\mathbb{G}_i} = \langle \langle S_{\mathbb{H}} \subseteq \mathcal{K} \times S, \Sigma, \Delta_{\mathbb{H}} \subseteq S_{\mathbb{H}} \times \Sigma \times S_{\mathbb{H}} \rangle, \Omega_{\mathbb{H}} \subseteq S_{\mathbb{H}}^{\omega} \rangle$  where:

- $(Q, q) \in S_{\mathbb{H}}$  iff  $Q \in \mathcal{K}$  and  $q \in Q$ ,
- $((Q, q), \sigma, (Q', q')) \in \Delta_{\mathbb{H}}$  iff  $(q, \sigma, q') \in \Delta$  and  $Q' = K(Q, \sigma, \rho(q'))$ ,
- Let consider the winning condition is a parity one. From the color function  $\tau$ , the new color function is defined by  $\tau_{\mathbb{H}}(Q, q \in Q) = \tau(\rho(Q))$ <sup>1</sup>. This transformation covers Büchi and co-Büchi games. For a reachability condition, the new states to reach are those with the first coordinate in an observation to reach.

This formalism induces a natural equivalence relation on states:  $(Q, q) \approx (Q', q')$  iff  $Q = Q'$ . We define the function  $Allow : S \times \mathcal{K} \rightarrow 2^{\Sigma}$  by:  $\sigma \in Allow(s, P)$  iff  $\forall q \in S_{\mathbb{H}}, (s, \sigma, q) \in \Delta_{\mathbb{H}} \implies q \in P$ . By extension, we define  $Allow^{\approx}((S, s), P) = \bigcap_{\{(S', s') \in S_{\mathbb{H}}\}} Allow(s', P)$ .

As the knowledge history can be obtained by the state history, Adam may use arbitrary strategies in this new game. Those strategies are directly mapped to the one of the initial game. On the other hand, Eve must only use strategies based on equivalence classes induced by  $\approx$ : she must only build her strategies on the first coordinate of states. We denote by  $T$  this one-to-one function mapping strategies in game  $\mathbb{G}_i$  to strategies in game  $\mathbb{H}_{\mathbb{G}_i}$ . It is immediate to see that  $\forall \varphi_1 \in \Phi_1, \forall \varphi_2 \in \Phi_2, \|\varphi_1, \varphi_2\|_{s_0}^{\mathbb{G}_i} = \|T(\varphi_1), T(\varphi_2)\|_{(\{s_0\}, s_0)}^{\mathbb{H}_{\mathbb{G}_i}}$ .

<sup>1</sup>The remark made in section “knowledge” proves that this new color is well-defined.

Considering the preceding result, the values in each state are the same in  $\mathbb{H}_{\mathbb{G}_i}$  and in  $\mathbb{G}_i$ . Actually, there is an immediate naïve algorithm mapping winning regions with an exponential complexity on the number of states in game  $\mathbb{G}_i$ .

If we reconsider the game of the bar man blind with boxing gloves using this reduction, we obtain the game presented in figure 10.

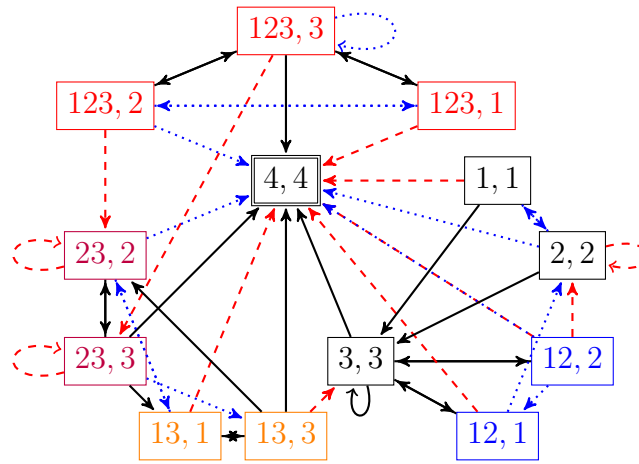


Figure 10: The knowledge version of the game of the bar man blind with boxing gloves. The red and dashed transitions correspond to reversing two opposite glasses, the blue and dotted ones to reverse two adjacent glasses and the black ones to reverse one glass.

### 5.5.1 Parity games

The parity condition covers the Büchi and the co-Büchi one. However, winning in a parity game with imperfect information may require infinite memory. This result can be shown using the example given in the section on concurrent games. Indeed, concurrent games are equivalent to semi-perfect information games, which are a subclass of games with imperfect information. Considering this, Figure 11 shows a game equivalent to the one defined in the concurrent section, which require infinite memory for Eve to win. In this game, Eve wins if and only if the state  $s_2$  is visited infinitely often and the state  $s_3$  is visited finitely often.

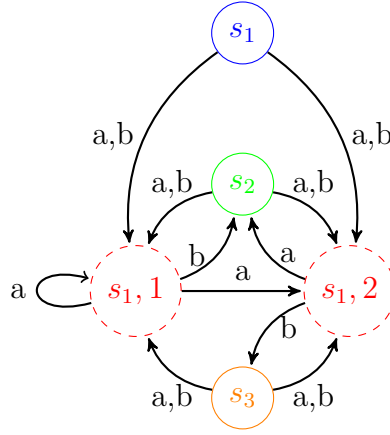


Figure 11: Example of a parity game with imperfect information which require infinite memory for Eve to win.

### 5.5.2 Solving Büchi games

We now consider the case the winning condition is a Büchi one. Let  $\mathcal{B} \subseteq S$  be the objective states. To this set maps a new set  $\mathcal{B}_{\mathbb{H}}$  in game  $\mathbb{H}_{\mathbb{G}_i}$  defined by  $(Q, q) \in \mathcal{B}_{\mathbb{H}}$  iff  $\tau(q) \in \mathcal{B}$ . Let us explain how to solve a Büchi games with imperfect information as described in [1]. The first task is to show that almost-sure winning strategies can be chosen positional. To do it, let define the increasing and bounded sequence of subset of states  $(Q_i)_i$  as follows:

- $Q_0 = W_1^{\mathbb{H}_{\mathbb{G}_i}} \cap \mathcal{B}_{\mathbb{H}}$  (recall that  $W_1^{\mathbb{H}_{\mathbb{G}_i}}$  is the winning region for Eve in game  $\mathbb{H}_{\mathbb{G}_i}$ ),
- $\forall i \in \omega, Q_{i+1} = Q_i \cup \{(Q, q) \in S_{\mathbb{H}} \mid \exists \sigma \in \text{Allow}^{\approx}((Q, q), W_1^{\mathbb{H}_{\mathbb{G}_i}})\}$   
such that  $\text{Post}_{\sigma}((Q, q)) \subseteq Q_i\}$

$(Q_i)_i$  being an increasing sequence on a finite set,  $\exists j \in \omega, \forall k \geq j, Q_k = Q_j$ . We denote by  $Q_{\infty}$  this greatest set.

Our aim is to show that  $W_1^{\mathbb{H}_{\mathbb{G}_i}} = Q_{\infty}$ . It is immediate that  $Q_{\infty} \subseteq W_1^{\mathbb{H}_{\mathbb{G}_i}}$ . To prove the converse inclusion, let us consider a state  $s \in X = W_1^{\mathbb{H}_{\mathbb{G}_i}} - Q_{\infty}$ . By definition,  $\forall \sigma \in \Sigma, \exists q \in X, (s, \sigma, q) \in \Delta_{\mathbb{H}}$ . We fix the strategy for Adam that chooses systematically one of these possible states  $q$ . Being on a state in  $X$ , we now consider a knowledge-based almost-sure winning strategy for Eve.

In this strategy, Eve must always choose a move in  $Allow^{\approx}((Q, q), W_1^{\mathbb{H}_{G_i}})$  where  $(Q, q)$  is the last state of the play, as if she does not Adam has a trivial counter-strategy that makes her living her winning region with a non zero probability. With those two fixed strategies, the play will stay in  $X$ . Moreover, as  $\mathcal{B}_{\mathbb{H}} \cap W_1^{\mathbb{H}_{G_i}} \subseteq Q_{\infty}$ , we have  $\mathcal{B}_{\mathbb{H}} \cap X = \emptyset$ . Hence this state is not in  $W_1^{\mathbb{H}_{G_i}}$ . By contradiction, the initial state was not in the winning region for Eve. Hence  $W_1^{\mathbb{H}_{G_i}} - Q_{\infty} = \emptyset$  what achieves to show that  $W_1^{\mathbb{H}_{G_i}} = Q_{\infty}$ .

Let us now consider the following strategy for Eve in  $Q_{\infty}$ : in state  $s$ , Eve chooses randomly uniformly one move in  $Allow^{\approx}(s, W_1^{\mathbb{H}_{G_i}})$ . This assures she will eventually reach a state in  $Q_0$  with probability one. When she is on  $Q_0$ , she is still in  $Q_{\infty}$  and therefore can reach a state in  $Q_0$  with probability one. This shows that if Eve has an almost-winning strategy from a state  $s_0$ , then she also has a positional almost-winning one from the same state. To conclude, we can use the result 1 which give an algorithm to find the winning region with a 2EXPTIME complexity on the number of states.

### 5.5.3 Solving co-Büchi games

Solving co-Büchi games with imperfect information was still an open question. This section presents a solution to this problem.

Let  $R$  be a subset of states. The sub game  $\mathbb{H}_{G_i}(R)$  is the game  $\langle\langle R, \Sigma, \Delta_{\mathbb{H}} \cap R \times \Sigma \times R \rangle, \Omega_{\mathbb{H}} \cap R^{\omega} \rangle$ . It is only defined if there is no dead end in it. In other words, the game  $\mathbb{H}_{G_i}(R)$  is the sub game of  $\mathbb{H}_{G_i}$  where the only states are those in  $R$ , and the transitions are those of  $\mathbb{H}_{G_i}$  with an initial and arrival state in  $R$ .

Let  $\mathcal{B}_{\mathbb{H}}$  be the co-Büchi objective (that is to say the states Eve does not want to visit infinitely often). We define the safety winning plays on  $\mathcal{B}_{\mathbb{H}}$  by  $\Omega_{safety} = (S_{\mathbb{H}} - \mathcal{B}_{\mathbb{H}})^{\omega}$ , the almost-sure winning region will be denoted  $safety(\mathcal{B}_{\mathbb{H}})$ . Strategies to almost-surely win a safety objective can be chosen positional. To prove this, let consider a game with imperfect information with a safety objective. The following transformation is performed on the game: any transition from a final state is deleted, and a self-loop transition is added from any final state, then final and non-final states are flipped. It is immediate to show that almost-surely winning the initial game is equivalent

to almost-winning the new game in which the objective is a Büchi one. We already shown that strategies in Büchi games with imperfect information can be chosen positional. As this strategy can be mapped directly to the initial game, this proves strategies to almost-surely win a game with a safety objective can be chosen positional.

Actually, the preceding construction can't be adapted directly to co-Büchi games. Meanwhile, the idea is to use a similar method, with an increasing construction of the sets  $(Q_i)_i$ , such that the initial set  $Q_0$  verifies  $Q_0 = \text{safety}(\mathcal{B}_{\mathbb{H}})$ . This construction is not large enough, as shown in Figure 12. In this game, state  $s_2$  is in  $\text{safety}(\mathcal{B}_{\mathbb{H}})$  but not state  $s_0$ . The construction of the sequence  $Q_i$  will bring state  $s_1$  into it but not state  $s_0$  as it is always possible for Adam to force to stay on it. However, this state is almost-surely (and even surely) winning for Eve. It is why the method described beneath is a bit more sophisticated.

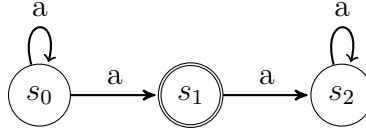


Figure 12: Example of a co-Büchi game with imperfect information where the adaptation of the construction proposed in the Büchi case is not sufficient. The information is complete for Eve and the only accepting state is  $s_1$ .

We introduce a function  $\rho : 2^{S_{\mathbb{H}}} \rightarrow (2^{S_{\mathbb{H}}})^{\omega}$  mapping any subset  $R$  of states to an increasing (hence converging) sequence of subsets of states  $(Q_i^R)_i$ , as we did in the Büchi case, inductively defined by:

- $Q_0^R = \text{safety}(\mathcal{B}_{\mathbb{H}})$ ,
- $\forall i \in \omega, Q_{i+1}^R = Q_i^R \cup \{(Q, q) \in S_H \cap R \mid \exists \sigma \in \text{Allow}^{\approx}((Q, q), W_1^{\mathbb{H}G_i(R)}) \text{ such that } \text{Post}_{\sigma}((Q, q)) \subseteq Q_i^R\}$ .

$(Q_i^R)_i$  being monotone on a finite set,  $\exists j \in \omega, \forall k \geq j, Q_k = Q_j$ . Let  $Q_{\infty}$  be this greatest set. We now define the function  $\rho_{\infty} : 2^{S_{\mathbb{H}}} \rightarrow 2^{S_{\mathbb{H}}}$  such that  $\rho_{\infty}(R) = Q_{\infty}^R$ .

This function allows us to introduce a new inductively built increasing (hence converging) sequence of subsets of states  $(Q_i)_i$ :

- $Q_0 = \rho_\infty(S_{\mathbb{H}})$ ,
- $\forall i \in \omega, Q_{i+1} = Q_i \cup \rho_\infty(S_{\mathbb{H}} - Q_i)$ . Note that by definition of  $Q_i$ , there is no dead end in the game  $\mathbb{H}_{G_i}(S_{\mathbb{H}} - Q_i)$ .

Once again, this sequence of subset is ultimately constant. Let  $Q_\infty$  be its limit. Going back to Figure 12, the initial set is  $Q_0 = \{s_1, s_2\}$ . Then  $Q_\infty = Q_1 = Q_0 \cup \{s_0\}$ .

More generally, the following assertion holds:  $Q_\infty = W_1^{\mathbb{H}_{G_i}}$  and Eve has a positional strategy to almost-surely win on  $Q_\infty$ .

To prove this, let consider the first assertion:  $Q_\infty \subseteq W_1^{\mathbb{H}_{G_i}}$  and Eve has a positional strategy to almost-surely win on  $Q_\infty$ . This can be shown by induction on the construction of  $Q_\infty$ :

- Let  $q$  be a state in  $Q_0$ . Either Eve is not in a state in *safety*( $\mathcal{B}_{\mathbb{H}}$ ), and as for Büchi games, she has a positional strategy consisting of playing uniformly randomly moves in  $Allow^\approx((Q, q), W_1^{\mathbb{H}_{G_i}(R)})$ . This strategy ensures that she will eventually reach a state in  $Q_0^{S_{\mathbb{H}}}$  with probability one. Being on a state in *safety*( $\mathcal{B}_{\mathbb{H}}$ ), she has a positional strategy to ensure she will never reach a state in  $\mathcal{B}_{\mathbb{H}}$  with probability one. Those two strategies can be merged to obtain a new positional strategy that makes Eve almost-surely win the game.
- Let  $q$  be a state in  $Q_{i+1} - Q_i$ . We suppose that all states in  $Q_i$  are winning for Eve in the almost-sure case with positional strategies. In the game  $\mathbb{H}_{G_i}(S_{\mathbb{H}} - Q_i)$ , Eve has a positional strategy to almost-surely win the game (reconsider the preceding step with the game  $\mathbb{H}_{G_i}(S_{\mathbb{H}} - Q_i)$ ). Back in the main game, she keeps the same strategy: each time she has to use this strategy, either Adam chooses a state in  $S_{\mathbb{H}} - Q_i$  so she can continue using this strategy, or he chooses a state in  $Q_i$  which is by hypothesis a state from which Eve has a positional almost-sure winning strategy.

The second assertion to prove is that  $W_1^{\mathbb{H}_{G_i}} \subseteq Q_\infty$ . By definition of  $Q_\infty$ , in the game  $\mathbb{H}_{G_i}(S_{\mathbb{H}} - Q_\infty)$ ,  $safety(\mathcal{B}_{\mathbb{H}} - Q_\infty) = \emptyset$ . Hence, by the Blackwell optimality theorem, Adam has a strategy such that in any state in  $S_{\mathbb{H}} - Q_\infty$ , the probability of going back to an accepting state is non zero. As there is a finite number of states, the smallest of these probabilities  $\eta$  is such that  $\eta > 0$ . And as this probability is totally independent of any event, Borel Cantelli Lemma concludes that the probability of visiting infinitely often states in  $S_{\mathbb{H}} - Q_\infty$  is greater or equal to  $\eta^{|S_{\mathbb{H}} - Q_\infty|} > 0$ .

This achieves the proof that co-Büchi games with imperfect informations almost-sure winning strategies can be chosen positional.

## 6 Concurrent games with imperfect information

The games with imperfect information introduced previously in this document are totally asymmetric as one player actually has a full information about the game. It is why it is natural to consider a generalized version of this game, where both players have incomplete information. In this section, we present a new game model taking this aspect into account and some results on them.

### 6.1 Definitions

A concurrent arena with imperfect information is a tuple  $\mathcal{A} = \langle S, \Sigma_1, \Sigma_2, \delta, O, \rho \rangle$  where:

- $S$  is the set of states,
- $\Sigma_1$  is the set of moves for Eve,
- $\Sigma_2$  is the set of moves for Adam,
- $\delta : S \times \Sigma_1 \times \Sigma_2 \rightarrow S$  is the transition total function on states,
- $O$  is the set of observations,
- $\rho : S \rightarrow O$  is the function mapping any state to its observation.

A play on a concurrent arena of imperfect information goes as follows: being in a state  $s \in S$ , both players choose simultaneously a move in their set of moves ( $\sigma_1$  and  $\sigma_2$ ). These choices define an unique new state  $s' = \delta(s, \sigma_1, \sigma_2)$ . This state maps to an observation  $\rho(s')$  which is given to both players. Then the iteration is repeated forever.

For the same reason as in games with imperfect information, the objectives are defined as sets of sequences of observations: let  $\Omega \in O^\omega$  be an objective, then  $\mathbb{G} = \langle \mathcal{A}, \Omega \rangle$  is a concurrent game with imperfect information. As they are exactly the same as in the imperfect information case, winning conditions will not be redefined there.

## 6.2 Strategies, knowledges

Using the same construction as in the imperfect information case, there are two sets of knowledges  $\mathcal{K}_1$  and  $\mathcal{K}_2$  belonging respectively to Eve and Adam. Those sets have no reason to be equal, as the information of each player depends on his moves.

Consecutively, strategies may be defined on sequences of knowledge. Formally, a strategy for Eve is a function  $\varphi_1 : \mathcal{K}_1^* \rightarrow \Sigma_1$  and a strategy for Adam is a function  $\varphi_2 : \mathcal{K}_2^* \rightarrow \Sigma_2$ . We will use  $\Phi_1$ ,  $\Phi_2$ ,  $\Phi_1^p$  and  $\Phi_2^p$  to denote the usual sets of strategies for Eve, strategies for Adam, positional strategies for Eve and positional strategies for Adam.

Almost-sure and limit-sure objectives are defined as in the previous cases. And as in previous cases, we will limit our study to almost-sure objective.

## 6.3 Positional strategies

This formalism is trivially larger than the one of concurrent games and games with imperfect information. Therefore, the parity condition may require infinite memory to assure the winning of a player.

For the Büchi case, restricting to uniformly distributed positional strategies is sufficient, a proof of this result is described beneath.

Let  $\varphi$  be a winning strategy for Eve on her winning region  $W$ , and  $\varphi_p$  be the positional strategy consisting of playing any move in  $Allow^\approx(s, W)$  while being on state  $s \in W$  where  $Allow^\approx(s, W)$  is defined as previously. By definition of  $\varphi$ ;

$$\forall \varphi_2 \in \Phi_2, \forall s_0 \in S, \Lambda_{\varphi, \varphi_2}^{s_0} \subseteq \Lambda_{\varphi_p, \varphi_2}^{s_0}. \quad (1)$$

We denote by  $\mathcal{B}$  the subset of accepting states.

For all state  $s \in W$ , we define  $path_s = \{p = p_1 \dots p_n \in S^* \mathcal{B} \mid \exists h \in S^* \text{ such that } p_1 \in support(\varphi(hs)) \text{ and } \forall 1 < i < n, p_i \in support(\varphi(hsp_1 \dots p_i))\}$ . Let  $m_s$  be the minimum length of a word in  $path_s$  that is to say:

$$m_s = \min_{p \in path_s} |p|.$$

Because of (1), the minimum of these lengths corresponds to a path compatible with  $\varphi_p$ .

We introduce:

$$M = \max_{s \in W} m_s.$$

We now consider that Eve is playing using the strategy  $\varphi_p$ , and we introduce the event: “during the next  $M$  moves, an accepting state will be visited”. This event has a probability strictly greater than zero. Actually, this probability is greater than the minimum (reached) of the probabilities to realize it being on a given state which is strictly greater than zero.

Therefore, Borel-Cantelli Lemma concludes that the probability for the event to happen infinitely often is one.

Finally, Theorem 1 concludes that if there exists a winning strategy on  $W$ , then there exists a uniformly distributed positional one.

## 7 Extension to pushdown models

### 7.1 Pushdown model

The pushdown model allows finite representation of infinite arenas. This process is made using a stack that extend the transition relation. Formally, to get a pushdown model, one should add to the preceding definitions a stack  $\eta$ , an alphabet  $\Sigma_\eta$  on the stack and a special character  $\perp$  to denote the bottom-of-stack symbol.

The stack is an element of  $\perp \Sigma_\eta^*$ . The stack allows three types of operations: **push**( $a$ ) maps a stack  $\perp \sigma_1 \dots \sigma_n$  to the stack  $\perp \sigma_1 \dots \sigma_n a$ , **switch**( $a$ ) maps a stack  $\perp \sigma_1 \dots \sigma_{n-1} \sigma_n$  to the stack  $\perp \sigma_1 \dots \sigma_{n-1} a$  and **pop**() maps a stack  $\perp \sigma_1 \dots \sigma_{n-1} \sigma_n$  to the stack  $\perp \sigma_1 \dots \sigma_{n-1}$ . The operations **switch**( $a$ ) and **pop**() have no effect on the stack  $\perp$ .

The top of the stack may be accessed with the function `top()` which maps a stack  $\sigma_1 \dots \sigma_n$  to the character  $\sigma_n \in \{\perp\} \cup \Sigma_\eta$ .

Transition relations are now redefined to take the stack into account. This can be seen as labeling edges by the characters from  $\{\perp\} \cup \Sigma_\eta$  (if they were already labeled, then they are labeled by both the ancient and the new labels). Moreover, to each edge is mapped an operation between the three presented before.

## 7.2 Decidability

A method to solve a pushdown game is to reduce it to a game with the same behavior (and easily transportable strategies) but with no stack. This method require an exponential blow-up of the number of states.

The process for this reduction will not be detailed here, one could find all the important information about this reduction in [21] and [14]. This reduction has no link with the type (for example imperfect information, or concurrent) of game used. Actually, even concurrent games with imperfect information may be reduced using this algorithm.

Therefore, all the results that have been presented in this document are compatible with a pushdown extension, the only difference will be the complexity that will have an extra exponential blow up.

# 8 Conclusion, further work

## 8.1 Conclusion

We have considered several models of games in this document, starting from classical turn based games to concurrent ones with imperfect information. Globally, the Büchi and co-Büchi cases are sufficiently simple to permit to restrict strategies to positional ones, leading to the decidability of several models.

On the other hand, parity games are quickly too complicated to be considered with this method.

For the pushdown case, it does not affect determinacy, but a serious drawback on the complexity of algorithms comes from an unavoidable exponential blow-up.

## 8.2 Further work

There are still several open questions about those games. The first one is on concurrent co-Büchi games with imperfect information that have not been treated yet.

Moreover, parity condition does not allow the use of positional strategies (not even finite memory ones). Nevertheless, infinite strategies have not yet been studied, even if a reduction could be imaginable. For instance, the history given by the number of times each color has been visited might be sufficient to find winning strategies.

Another important question that has not been treated in this document is the case of limit-sure objectives.

In conclusion, there is still a lot of work to be done on games with imperfect information. . .

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## A Appendix

### A.1 Borel-Cantelli lemma

Let  $(X_i)_i$  be an infinite sequence of events in a probability space of measure  $P$ . Then the Borel-Cantelli lemma states that:

$$\text{if } \sum_{i=0}^{+\infty} P(X_i) < +\infty \text{ then } P(\limsup_{i \rightarrow +\infty} X_i) = 0,$$

$$\text{if } \sum_{i=0}^{+\infty} P(X_i) = +\infty \text{ and if the events are mutually independent}$$

$$\text{then } P(\limsup_{i \rightarrow +\infty} X_i) = 1.$$

### A.2 Carathéodory extension theorem

**Theorem 2.** *Let  $\Omega$  be a set. Let  $R$  be a ring in  $2^\Omega$  and  $\mu_R$  a  $\sigma$ -finite measure on it. Then the Carathéodory extension theorem states that there exist a unique extension of  $\mu_R$  on the  $\sigma$ -algebra generated by  $R$ .*