Stability Index of Interaction forms

Joseph Abdou *

December 15, 2008

Abstract An interaction form is an abstract model of interaction based on a description of power distribution among agents over alternatives. A solution known as the settlement set is defined at any preference profile. Necessary and sufficient conditions for stability, that is the existence of settlements, are established. A Stability Index that plays a role similar to that of the Nakamura Number is defined. It measures, loosely speaking, the complexity of those configurations that prevent a settlement. To any strategic game form one can associate an interaction form in such a way that given an equilibrium concept (e.g. Nash or strong Nash) and a preference profile, settlements of the interaction form are precisely the equilibrium outcomes of the resulting game. As a consequence we have necessary and sufficient conditions for the solvability of game forms. The paper provides a localization of the index in case of unstability.

Keywords: Interactive Form, Stability Index, Nash Equilibrium, Strong Equilibrium, Solvability, Consistency, Simple Game, Effectivity Function, Acyclicity, Nakamura Number.

JEL Classification: C70, D71 **AMS Classification:** 91A44

*Ecole d'Economie de Paris, Paris 1, CNRS, 106-112 boulevard de l'Hôpital 75647 Paris Cedex 13 - France; email: abdou@univ-paris1.fr

Introduction

Two basic models of interaction between rational players have been historically distinguished : one is strategic, the other based on power description. In the strategic model, every agent is represented by an explicit set of available actions; the interaction is described as the mechanism that transforms the joint actions into some outcome. The power-based model presents a description of the agents power or their aggregates; the interaction is the effective exercice of that power. Instances of this model are cooperative or coalitionnal games, simple games and effectivity functions. The advantage of the strategic model is that it is totally explicit: available actions, information and rules are commonly known. However politics, diplomacy, international relations and warfare do not exhibit totally explicit sets of actions and rules. Parties, unions, coalitions have some power to force issues or to oppose them. Countries, regional or global powers do not obey formal rules in prevailing conflicts.

In this paper we present a model of interaction based on power distribution among agents, a model general enough so that to allow equally for a representation of classical coalitional models as well as the essential features of strategic ones. Let N be a set of agents (also called players, individuals) and let A be a set of states (also called alternatives, issues). Our approach is founded on the general idea that given some prevailing state, agents dispose of some power to oppose that state, that is to disrupt it if they have an interest to do so. We shall illustrate informally our arguments with an example in politics by taking the case of a government formation in some State. Players are the atomic entities that are endowed with autonomous wish and will. In our example players may be parties, influential group, etc... A coalition is a subset of players with coordinated action. A government (to be formed) is any element of A. We start by considering the so-called simple game model. In a simple game, power is withheld by some set $\mathcal{W}$ of winning coalitions: precisely, a winning coalition has the power to oppose any current or proposed scenario and to propose any other, whereas a loosing coalition can oppose no government at all. Whether a winning coalition will object and act in consequence depends on its actual preferences. If no objection is formulated, the government is adopted. In technical terms the outcome is in the core of the simple game given the preferences.

Though simple games can fairly model some decision mechanisms, like weighted majority voting in some institutions, it is too simple to describe political issues underlying a government formation in most countries. This is because in that model a coalition is either absolutely powerful or totally powerless. An effectivity function E allows a more general distribution of power among the coalitions. In our example, if $B \in E(S)$ where B is a subset of possible goverments and S is a coalition, then the latter can upset a by threatening to form some other goverment in B but has no enough power to force precisely one alternative. However here too, the power of a coalition does not depend on the current state a, in other words the model takes into account for any coalition, only that part of power that is common at all states. This is a significant restriction in the model since we think that in most interactions, a coalition may achieve something if the current situation is a and something else if it were b. The idea is then to allow for an effectivity power depending on the state a. This is the local effectivity function. Furthermore, the same outcome can be implemented in various ways. For instance, assume that (S_1, S_2, S_3) are concerned by goverment formation, and that many scenarios may lead to a, then in one scenario S_2 can upset it by proposing B_2, while in the other coalition S_3 can upset it by proposing B_3. Therefore we may consider the confederation (S_2, S_3) as the opposition actor. Although the action in not coordinated among the coalitions of the confederation, there is at least a collusion to oppose a. Taking into account this idea compels us to introduce the concept of interaction form. In an interaction form we take into account the dependence of the interaction power on the actual state in all possible scenarios that lead to that state and the reactive power of any confederation that contributes to it.

Given a preference profile for players, states that no confederation have any interest to upset is called a settlement at that profile. The interaction form is said to be stable if it admits at least one settlement at any preference profile. An interaction form can be associated to any strategic game form together with a given equilibrium concept. The idea of interaction form is similar to that of effectivity structures introduced by Abdou and Keiding (2003). The avantages of the present model are (1) that it allows for the representation of various equilibria concepts within the *same* interaction form, whereas the other is specific to one equilibrium concept, (2) that in the current model, operations like projections *faithfully* reflect the change in the underlying confederation and (3) most importantly, since only the interaction form associated to some game form and some equilibrium concept is relevant for stability, a *comparison* between different procedures or mechanisms becomes possible regarding stability. An interaction form can thus be viewed as an intrinsic representation of power without a direct reference to the strategies or the equilibrium concept.

Now assume that the interaction form is not stable as it is indeed so often the case in political life. Our idea is to define a measure of unstability and for that purpose we introduce the stability index. The latter is a number that may be set to infinity in case of stability and that measures the difficulty to exclude settlements in the society. If this number is low, for instance two, then a simple split in the society with strong opposition power on each side can lead, at polarized preferences, to a stalemate . If the index is high then unless agents possess some intricate preference profile, a settlement can be reached. The stability Index introduced in this paper plays a similar role as the Nakamura Number for simple games (Nakamura 1979), the difference being that the Nakamura number is defined on the winning coalition structure only whereas the stability index depends on the whole interaction form.

The paper is organized as follows. In section 1 we review some power - based models grouped under the name of coalitional forms and where stability, acyclicity and index are introduced and briefly studied. Section 2 is devoted to the general model of interaction form. The notions of settlement, stability, cycle, stability index, projections and restrictions are defined. A characterisation of stability is given as well as a localization of the index. In section 3 we present applications of the main results to strategic game forms. We conclude our study by some remarks.

1 Power distibutions in coalitional forms

In this section we review some models of interaction based on a description of power distribution in a society with no explicit mention of any strategic mechanism that lies behind that power distribution. We shall see in section 3 how, starting from a strategic mechanism (i.e. a game form) and an equilibrium concept, one can derive a description of the underlying power distribution. The notions that we present in this section, have in common that only the *independent* power held by a coalition is represented. Ranked by degree of complexity those are: simple games, effectivity functions and local effectivity functions. On each we define a natural solution concept and the corresponding notion of stability index.

1.1 Basic notations

Throughout this paper we shall consider a finite set N the elements of which are called players or agents and a finite set A the elements of which are called alternatives or states. We make use of the following notational conventions : For any set D, we denote by $\mathcal{P}(D)$ the set of all subsets of D and by $\mathcal{P}_0(D) = \mathcal{P}(D)\backslash\{\emptyset\}$ the set of all non-empty subsets of D. Elements of $\mathcal{P}_0(N)$ are called *coalitions*. $N\backslash S$ is denoted S^c. Similarly if $B \in \mathcal{P}(A)$, $A\backslash B$ is denoted B^c. $L(A)$ will denote the set of all linear orders on A (that is all binary relations on A which are complete, transitive, and antisymmetric). If $R \in L(A)$, and $a, b \in A, a \neq b$, $a\ R\ b$ means that a is better than b in the linear order R. A preference profile (over A) is a map from N to $L(A)$, so that a preference profile is an element of $L(A)^N$. For every preference profile $R_N \in L(A)^N$ and $S \in \mathcal{P}_0(N)$ we put

$$P(a, S, R_N) = \{b \in A \mid\ b \neq a,\ b\ R^i\ a,\ \forall i \in S\}$$

(so that $P(a, S, R_N)$ consists of all the outcomes considered to be better than a by all members of the coalition S), and $P^c(a, S, R_N) = A\backslash P(a, S, R_N)$.

1.2 Simple games and the Nakamura number

A subset $\mathcal{W}$ of $\mathcal{P}_0(N)$ is called a *simple game* (Nakamura 1979). A coalition in $\mathcal{W}$ is a *winning coalition*, and a coalition not in $\mathcal{W}$ is a loosing coalition. Given some alternative set A, the action of $\mathcal{W}$ on A can be described as follows. Any coalition $S \in \mathcal{W}$ can react to any current state by imposing any other one. With this interpretation in mind, the following definition is in order. Let $R_N \in L(A)^N$. An alternative $a \in A$ is *dominated* at R_N if there exists $S \in \mathcal{W}$ such that $P(a, S, R_N) \neq \emptyset$. The *core* of $(\mathcal{W}, A)$ at R_N is the set of undominated alternatives. It is denoted $C(\mathcal{W}, A, R_N)$. $\mathcal{W}$ is *stable* on A if $C(\mathcal{W}, A, R_N) \neq \emptyset$ for all $R_N \in L(A)^N$.

Definition 1.1 A family $(S_1, \ldots, S_r)$ where $S_k \in \mathcal{W}$ $(k = 1 \ldots, r)$ has empty intersection if $\cap_{k=1}^{k=r} S_k = \emptyset$. The natural number r is the length of the family. The *Nakamura Number* of $\mathcal{W}$ denoted $\nu(\mathcal{W})$ is defined as the minimum length of all families with empty intersection. If $\mathcal{W}$ has no family with empty intersection then we set $\nu(\mathcal{W}) = +\infty$.

The following result was proved in Nakamura (1979) :

Theorem 1.2 *$\mathcal{W}$ is stable on A if and only if $|A| < \nu(\mathcal{W})$.*

The notion of family with empty intersection and the Nakamura number depend only on $\mathcal{W}$. We are going to introduce similar notions that reflect the action of $\mathcal{W}$ on A. The notion of cycle will be at the heart of the study of stability of far more general models. In this particular framework it takes a relatively simple form.

Definition 1.3 Let $\mathcal{W}$ be a simple game and let A be an alternative set. A family $(S_1, \ldots, S_r)$ where $S_k \in \mathcal{W}$ $(k = 1 \ldots, r)$ with empty intersection is a *cycle* in $(\mathcal{W}, A)$ if $r \leq |A|$. The *length* of the cycle is the number r. We sometimes refers to it as an r-cycle. The *stability index* of $(\mathcal{W}, A)$, denoted $\sigma_{(\mathcal{W},A)}$ is the minimal length of a cycle in $(\mathcal{W}, A)$. If no cycle exists in $(\mathcal{W}, A)$ then we set $\sigma_{(\mathcal{W},A)} = +\infty$.

It is easy to see that $\mathcal{W}$ is stable on A if and only if $(\mathcal{W}, A)$ has no cycle. Moreover we have the following:

$$\begin{aligned} \sigma_{(\mathcal{W},A)} &= \nu(\mathcal{W}) \ \text{ if } \ \nu(\mathcal{W}) \leq |A| \qquad (1) \\ &= +\infty \quad \text{ if } \ \nu(\mathcal{W}) > |A| \qquad (2) \end{aligned}$$

1.3 Effectivity Functions

In a simple game only coalitions in $\mathcal{W}$ have the power to oppose an alternative. Moreover this power is very sharp since a winning coalition can reach any alternative and a loosing coalition has no power at all. In order to describe an interaction where the repartition of power among coalitions is more general (for instance proportional) one can consider effectivity functions. In Moulin and Peleg (1982) this notion has been introduced in relation with strong implementation of social choice correspondences, in Abdou (1982) an equivalent notion was defined as a generalization of simple games and veto functions. One must view an effectivity function as an abstract version of a coalitional (i.e. cooperative) game.

Definition 1.4 An *Effectivity function* on (N, A) is a mapping $E : \mathcal{P}(N) \to \mathcal{P}(\mathcal{P}_0(A)$ such that :

(i) $E(\emptyset) = \emptyset$

(ii) $B \in E(S), B \subset B' \Rightarrow B' \in E(S)$

The interpretation of $B \in E(S)$ is as follows: whatever is the proposed or actual state, coalition S has the power to put the issue in B. The action that leads to B is not explicit in this model. Let $R_N \in L(A)^N$. An alternative $a \in A$ is *dominated* at R_N if there exists $S \in \mathcal{P}_0(N)$ such that $P(a, S, R_N) \in E(S)$. The *core* of E at R_N is the set of undominated alternatives. It is denoted $C(E, R_N)$. E is *stable* if $C(E, R_N) \neq \emptyset$ for all $R_N \in L(A)^N$. In order to study stability one is lead to the following:

Definition 1.5 An E- family is an r- tuple $((B_1, S_1), \ldots, (B_r, S_r))$ where $r \geq 1$, $S_k \in \mathcal{P}_0(N)$, $B_k \in E(S_k)$ $(k = 1, \ldots, r)$. An E- family is a *cycle* if there exists an r- tuple $(U_1, \ldots, U_r)$ where $U_k \in \mathcal{P}_0(A)$ $(k = 1, \ldots, r)$ such that : (i) $\cup_{k=1}^{r} U_k = A$,

(ii) For any $\emptyset \neq J \subset \{1, \ldots, r\}$ such that $\cap_{k \in J} S_k \neq \emptyset$, there exists $k \in J$ such that for all $l \in J$: $U_k \cap B_l = \emptyset$

$(U_1, \ldots, U_r)$ is the *basis* of the cycle and r its *length*. $E[\cdot]$ is *acyclic* if it has no cycle.

The length of a cycle is greater or equal to 2. Moreover any cycle with basis $(U_1, \ldots, U_r)$ gives rise to a shorter or an equal length cycle where the basis is a partition of A. Put $\tilde{U}_1 = U_1$ and $\tilde{U}_k = U_k \setminus \cup_{l=1}^{k-1} U_l$ $(2 \leq k \leq r)$, and remove the indices k corresponding to empty $\tilde{U}_k$. It follows that any cycle gives rise to a cycle of length not exceeding A.

The action of a simple game $\mathcal{W}$ on a set A can be represented canonically as an effectivity function. This is done as follows: If $S \in \mathcal{W}$ put $E(S) := P_0(A)$, if $S \notin W$ put $E(S) := \emptyset$. Clearly the core of E and the core of $\mathcal{W}$ coincide for every preference profile. Moreover to every cycle in $\mathcal{W}$ one can associate some cycle in E of the same length as follows: Let

$(S_1, \ldots, S_r)$ be a cycle in $(\mathcal{W}, A)$ and let $A = \{a_1, \ldots, a_p\}$, then we have a cycle $((B_1, S_1), \ldots, (B_r, S_r))$ in E with basis $(U_1, \ldots, U_r)$ if we put: $B_k = \{a_k\}$ $(k = 1, \ldots, r)$, $U_1 = \{a_k, \ldots, a_p\}$, $U_k = \{a_{k-1}\}$ $(k = 2, \ldots, r)$. Conversely any cycle $((B_1, S_1), \ldots, (B_r, S_r))$ in E with a partition basis $(U_1, \ldots, U_r)$ is such that $r \leq |A|$ and $\cap_{k=1}^r S_k = \emptyset$.

On the other hand given an effectivity function E, one can define the simple game induced by E, namely $\mathcal{W}$ is defined as the set of all coalitions S such that $E(S) = \mathcal{P}_0(A)$. Here we provide two examples of relatively easy cycles that appear in the study of stability of effectivity functions:

Example 1.6 (a) An an r- tuple $((B_1, S_1), \ldots, (B_r, S_r))$ where $r \geq 1$, $S_k \in \mathcal{P}_0(N)$, $B_k \in E(S_k)$ $(k = 1, \ldots, r)$, $B_k \cap B_l = \emptyset$ $(k \neq l)$ and $\cap_{k=1}^r S_k = \emptyset$, is a cycle. We have a basis by putting: $U_1 = A \setminus \cup_{k=2}^r U_k$ and $U_k = B_{k-1}, k = 2, \ldots, r$.

(b) An an r- tuple $((B_1, S_1), \ldots, (B_r, S_r))$ where $r \geq 1$, $S_k \in \mathcal{P}_0(N)$, $B_k \in E(S_k)$ $(k = 1, \ldots, r)$, $S_k \cap S_l = \emptyset$ $(k \neq l)$ and $\cap_{k=1}^r B_k = \emptyset$, is a cycle. By putting $U_k = B_k^c$ $(k = 1, \ldots, r)$ we have a basis.

A selection of $\mathcal{P}_0(\{1, \ldots, r\})$ is a map $\theta : \mathcal{P}_0(\{1, \ldots, r\}) \to \{1, \ldots, r\}$ such that $\theta(J) \in J$, for all $J \in \mathcal{P}_0(\{1, \ldots, r\})$. Let Σ_r be the set of all selections of $\mathcal{P}_0(\{1, \ldots, r\})$. To any E- family $((B_1, S_1), \ldots, (B_r, S_r))$, $\theta \in \Sigma_r$ and $k \in \{1, \ldots, r\}$ we associate

$$\mathcal{J}_k := \{J \in \mathcal{P}_0(\{1, \ldots, r\}) | \cap_{j \in J} S_j \neq \emptyset, \theta(J) = k\}$$

$$\mathcal{A}_k^\theta((B_1, S_1), \ldots, (B_r, S_r)) := \bigcup_{J \in \mathcal{J}_k} \bigcup_{j \in J} B_j$$

Proposition 1.7 *E is acyclic if and only if for any E- family $((B_1, S_1), \ldots, (B_r, S_r))$ and any $\theta \in \Sigma_r$, $\cap_{k=1}^r \mathcal{A}_k^\theta \neq \emptyset$.*

Proof. Let $((B_1, S_1), \ldots, (B_r, S_r))$ be a cycle with basis $(U_1, \ldots, U_r)$. We define $\theta \in \Sigma_r$ as follows: By property (ii) of definition 1.5, we put $\theta(J) = k$ if $\cap_{j \in J} S_j \neq \emptyset$ and $\theta(J) \in J$ arbitrarily if $\cap_{j \in J} S_j = \emptyset$. It follows that for all $J \in \mathcal{J}_k$, $U_k \cap (\cup_{j \in J} B_j) = \emptyset$ or equivalently: $U_k \cap \mathcal{A}_k^\theta = \emptyset$. Since $(U_1, \ldots, U_r)$ is a covering of A, we have that $\cap_{k=1}^r \mathcal{A}_k^\theta = \emptyset$. Conversely Assume that $\cap_{k=1}^r \mathcal{A}_k^\theta = \emptyset$. If for all $k = 1, \ldots, r$, $\mathcal{A}_k^\theta \neq A$, then it is easy to verify that the E-family $((B_1, S_1), \ldots, (B_r, S_r))$ is a cycle with basis $(A \setminus A_1^\theta, \ldots, A \setminus A_r^\theta)$. If some of the indices k are such that $\mathcal{A}_k^\theta = A$, then by removing them from the E- family and renumbering, we have again a cycle. □

Regarding stability, we have the following result which was first proved by Keiding (1985), see also Abdou and Keiding (1991) theorem 5.3 :

Theorem 1.8 *E is stable if and only if E is acyclic.*

Definition 1.9 The *stability index* of E, denoted $\sigma(E)$, is the minimal length of a cycle in E. This index is set to $+\infty$ if E is acyclic.

Going back to $(\mathcal{W}, A)$ and the associated effectivity function E one can easily see from the discussion preceding examples 1.6 that the stability index of $(\mathcal{W}, A)$ and E coincide. In general since one can consider only cycles based on partitions of A, it follows that either $2 \leq \sigma(E) \leq |A|$ or $\sigma(E) = +\infty$. In the rest of this section we study stability and index of a particularly interesting class of effectivity functions and for that purpose we need some definitions:

An effectivity function E is said to be:

monotonic w.r.t. players if for all $S, T \in \mathcal{P}_0(N)$,

$$S \subset T \Rightarrow E(S) \subset E(T), \tag{3}$$

regular if for all $S_1 \in \mathcal{P}_0(N), S_2 \in \mathcal{P}_0(N)$,

$$S_1 \cap S_2 = \emptyset,\ B_1 \in E(S_1),\ B_2 \in E(S_2) \Rightarrow B_1 \cap B_2 \neq \emptyset, \tag{4}$$

maximal if for all $S \in \mathcal{P}_0(N)$, $B \in \mathcal{P}_0(A)$,

$$B^c \notin E(S^c) \Longrightarrow B \in E(S), \tag{5}$$

superadditive if for all $S_1 \in \mathcal{P}_0(N), S_2 \in \mathcal{P}_0(N)$,

$$S_1 \cap S_2 = \emptyset,\ B_1 \in E(S_1),\ B_2 \in E(S_2) \Rightarrow B_1 \cap B_2 \in E(S_1 \cup S_2), \tag{6}$$

subadditive if for all $S_1 \in \mathcal{P}_0(N), S_2 \in \mathcal{P}_0(N)$,

$$B_1 \cap B_2 = \emptyset,\ B_1 \in E(S_1),\ B_2 \in E(S_2) \Rightarrow B_1 \cup B_2 \in E(S_1 \cap S_2). \tag{7}$$

Effectivity functions that are derived from strategic game forms (See section 3), and that play a role in the study of strong Nash solvavility satisfy some of these properties among which maximality is most important. In the case of maximal effectivity functions we have the following clear cut result that can be deduced from Abdou (1982) and Peleg (1984) (Theorem 6.A.9).

Theorem 1.10 *Assume that E is maximal. E is stable if and only if E is superadditive and subadditive.*

Moreover in case of unstability we have the following localization of the index:

Theorem 1.11 *(i) $\sigma(E) = 2$ if and only if E is not regular.*

(ii) Assume that E is maximal. Then $\sigma(E) \in \{2, 3, +\infty\}$.

Proof. (i) Let $((S_1, B_1), (S_2, B_2))$ be a 2-cycle with basis (U_1, U_2). It follows that $S_1 \cap S_2 = \emptyset$, $B_1 \subset U_1^c$ and $B_2 \subset U_2^c$ so that $B_1 \cap B_2 \subset U_1^c \cap U_2^c = \emptyset$. This contradicts regularity. Conversely if $((S_1, B_1), (S_2, B_2))$ is an E -familly such that $S_1 \cap S_2 = \emptyset$ and $B_1 \cap B_2 = \emptyset$ then (B_2^c, B_1^c) is a basis for a cycle.

(ii) In view of Theorem 1.10, a maximal effectivity function is stable if and only if it is subadditive and superadditive. If E is not superadditive then there exists $S_1, S_2 \in \mathcal{P}_0(N)$, $B_1, B_2 \in \mathcal{P}_0(A)$ such that $S_1 \cap S_2 = \emptyset$, $B_1 \in E(S_1)$, $B_2 \in E(S_2)$ and $B_1 \cap B_2 \notin E(S_1 \cup S_2)$. Put $S_3 = (S_1 \cup S_2)^c$, $B_3 = (B_1 \cap B_2)^c$. By maximality $B_3 \in S_3$ so that $((S_1, B_1), (S_2, B_2), (S_3, B_3))$ is a cycle with basis (B_1^c, B_2^c, B_3^c). A similar proof can be done if E is not subadditive. Therefore $\sigma(E) \leq 3$. □

1.4 Local effectivity functions

The distribution of power described in an effectivity function is global. A coalition is effective to some subsets of alternatives or not. In the study of game form solvability it is useful to introduce the idea that the power of a coalition depends on the starting point or the actual position. This is the reason why it was first introduced in Abdou (1995), as an object related to a game form. In any case, when modeling political or economical interaction, it is highly realistic that acting (or reacting) power of the agents depends on the current situation. This is why we introduce the following:

Definition 1.12 *A local effectivity function is a family* $(E[U], U \in \mathcal{P}_0(A))$ *where for any* $U \in \mathcal{P}_0(A)$, $E[U] : \mathcal{P}(N) \to \mathcal{P}(\mathcal{P}_0(A))$ *the following conditions are satisfied:*

(i) $E[U](\emptyset) = \emptyset$

(ii) $B \in E[U](S), B \subset B' \Rightarrow B' \in E[U](S)$

(iii) $U \subset V \Rightarrow E[V](S) \subset E[U](S)$

The formula $B \in E[U](S)$ can be interpreted as follows: When the current state is in U, coalition S can adapt its response in order to realize some state in B. Let $R_N \in L(A)^N$. An alternative $a \in A$ is *dominated* at R_N if there exists $U \in \mathcal{P}_0(A)$, $S \in \mathcal{P}_0(N)$ such that $a \in U$ and $P(a, S, R_N) \in E[U](S)$. The *core* of $E[\cdot]$ at R_N is the set of undominated alternatives. It is denoted $C(E[\cdot], R_N)$. $E[\cdot]$ is *stable* if $C(E[\cdot], R_N) \neq \emptyset$ for all $R_N \in L(A)^N$.

Definition 1.13 An $E[\cdot]$- family is a r- tuple $((U_1 B_1, S_1), \ldots, (U_r, B_r, S_r))$ where $r \geq 1$, $S_k \in \mathcal{P}_0(N)$, $U_k \in \mathcal{P}_0(A)$, $B_k \in E[U_k](S_k)$ $(k = 1, \ldots, r)$. A *cycle* in $E[\cdot]$ is an $E[\cdot]$- family such that: (i) $\cup_{k=1}^r U_k = A$,

(ii) For any $\emptyset \neq J \subset \{1, \ldots, r\}$ such that $\cap_{k \in J} S_k \neq \emptyset$ there exists $k \in J$ such that for all $l \in J$, $U_k \cap B_l = \emptyset$

$(U_1, \ldots, U_r)$ is the *basis* of the cycle and r its *length.* $E[\cdot]$ is *acyclic* if it has no cycle. The *stability index* of $E[\cdot]$, denoted $\sigma(E[\cdot])$, is the minimal length of a cycle in $E[\cdot]$. This index is set to $+\infty$ if $E[\cdot]$ is acyclic.

Example 1.14 A *local simple game* on (N, A) is defined as a collection $(\mathcal{W}_a, a \in A)$ where $\mathcal{W}_a \in \mathcal{P}_0(N)$, $\emptyset \notin \mathcal{W}_a$ $(a \in A)$. For $U \in \mathcal{P}_0(A)$, put $\mathcal{W}[U] := \cap_{a \in U} \mathcal{W}_a$. Define a cycle for $(\mathcal{W}_a, a \in A)$ of length r as an $2r$-tuple $(U_1, S_1, \ldots, U_r, S_r)$ where $U_k \in \mathcal{P}_0(A)$, $S_k \in W[U_k]$ $(k = 1, \ldots, r)$, $\cup_{k=1}^r U_k = A$ and $\cap S_{k=1}^r = \emptyset$. A local effectivity function that reflects the same power distribution is defined by $E[U](S) = \mathcal{P}_0(A)$ if $S \in \mathcal{W}[U]$ and $E[U](S) = \emptyset$ if $S \notin \mathcal{W}[U]$. One can prove that any cycle in $(\mathcal{W}_a, a \in A)$ gives rise to a cycle in E and conversely.

Using the notations of subsection 1.3 we have a characterization of cycles a local effectivity functions, where, for simplicity, $\mathcal{A}_k^\theta((S_1, B_1), \ldots, (S_r, B_r))$ is denoted $\mathcal{A}_k^\theta$.

Proposition 1.15 *An $E[\cdot]$- family $((U_1 B_1, S_1), \ldots, (U_r, B_r, S_r))$ is a cycle if and only if $\cup_{k=1}^r U_k = A$ and if there exists $\theta \in \Sigma_r$ such that $U_k \cap \mathcal{A}_k^\theta = \emptyset$ $(k = 1, \ldots, r)$.*

The proof is an easy adaptation of that of proposition 1.7. Moreover we can state the following:

Theorem 1.16 *$E[\cdot]$ is stable if and only it is acyclic.*

The proof of a far more general result will be given in Theorem 2.5. Given an effectivity function E one can define "canonically" a local effectivity function that reflects the power distribution of E by positing, for $U \in \mathcal{P}_0(A)$, $S \in \mathcal{P}_0(N)$, $E[U](S) := E(S)$. It is clear that the core corespondences of $E[\cdot]$ and E coincide and that cycles in E and $E[\cdot]$ are the same. Conversely, starting from $E[\cdot]$, one can "extract", consistently with our interpretation, an effectivity function E_0 as follows: for $S \in \mathcal{P}_0(N)$ put $E_0(S) := E[A](S)$. Besides E_0, there is a second mapping $E_\xi : \mathcal{P}(N) \to \mathcal{P}(\mathcal{P}_0(A))$ that plays a special role in the study of stability of $E[\cdot]$. It is defined by $E_\xi(\emptyset) = \emptyset$ and for $S \in \mathcal{P}_0(S)$, by :

$$E_\xi(S) := \{B \in \mathcal{P}_0(A) \mid B = A \,\text{or}\, \exists a \notin B, B \in E[a](S)\} \tag{8}$$

Note that properties (i) and (ii) of definition 1.4 are satisfied by E_0, whereas E_ξ does not necessarily satisfy property (ii) (monotonicity w.r.t. alternatives). E_0 is called the (global) effectivity function, and E_ξ is called the *exact* "effectivity" function induced by the local effectivity function $E[\cdot]$. Clearly $C(E[\cdot], R_N) \subset C(E_0, R_N)$ for all $R_N \in L(A)^N$ and any cycle in E_0 is a cycle in $E[\cdot]$. Moreover one has:

Lemma 1.17 *If $E_\xi = E_0$ then:*

(i) $C(E_0, R_N) = C(E[\cdot], R_N)$ for all $R_N \in L(A)^N$,

(ii) $E[\cdot]$ and E_0 have the same cycles.

Proof. If a is dominated in $E[\cdot]$ at R_N, then for some $S \in \mathcal{P}_0(N)$ $P(a, S, R_N) \in E[a](S)$. Since $a \notin P(a, S, R_N)$, one has $P(a, S, R_N) \in E_\xi(S) = E_0(S)$ so that a is dominated in E_0. Likewise if $((U_1 B_1, S_1), \ldots, (U_r, B_r, S_r))$ is a cycle in $E[\cdot]$, then in particular $U_k \cap B_k = \emptyset$, $U_k \neq \emptyset$ and since $B_k \in E[U_k](S_k)$ it follows that $B_k \in E_\xi(S_k) = E_0(S_k)$, so that this is also a cycle in E_0.

Lemma 1.18 *Assume that E_0 is maximal and $E_\xi \neq E_0$. Then:*

(i) then there exists a cycle of length at most 3 in $E[\cdot]$,that is not a cycle in E_0

(ii) there exists $R_N \in L(A)^N$ such that $C(E[\cdot], R_N) = \emptyset$, if further E_0 is monotonic w.r.t. players then also $|C(E_0, R_N)| = 1$.

Proof. (i) If $B \in E_\xi(S)$ and $B \notin E_0(S)$, then there exists $a \in B^c$ such that $B \in E[a](S)$ and $B^c \in E_0(S^c)$. Put $S_1 = S, S_2 = S^c, S_3 = N$, $B_1 = B, B_2 = B^c, B_3 = \{a\}$, $U_1 = \{a\}, U_2 = B, U_3 = B^c \setminus \{a\}$. In $E[\cdot]$ this defines a $3-$cycle if $U_3 \neq \emptyset$, and a 2 -cycle if $U_3 = \emptyset$, that is not a cycle in E_0.

(ii) Define a profile R_N such that: for $i \in S$, $B\ R^i\ \{a\}\ R^i\ B^c \setminus \{a\}$ and for $i \in S^c$, $\{a\}\ R^i\ B^c \setminus \{a\}\ R^i\ B$. If $b \in B$ then $P(b, S^c, R_N) \supset B^c \in E_0(S^c)$, so that b is dominated in E_0. If $b \in B \setminus \{a\}$, then $P(b, N, R_N) \supset \{a\} \in E_0(N)$. Indeed by maximality of E_0, $E_0 = \mathcal{P}_0(A)$. Therefore one has $C(E_0, R_N) \subset \{a\}$. Now $P(a, S, R_N) = B \in E[a](S)$ implies that a is dominated in $E[\cdot]$, and since $C(E[\cdot], R_N) \subset C(E_0, R_N)$ we have $C(E[\cdot], R_N) = \emptyset$. If moreover E_0 is monotonic w.r.t. players , then for $T \in \mathcal{P}_0(N)$, $T \subset S$ we have $P(a, T, R_N) = B \notin E_0(S)$ and for $T \cap S^c \neq \emptyset$ we have $P(a, T, R_N) = \emptyset$ so that a is not dominated in E_0. □

As an immediate consequence of assertion (ii) of lemma 1.18 we have the following:

Proposition 1.19 *(i) Assume that E_0 is maximal and monotonic w.r.t. players, then $E_\xi = E_0$ if and only if $C(E_0, R_N) = C(E[\cdot], R_N)$ for all $R_N \in L(A)^N$.*

(ii) Assume that E_0 is maximal then $E_\xi = E_0$ if and only if E_0 and $E[\cdot]$ have the same cycles.

We summarize the main results concerning stability in the following:

Theorem 1.20 *(i) Assume that E_0 is maximal. Then $E[\cdot]$ is stable if and only if E_0 is stable and $E_\xi = E_0$. Moreover in that case $C(E_0, R_N) = C(E[\cdot], R_N)$ for all $R_N \in L(A)^N$.*

Proof. If $E[\cdot]$ is stable then E_0 is stable and by the first part of assertion (ii) of lemma 1.18 one has $E_\xi = E_0$. Conversely if $E_\xi = E_0$, then $C(E_0, R_N) = C(E[\cdot], R_N)$ for all $R_N \in L(A)^N$ by lemma 1.17 assertion (i) and since E_0 is stable it follows that $E[\cdot]$ is stable. □

In case of instability one can obtain a localization of the index. For that purpose we assert the following:

Theorem 1.21 *(i) If E_0 is not regular then $\sigma(E[\cdot]) = 2$.*

(ii) Assume that E_0 is maximal. Then $\sigma(E[\cdot]) \in \{2, 3, +\infty\}$

Proof. If $E[\cdot]$ is not stable and E_0 stable, then by theorem 1.20 $E_\xi \neq E_0$ and by lemma 1.18 assertion (i) $E[\cdot]$ has a cycle of length ≤ 3, so that $\sigma(E[\cdot]) \leq 3$. If E_0 is not stable then by theorem 1.11 $\sigma(E_0) \leq 3$ and since $\sigma(E[\cdot]) \leq \sigma(E_0)$ we conclude again that $\sigma(E[\cdot]) \leq 3$. If E_0 is not regular then $\sigma(E[\cdot]) \leq \sigma(E_0) = 2$ so that $\sigma(E[\cdot]) = 2$.

2 Interactive forms

In this section we present a general model of interaction. The elements of A are viewed as social situations or states. At any state $a \in A$ we dispose of a description of the acting power of the agents in the society. The acting power which depends generally on a is represented by a set of interaction arrays. Thus if the state of the society is a, some individuals or coalitions can move or threat to move to other states upsetting therefore the state a. Thus power is described as a multipolar force that can be used to upset a status quo. Formally we define the following:

An *interaction array* on (N, A) is a mapping $\varphi : \mathcal{P}_0(N) \to \mathcal{P}(A)$. Let $\Phi = \Phi(N, A)$ be the set of all interaction arrays. We endow $\Phi(N, A)$ with the partial order $\leq$ where $\varphi \leq \varphi'$ if and only if $\varphi(S) \subset \varphi'(S)$ for all $S \in \mathcal{P}(N)$. By *confederation* $\mathcal{S}$ we mean a non empty subset of $\mathcal{P}_0(N)$. The *support* of φ denoted $[\varphi]$ is the confederation formed by all coalitions $S \in \mathcal{P}_0(N)$ such that $\varphi(S) \neq \emptyset$. We denote by $\Phi_0(N, A)$ the subset of interaction arrays with non empty support.

Definition 2.1 An *interaction form* over (N, A) is a mapping $\mathcal{E}$ from $\mathcal{P}_0(A)$ to subsets of $\Phi_0(N, A)$ satisfying the following conditions:
(i) $\varphi \leq \varphi', \varphi \in \mathcal{E}[U] \Rightarrow \varphi' \in \mathcal{E}[U]$
(ii)$U \subset V \Rightarrow \mathcal{E}[V] \subset \mathcal{E}[U]$

We may think of an interaction array in $\mathcal{E}[U]$ as a description of an availlable move of the agents given any state in U. To interpret the statement $\varphi \in \mathcal{E}(\{a\})$, one has to imagine that a may occur in different scenarios that are not directly explicited in the model; any scenario leading to state a

arouses some coalition S that objects by threatening to drive the outcome into $\varphi(S)$. The set of all such coalitions is the active confederation of φ. The interaction array is the result of a *disjunctive* move of the coalitions, that is the surge of some objecting coalition S is not concomitant to that of another coalition. Within a coalition action is coordinated, not within a confederation. Rather when some confederation becomes active at a this activation must be understood as a *collusion* of interests between its components. Our model is universal in the sense that we allow *a priori* all coalitions to react to some state in U. Nevertheless, the fact that $\varphi(S) = \emptyset$ for some S means that coalition S is inhibited or desactivated and therefore that the power represented by φ holds without the participation of S. Therefore the support of φ is in fact the *active* confederation behind φ. Whether coalitions in the confederation have a real interest to make their move depends on the actual preferences. This is why we introduce the following:

Definition 2.2 An alternative a is *dominated* at the preference profile R_N if there exists some $U \in \mathcal{P}(A), U \ni a$, and some $\varphi \in \mathcal{E}(U)$ such that $\varphi(S) \subset P(a, S, R_N)$ for all $S \in \mathcal{P}_0(N)$. The alternative a is a *settlement* at R_N if it is not dominated at R_N. The set of all settlements at R_N will be denoted: $SET(\mathcal{E}, R_N)$.

It follows from the definition that the absence of a settlement at some preference profile can be expressed as an impasse, a deadlock or a stalemate. Stability, therefore,is a highly desirable property for an interaction form. The following subsection is devoted to the conditions under which that property is met.

2.1 Stability and acyclicity

Let $\mathcal{E}$ be an interaction form.

Definition 2.3 An $\mathcal{E}$ - *family* is any r-tuple $((U_1, \varphi_1), \ldots, (U_r, \varphi_r))$ where: $U_k \in \mathcal{P}_0(A)$, $\varphi_k \in \mathcal{E}[U_k]$ $(k = 1, \ldots, r)$. An $\mathcal{E}$ -family is a *cycle* in $\mathcal{E}$ if it satisfies : (i) $\cup_{k=1}^{r} U_k = A$,
(ii) For any $i \in N$ and $\emptyset \neq J \subset \{1, \ldots, r\}$ there exists $k \in J$ such that for all $l \in J$, $S \ni i$ one has : $U_k \cap \varphi_l(S) = \emptyset$.

The covering $(U_1, \ldots, U_r)$ will be called the *basis* of the cycle. The natural number r is the *length* of the cycle. Such a cycle will be called an r- cycle. The interaction form $\mathcal{E}$ is said to be *acyclic* if there are no cycles (strict cycles) in $\mathcal{E}$.

In order to state a characterization of cycles we recall, from subsection 1.3, that Σ_r is the set of all selections of $\mathcal{P}_0(\{1, \ldots, r\})$. To any

$(\varphi_1, \ldots, \varphi_r) \in \Phi(N, A)^r$, $\theta \equiv (\theta^1, \ldots, \theta^n) \in (\Sigma_r)^n$ and $k \in \{1, \ldots, r\}$ we associate:

$$\mathcal{J}_k^i := \{J \in \mathcal{P}_0(\{1, \ldots, r\}) | \theta^i(J) = k\}$$

$$\mathcal{A}_k^\theta(\varphi_1, \ldots, \varphi_r) := \bigcup_{i=1}^n \bigcup_{J \in \mathcal{J}_k^i} \bigcup_{l \in J} \bigcup_{S \ni i} \varphi_l(S)$$

For simplicity, in what follows $\mathcal{A}_k^\theta(\varphi_1, \ldots, \varphi_r)$ is denoted $\mathcal{A}_k^\theta$:

Proposition 2.4 *An $\mathcal{E}$- family $((U_1, \varphi_1), \ldots, (U_r, \varphi_r))$ is a cycle if and only if $\cup_{k=1}^r U_k = A$ and if there exists $\theta = (\theta^1, \ldots, \theta^n) \in (\Sigma_r)^n$ such that for all $k \in \{1, \ldots, r\}$ one has : $U_k \cap \mathcal{A}_k^\theta = \emptyset$.*

Proof. Let $((U_1, \varphi_1), \ldots, (U_r, \varphi_r))$ be a cycle with basis $(U_1, \ldots, U_r)$. Let $i \in N$. We define $\theta^i \in \Sigma_r$ as follows: By property (ii) of definition 2.3, we put $\theta^i(J) = k$ if for all $l \in J$, $S \ni i$ we have $U_k \cap \varphi_l(S) = \emptyset$. It follows that $U_k \cap \mathcal{A}_k^\theta = \emptyset$ $(k = 1 \ldots, r)$. Conversely if the latter relations hold then property (ii) of definition 2.3 is clearly satisfied. □

Here is the main result on stability of interaction forms. It generalizes similar results on effectivity functions (Theorem 1.8) and local effectivity functions (Theorem 1.16). The proof that we present is an adaptation of a similar one that appeared in Abdou and Keiding (2003).

Theorem 2.5 *$\mathcal{E}$ is stable if and only if $\mathcal{E}$ is acyclic.*

Proof. We first prove that the existence of a cycle in $\mathcal{E}$ entails the existence of a cycle the basis of which is a partition of A. Let $((U_1, \varphi_1) \ldots (U_r, \varphi_r))$ be a cycle with minimal length. If we put $\tilde{U}_1 = U_1$, $\tilde{U}_j = U_j \backslash (U^1 \cup \cdots \cup U_{j-1})$ for $2 \leq j \leq r$, then replacing U^j by $\tilde{U}^j$ results in a cycle with the desired property.

For $\varphi \in \Phi(N, A)$ and $i \in N$ we put $R^i(\varphi) = \cup_{S \ni i} \varphi(S)$. Assume that $((U_1, \varphi_1), \ldots, (U_r, \varphi_r))$ is a cycle and that its basis is a partition of A. We shall exhibit a preference profile $Q_N \in L(A)^N$ such that all alternatives are dominated in $\mathcal{E}$ at Q_N. Let $i \in N$ be fixed. We define a permutation $(k_1^i, \ldots, k_r^i)$ of $\{1, \ldots, r\}$ as follows : By property (ii) of the cycle there exists $k_1^i \in \{1, \ldots, r\}$ such that $U_{k_1^i} \cap R^i(\varphi_l) = \emptyset$ for all indices $l \in \{1, \ldots, r\}$, and by induction, there exists $k_s^i \in \{1, \ldots, r\} \backslash \{k_1^i, \ldots, k_{s-1}^i\}$, $s = 2, \ldots, r$ such that $U_{k_s^i} \cap R^i(\varphi_l) = \emptyset$ for $l \in \{1, \ldots, r\} \backslash \{k_1^i, \ldots, k_{s-1}^i\}$. We define a preference relation $Q_i \in L(A)$ such that $x\ Q_i\ y$ if $x \in U_{k_s^i}$, $y \in U_{k_t^i}$, and $s > t$, and such that the restriction of Q_i to each U_k is arbitrary. This relation is well defined since $(U_1, \ldots, U_r)$ is a partition of A. Let $a \in A$ and let $k \in \{1, \ldots, r\}$ such that $a \in U_k$, then by construction $k = k_t^i$ for some t, $R^i(\varphi_k) \cap U_{k_s^i} = \emptyset$ for $s = 1, \ldots, t$. It follows that $R^i(\varphi_k)$ is a subset of $\bigcup_{s=t+1}^r U_{k_s^i}$ (if $s = r$ this set is empty !). By construction of Q_i the latter set is included in $P(a, \{i\}, Q_N)$. Therefore $R^i(\varphi_k) \subset P(a, \{i\}, Q_N)$ and for all $S \neq \emptyset$:

$$\varphi_k(S) \subset \bigcap_{i \in S} R^i(\varphi_k) \subset \bigcap_{i \in S} P(a, \{i\}, Q_N) = P(a, S, Q_N)$$

Since $a \in U_k$ and $\varphi_k \in \mathcal{E}[U_k]$ it follows that a is dominated at Q_N.

Now we prove the converse. Assume that $\mathcal{E}$ is not stable, then there is a preference profile $Q_N \in L(A)^N$ at which all alternatives in A are dominated in $\mathcal{E}$. For $a \in A$ let $\varphi_a \in \Phi$ be defined by $\varphi_a(S) = P(a, S, Q_N)$ for all $S \neq \emptyset$. There exists $U_a \in \mathcal{P}_0(A)$ such that $a \in U_a$ and $\varphi_a \in \mathcal{E}[U_a]$. One can take $U_a = \{a\}$ (property (iii) of definition 2.1). Let $A = \{a_1 \ldots, a_p\}$. We put $U_k = \{a_k\}$ $(k = 1, \ldots, p)$ $\varphi_k = \varphi_{a_k}$ Then $((U_k, \varphi_k), (k = 1, \ldots, p))$ is a cycle. Let $i \in N$ and let $\emptyset \neq J \subset \{1, \ldots, p\}$. Choose $k \in J$ such that $a_l \; R_i \; a_k$ for all $l \in J$. Then $\{a_k\} \cap P(a_l, i, Q_N) = \emptyset$ for all $l \in J$, so that $\{a_k\} \cap \varphi_l(S) = \emptyset$ for all $S \ni i$. This is precisely condition (ii) of definition 2.3. □

2.2 Stability index of interaction forms

In order to obtain a measure of unstability when the latter occurs, we introduce:

Definition 2.6 The *stability index* of $\mathcal{E}$, denoted $\sigma(\mathcal{E})$, is the minimal length of a cycle in $\mathcal{E}$. This number is set to $+\infty$ if $\mathcal{E}$ is acyclic.

Let $f : A \to A'$ be a map. If $\varphi' \in \Phi(N, A')$ we denote $f^{-1} \circ \varphi'$, the element φ of $\Phi(N, A)$ defined by $\varphi(S) = (f^{-1} \circ \varphi')(S)$ for all $S \in \mathcal{P}_0(N)$. For any interaction form $\mathcal{E}$ on (N, A) we define the interaction form $\mathcal{E}^f$ on (N, A') as follows: For $U' \in \mathcal{P}_0(A')$:

$$\mathcal{E}^f[U'] = \{\varphi' \in \Phi(N, A') |\, f^{-1} \circ \varphi' \in \mathcal{E}[f^{-1}(U')]\}$$

The $\mathcal{E}^f$- family $((U'_1, \varphi'_1), \ldots, (U'_r, \varphi'_r))$, is a cycle of $\mathcal{E}^f$ if and only if the $\mathcal{E}$- family $((f^{-1}(U'_1), f^{-1} \circ \varphi'_1), \ldots, (f^{-1}(U'_r), f^{-1} \circ \varphi'_r))$ is a cycle. It follows that if $\mathcal{E}$ is acyclic then $\mathcal{E}^f$ is acyclic.

Let $((U_1, \varphi_1), \ldots, (U_r, \varphi_r))$ be a cycle of $\mathcal{E}$ based on the partition $(U_1, \ldots, U_r)$. Let A' be some set with r elements $A' := \{u_1, \ldots, u_r\}$ and let $f : A \to A'$ be defined by $f(a) = u_k$ for $a \in U_k$. Define $\varphi' \in \Phi(N, A')$ by putting $\varphi'_k(S) := f(\varphi_k(S))$ $(S \in \mathcal{P}_0(N))$. For any $S \in \mathcal{P}_0(N)$ and $k, l \in \{1, \ldots, r\}$ one has $U_k \cap \varphi_l(S) = \emptyset$ if and only if $\{u_k\} \cap f(\varphi_l(S)) = \emptyset$. It follows that $(\{(u_1\}, \varphi'_1), \ldots, (\{u_r\}, \varphi'_r))$ is a cycle of $\mathcal{E}^f$ based on the partition $(\{u_1\}, \ldots, \{u_r\})$. Therefore we have the following characterization of the index:

Theorem 2.7 *The index of an unstable interaction form $\mathcal{E}$ is the integer σ such that:*

(i) If $|A'| \geq \sigma$ there exists a mapping $f : A \to A'$ such that $\mathcal{E}^f$ is unstable.

(ii) If $|A'| < \sigma$ then for any mapping $f : A \to A'$, $\mathcal{E}^f$ is stable

This characterization provides an interpretation of the stability index. Assume that an interaction form is unstable with a stability index σ, then merging some social states (or alternatives) results in a decrease of the number of alternatives and a transformation of the interaction form in a way that respects power distribution. This is the interpretation of the transformation $\mathcal{E} \to \mathcal{E}^f$. This transformation may occur, for instance, when the agents do not distinguish any more between two previously different alternatives. If the number of the new alternatives is less than σ then the new interaction form will be stable. If $\sigma = 2$ alternatives can be partitioned into two aggregates, or two major issues, over which the society can be opposed, and the power of agents or institutions allowed by the rules is such that either issue can be opposed and neither can be forced.

2.3 Projections and restrictions

As explained above, since an interaction array is defined on all coalitions, the model allows *a priori* the surge of any confederation. Now it may be the case that institutionnally (by law or structural impossibilities) some coalitions are not allowed to form, that is only coalitions in some $\mathcal{S} \subset \mathcal{P}_0(N)$ can actually be active. For instance in a legislature (Senate, House of representatives) only some coalitions are practically possible. The definitions can be adapted in consequence. An alternative a is $\mathcal{S}$-*dominated* at the preference profile R_N if there exists some $U \in \mathcal{P}_0(A), U \ni a$, and some $\varphi \in \mathcal{E}(U)$ such that $[\varphi] \subset \mathcal{S}$, $\varphi(S) \subset P(a, S, R_N)$ for all $S \in \mathcal{S}$. The alternative a is an $\mathcal{S}$ -*settlement* at R_N if it is not $\mathcal{S}$-dominated at R_N. The set of all $\mathcal{S}$-settlements at R_N will be denoted: $SET_{\mathcal{S}}(\mathcal{E}, R_N)$. In order to cope with this situation without changing our model, we would like to define an interaction form that reflects the activity of confederation $\mathcal{S}$. This is precisely the role of an operation called projection.

The *projection* of $\mathcal{E}$ on $\mathcal{S}$ is the interaction form $\mathcal{E}(\mathcal{S})$ defined by :

$$\mathcal{E}(\mathcal{S})[U] := \{\varphi \in \Phi \mid \exists \varphi' \in \mathcal{E}[U], [\varphi'] \subset \mathcal{S}, \varphi' \leq \varphi\} \tag{9}$$

Clearly one has: $SET_{\mathcal{S}}(\mathcal{E}, R_N) = SET(\mathcal{E}(\mathcal{S}), R_N)$. Moreover if $\mathcal{T} \subset \mathcal{S}$ is another confederation one has $\mathcal{E}(\mathcal{T}) = \mathcal{E}(\mathcal{S})(\mathcal{T})$.

As explained earlier an interaction array represents a disjunctive and uncoordinated action of coalitions now limited to those belonging to $\mathcal{S}$. Now consider the case of a legislative body composed of two Chambers. If a, a confidence motion for instance, is disrupted by a confederation, say $\mathcal{S}$, in Chamber 1, then a is discarded. If a passes Chamber 1 unopposed then it has to be presented to Chamber 2, where the sovereign confedation is $\mathcal{T}$. Any nonempty set $\mathfrak{F}$ of confederations will be called a *confederation structure* (Example the Congress : Senate and House of representatives, where some proposal is submitted successively to both legislatures). In a confederation structure, each confederation acts independently of other confederations.

The notions of $\mathfrak{F}$- settlement and $\mathfrak{F}$- stability are defined in consequence: An alternative a is $\mathfrak{F}$-*dominated* at the preference profile R_N if there exists some $\mathcal{S} \in \mathfrak{F}$ and some $U \in \mathcal{P}_0(A), U \ni a$ and $\varphi \in \mathcal{E}(\mathcal{S})[U]$ such that $[\varphi] \subset \mathcal{S}$, $\varphi(S) \subset P(a, S, R_N)$ for all $S \in \mathcal{S}$. The alternative a is an $\mathfrak{F}$ -*settlement* at R_N if it is not $\mathfrak{F}$-dominated at R_N. The set of $\mathfrak{F}$ -settlements at R_N will be denoted: $SET_{\mathfrak{F}}(\mathcal{E}, R_N)$. The *restriction* of $\mathcal{E}$ to $\mathfrak{F}$ is defined by:

$$\mathcal{E}_{\mathfrak{F}}[U] := \bigcup_{\mathcal{S} \in \mathfrak{F}} \mathcal{E}(\mathcal{S})[U] \tag{10}$$

It is clear from the definition that: $SET_{\mathfrak{F}}(\mathcal{E}, R_N) = SET(\mathcal{E}_{\mathfrak{F}}, R_N) = \cap_{\mathcal{S} \in \mathfrak{F}} SET(\mathcal{E}(\mathcal{S}), R_N)$. Moreover if $\mathfrak{G} \subset \mathfrak{F}$ then $\mathcal{E}_{\mathfrak{G}} = (\mathcal{E}_{\mathfrak{F}})_{\mathfrak{G}}$.

Example 2.8 a) Let $\mathcal{S} = \mathcal{N} = \{\{i\} | i \in N\}$. $\mathcal{N}$ is the confederation corresponding to the situation where only individuals have some power. A settlement for $\mathcal{N}$ is similar to a Nash equilibrium outcome.

a) Let $\mathcal{S} = \mathcal{P}_0(N)$. $\mathcal{S}$ is the confederation corresponding to the situation where all coalitions have some power. A settlement for $\mathcal{S}$ is similar to a strong Nash equilibrium outcome.

c) Let $\mathfrak{F} \equiv \mathfrak{P}_1 = \{\{S\} \mid S \in \mathcal{P}_0(N)\}$. $\mathfrak{P}_1$ is a confederation structure, where every confederation is a single coalition. A settlement in this case is similar to an element of the local core (section 1.4).

One of the advantages of our present model of interaction forms compared to the one proposed in Abdou and Keiding (2003) is that it allows for projections and restrictions in a way that will prove relevant in the case of interaction forms derived from strategic game forms. We shall see that using the interaction form drawn from a strategic game form we can compute strong Nash equilibria outcomes ($\mathcal{S} = \mathcal{P}_0(N)$). By projection to the confederation $\mathcal{N}$ we can compute Nash outcomes. By restriction to the confederation structure $\mathfrak{P}_1$ we have the exact core.

2.4 Induced effectivity

Now we show how interaction forms generalize local effectivity functions and effectivity functions, and how conversely they induce them. To any local effectivity function $E[\cdot]$ (definition 1.12), one can associate "canonically" an interaction form as follows:

$$\mathcal{E}[U] = \{\varphi \in \Phi | \exists S \in \mathcal{P}_0(N) : \varphi(S) \in E[U](S)\} \tag{11}$$

Reciprocally to any interaction form $\mathcal{E}$, the restriction of $\mathcal{E}$ to the confederation structure $\mathfrak{P}_1 = \{\{S\} \mid S \in \mathcal{P}(N)\}$, denoted $\mathcal{E}_1$ "is" the local effectivity function induced by $\mathcal{E}$. In fact we shall put for $S \in \mathcal{P}_0(N)$ and $U \in \mathcal{P}_0(A)$:

$$E_1[U](S) = \{B \in \mathcal{P}_0(A) \mid \exists \varphi \in \mathcal{E}[U], [\varphi] = \{S\}, \varphi(S) = B\} \tag{12}$$

and $E_1[U](\emptyset) = \emptyset$. One has:

$$\mathcal{E}_1[U] = \{\varphi \in \Phi | \exists S \in \mathcal{P}_0(N) : \varphi(S) \in E_1[U](S)\} \tag{13}$$

The induced (global) effectivity function is defined by:

$$\begin{aligned} E_0(S) &= E_1[A](S) \\ &= \{B \in \mathcal{P}_0(A) \mid \exists \varphi \in \mathcal{E}_1[A], [\varphi] = \{S\}, \varphi(S) = B\} \\ &= \{B \in \mathcal{P}_0(A) \mid \exists \varphi \in \mathcal{E}[A], [\varphi] = \{S\}, \varphi(S) = B\} \end{aligned} \tag{14}$$

and the corresponding interaction form is denoted $\mathcal{E}_0$, where for any $U \in \mathcal{P}_0(A)$:

$$\mathcal{E}_0[U] = \mathcal{E}_1[A] \tag{15}$$

$$= \{\varphi \in \Phi | \exists S \in \mathcal{P}_0(N) : \varphi(S) \in E_0(S)\} \tag{16}$$

It is easy to see that, for all profiles R_N, on has: $SET(\mathcal{E}_1, R_N) = C(E_1[\cdot], R_N)$ and $SET(\mathcal{E}_0, R_N) = C(E_0, R_N)$. Furthermore, one can show that to any r-cycle in $E_1[\cdot]$ one can associate an r-cycle in $\mathcal{E}_1$ and conversely. It follows that the definitions 1.13 and and 2.6 of index are consistent.

2.5 Localization of the index and exactness

In this subsection we shall compute or at least localize the stability index of a given interaction form. For that purpose we devise some structural properties on interaction forms. Some of those properties can be described using the global effectivity function E_0, others need higher level restrictions of the interaction form. We recall that a subset of coalitons $\mathcal{S} \subset \mathcal{P}_0(N)$ is called a confederation. A confederation $\mathcal{S}$ is said to be *admissible* when $|\mathcal{S}| = 1$ or when $|\mathcal{S}| \geq 2$, $N \notin \mathcal{S}$ and for all $S, T \in \mathcal{S}$, $S \neq T$ we have $S \cup T = N$. Let $\mathcal{E}$ be an interaction form. Let $\mathfrak{P}$ be the set of all admissible confederations. For every integer $r \geq 1$, let $\mathfrak{P}_r$ be the confederation structure composed of those $\mathcal{S} \in \mathfrak{P}$ with $|\mathcal{S}| \leq r$, and let $\mathcal{E}_r \equiv \mathcal{E}_{\mathfrak{P}_r}$ be the restriction of $\mathcal{E}$ to $\mathfrak{P}_r$. In particular $\mathcal{E}_1$ is the restriction to the confederation structure defined in example 2.8(c). We also recall that E_0 and $\mathcal{E}_0$ are defined from $\mathcal{E}$ by (14) and (15) respectively. Given that the cardinal of N is n, it follows that: $\mathfrak{P}_1 \subset \cdots \subset \mathfrak{P}_n = \mathfrak{P}$ and $\mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n \subset \mathcal{E}$ and:

$$C(E_0, R_N) = SET(\mathcal{E}_0, R_N) \supset SET(\mathcal{E}_1, R_N) \supset \cdots \supset SET(\mathcal{E}_n, R_N) \supset SET(\mathcal{E}, R_N) \tag{17}$$

For any confederation $\mathcal{S}$, we define the following sets, where $R(\varphi) := \cup_{S \in \mathcal{P}_0(N)} \varphi(S)$:

$$\mathbf{E}_0(\mathcal{S}) = \{\varphi \in \Phi | \ [\varphi] \subset \mathcal{S} \ \text{ and } \ \exists S \in \mathcal{S}, \varphi(S) \in E_0(S)\} \tag{18}$$

$$\mathbf{E}_\xi(\mathcal{S}) = \{\varphi \in \Phi | \ [\varphi] \subset \mathcal{S} \ \text{ and } \ R(\varphi) = A \ \text{ or } \ \exists a \notin R(\varphi), \varphi \in \mathcal{E}[a]\} \tag{19}$$

$$\mathbf{D}(\mathcal{S}) = \{\varphi \in \Phi | \ [\varphi] \subset \mathcal{S} \ \text{ and } \ S \neq T \Rightarrow \varphi(S) \cap \varphi(T) = \emptyset\} \tag{20}$$

Lemma 2.9 *For any confederation $\mathcal{S}$,one has:* $\mathbf{E}_0(\mathcal{S}) \subset \mathbf{E}_\xi(\mathcal{S})$

Proof. Let $\varphi \in \mathbf{E}_0(\mathcal{S})$. Either $R(\varphi) = A$ or there exists $a \notin R(\varphi)$. In the latter case, $\varphi \in \mathcal{E}_0(\mathcal{S}) \subset \mathcal{E}[a](S)$, so that $\varphi \in \mathbf{E}_\xi(\mathcal{S})$.

Definition 2.10 Let $r \geq 1$. $\mathcal{E}$ is r *-exact* if for all $\mathcal{S} \in \mathfrak{P}_r$ one has:

$$\mathbf{E}_\xi(\mathcal{S}) \cap \mathbf{D}(\mathcal{S}) = \mathbf{E}_0(\mathcal{S}) \cap \mathbf{D}(\mathcal{S}) \tag{21}$$

Lemma 2.11 *(i) Assume E_0 is maximal. If $\mathcal{E}$ is not r- exact, then $\mathcal{E}_r$ has a cycle of length $\leq r+2$.*

(ii) Assume E_0 is maximal. If $\mathcal{E}$ is not r- exact, then there exists some $R_N \in L(A)^N$ such that $SET(\mathcal{E}_r, R_N) = \emptyset$, if further E_0 is superadditive we have $|C(E_0, R_N)| = 1$

Proof. If $\mathcal{E}$ is not r- exact, then there exists, a confederation $\mathcal{S} \in \mathfrak{P}_r$, $a \in A$, $\varphi \in \Phi$ such that $\varphi \in \mathcal{E}[a](\mathcal{S})$, $a \notin R(\varphi)$, and for all $S \in \mathcal{S}$, $\varphi(S) \notin E_0(S)$. Let $\mathcal{S} = \{S_1, \ldots, S_r\}$. Let $T_k = S_k^c$ $(k = 1, \ldots, r)$, $T_0 = N \setminus \cup_1^r T_k$, $B_k = \varphi(S_k)$. Since E_0 is maximal we have that $B_k^c \in E_0(T_k)$. We construct an $\mathcal{E}_r$-family $((U_0, \varphi_0), \ldots, (U_{r+1}, \varphi_{r+1}))$ as follows. Let $U_0 := B_0 \setminus \{a\}$, $U_k := B_k$ $(k = 1, \ldots, r)$, $U_{r+1} := \{a\}$, $[\varphi_0] = \{N\}$, $\varphi_0(N) := \{a\}$, $[\varphi_k] = \{T_k\}$, $\varphi_k(T_k) := B_k^c$ $(k = 1, \ldots, r)$, $\varphi_{r+1} := \varphi$. If $B_0 \setminus \{a\} \neq \emptyset$ this defines a cycle of length $r+2$. If $B_0 \setminus \{a\} = \emptyset$ we can remove index 0 and thus have a cycle of length $r+1$.

(ii) we construct a profile $R_N = (R_i)$ with the following properties:

$$\begin{array}{lll} (i \in T_k, k \neq 0) & : A \setminus B_0 \cup B_k \;\; R_i \;\; \{a\} \;\; R_i \; B_0 \setminus \{a\} \, R_i \;\; B_k \\ (i \in T_0) & : \{a\} \qquad\qquad\quad R_i \; A \setminus B_0 \, R_i \; B_0 \setminus \{a\} \end{array}$$

An alternative $b \in B_k$ where $k \in \{1, \ldots, r\}$ is dominated in E_0 since $B_k^c \in E_0(T_k)$ and $B_k^c \subset P(b, T_k, R_N)$. An alternative $b \in B_0 \subset \{a\}$ is dominated in E_0 since by maximality of E_0, $\{a\} \in E_0(N)$ and $\{a\} \subset P(b, N, R_N)$. It follows that $C(E_0, R_N) \subset \{a\}$. For $k = 1, \ldots, r$, $S_k = \cup_{l \neq k} T_l$, $P(a, S_k, R_N) = \cap_{l \neq k} P(a, T_l, R_N) = \cap_{l \neq k} A \setminus (B_0 \cup B_l) = B_k = \varphi(S_k)$. So that a is not a settlement in $\mathcal{E}_r$. Since $SET(\mathcal{E}_r, R_N) \subset C(E_0, R_N)$, it follows that $SET(\mathcal{E}_r, R_N) = \emptyset$. If furthermore E_0 is superadditive, one can prove that $C(E_0, R_N) = \{a\}$. $\square$

We thus have an easy Characterization of r- exactness when E_0 is maximal and superadditive:

Proposition 2.12 *Assume that E_0 is maximal and superadditive, then $\mathcal{E}$ is r-exact if and only if $SET(\mathcal{E}_r, R_N) = C(E_0, R_N)$ for all $R_N \in L(A)^N$.*

Since stability of $\mathcal{E}$ implies stability of $\mathcal{E}_r$ for $1 \leq r \leq n$, the following provides necessary conditions for stability of $\mathcal{E}$ with maximal E_0:

Theorem 2.13 *Assume that E_0 is maximal. Then for $1 \le r \le n$, $\mathcal{E}_r$ is stable if and only if E_0 is stable and $\mathcal{E}$ is r-exact. Moreover in this case $SET(\mathcal{E}_r, R_N) = C(E_0, R_N)$ for all $R_N \in L(A)^N$.*

Proof. An easy consequence of lemma 2.11. □

As a second corollory of lemma 2.11, we obtain the following partial localization of the index in case of unstable $\mathcal{E}$:

Theorem 2.14 *Assume E_0 is maximal :*
(i) If E_0 is not stable then $\sigma(\mathcal{E}) \le 3$
(ii) If E_0 is stable but $\mathcal{E}$ is not r-exact then $\sigma(\mathcal{E}) \le r + 2$.

3 Stability index of strategic game forms

A *game* is an array $\Gamma = (X_1, \ldots, X_n; Q_1, \ldots, Q_n)$, where for each $i \in N = \{1, \ldots, n\}$, X_i is a non-empty set of strategies of player i, and Q_i is a quasi-order (complete, transitive, reflexive binary relation) on $X_N = \prod_{i \in N} X_i$. We denote by Q_i° the strict binary relation induced by Q_i For every coalition $S \in \mathcal{P}_0(N)$, the product $\prod_{i \in S} X_i$ is denoted X_S (by convention $X_\emptyset$ is the singleton $\{\emptyset\}$). Let $\mathcal{S}$ be a confederation. A strategy array $x_N \in X_N$ is an $\mathcal{S}$*-equilibrium* of the game Γ if there is no coalition $S \in \mathcal{S}$ and $y_S \in X_S$ such that $(y_S, x_{S^c})\; Q_i^\circ\; x_N$ for all $i \in S$.

We consider a *game form* $G = (X_1, \ldots, X_n, A, g)$ where X_i is the *strategy set* of player i, $(i \in N)$ and $g : \prod_{i \in N} X_i \to A$ is the *outcome function*. We shall assume that g is onto. If $x_N \in X_N$, the notation $g(x_S, X_{S^c})$ stands for $\{g(x_S, y_{S^c}) \mid y_{S^c} \in X_{S^c}\}$ if $S \neq \emptyset$ and for $g(X_N)$ if $S = \emptyset$. For each preference profile $R_N \in L(A^N)$, the game form G induces a game $(X_1, \ldots, X_n; Q_1, \ldots, Q_n)$ with the same strategy spaces as in G and with the Q_i defined by: $x_N\, Q_i\, y_N$ if and only if $g(x_N)\, R_i\, g(y_N)$ for $x_N, y_N \in X_N$. We denote this game by (G, R_N).

Let $\mathcal{S}$ be a confederation. We say that $a \in A$ is an $\mathcal{S}$*-equilibrium outcome* of (G, R_N) if there is an $\mathcal{S}$-equilibrium of (G, R_N) $x_N \in X_N$ with $g(x_N) = a$. The game form G is said to be *solvable in* $\mathcal{S}$*-equilibrium* or $\mathcal{S}$*-solvable*, if for each preference profile $R_N \in Q(A)^N$, the game (G, R_N) has an $\mathcal{S}$-equilibrium. In particular, when $\mathcal{S} = \mathcal{N} = \{\{1\}, \ldots, \{n\}\}$, an $\mathcal{S}$-equilibrium is a Nash equilibrium. Similarly, when $\mathcal{S} = \mathcal{P}_0(N)$, an $\mathcal{S}$-equilibrium is a strong Nash equilibrium.

For $R_N \in L(A)^N$ a preference profile, an alternative a is β- dominated in (G, R_N) if there exists some $S \in \mathcal{P}_0(N)$ such that for any $x_{S^c} \in X_{S^c}$, there exists some $y_S \in X_S$ such that $g(y_S, x_{S^c}) \in P(a, S, R_N)$. The β - core of (G, R_N) is the set of all alternatives that are not β dominated in (G, R_N) . It is denoted $C_\beta(G, R_N)$. G is said to be stable if $C_\beta(G, R_N)$ is nonempty for all profiles R_N. Let $\mathfrak{F}$ is a confederation structure. An alternative a is in the $\mathfrak{F}$ - core of (G, R_N) if for all $\mathcal{S} \in \mathfrak{F}$, a is an $\mathcal{S}$-equilibrium outcome of (G, R_N). It will be denoted $C_{\mathfrak{F}}(G, R_N)$, G is said to be $\mathfrak{F}$-stable if $C_{\mathfrak{F}}(G, R_N)$ is nonempty for all R_N. In order to be consistent with the literature on this subject, we shall call r- exact core $(r \geq 1)$ the $\mathfrak{F}$-core corresponding to $\mathfrak{F} = \mathfrak{P}_r$.

Given the game form $G = (X_1, \ldots, X_n, A, g)$ the *β-interaction form* (over (N, A)) associated with G is the interaction form $\mathcal{E}_\beta^G$ defined as follows: For $U \in \mathcal{P}_0(A)$:

$$\mathcal{E}_\beta^G[U] =$$

$$\{\varphi \in \Phi(N, A) \mid \forall y_N \in g^{-1}(U), \exists S \in \mathcal{P}_0(N), \exists x_S \in X_S : g(x_S, y_{S^c}) \in \varphi(S)\} \quad (22)$$

The interaction form $\mathcal{E}_\beta^G$ satisfies a remarkable property : for $U \in \mathcal{P}_0(A)$

$$\mathcal{E}_\beta^G[U] = \bigcap_{a \in U} \mathcal{E}_\beta^G[a] \quad (23)$$

For any confederation $\mathcal{S}$, let $\mathcal{E}_\beta^G(\mathcal{S})$ be the projection of $\mathcal{E}_\beta$ to $\mathcal{S}$. When $\mathcal{S} = \{S\}$, instead of $\mathcal{E}_\beta^G(\{S\})$, we consider $E_\beta^G[\cdot](S)$, where $E_\beta^G[\cdot]$ is the local effectivity function associated to G precisely: for $S \in \mathcal{P}_0(N)$:

$$E_\beta^G[U](S) = \{B \in \mathcal{P}_0(A) \mid \forall y_N \in g^{-1}(U), \exists x_S \in X_S : g(x_S, y_{S^c}) \in B\} \quad (24)$$

The classical effectivity function E_β^G, is such that:

$$E_\beta^G(S) = \{B \in \mathcal{P}_0(A) \mid \forall y_N \in X_N, \exists x_S \in X_S : g(x_S, y_{S^c}) \in B\} = E_\beta^G[A](S) \quad (25)$$

The following result can be considered as the *raison d'être* of the introduction of the interaction form $\mathcal{E}_\beta^G$:

Lemma 3.1 *Let $G = (X_1, \ldots, X_n, A, g)$ be a game form. For any confederation $\mathcal{S}$, the set of $\mathcal{S}$-equilibrium outcomes of (G, R_N) coincides with the settlement set of $\mathcal{E}_\beta^G(\mathcal{S})$ at R_N. Therefore G is $\mathcal{S}$- solvable if and only if $\mathcal{E}_\beta^G(\mathcal{S})$ is stable.*

For any confederation structure $\mathfrak{F}$, the $\mathfrak{F}$- core of (G, R_N) coincides with the settlement set of $(\mathcal{E}_\beta^G)_{\mathfrak{F}}$ at R_N. Therefore G is $\mathfrak{F}$-stable if and only if $(\mathcal{E}_\beta^G)_{\mathfrak{F}}$ is stable.

It is easy to see that E_β^G is maximal and monotonic w.r.t. alternatives. G is said to be *tight* if E_β^G is regular (see subsection 1.3) Moreover if E_β^G regular, than it is superadditive. The following result is then a consequence of theorem 1.10:

Theorem 3.2 *G is stable for the* β*- core if and only if* E_β^G *is superadditive (or regular) and subadditive.*

For strong solvability we have the following result which is a consequence of theorem 2.13:

Theorem 3.3 *(i) If G is strongly solvable then* E_β^G *is superadditive (or regular), subadditive and* $\mathcal{E}_\beta^G$ *is r-exact for* $1 \leq r \leq n$*.*

(ii) If $n = 2$ *G is strongly solvable if and only if* E_β^G *is regular and* $\mathcal{E}_\beta^G$ *is 2-exact.*

Define the stability index of $(G, \mathcal{S})$ as the stability index of $\mathcal{E}_\beta^G(\mathcal{S})$. It will be denoted $\sigma(G, \mathcal{S})$. Similarly if $\mathfrak{F}$ is a confederation structure, we denote $\sigma(G, \mathfrak{F})$, the corresponding stability index. Then as a consequence of 2.14:

Theorem 3.4 *Assume that G is not strongly solvable and let* σ *be its index for strong Nash Equilibrium then:*

$\sigma = 2$ *if and only if* E_β^G *is not regular*

$\sigma = 3$ *if* E_β^G *is regular but not subadditive*

$\sigma \leq r + 2$ *if* $\mathcal{E}_\beta^G$ *is not r-exact* $(1 \leq r \leq n)$

Now we give a localization of the index for some classes of games, that can be obtained as corollaries from known results in the literature.

Theorem 3.5 *The Nash stability index of a two -player game form is either* 2 *or* $+\infty$

This is a corollary of the fact that for these game forms Nash solvability is equivalent to tightness. See Gurvich (1975, 1989) or Abdou (1995). If we consider the class of rectangular game forms - i.e. such that for any $a \in A$, $g^{-1}(a) = \prod_{i=1}^n Y_i$, for some $Y_i \subset X_i, (i = 1, \ldots, n)$ - (Gurvich, 1978 and Abdou, 1995, 2000) one has a similar characterization for Nash solvability by tightness. It does not follow however that the Nash-Stability index of a non solvable rectangular game form is 2 in case of instability. The only property that we can assert in this case is that its core index 2. The strong Nash index for rectangular game forms has a simple characterization.

Theorem 3.6 *Let G be a rectangular game form.*

(i) If G is strongly solvable then G is essentially a one-player game form.

(ii) If G is not strongly solvable then the strong Nash index is either 2 *or* 3*.*

Proof: The first assertion is theorem 4.7 of Abdou (2000) which asserts that any rectangular game form such that $\mathcal{E}_\beta^G$ is 1-exact is essentially a one-player game form. Assume that G is not strongly solvable. It follows in particular that $\mathcal{E}_\beta^G$ is not 1- exact, then by theorem 3.4 (iii) the index is less than $1 + 2 = 3$, and in fact is equal 3 if E_β^G is regular.

4 Concluding remarks

The model of interaction form as a description of power distribution of a set of agents N over a set of alternatives A encompasses aspects of the so-called cooperative and strategic models. The settlement set defined at a preference profile reflects the alternatives that have some likelihood to emerge given the power of active confederations.. Any game form coupled with a classical equilibrium concept gives rise to some interaction form. Interactive forms defined on the same sets of agents and alternatives can be compared with each other. In particular they can be compared with respect to their stability, a main issue in political science and social choice. Stability is proven to be equivalent to acyclicity. Sovability of strategic game forms is thus reduced to a problem of acyclicity. In order to give an idea of the nature of unstable interaction forms, we defined a stability index that generalizes the Nakamura number originally defined only for simple games. Although we succeeded to localize this number in many cases, using structural properties, like superadditivity, subadditivity and r- exactness, many questions about the stability index are still open.

References

Abdou, J., 1982. Stabilité de la fonction veto, Cas du veto maximal. Math. Sci. Hum. 80, 39-63.

Abdou, J., 1995. Nash and strongly consistent two-player game forms. Int. J. Game Theory 24, 345-356.

Abdou, J., 1998. Rectangularity and tightness: a normal form characterization of perfect information extensive game forms. Math. of oper. research 23, No 3, 553-567.

Abdou, J., 2000. Exact stability and its applications to strong solvability. Math. Social Sciences 39, 263 - 275.

Abdou, J. and Keiding H., 1991. Effectivity functions in social choice. Kluwer Academic Publishers, Dordrecht.

Abdou, J. and Keiding H., 2003. On necessary and sufficient conditions for solvability of game forms. Math. Social Sciences 46, 243 - 260.

Keiding, H., 1985. Necessary and sufficient conditions for stability of effectivity functions, Int. J. Game Theory 14, 93 - 101.

Gurvich, V.A., 1975. Solvability in pure strategies. Zh. Vychisl. Mat. i Mat. Fiz. 15, 358-371; English transl. in USSR Comput. Math. and Math. Phys. 15.

Gurvich, V.A., 1978. Application of Boolean functions and contact schemes in game theory. Candidate's dissertation Moskow, Fiz-Tekn. Inst. Russian.

Gurvich, V.A., 1989. Equilibrium in pure strategies. Soviet Math. Dokl. 38, 597-602.

Moulin, H. and Peleg, B., 1982. Cores of effectivity functions and implementation theory. J. Math. Econ. 10, 115-162.

Nakamura, K. (1979), The vetoers in a simple game with ordinal preferences, Int. J. Game Theory 8, issue 1, 55-61

Peleg, B., 1984. Game theoretic analysis of voting in committees. Cambridge University Press, Gambridge.