

Transport equations with rough force fields and applications to the Vlasov-Poisson equation.

Delphine Salort *

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Abstract

The aim of this article is to give new dispersive tools for certain kinetic equations. As an application, we study the three dimensional Vlasov-Poisson equation for initial data having strictly less than six moments in $L^1_{x,\xi}$ where the non linear term E is a priori rough. We prove via new dispersive effect that in fact the force field E is smooth in space at the cost of a localisation in a ball and an averaging in time. We deduce new conditions to bound the density ρ in L^∞ and to have existence and uniqueness of global weak solutions of the Vlasov-Poisson equation with bounded density for initial data having strictly less than 6 moments in $L^1_{x,\xi}$. The proof is based on a new approach which consists in establishing a priori moment effects on the one hand for linear transport equations with rough force fields and on the other hand along the trajectories of the Vlasov-Poisson equation.

L'objectif de cet article est de donner de nouveaux outils dispersifs pour certaines équations cinétiques. Comme application, on étudie l'équation de Vlasov-Poisson en dimension 3 pour des données initiales ayant strictement moins de six moments dans $L^1_{x,\xi}$ où le terme non linéaire est a priori peu régulier. On prouve, via de nouveaux effets dispersifs que, en fait, le terme de force E est régulier en espace quitte à se localiser sur une boule en espace et à intégrer en temps. On en déduit de nouvelles conditions pour que la densité ρ soit dans L^∞ et pour obtenir existence et unicité de solutions faibles de l'équation de Vlasov-Poisson avec densité bornée pour des données initiales ayant strictement moins de six moments dans $L^1_{x,\xi}$. La preuve est basée sur une nouvelle approche qui consiste à établir des effets de moments a priori d'une part pour des équations de transport avec des termes de force peu réguliers et d'autre part le long des trajectoires de l'équation de Vlasov-Poisson.

1 Introduction

In this article, we study the three dimensional Vlasov-Poisson equation given by

$$\begin{cases} \partial_t f + \xi \nabla_x f \pm E \cdot \nabla_\xi f = 0 \\ f(0, x, \xi) = f^{in}(x, \xi) \end{cases} \quad (1)$$

where f^{in} is a positive measurable function, with $E = \varphi * \rho$,

$$\varphi(x) = \frac{x}{|x|^3} \quad \text{and} \quad \rho(t, x) = \int f(t, x, \xi) d\xi.$$

*Laboratoire Analyse, Géométrie and Applications, Institut Galilée, Université Paris 13, 99 avenue J.B. Clément 93430 Villetaneuse, FRANCE; salort@math.univ-paris13.fr

This equation models the evolution of a system of particles in gravitational (for sign $-$) or coulombian (for sign $+$) interactions. The solution $f(t, x, \xi)$ models the microscopic density of particles which are, at time t , at position x with velocity ξ , and $\rho(t, x)$ models the probability of finding a particle at time t at position x . Finally $E(t, x)$ models the electrostatic or gravitational (depending on the sign) potential created by ρ .

Let $x \in \mathbb{R}$. In this article, $x + 0$ and $x - 0$ denote $x + \varepsilon$ and $x - \varepsilon$ respectively where $\varepsilon > 0$ can be taken arbitrarily small.

The aim of this article is on the one hand to prove new dispersive estimates and on the other hand to give an application of this estimates by proving existence and uniqueness of weak solutions of equation (1) in the case where the initial data has m moments in $L^1_{x,\xi}$, where here $m < 6$ is under the critical number of moments to have a smooth non linear term E (see Theorem 2). Before stating our results, let us first recall known results on the Vlasov-Poisson equation.

1.1 Known results.

1.1.1 Existence of weak solutions.

The main ingredients to prove the existence of weak solutions of Vlasov-Poisson equation (1) are the following a priori estimates inspired by corresponding laws of physics applied to equation (1):

- mass conservation and the Liouville principle *i.e.* for all $t \in \mathbb{R}$, for all $p \in [1, +\infty]$

$$\|f(t)\|_{L^p_{x,\xi}} = \|f^{in}\|_{L^p_{x,\xi}}$$

- energy conservation

$$\int \frac{|\xi|^2}{2} f(t, x, \xi) dx d\xi \pm \int \frac{1}{2} |E(t, x)|^2 dx = C.$$

More precisely, under the natural assumptions

$$f^{in} \in L^1_{x,\xi} \cap L^\infty_{x,\xi} \quad \text{and} \quad \int \frac{|\xi|^2}{2} f^{in}(x, \xi) dx d\xi < +\infty \quad (2)$$

A. A. Arsen'ev in [1], [2], E. Hörst and R. Hunze in [17] proved existence of global weak solutions. Global existence of renormalized solutions of (1) has been established by R. J. DiPerna and P.-L. Lions (see [11] and [10]) for more general initial data which only satisfy the minimal regularity $f^{in} \in L^1_{x,\xi}$, $f^{in} \log(f^{in}) \in L^1_{x,\xi}$ and $|\xi|^2 f^{in} \in L^1_{x,\xi}$. Let us mention that in this article, we only consider weak solutions of (1) such that f^{in} satisfies conditions (2). The next question is therefore to obtain uniqueness of such solutions.

1.1.2 Uniqueness result.

A way to prove uniqueness of weak solutions to the Vlasov-Poisson equation (1) (which we will make use of later on) is to use the following sufficient condition given by G. Loeper in [13].

Theorem 1 [13]

Let f^{in} be a bounded positive measure; given $T > 0$, there exists at most one weak solution to the Vlasov-Poisson equation (1) such that

$$\rho \in L^\infty([0, T] \times \mathbb{R}^3).$$

Unfortunately, the a priori estimates given above are not sufficient to prove that $\rho \in L^\infty([0, T] \times \mathbb{R}^3)$. So, to use Theorem 1, we need more information on the initial data. Such information is given by the propagation of moments which is the following result of B. Perthame and P.-L. Lions in [18].

Theorem 2 [18]

Assume that $f^{in} \in L^\infty_{x,\xi}$ and that for $m > 3$

$$\|(1 + |\xi|)^{m_0} f^{in}\|_{L^1_{x,\xi}} < +\infty \quad \text{for all } m_0 < m.$$

Then, a weak solution of the Vlasov-Poisson equation (1) exists such that for all $T > 0$

$$\sup_{t \in [0, T]} \|(1 + |\xi|)^{m_0} f(t)\|_{L^1_{x,\xi}} < C(T) \quad \text{and}$$

$$E \in \mathcal{C}(\mathbb{R}^+)(L^q(\mathbb{R}^3)) \quad \text{if } \frac{3}{2} < q < \frac{3(3+m)}{6-m} \quad \text{and } m < 6$$

$$E \in \mathcal{C}(\mathbb{R}^+)(C^\alpha(\mathbb{R}^3)) \quad \text{if } \alpha < \frac{m-6}{3+m} \quad \text{and } m > 6.$$

In particular, in case $m > 6$, E is smooth and belongs to $L^\infty([0, T] \times \mathbb{R}^3)$ for all $T > 0$, and the characteristics (X, V) of the Vlasov-Poisson equation (1) are small perturbations of those given by the free transport equation. This perturbation of the characteristics by the free transport equation is the key point to obtain a condition on the initial data such that $\rho \in L^\infty([0, T] \times \mathbb{R}^3)$. Indeed,

$$X(t, x, \xi) = x + t\xi + R_1(t, x, \xi) \quad \text{and} \quad V(t, x, \xi) = \xi + R_2(t, x, \xi)$$

where here

$$R_1(t, x, \xi) = \int_0^t (t-s)E(s, X(s, x, \xi))ds \quad \text{and} \quad R_2(t, x, \xi) = \int_0^t E(s, X(s, x, \xi))ds.$$

They deduce the following control for all $T > 0$ and $t \in [0, T]$

$$\|R_1(t)\|_{L^\infty_{x,\xi}} \leq |t|^2 \|E\|_{L^\infty([0, T] \times \mathbb{R}^3)} \quad \text{and} \quad \|R_2(t)\|_{L^\infty_{x,\xi}} \leq |t| \|E\|_{L^\infty([0, T] \times \mathbb{R}^3)}. \quad (3)$$

This control on E and on the characteristics allows them to prove the following result which in turn implies sufficient conditions on the initial data to show that the density ρ belongs to $L^\infty([0, T] \times \mathbb{R}^3)$.

Theorem 3 [18]

Assume that $f^{in} \in L_{x,\xi}^\infty$ and that

$$\|(1 + |\xi|)^{6+0} f^{in}\|_{L_{x,\xi}^1} < +\infty.$$

Assume furthermore that for all $R > 0$ and for all $T > 0$

$$\text{supess} \left\{ f^{in}(y + t\xi, w), |x - y| \leq Rt^2, |\xi - w| \leq Rt \right\} \in L^\infty([0, T] \times \mathbb{R}_x^3(L_\xi^1)). \quad (4)$$

Then, a weak solution of the Vlasov-Poisson equation (1) exists such that for all $T > 0$, $\rho \in L^\infty([0, T] \times \mathbb{R}_x^3)$.

Let us mention that we can also obtain global existence and uniqueness of solutions to the Vlasov-Poisson equation (1) by way of classical solutions. We refer to the articles by J. Batt in [5], J. Batt and G. Rein in [6], C. Bardos and P. Degond in [4], E. Hörst in [14], [15] and [16], K. Pfaffelmoser in [20] and by J. Schaeffer in [22] (see also the article by B. Perthame [19] and references therein).

To give new conditions on the initial data to obtain bounded density (and hence uniqueness of such solutions), we introduce a new approach to the study of Vlasov-Poisson equation (1). One of the key arguments we used involved a precise study of characteristics using moment effects for transport equations with rough force fields. This approach brought new answers to the question stated above by relaxing constraints on the moments of the initial data. Let us mention that I. Gasser, P.-E. Jabin and B. Perthame in [12] studied the Vlasov-Poisson equation by using moment effects established for the free transport equation.

1.2 Results and outline of the proof.**1.2.1 Main result.**

The main result is the following.

Theorem 4 Let $\infty+ > p \geq 3$ and $f^{in} \in L_{x,\xi}^\infty$. Then, $m(p) < 6$ exists such that if

$$\|(1 + |\xi|)^{m(p)} f^{in}\|_{L_{x,\xi}^1} < +\infty$$

and if for all $T_0 > 0$, $R > 0$,

$$\text{supess} \left\{ f^{in}(y + t\xi, w), |x - y| \leq R|t|^{\frac{1}{p'}+1}, |\xi - w| \leq R|t|^{\frac{1}{p'}} \right\} \in L^\infty([0, T_0] \times \mathbb{R}_x^3(L_\xi^1)) \quad (5)$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ then, a unique solution of the Vlasov-Poisson equation (1) exists such that the density of probability $\rho \in L^\infty([0, T_0] \times \mathbb{R}_x^3)$ for all $T_0 > 0$.

Remark 1 • Theorem 4 allows us to control $\rho \in L^\infty([0, T_0] \times \mathbb{R}^3)$ for all $T_0 > 0$, for bounded initial data satisfying condition (5) and which have $m(p) < 6$ moments in $L_{x,\xi}^1$. Since the control of characteristics is worse than if $E \in L^\infty([0, T_0] \times \mathbb{R}^3)$, one is forced

to add an additional constraint when compared to condition (4), namely condition (5). However, in terms of decay with respect to ξ , conditions (4) and (5) are weak. Indeed, we remark that conditions (4) and (5) hold as soon as

$$|f^{in}(x, \xi)| \leq \frac{C}{(1 + |\xi|)^{3+0}}.$$

1.2.2 Outline of the Proof.

For the following computations to be licit, we would have to consider a perturbation of the Vlasov-Poisson equation in which the Coulomb potential E is regularized. But in the course of the proof of Theorem 4 we shall see that the norms involved allow us to pass to the limit, retaining the desired estimates by standard arguments. Therefore we assume from now on that we are dealing with a smooth potential $E \in C_t(C_b^\infty(\mathbb{R}^3))$.

The strategy adopted to obtain Theorem 4 is to prove that the characteristics given by the Vlasov-Poisson equation (1) are a small perturbation of those given by the free transport equation (see Theorem 6 below). To do this we first control the force field E in spaces of type C^α (see Theorem 5 below for more details).

Estimates on the potential E . There are two distinct results in this section. The first one is local in time and allows us to control E in spaces of type C^α with $\alpha < 1$ using strictly less than 6 moments in $L_{x,\xi}^1$ on the initial data. The crux of the proof of this result is the study of effect moments for the linear transport equation with rough force fields (see section 2). The second one gives a result for arbitrarily large time. These estimates are obtained by combining the global in time results obtained by P.-L. Lions and B. Perthame in [18] and the local in time results obtained in this article.

We introduce the following definition.

Definition 1 Let $E : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a map. Let $T_0 > 0$ and $p \geq 1$. We set

$$S_{p,T_0}(E) = \sup_{|B| \leq 1} \|E\|_{L_{T_0}^p(L^\infty(B))}$$

where B denotes a ball of \mathbb{R}^3 of size 1. When there is no doubt which E is being considered, we will refer to S_{p,T_0} as $S_{p,T_0}(E)$. Let $f \in L_{x,\xi}^1$, $m \geq 0$ and $p \geq 1$. We define

$$\|f\|_{L_{x,\xi}^{1,m}} = \|(1 + |\xi|)^m f\|_{L_{x,\xi}^1}.$$

Theorem 5 Let $f^{in} \in L_{x,\xi}^\infty$ and let $\infty+ > p \geq 3$. Then, $m(p) < 6$ exists such that if

$$\|(1 + |\xi|)^{\frac{3p-1}{pp'}+0} f^{in}\|_{L_{x,\xi}^{p+0}} < +\infty \quad \text{and} \quad \|f^{in}\|_{L_{x,\xi}^{1,m(p)}} < +\infty.$$

Then, $T_0 > 0$ and $C > 0$ exist such that

$$\sup_{|B| \leq 1} \|E\|_{L_{T_0}^p(C^{1-\frac{3}{p+0}}(B))} \leq C \tag{6}$$

where B denotes a ball of \mathbb{R}^3 of size 1.

Let $3 \leq p < +\infty$. Then, $m(p) < 6$ exists such that if

$$\|f^{in}\|_{L_{x,\xi}^{1,m(p)}} < +\infty,$$

then, for all $T_0 > 0$ a constant C exists such that the following estimate holds

$$\sup_{|B| \leq 1} \|E\|_{L_{T_0}^p(L^\infty(B))} \leq C. \quad (7)$$

Remark 2 • Estimate (6) allows us in particular to control E in spaces of type C^α where $\alpha = 1 - 0$ for relatively general initial data. Consequently, the estimate holds a priori only for small enough time intervals.

- The estimate (7) is obtained by interpolation between the global in time results obtained by P.-L. Lions and B. Perthame in [18] and estimate (6).

Perturbation of the characteristics given by the free transport equation. The following Theorem gives some a priori estimates (i.e. we prove those estimates for the approximate system of (1) where the characteristics are well defined and where all the computations are licit) on the characteristics of the Vlasov-Poisson equation (1).

Theorem 6 Let $f^{in} \in L_{x,\xi}^1 \cap L_{x,\xi}^\infty$, and $\infty+ > p \geq 3$. Then $m(p) < 6$ exists such that if

$$\|f^{in}\|_{L_{x,\xi}^{1,m(p)}} < +\infty,$$

then, for all $T_0 > 0$, a constant C exists such that for all $(x, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3$ and $t \in [0, T_0]$ the following a priori estimates hold

$$|X(t, x, \xi) - x - t\xi| \leq C|t|^{\frac{1}{p'}+1} \quad \text{and} \quad |V(t, x, \xi) - \xi| \leq C|t|^{\frac{1}{p'}} \quad (8)$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and where (X, V) are the characteristics of the equation (1).

Let us mention that Theorem 6 directly implies Theorem 4. Indeed we follow the strategy adopted by P.-L. Lions and B. Perthame in [18]. We observe using Theorem 6 that

$$f(t, x, \xi) \leq \sup_{(y,w)} \{f^{in}(y + t\xi, w), |y - x| \leq R|t|^{1+\frac{1}{p'}}, |w - \xi| \leq R|t|^{\frac{1}{p'}}\}$$

and so $\rho \in L^\infty([0, T_0] \times \mathbb{R}^3)$.

The difficulty in proving Theorem 6 lies in the fact that Theorem 5 does not give control over E in $L^\infty([0, T] \times \mathbb{R}^3)$; so we cannot follow the same strategy as that adopted by P.-L. Lions and B. Perthame in [18] to prove that the trajectories are a small perturbation of those given by the free transport equation. To obtain a good approximation of the characteristics arising from free transport, we will prove the following weaker condition using moment effects

$$\sup_{(x_0, \xi_0) \in \mathbb{R}^6} \int_0^t |E(s, X(s, x_0, \xi_0))|^p ds < +\infty \quad \text{where} \quad p \geq 1. \quad (9)$$

We shall see that to obtain an estimate such as (9), bounded initial data with strictly less than 6 moments in $L^1_{x,\xi}$ are allowed. Conversely, estimate (9) gives us weaker control on R_1 and R_2 when t is small than was obtained by P.-L. Lions and B. Perthame in [18] (see estimate (3)).

The remainder of the article is organized as follows.

- In the second section, we study linear transport equations with a rough force field. This study is crucial to prove the theorems in this article.
- In the third section, we use the results obtained in the second section to prove Theorem 5.
- The last section is devoted to the proof of Theorem 6 (and hence Theorem 4 using the remark made above). This proof is based on the preceding section's results and on new moment effects along the characteristics of the Vlasov-Poisson equation.

2 Linear transport equation with rough force field.

In this section, we assume that we are in dimension $d \geq 1$ and we consider the linear transport equation with a force field $F(t, x)$ given by

$$\begin{cases} \partial_t f + \xi \nabla_x f + F \cdot \nabla_\xi f = 0 \\ f(0, x, \xi) = f^{in}(x, \xi). \end{cases} \quad (10)$$

Later on E is substituted with F , in equation (10) to get equation (1). In particular, in this section, we never use the fact that, for the Vlasov-Poisson equation (1), E can be written explicitly in terms of the density ρ .

Remark 3 Here, the force field $F(t, x) \in C_t(C_b^\infty(\mathbb{R}^d))$ is assumed to be smooth; but we call it transport equation with rough force fields because all the estimates we establish involve weak norms in F which we describe below.

If the force field F is smooth enough, then there exists a unique solution of equation (10) which may be written explicitly in terms of the initial data by

$$f(t, x, \xi) = f^{in}(X(t, 0, x, \xi), V(t, 0, x, \xi))$$

where for all $(t_1, t_2) \in \mathbb{R}^2$, $(X(t_1, t_2, x, \xi), V(t_1, t_2, x, \xi))$ is the solution of the system

$$\begin{cases} \dot{X}(t_2) = V(t_2, X(t_2)) \\ \dot{V}(t_2) = F(t_2, X(t_2)) \end{cases} \quad (11)$$

with

$$X(t_1, t_1, x, \xi) = x \quad \text{and} \quad V(t_1, t_1, x, \xi) = \xi$$

as initial data.

In this section we endeavour to study this equation qualitatively using only weak norms on the force field F . More precisely, given $p \geq 1$, and $T > 0$, constants involving F will only depend on

$$S_{p,T} = \sup_{|B| \leq 1} \|F\|_{L^p_{[0,T]}(L^\infty(B))} \quad (12)$$

where B is a ball of \mathbb{R}^d of size 1.

We will first explain what are the motivations behind the study of this equation to show Theorems 5 and 6, then proceed to state the results obtained on equation (10) and finally prove them.

2.1 Motivations.

Reasons to study this equation to prove Theorems 5 and 6 are twofold.

- First, it provides us information on the characteristics in terms of the smoothness of F , which will be later useful to prove theorem 6 and gain uniform control on the characteristics.
- Also, such a study enables us to control the force field E which is directly related to the density ρ by studying ρ , seen as the density corresponding to the solution of the linear transport equation (10). This allows us to apply the study made on (10) and hence prove Theorem 5.

More precisely, we focus here on the study of the characteristics and on moment effects regarding equation (10). Indeed, one of the key estimates that we have to prove in order to obtain Theorem 5 is that for all $p \geq 3$, a constant C exists such that

$$\|E\|_{L_T^p(L_B^\infty)} \leq C(|B|).$$

To control $\|E\|_{L_T^p(L_B^\infty)}$, the idea is the following. First, we split the force field E in two parts. Let $\gamma \in \mathcal{D}(\mathbb{R}^3)$ be a function such that $0 \leq \gamma \leq 1$ and $\gamma \equiv 1$ on $B(0, \frac{1}{4})$. We write for all $(t, x) \in \mathbb{R} \times \mathbb{R}^3$

$$E(t, x) = E^1(t, x) + E^2(t, x)$$

with

$$E^1(t, x) = \int \varphi(y)(1 - \gamma(y))\rho(t, x - y)dy.$$

This splitting has been used for the study of the Vlasov-Poisson equation by P.-L. Lions and B. Perthame in [18] and by F. Castella in [8].

Study of E^1 . E^1 is the convolution between a smooth function and the density of probability ρ . Hence, E^1 is easily estimated by

$$\|E^1\|_{L^\infty(\mathbb{R} \times \mathbb{R}^3)} \leq \sup_{(t,x) \in \mathbb{R} \times \mathbb{R}^3} \left| \int \varphi(y)(1 - \gamma(y))\rho(x - y)dy \right| \leq C\|f^{in}\|_{L_{x,\xi}^1}. \quad (13)$$

Study of E^2 . E^2 is harder to deal with because it is the convolution between ρ and the function $\gamma\varphi$ which becomes rough near 0. To tackle this difficulty, we shall make use of the fact that for each $x \in \mathbb{R}^3$ the density function $\rho(\cdot)\gamma(x-\cdot)$ involved is truncated in space to a ball of fixed

size. More precisely, we observe that if the trajectories do not remain for too long in a compact set, then truncation in space of the density ρ (recall that ρ can be written using characteristics) provides additional moments over Hölder inequalities, averaging over time. Let us note that moment effects have been established for the free transport equation for the Euclidean metric (see B. Perthame [19] and references therein) and for non-trapped metrics in [21]. The setting is different here since we add a rough force field term to the transport equation. Indeed, we only have an a priori estimate on E in the space $L_t^p(L_B^\infty)$ where B is a ball of fixed size. To address the difficulty of a rough force field, we use estimates on the characteristics in terms of measures, which do not require assumptions on the smoothness of these characteristics (see [21] for a similar strategy).

Remark 4 *We estimate E over balls and not over the whole space \mathbb{R}^3 . This is due to the fact that moment effects disappear when we consider $\sup_{x \in \mathbb{R}^3} \rho(\cdot) \gamma(x \cdot)$.*

2.2 Results obtained.

The fundamental properties that we will use on the transport equation (10) are given by the following propositions. The first one gives the following approximation on the characteristics.

Proposition 1 *Let $T > 0$ and $p \geq 1$. Then a constant C exists such that , for all $(t_1, t_2) \in [0, T]^2$, for all $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$,*

$$|X(t_1, t_2, x, \xi) - x + (t_1 - t_2)\xi| \leq C(1 + S_{p,T})^{1+\frac{1}{p}}(1 + |\xi|)^{\frac{1}{p}} \quad (14)$$

and

$$|V(t_1, t_2, x, \xi) - \xi| \leq C(1 + S_{p,T})^{1+\frac{1}{p}}(1 + |\xi|)^{\frac{1}{p}}. \quad (15)$$

Remark 5 • *This proposition gives us an approximation of the characteristics by those from the free transport equation which is precise enough to allow us to obtain moment effects and propagation of moments on the solution of (10).*

- *However this approximation of the characteristics by those of the free transport equation gets worse as ξ grows, and it is still a long way from providing enough control over characteristics to prove Theorem 6. This first approximation will nevertheless be crucial to prove Theorem 6.*

The second proposition establishes that the solution effectively propagates moments, and a local in time moment effect.

Proposition 2 *Let $T > 0$, $p \geq 1$ and $\alpha \geq 0$. Then, for all $q \geq 1$, a constant C exists such that for all $t \in [0, T]$ the following estimate holds*

$$\|(1 + |\xi|)^\alpha f(t)\|_{L_{x,\xi}^q} \leq C(1 + S_{p,T})^{\alpha(1+\frac{1}{p})} \|(1 + |\xi|)^\alpha f^{in}\|_{L_{x,\xi}^q}.$$

Moreover, the following moment effects occur. Let $\gamma \in \mathcal{D}(\mathbb{R}^d)$. Then, for all $(q, p) \geq 1$, for all $\alpha \geq 0$, a constant C exists such that for any ball $B \subset \mathbb{R}^d$ of size 1 the following estimate holds

$$\left\| \sup_{x \in B} \gamma(x \cdot) (1 + |\xi|)^{\alpha + \frac{1}{qp'}} f(t) \right\|_{L_{T,x,\xi}^q} \leq C(1 + S_{p,T})^{\left(\frac{\alpha q + 1}{q}\right)(1 + \frac{1}{p})} \|(1 + |\xi|)^\alpha f^{in}\|_{L_{x,\xi}^q}. \quad (16)$$

Remark 6 *We gain $\frac{1}{qp'}$ moments in L^q over the Hölder inequalities and propagation of moments.*

2.3 Qualitative study of equation (10).

This part is devoted to the proof of Propositions 1 and 2.

2.3.1 Proof of Proposition 1.

We first study characteristics on small time intervals whose length is inversely proportional to $|\xi|$, to make sure that $X(t, x, \xi)$ stays in a ball of size 1. Then, using a time splitting with respect to the velocity, we prove that the characteristics (X, V) are a perturbation of those given by the free transport equation as stated in Proposition 1.

Remark 7 *Let us note that this splitting in time with respect to the velocity was inspired by a similar idea of splitting in time and frequency or in time and velocity which has been used in many contexts. The time-frequency splitting was introduced by H. Bahouri and J.-Y. Chemin in [3] for the wave equation, then used in the article by N. Burq, P. Gérard and N. Tzvetkov in [7] for the Schrödinger equation and in [21] for the Liouville equation.*

Local study in time with respect to the velocity of characteristics. The following Lemma holds

Lemma 1 *Let $p \geq 1$ and $T > 0$. Then, for all $(t_0, t_1, \xi) \in [0, T] \times [0, T] \times \mathbb{R}^d$, for all $t_2 \in [0, T]$ with $|t_2 - t_0| \leq \min(\frac{1}{4(T+1)[1+S_{p,T}]}, \frac{1}{4|\xi|})$, for all $x \in \mathbb{R}^d$*

$$X(t_1, t_2, x, \xi) \in B(X(t_1, t_0, x, \xi), 1).$$

Proof of Lemma 1. Let $p \geq 1$, $T > 0$ and $(t_0, t_1, \xi) \in [0, T] \times [0, T] \times \mathbb{R}^d$. Let

$$T_0(t_0, t_1, \xi) = \sup \left\{ |t_2 - t_0|, X(t_1, t_2, x, \xi) \in B(X(t_1, t_0, x, \xi), 1) \right\}.$$

As $F \in C_t(C_b^\infty(\mathbb{R}^d))$, we have $T_0(t_0, t_1, \xi) > 0$. Applying Taylor's formula at order two with integral remainder, we obtain for all $(t_1, t_2, x, \xi) \in [0, T] \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$

$$X(t_1, t_2, x, \xi) = x + (t_2 - t_1)\xi + \int_{t_1}^{t_2} (t_2 - s)F(s, X(t_1, s, x, \xi))ds.$$

We deduce that

$$X(t_1, t_2, x, \xi) - X(t_1, t_0, x, \xi) = (t_2 - t_0)\xi + (t_2 - t_0) \int_{t_1}^{t_0} F(s, X(s))ds + \int_{t_0}^{t_2} (t_2 - s)F(s, X(s))ds.$$

We deduce that insofar as $|t_2 - t_0| \leq \min(\frac{1}{4(T+1)[1+S_{p,T}]}, \frac{1}{4|\xi|})$,

$$\left| X(t_1, t_2, x, \xi) - X(t_1, t_0, x, \xi) \right| \leq \frac{3}{4}$$

which proves Lemma 1. □

Splitting in time with respect to the velocity. We have

$$X(t_1, t_2, x, \xi) = x + (t_2 - t_1)\xi + \int_{t_1}^{t_2} (t_2 - s)F(s, X(t_1, s, x, \xi))ds \quad \text{and}$$

$$V(t_1, t_2, x, \xi) = \xi + \int_{t_1}^{t_2} F(s, X(t_1, s, x, \xi))ds.$$

To obtain Proposition 1, it suffices to prove that for all $p \geq 1$, for all $T > 0$, a constant C exists such that for all $(t_1, t_2, x, \xi) \in [0, T] \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ the following estimate holds

$$\left| \int_{t_1}^{t_2} F(s, X(t_1, s, x, \xi))ds \right| \leq C(1 + S_{p,T})^{1+\frac{1}{p}}(1 + |\xi|)^{\frac{1}{p}}. \quad (17)$$

To prove (17), we split the integral in small time intervals of length less than $\min(\frac{1}{4(T+1)[1+S_{p,T}]}, \frac{1}{4|\xi|})$ for which we know that $X(t_1, t_2, x, \xi)$ stays in a ball of size 1. We obtain

$$\left| \int_{t_1}^{t_2} F(s, X(t_1, s))ds \right| \leq \int_{t_1}^{t_2} |F(s, X(t_1, s))|ds = \sum_{k=0}^N \int_{t^k}^{t^{k+1}} |F(s, X(t_1, s))|ds$$

where $N \sim \frac{1}{\min(\frac{1}{4(T+1)[1+S_{p,T}]}, \frac{1}{4|\xi|})}$ and $|t^k - t^{k+1}| \sim \min(\frac{1}{4(T+1)[1+S_{p,T}]}, \frac{1}{4|\xi|})$. On each time interval $[t^k, t^{k+1}]$, the trajectory X remaining in a ball of size 1, we deduce that

$$\int_{t^k}^{t^{k+1}} |F(s, X(t_1, s))|ds \leq |t^k - t^{k+1}|^{\frac{1}{p}} S_{p,T}.$$

Hence we have

$$\int_{t_1}^{t_2} |F(s, X(t_1, s))|ds \leq \sum_{k=0}^N |t^k - t^{k+1}|^{\frac{1}{p}} S_{p,T} \leq C(1 + S_{T,p})^{1+\frac{1}{p}}(1 + |\xi|)^{\frac{1}{p}}$$

and Proposition 1 follows. \square

2.3.2 Proof of Proposition 2.

The proof of Proposition 2 follows from the properties of the characteristics in Proposition 1.

Propagation of moments. For all $p \geq 1$, the following equality holds

$$\|(1 + |\xi|)^\alpha f\|_{L_{x,\xi}^p} = \|(1 + |\xi|)^\alpha f^{in}(X, V)\|_{L_{x,\xi}^p}.$$

Using inequality (15), we obtain

$$\|(1 + |\xi|)^\alpha f^{in}(X, V)\|_{L_{x,\xi}^p} \leq C\|(1 + |V|)^\alpha f^{in}(X, V)\|_{L_{x,\xi}^p}.$$

Making the change of variables $(X, V) \rightarrow (x, \xi)$ we get propagation of moments as in Proposition 2.

Moment effects. Let us first prove the following lemma which states that the trajectories do not remain too long a time in a compact set and then prove estimate (16) of Proposition 2.

Lemma 2 Let $T > 0$ and $p \geq 1$. Then, a constant C exists such that for all $\beta > 0$, for all $t_1 \in [0, T]$, for all $|\xi| \geq 1$,

$$\mu\left\{t_2 \in [0, T], X(t_1, t_2, x, \xi) \in B(\beta)\right\} \leq \frac{(1 + S_{p,T})^{1+\frac{1}{p}} C(1 + \beta)}{(1 + |\xi|)^{\frac{1}{p'}}$$

where μ denotes the Lebesgue measure.

Proof of Lemma 2. Let t_0 and t_1 in $[0, T]$. According to estimate (14) of Proposition 1, a constant C exists such that for all $t_2 \in [0, T]$ the following estimate holds

$$|X(t_1, t_2, x, \xi) - X(t_1, t_0, x, \xi)| \geq |t_2 - t_0||\xi| - C(1 + S_{p,T})^{1+\frac{1}{p}}(1 + |\xi|)^{\frac{1}{p}}$$

which proves Lemma 2. □

Let $q \geq 1$, $\alpha \geq 0$ and B a ball of size 1. Let us define

$$D = \left\| \sup_{y \in B} \gamma(y - \cdot) (1 + |\xi|)^\alpha f \right\|_{L^q_{T,x,\xi}}^q.$$

$$D = \int_0^T \int_x \int_\xi (1 + |\xi|)^{\frac{1}{p'}} (1 + |\xi|)^{\alpha q - \frac{1}{p'}} (f^{in})^q(X, V) \left(\sup_{y \in B} \gamma(y - \cdot) \right)^q dt dx d\xi.$$

Making the change of variables $(X, V) \rightarrow (x, \xi)$ which is a diffeomorphism of Jacobian equal to 1, we obtain

$$D = \int_0^T \int_x \int_\xi (1 + |V|)^{\frac{1}{p'}} (1 + |V|)^{\alpha q - \frac{1}{p'}} (f^{in})^q(x, \xi) \left(\sup_{y \in B} \gamma(y - X) \right)^q dt dx d\xi.$$

Estimate (15) implies that

$$D \leq C(T, p)(1 + S_{p,T})^{\alpha q(1 + \frac{1}{p})} \int_0^T \int_x \int_\xi (1 + |\xi|)^{\frac{1}{p'}} (1 + |\xi|)^{\alpha q - \frac{1}{p'}} (f^{in})^q(x, \xi) \left(\sup_{y \in B} \gamma(y - X) \right)^q dt dx d\xi.$$

We split the right hand side of the above expression in two parts. In the first part given by

$$D_1 = \int_0^T \int_x \int_{B(0,1)} (1 + |\xi|)^{\frac{1}{p'}} (1 + |\xi|)^{\alpha q - \frac{1}{p'}} (f^{in})^q(x, \xi) \left(\sup_{y \in B} \gamma(y - X) \right)^q dt dx d\xi,$$

we integrate in ξ over the unit ball. We deduce immediately that

$$D_1 \leq C \|f^{in}\|_{L^q_{x,\xi}}^q.$$

The second part is given by

$$D_2 = \int_0^T \int_x \int_{cB(0,1)} (1 + |\xi|)^{\frac{1}{p'}} (1 + |\xi|)^{\alpha q - \frac{1}{p'}} (f^{in})^q(x, \xi) \left(\sup_{y \in B} \gamma(y - X) \right)^q dt dx d\xi.$$

Using Lemma 2 which states that trajectories do not stay a long time in a compact, we deduce by first integrating with respect to time at fixed x and fixed $\xi \in {}^cB(0,1)$ that a constant C exists such that

$$\int_0^T \left(\sup_{y \in B} \gamma(y - X(s, 0, x, \xi)) \right)^q ds \leq \frac{C(1 + S_{p,T})^{1+\frac{1}{p}}}{(1 + |\xi|)^{\frac{1}{p'}}}.$$

Hence we deduce that

$$D_2 \leq C(1 + S_{p,T})^{1+\frac{1}{p}} \|(1 + |\xi|)^{\alpha - \frac{1}{p'q}} f^{in}\|_{L_{x,\xi}^q}^q.$$

Putting it all back together, we obtain that

$$D \leq C(1 + S_{p,T})^{(\alpha q + 1)(1 + \frac{1}{p})} \|(1 + |\xi|)^{\alpha - \frac{1}{p'q}} f^{in}\|_{L_{x,\xi}^q}^q$$

which ends the proof of the moment effects in Proposition 2. \square

We now use the preceding study of the linear transport equation (10) to prove Theorems 5 and 6.

3 Estimates on the potential E (proof of Theorem 5).

The aim of this section is to prove Theorem 5. First, we prove estimates on

$$\|E\|_{L_{T_0}^p(L_B^\infty)}$$

using the moment effects in Proposition 2. Then we prove inequality (6) of Theorem 5 using Littlewood-Paley theory. Finally, we prove estimate (7) by combining estimate (6) and the global in time results obtained by P.-L. Lions and B. Perthame in Theorem 2.

3.1 Estimate on $\|E\|_{L_{T_0}^p(L_B^\infty)}$.

Let $\gamma \in \mathcal{D}(\mathbb{R}^3)$ with $\gamma(0) = 1$. We split E in a regular part E^1 and a rough part E^2 where

$$E^1(t, x) = \int \varphi(y)(1 - \gamma)(y)\rho(t, x - y)dy.$$

We have seen that E^1 can be estimate by (13). Hence it is enough to control the rough part E^2 . We have

$$|E^2(t, x)| \leq \left| \int \tilde{\gamma}(y)\varphi(y)\gamma(y)\rho(x - y)dy \right| \leq C\|\tilde{\gamma}\varphi\|_{L^{p'-0}}\|\gamma(x-\cdot)\rho\|_{L^{p+0}}.$$

As $p \geq 3$, we know that a constant C exists such that

$$\|\tilde{\gamma}\varphi\|_{L^{p'-0}} \leq C,$$

we deduce that

$$\left| \int \varphi(y)\gamma(y)\rho(x - y)dy \right| \leq C\|\gamma(x-\cdot)\rho\|_{L^{p+0}}.$$

Using Hölder inequalities, we deduce that

$$\|\gamma(x-\cdot)\rho\|_{L^{p+0}} = \|\gamma(x-\cdot)f\|_{L_x^{p+0}(L_\xi^1)} \leq C\|\gamma(x-\cdot)(1+|\xi|^{\frac{3}{p'}+0})f\|_{L_{x,\xi}^{p+0}}.$$

Let B be a ball such that $|B| \leq 1$. Then we have

$$\|E^2\|_{L_{T_0}^{p+0}(L_B^\infty)} \leq C\|\sup_{x \in B} \gamma(x-\cdot)(1+|\xi|^{\frac{3}{p'}+0})f\|_{L_{T_0,x,\xi}^{p+0}}.$$

We now apply Proposition 2 with $q = p + 0$ and $\alpha = \frac{3}{p'} + 0$ and we obtain

$$\|E^2\|_{L_{T_0}^{p+0}(L_B^\infty)} \leq C(1 + S_{p,T})^{(3-\frac{2}{p})(1+\frac{1}{p})+0} \|(1+|\xi|)^{\frac{3p-1}{pp'}+0} f^{in}\|_{L_{x,\xi}^{p+0}}$$

hence

$$\sup_{|B| \leq 1} \|E^2\|_{L_{T_0}^p(L^\infty(B))} \leq C(1 + S_{p,T})^{(3-\frac{2}{p})(1+\frac{1}{p})+0} (T_0)^{0+0} \|(1+|\xi|)^{\frac{3p-1}{pp'}+0} f^{in}\|_{L_{x,\xi}^{p+0}}. \quad (18)$$

As $(T_0)^{0+0} \rightarrow 0$ when $T_0 \rightarrow 0$, and combining estimate (18) with estimate (13), we deduce by a bootstrap argument that if T_0 is small enough then $S_{p,T_0}(E) < +\infty$. \square

3.2 Proof of estimate 6.

Let $f \in \mathcal{C}^\alpha(\mathbb{R}^d)$ where $0 < \alpha < 1$. Then, a constant $C_\alpha > 0$ exists such that

$$\|f\|_{\mathcal{C}^\alpha(\mathbb{R}^d)} \leq C_\alpha \sup_{q \in \mathbb{N}} 2^{q\alpha} \|\Delta_q f\|_{L^\infty(\mathbb{R}^d)}$$

where Δ_q is an operator of frequency localization in a ring of size 2^q (see for example the article by J.-Y. Chemin [9] for a precise definition of Δ_q and for the proof of the above result). Concerning the smooth part E^1 of the force field E , we have

$$\|E^1\|_{L_{[0,T_0]}^\infty(\mathcal{C}^1(\mathbb{R}^3))} \leq C\|f^{in}\|_{L_{x,\xi}^1}. \quad (19)$$

Let $\psi \in \mathcal{D}(B_2)$ where B_2 is a ball of size 2. Let us estimate ψE^2 . Let $q \geq 1$, then

$$\Delta_q(\psi E^2)(x) = \psi \Delta_q E^2(x) + [\Delta_q(\psi E^2) - \psi \Delta_q E^2](x).$$

• **Estimate on $\Delta_q(\psi E^2) - \psi \Delta_q E^2$.** The following lemma holds.

Lemma 3 *Let $f^{in} \in L_{x,\xi}^\infty$ and $\nu \in]0, 1[$. Then, $m(\nu) < 6$ exists such that if*

$$\|f^{in}\|_{L_{x,\xi}^{1,m(\nu)}} < +\infty,$$

then, for all $T_0 > 0$, a constant C exists such that for all $q \geq 0$

$$\|\Delta_q(\psi E^2) - \psi \Delta_q E^2\|_{L^\infty([0,T_0])(L^\infty(\mathbb{R}^3))} \leq C2^{-q(1-\nu)}.$$

Proof of Lemma 3. Let us first prove the following standart lemma.

Lemma 4 *Let a, b be two functions. Then, a constant C exists such that for all $q \geq 0$, for all $r \in]1, +\infty]$,*

$$\left\| \Delta_q(ab) - a\Delta_q b \right\|_{L^\infty(\mathbb{R}^d)} \leq C 2^{-q + \frac{dq}{r}} \|a\|_{C^1} \|b\|_{L^r(\mathbb{R}^d)}.$$

Proof of Lemma 4. There exists a function $h \in \mathcal{S}(\mathbb{R}^d)$ such that for all $x \in \mathbb{R}^d$ the following equality holds

$$\Delta_q(ab) - a\Delta_q b(x) = 2^{qd} \int h(2^q(x-y)) \frac{(a(x) - a(y))}{|x-y|} |x-y| b(y) dy.$$

We have

$$|\Delta_q(ab) - a\Delta_q b(x)| \leq C 2^{-q} \|a\|_{C^1} 2^{qd} \int 2^q |x-y| |h(2^q(x-y)) b(y)| dy.$$

Using Hölder inequalities, we obtain Lemma 4. □

We deduce from Lemma 4 and Theorem 2 that for all $\nu \in]0, 1[$, $m(\nu) < 6$ exists such that if

$$\|f^{in}\|_{L_{x,\xi}^{1,m(\nu)}} < +\infty,$$

then, for all $T_0 > 0$, a constant C exists such that for all $q \geq 0$

$$\left\| \Delta_q(\psi E^2) - \psi \Delta_q E^2 \right\|_{L_{[0,T_0]}^\infty(L^\infty(\mathbb{R}^3))} \leq C 2^{-q(1-\nu)} \quad (20)$$

which proves Lemma 3. □

• **Estimate on $\psi \Delta_q E^2$.** The following Lemma holds

Lemma 5 *Let $3 \leq p < +\infty$ and $f^{in} \in L_{x,\xi}^\infty$. Then $m(p) < 6$ exists such that if*

$$\|f^{in}\|_{L_{x,\xi}^{1,m(p)}} < +\infty$$

then, for all $T_0 > 0$ a constant C exists such that for all $q \in \mathbb{N}$, for all ball $B \subset \mathbb{R}^3$ of size 1 and $t \in [0, T_0]$

$$\|\Delta_q E^2(t)\|_{L^\infty(B)} \leq C 2^{q(-1 + \frac{3}{p+0})} \left[1 + \sup_{x \in B_2} \|\tilde{\gamma}(x-\cdot)\rho(t)\|_{L^{p+0}} \right]$$

where $|B_2| = 2$.

Proof of Lemma 5. We have

$$\psi \Delta_q E^2(x) = \psi \left[\Delta_q(\gamma\varphi) * \rho \right](x).$$

We split $\Delta_q E^2(t, x)$ as follows

$$\Delta_q E^2(t, x) = \int \Delta_q(\gamma\varphi)(y)\rho(t, x - y)dy = I_{1,q}(t, x) + I_{2,q}(t, x)$$

where

$$I_{1,q}(t, x) = \int \gamma\Delta_q(\varphi)(y)\rho(t, x - y)dy.$$

- *Estimate of $I_{1,q}$.* We can write

$$I_{1,q}(t, x) = \int \gamma\Delta_q(\varphi)(y)\tilde{\gamma}\rho(t, x - y)dy.$$

and where $\tilde{\gamma} \in C_c^\infty(\mathbb{R}^3)$ with $\tilde{\gamma} \equiv 1$ on $B(0, \frac{1}{2})$. Let $\infty+ > p \geq 3$. Using Hölder inequalities, we deduce that for all $(t, x) \in \mathbb{R} \times \mathbb{R}^3$ and $q \geq 1$

$$|I_1(t, x)| \leq \|\gamma\Delta_q(\varphi)\|_{L^{(p+0)'}} \|\tilde{\gamma}(x-\cdot)\rho(t)\|_{L^{p+0}}.$$

The following lemma holds.

Lemma 6 *Let $\gamma \in \mathcal{D}(\mathbb{R}^3)$. Let $1 \leq r \leq \frac{3}{2} + 0$. Then a constant C exists such that for all $q \geq 1$*

$$\left\| \Delta_q\left(\frac{x}{|x|^3}\gamma\right) - \Delta_q\left(\frac{x}{|x|^3}\right)\gamma \right\|_{L^r} \leq C2^{q(-1+0)}. \quad (21)$$

Let $p' \in [1, \frac{3}{2}[$. Then, a constant C which only depends on the measure of the support of γ , p' and $\|\gamma\|_{C^1}$ exists such that for all $q \geq 1$

$$\left\| \gamma\Delta_q\left(\frac{x}{|x|^3}\right) \right\|_{L^{p'}} \leq C2^{q(-1+\frac{3}{p}+0)} \quad (22)$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof of Lemma 6. Assume for example that $\gamma \in \mathcal{D}(B(0, 1))$. We first prove estimate (21). We set

$$\left[\Delta_q\left(\frac{x}{|x|^3}\gamma\right) - \Delta_q\left(\frac{x}{|x|^3}\right)\gamma \right] = S(x).$$

There exists a function $h \in \mathcal{S}(\mathbb{R}^3)$ such that

$$S(x) = 2^{qd} \int h(2^q(x - y))[\gamma(x) - \gamma(y)]\frac{y}{|y|^3}dy.$$

Let $a \in \mathcal{D}(\mathbb{R}^3)$ with $a \equiv 1$ on the ball $B(0, 2)$. We split S in two parts

$$S(x) = S_1(x) + S_2(x)$$

with

$$S_1(x) = 2^{qd} \int h(2^q(x - y))[\gamma(x) - \gamma(y)]a(y)\frac{y}{|y|^3}dy.$$

Study of S_1 . We have

$$S_1(x) = 2^{q(d-1)} \int h(2^q(x-y)) \frac{(\gamma(x) - \gamma(y))}{|x-y|} 2^q |x-y| a(y) \frac{y}{|y|^3} dy. \quad (23)$$

We deduce that

$$|S_1(x)| \leq 2^{q(d-1)} \|\gamma\|_{C^1} \int |h|(2^q(x-y)) 2^q |x-y| a(y) \frac{y}{|y|^3} dy.$$

Using Young inequalities, we get

$$\|S_1\|_{L^1} \leq C 2^{-q} \left\| a \frac{y}{|y|^3} \right\|_{L^1} \leq C 2^{-q}. \quad (24)$$

We deduce also by Young inequalities that

$$\|S_1\|_{L^{\frac{3}{2}+0}} \leq C 2^{q(-1+0)} \left\| a(y) \frac{y}{|y|^3} \right\|_{L^{\frac{3}{2}-0}} \leq C 2^{q(-1+0)}. \quad (25)$$

Interpolate estimates (25) and estimate (24), we deduce that for all $r \in [1, \frac{3}{2} + 0]$

$$\|S_1\|_{L^{\frac{3}{2}+0}} \leq C 2^{q(-1+0)}.$$

Study of S_2 . We have

$$S_2(x) = 2^{qd} \int h(2^q(x-y)) (\gamma(x) - \gamma(y)) (1 - a(y)) \frac{y}{|y|^3} dy.$$

Here, in the integral, the truncated function $1 - a$ implies that y remains outside the ball $B(0, 2)$. So we have $\gamma(y) \equiv 0$ in the integral. If moreover $x \in {}^c B(0, 1)$ we also have $\gamma(x) = 0$ because the support of γ is contained in the ball $B(0, 1)$. Hence S_2 is compactly supported in the ball $B(0, 1)$. So we have

$$\|S_2\|_{L^1} \leq C \|S_2\|_{L^2}.$$

Multiplying above and below by $|x-y|$, we obtain

$$|S_2(x)| \leq \|\gamma\|_{C^1} 2^{q(d-1)} \int |h|(2^q(x-y)) 2^q |x-y| \left| (1 - a(y)) \frac{y}{|y|^3} \right| dy.$$

Applying Young inequalities, we obtain that for all $r \in [1, 2]$

$$\|S_2\|_{L^r} \leq C \|S_2\|_{L^2} \leq C 2^{-q} \left\| (1 - a(y)) \frac{y}{|y|^3} \right\|_{L^2} \leq C 2^{-q}$$

which conclude the proof of estimate (21).

Let us now prove estimate (22). We deduce from estimate (21) that a constant C exists such that for all $q \geq 1$

$$\left\| \gamma \Delta_q \left(\frac{x}{|x|^3} \right) \right\|_{L^{\frac{3}{2}-0}} \leq \left\| \Delta_q \left(\frac{x}{|x|^3} \right) \gamma \right\|_{L^{\frac{3}{2}-0}} + \left\| \Delta_q \left(\gamma \frac{x}{|x|^3} \right) \right\|_{L^{\frac{3}{2}-0}} \leq C + \left\| \gamma \frac{x}{|x|^3} \right\|_{L^{\frac{3}{2}-0}} < +\infty,$$

by interpolation, we reduce the proof of estimate (22) to the case $p' = 1$. A constant C exists such that for all $q \geq 1$

$$\left\| \gamma \Delta_q \left(\frac{x}{|x|^3} \right) \right\|_{L^1} \leq \left\| \Delta_q \left(\frac{x}{|x|^3} \right) \right\|_{L^1} \leq C 2^{-q}$$

which proves estimate (22) and Lemma 6. \square

Let B be a ball of size 1. Taking $\psi \in \mathcal{D}(B_2)$ such that $\psi \equiv 1$ on B , we deduce from Lemma 6 that for all $3 \leq p < +\infty$ a constant C exists such that for all $q \geq 1$, for all $t \in \mathbb{R}$

$$\|I_{1,q}(t)\|_{L^\infty(B)} \leq \|\psi I_{1,q}\|_{L^\infty(\mathbb{R}^3)} \leq C 2^{q(-1+\frac{3}{p+0})} \sup_{x \in B_2} \|\tilde{\gamma}(x-\cdot)\rho(t)\|_{L^{p+0}}. \quad (26)$$

• *Estimate of $I_{2,q}$.* We have

$$I_{2,q}(t, x) = \int [\Delta_q(\gamma\varphi) - \gamma\Delta_q(\varphi)](y)\rho(t, x-y)dy.$$

Applying Hölder inequalities, we deduce that a constant C exists such that for all $t \in \mathbb{R}$,

$$\|I_{2,q}(t)\|_{L^\infty(\mathbb{R}^3)} \leq C \|\Delta_q(\gamma\varphi) - \gamma\Delta_q(\varphi)\|_{L^{\frac{3}{2}+0}} \|\rho(t)\|_{L^{3-0}}. \quad (27)$$

Applying estimate (21) in estimate (27), we deduce that

$$\|I_{2,q}(t)\|_{L^\infty(\mathbb{R}^3)} \leq C 2^{q(-1+0)} \|\rho(t)\|_{L^{3-0}}.$$

As $3-0 < 3$, using Theorem 2, we deduce that $m < 6$ exists such that if

$$\|f^{in}\|_{L_{x,\xi}^{1,m}} < +\infty,$$

then, for all $T_0 > 0$ a constant C exists such that for all $t \in [0, T_0]$

$$\|\rho(t)\|_{L^{3-0}} \leq C.$$

We deduce that for all $T_0 > 0$ a constant C exists such that for all $t \in [0, T_0]$

$$\|I_{2,q}(t)\|_{L^\infty(\mathbb{R}^3)} \leq C 2^{q(-1+0)}. \quad (28)$$

Combining estimate (28) with estimate (26), we deduce that for all $\infty+ > p \geq 3$, $m(p) < 6$ exists such that if

$$\|f^{in}\|_{L_{x,\xi}^{1,m(p)}} < +\infty,$$

then, for all $T_0 > 0$ a constant C exists such that for all $q \geq 1$, and $t \in [0, T_0]$

$$\|\psi \Delta_q E^2(t)\|_{L^\infty(\mathbb{R}^3)} \leq 2^{q(-1+\frac{3}{p+0})} C \left[1 + \sup_{x \in B_2} \|\tilde{\gamma}(x-\cdot)\rho(t)\|_{L^{p+0}} \right]. \quad (29)$$

To treat the case $q = 0$, we remark that

$$\|\Delta_0 E^2\|_{L^\infty(\mathbb{R}^3)} \leq C \|E^2\|_{L^2(\mathbb{R}^3)}.$$

Using Theorem 2, we deduce that $m(2) < 6$ exists such that if

$$\|f^{in}\|_{L_{x,\xi}^{1,m(2)}} < +\infty,$$

then, for all $T_0 > 0$, a constant C exists such that

$$\|\Delta_0 E^2\|_{L_{[0,T_0]}^\infty(L^\infty(\mathbb{R}^3))} \leq C \|E^2\|_{L_{[0,T_0]}^\infty(L^2(\mathbb{R}^3))} \leq C. \quad (30)$$

which conclude the proof of Lemma 5. \square

Combining estimates (29), (30) and Lemma 3, we deduce that for all $\infty+ > p \geq 3$, $m(p) < 6$ exists such that if

$$\|f^{in}\|_{L_{x,\xi}^{1,m(p)}} < +\infty,$$

then, for all $T_0 > 0$, a constant C exists such that for all $t \in [0, T_0]$

$$\|E^2(t)\|_{(C^{1-\frac{3}{p+0}}(B))} \leq C \left(1 + \sup_{x \in B_2} \|\tilde{\gamma}(x-\cdot)\rho(t)\|_{L^{p+0}}\right)$$

where the size of B_2 is 2. If $T_0 > 0$ is small enough, then $S_{p,T_0} < +\infty$ and we can apply moment effects in Proposition 2. We deduce that for all $\infty+ > p \geq 3$, $m(p) < 6$ exists such that if

$$\|f^{in}\|_{L_{x,\xi}^{1,m(p)}} < +\infty$$

and if T_0 is small enough then a constant C exists such that

$$\sup_{|B| \leq 1} \|E^2\|_{L_{T_0}^p(C^{1-\frac{3}{p+0}}(B))} \leq C \left[\|(1+|\xi|)^{\frac{3p-1}{pp'}+0} f^{in}\|_{L_{x,\xi}^{p+0}} + 1 \right]. \quad (31)$$

Combining estimates (31) and (19) we deduce estimate (6). \square

3.3 Proof of estimate (7).

The problem with estimate (18) is that on the right hand side the exponent of S_{p,T_0} is strictly bigger than $1 - 0$. So we cannot control S_{p,T_0} for arbitrarily large time. The idea to prove estimate (7) is to localize E in frequency using the Littlewood-Paley decomposition and to estimate $\Delta_k E$ on the one hand with the results obtained by P.-L. Lions and B. Perthame in which the constant S_{p,T_0} does not appear and on the other hand with our result. This allows us by interpolation to estimate S_{p,T_0} for all $T_0 > 0$. In particular, the important fact here is the information given by estimate (6). Thus we obtain an estimate of E in a space of type C^α which is better than L^∞ . Moreover, if $p \in]3, \frac{10+\sqrt{88}}{6}[$, then in estimate (6), the loss of moments in $L_{x,\xi}^1$ is strictly less than 6. Let us now explain how we can interpolate our results with those of Theorem 2 from P.-L. Lions and B. Perthame.

Let B be a ball of size 1. For all $\theta \in [0, 1]$, for all $k \in \mathbb{N}$, the following estimate holds

$$\|\Delta_k E\|_{L^\infty(B)} \leq \|\Delta_k E\|_{L^\infty(B)}^\theta \|\Delta_k E\|_{L^\infty(B)}^{1-\theta}. \quad (32)$$

There are two ways to estimate $\|\Delta_k E\|_{L^\infty(B)}$.

- The first one uses the results from P.-L. Lions and B. Perthame and Bernstein inequalities (see the article by J.-Y. Chemin in [9]). We obtain that for all $q \in]\frac{3}{2}, +\infty[$, $m(q) < 6$ exists such that if

$$\|f^{in}\|_{L_{x,\xi}^{1,m(q)}} < +\infty,$$

then, for all $T > 0$ a constant C exists such that for all $k \in \mathbb{N}$

$$\|\Delta_k E\|_{L^\infty([0,T] \times \mathbb{R}^3)} \leq C 2^{\frac{3}{q}} \|\Delta_k E\|_{L_{[0,T]}^\infty(L^q)} \leq C 2^{\frac{3}{q}}. \quad (33)$$

Estimate (33) is good for two reasons.

- First, it holds for initial data having strictly less than 6 moments in $L_{x,\xi}^1$.
- Secondly, the bad term S_{p,T_0} does not appear.

The only problem of this estimate is that the norm in which E is expressed is too rough to control E in L^∞ .

- The second way to estimate $\|\Delta_k E\|_{L^\infty(B)}$ is to use our approach which makes it possible to control E in spaces of type C^α using strictly less than 6 moments in $L_{x,\xi}^1$ if $\alpha > 0$ is small enough. More precisely, Lemma 5 gives us for all $p \in]3, \frac{10+\sqrt{88}}{6}[$ the existence of $m(p) < 6$ such that if

$$\|f^{in}\|_{L_{x,\xi}^{1,m(p)}} < +\infty,$$

then, for all $T_0 > 0$ a constant C exists such that for all $t \in [0, T_0]$

$$\|\Delta_k E^2(t)\|_{L^\infty(B)} \leq C 2^{q(-1+\frac{3}{p+0})} \left[\sup_{x \in B_2} \|\tilde{\gamma}(x-\cdot)\rho(t)\|_{L^{p+0}} + 1 \right] \quad (34)$$

where B_2 a ball of size 2. Combining estimates (34) and (33) with q big enough, and applying estimate (32) with θ small enough such that the exponent of S_{p,T_0} which will appear later in estimates being strictly less than 1 i.e with $\theta = \left((3 - \frac{2}{p})(1 + \frac{1}{p}) + 0 \right)^{-1}$, we deduce that for all $p \in]3, \frac{10+\sqrt{88}}{6}[$, $m(p) < 6$ exists such that if

$$\|f^{in}\|_{L_{x,\xi}^{1,m(p)}} < +\infty,$$

then, for all $T_0 > 0$ a constant C exists such that for all $t \in [0, T_0]$

$$\|\Delta_k E(t)\|_{L^\infty(B)} \leq C 2^{-k(0+0)} \left(\sup_{x \in B_2} \|\tilde{\gamma}(x-\cdot)\rho(t)\|_{L^{p+0}}^\theta + 1 \right). \quad (35)$$

Combining Lemma 3 and estimate (35) we obtain that for all $p \in]3, \frac{10+\sqrt{88}}{6}[$, $m(p) < 6$ exists such that if

$$\|f^{in}\|_{L_{x,\xi}^{1,m(p)}} < +\infty,$$

then, for all $T_0 > 0$ a constant C exists that for all $t \in [0, T_0]$

$$\|E(t)\|_{C^{0+0}(B)} \leq C \left(\sup_{x \in B_2} \|\tilde{\gamma}(x-\cdot)\rho(t)\|_{L^{p+0}}^\theta + 1 \right).$$

Remark 8 Here, we must be careful of the fact that the norm on E in C^{0+0} is taken over a ball and not over the whole space \mathbb{R}^3 which is a priori a problem to identify the space C^α with $\alpha > 0$ using Littlewood-paley decomposition. But we have shown in Lemma 3 that in fact it is not the case.

Now by integrating over time, we can apply the moment effects in Proposition 2 obtained for the transport equation with rough force fields (10) to control the term

$$\sup_{x \in B_2} \|\tilde{\gamma}(x \cdot \cdot) \rho(t)\|_{L^{p+0}}^\theta \leq C \sup_{x \in B_2} \|\gamma(x \cdot \cdot) (1 + |\xi|)^{\frac{3}{p'}+0} f(t)\|_{L_{x,\xi}^{p+0}}^\theta$$

with norms on the initial data using strictly less than 6 moments whenever we assume that $p \in]3, \frac{10+\sqrt{88}}{6}[$. As $\theta = \left((3 - \frac{2}{p})(1 + \frac{1}{p}) + 0 \right)^{-1}$, we deduce that for all $p \in [3, \frac{10+\sqrt{88}}{6}[$, $m(p) < 6$ exists such that if

$$\|f^{in}\|_{L_{x,\xi}^{1,m(p)}} < +\infty,$$

then, for all $T_0 > 0$ a constant C exists that for all $t \in [0, T_0]$

$$S_{p,T_0}(E) \leq C(1 + S_{p,T_0}(E))^{1-0}.$$

This concludes the proof of estimate (7) if $p \geq 3$ is small enough. To obtain estimate (7) for all $p \in [3, +\infty[$, we have to use an interpolation argument which is detailed at the end of this article right after the proof of Lemma 8. \square

4 Proof of Theorem 6.

The idea behind our proof of Theorem 6 is to use the control on the trajectories given by Proposition 1 during the study of the linear transport equation with a rough force field (10). Even though this control is rather crude since the approximation gets worse as ξ grows, it provides enough information to ensure that two trajectories with different initial velocity will end up far from each other after a given time. This allows us to prove moment effects on the solution along each trajectory and to have the following control on E which gives Theorem 6 directly.

Proposition 3 Let $f^{in} \in L_{x,\xi}^\infty$ and $p \geq 3$. Then $m(p) < 6$ exists such that if

$$\|f^{in}\|_{L_{x,\xi}^{1,m(p)}} < \infty,$$

then, for all $T > 0$, a constant C exists such that the following estimate holds

$$\sup_{(x_0, \xi_0) \in \mathbb{R}^6} \|E^2(s, X(s, x_0, \xi_0))\|_{L^p([0, T])} \leq C. \quad (36)$$

Proof of Proposition 3. To prove Proposition 3 we first treat the case $p = 3$ and then we study the general case.

- *Case $p = 3$.* By taking $m(3) < 6$ big enough, we know from Theorem 5 that for all $T_0 > 0$, $S_{3,T_0} < +\infty$. So we can apply to the characteristics of the Vlasov-Poisson equation (1) all the

results obtained for the transport equation (10).

Let $(x_0, \xi_0) \in \mathbb{R}^6$. We have

$$|E^2(s, X(s, x_0, \xi_0))| \leq C \|\gamma(X(s, x_0, \xi_0) - \cdot)\rho\|_{L^{3+0}}. \quad (37)$$

We are going to gain moments on the initial data using the fact that

$$s \rightarrow X(s, 0, x, \xi) - X(0, s, x_0, \xi_0)$$

do not stay a long time in a compact using the approximation on trajectories given by Proposition 1. To understand what happens, let us consider a simpler case where the characteristic X is that given by the free transport equation. We are reduced to studying under which assumptions we can say that the function

$$\tilde{X}(s, x, \xi) : s \rightarrow x - x_0 + s(\xi + \xi_0)$$

does not remain a long time in a compact set. We observe in particular that the closer ξ gets to $-\xi_0$, the longer the trajectory \tilde{X} stays in a compact set. In our context, we must furthermore take into consideration the fact that the trajectories are a perturbation of those given by the free transport equation by a factor $|\xi|^{\frac{1}{3}}$. Hence, we are going to split the integral $\int f(t, x, \xi)d\xi = \rho(t, x)$ in two areas defined by an unknown parameter $1 > \alpha \geq \frac{1}{3}$ which will be optimized later on.

- The first area is localized on a ball $B(-\xi_0, \beta|\xi_0|^\alpha)$ where β is a constant taken large enough. In this area, dispersion is bad. Hence we must rely on the fact that the size of the ball where velocities remains is relatively small. Hence, here the estimate is done by using Hölder inequalities.
- The second area is localized outside the ball $B(-\xi_0, \beta|\xi_0|^\alpha)$. In this area where ξ is far enough from $-\xi_0$, we can use some moment effects to control the term of estimate (37).

Definition 2 Let $(x_0, \xi_0) \in \mathbb{R}^3 \times \mathbb{R}^3$. Let us define

$$A_{\xi_0} = B(-\xi_0, \beta|\xi_0|^\alpha)$$

where β is a large enough constant and $B_{\xi_0} = {}^c A_{\xi_0}$. Let $A \subset \mathbb{R}^3$ be a subset of the velocity space. We then define

$$\Gamma_A(x_0, \xi_0) = \int_0^T \int_x \int_A (1 + |\xi|)^{6+0} (f^{in})^{3+0}(x, \xi) \gamma(X(s, 0, x, \xi) - X(0, s, x_0, \xi_0)) dt dx d\xi.$$

In the following, we assume that $|\xi_0| \geq 1$. To deal with the case $|\xi_0| \leq 1$, splitting of the integral is unnecessary since the same strategy as the adopted below for the area with dispersion may be applied.

We split the density ρ as follows

$$\rho(t, x) = \rho^1(t, x) + \rho^2(t, x) \quad \text{with} \quad \rho^1(t, x) = \int_{A_{\xi_0}} f(t, x, \xi) d\xi.$$

Area A_{ξ_0} , without dispersion. Over A_{ξ_0} , dispersive effects are not needed. Using Hölder estimates we obtain that

$$\|\gamma(X(s, x_0, \xi_0) - \cdot)\rho^1\|_{L^{3+0}(\mathbb{R}^3)} \leq C\|f\|_{L_{\mathbb{R}^3 \times A_{\xi_0}}^{3+0}} (1 + |\xi_0|)^{2\alpha} \leq C\|(1 + |\xi|)^{2\alpha} f\|_{L_{\mathbb{R}^3 \times A_{\xi_0}}^{3+0}}.$$

Using the propagation of moments given by Proposition 2, we obtain

$$\|\gamma(X(s) - \cdot)\rho^1\|_{L^{3+0}(\mathbb{R}^3)} \leq C\|(1 + |\xi|)^{2\alpha} f^{in}\|_{L_{x,\xi}^{3+0}}. \quad (38)$$

We now have to estimate

$$\|\gamma(X(s, x_0, \xi_0) - \cdot)\rho^2\|_{L^{3+0}([0, T_0] \times \mathbb{R}^3)}.$$

Applying Hölder inequalities and the change of variable $(X, V) \rightarrow (x, \xi)$, as has been done previously, we deduce that

$$\|\gamma(X(s, x_0, \xi_0) - \cdot)\rho^2\|_{L^{3+0}([0, T_0] \times \mathbb{R}^3)} \leq C\Gamma_{V^{-1}({}^c A_{\xi_0})}(x_0, \xi_0). \quad (39)$$

Since V is a perturbation of magnitude $|\xi|^{1/3}$ of the identity, we deduce that in integral (39) we can replace the velocity domain of integration $V^{-1}({}^c A_{\xi_0})$ with the domain ${}^c A_{\xi_0}$ at the cost of a decrease in the constant β . We are reduced to estimating $\Gamma_{{}^c A_{\xi_0}}(x_0, \xi_0) = \Gamma_{B_{\xi_0}}(x_0, \xi_0)$.

Area B_{ξ_0} , with dispersion. In the case where ξ remains in B_{ξ_0} , the following Lemma gives us a dispersive effect and moment effects which allows us to control E along trajectories.

Lemma 7 *A constant C exists such that for all $(x_0, \xi_0) \in \mathbb{R}^3 \times \mathbb{R}^3$, $\xi \in B_{\xi_0}$*

$$\mu\left\{s, \tilde{X}(s, x, \xi) = X(s, 0, x, \xi) - X(0, s, x_0, \xi_0) \in B(0, 1)\right\} \leq \frac{C}{\max(|\xi|, |\xi_0|)^{\alpha - \frac{1}{3}}} \quad (40)$$

where μ denotes the Lebesgue measure, and

$$\Gamma_{B_{\xi_0}}(x_0, \xi_0) \leq C\|(1 + |\xi|)^{2 + \frac{1}{9} - \frac{\alpha}{3}} f^{in}\|_{L_{x,\xi}^{3+0}}.$$

Proof of Lemma 7. To simplify calculations, we assume that $x = x_0$; the case when $x \neq x_0$ add essentially a translation, which does not affect dispersion phenomena. Using Proposition 1, we obtain

$$\left|\tilde{X}(s, x, \xi)\right| \geq s|\xi + \xi_0| - C\max(|\xi|, |\xi_0|)^{\frac{1}{3}}.$$

As $\xi \in B_{\xi_0}$, we have

$$|\xi + \xi_0| \geq C\max(|\xi_0|, |\xi|)^\alpha.$$

Indeed, $|\xi + \xi_0| = R$ where $R \geq |\xi_0|^\alpha$. Two possibilities arise. Either, $R \geq 2|\xi_0|$ which implies that $|\xi| \sim R$ and in turn that

$$|\xi + \xi_0| \geq C\max(|\xi_0|, |\xi|)^\alpha.$$

Or $R \leq 2|\xi_0|$, then $|\xi| \leq 3|\xi_0|$, and so

$$|\xi + \xi_0| = R \geq |\xi_0|^\alpha \geq C \max(|\xi_0|, |\xi|)^\alpha.$$

We deduce that

$$\left| \tilde{X}(s, x, \xi) \right| \geq s \max(|\xi_0|, |\xi|)^\alpha - C \max(|\xi|, |\xi_0|)^{\frac{1}{3}}$$

which proves the first point of Lemma 7. By first integrating with respect to time over $\Gamma_{B_{\xi_0}}(x_0, \xi_0)$ and then applying estimate (40), we prove the second part of Lemma 7. \square

Optimizing the parameter α such that the loss of moments between 2α and $2 + \frac{1}{9} - \frac{\alpha}{3}$ are optimal, we obtain $\alpha = \frac{19}{21}$; which proves Proposition 3 for $p = 3$.

• *General case.* The proof of this estimate follows by interpolation between the estimate of Proposition (3) with $p = 3$ and the estimate on E^2 given by the following Lemma.

Lemma 8 *For all $T_0 > 0$, a constant C exists such that the following estimate holds*

$$\|E^2\|_{L^\infty_{[0, T_0]}(L^\infty(\mathbb{R}^3))} \leq C(T_0, S_{3, T_0}) \|(1 + |\xi|)^{2+0} f^{in}\|_{L^3_{x, \xi}}. \quad (41)$$

Proof of Lemma 8. Let $t \in [0, T_0]$. Using Hölder inequalities, we deduce that

$$\|E^2(t)\|_{L^\infty_x} \leq C \|\gamma \rho\|_{L^{3+0}} \|\varphi\|_{L^{\frac{3}{2}-0}} \leq C \|(1 + |\xi|)^{2+0} f\|_{L^3_{x, \xi}}.$$

Using the fact that the solution of equation (1) propagates moments (see Proposition 2), we deduce that

$$\|E^2\|_{L^\infty([0, T] \times \mathbb{R}^3)} \leq C \|(1 + |\xi|)^{2+0} f^{in}\|_{L^3_{x, \xi}}$$

which concludes the proof of Lemma 8. \square

We introduce the following definition which will be useful during interpolation.

Definition 3 *Let $\psi \in \mathcal{D}(\mathbb{R}^3)$ and $\varphi \in \mathcal{D}(\mathbb{R}^3 \setminus \{0\})$ such that*

$$\psi(\cdot) + \sum_{k \in \mathbb{N}^*} \varphi(2^{-k} \cdot) \equiv 1.$$

Let f be a function. For $k \in \mathbb{N}$, we define the following operators of localization in a ring of size 2^k by

$$T_0 f = f \psi \quad \text{and} \quad T_k f = \varphi(2^{-k} \cdot) f.$$

$$E^2(t, x) = \sum_{k \in \mathbb{N}} E_k^2(t, x) \quad \text{where} \quad E_k^k(t, x) = \nabla(\Delta)^{-1} \rho_k$$

with

$$\rho_k(t, x) = \int T_k f(t, x, \xi) d\xi$$

where here the localization obtained with T_k is done with respect to velocity. As the estimate (36) holds for $p = 3$, we deduce that the characteristics are a perturbation of those given by

the transport equation. In particular, following the proof given for $p = 3$, we deduce that a constant C exists (which depends on S_{3,T_0} , T_0) such that for all $k \in \mathbb{N}$

$$\sup_{(x_0, \xi_0) \in \mathbb{R}^6} \|E_k^2(s, X(s, x_0, \xi_0))\|_{L_{T_0}^3} \leq C \|(1 + |\xi|)^{2\alpha+0} \tilde{T}_k f^{in}\|_{L^{3+0}}$$

and

$$\|E_k^2\|_{L_{T_0}^\infty(L^\infty(\mathbb{R}^3))} \leq C \|(1 + |\xi|)^{2+0} \tilde{T}_k f^{in}\|_{L^{3+0}}$$

where \tilde{T}_k is an operator of localization in velocity in a ring of size 2^k . As the characteristics of the Vlasov-Poisson equation are a small perturbation of those given by the free transport equation (see Proposition 3 with $p = 3$), we deduce that for all $\beta \geq 0$

$$\|(1 + |\xi|)^\beta T_k f^{in}\|_{L^{3+0}} \sim 2^{k\beta} \|\tilde{T}_k f^{in}\|_{L^{3+0}}.$$

Using the fact that

$$\sup_{(x_0, \xi_0) \in \mathbb{R}^6} \|E_k^2(s, X(s, x_0, \xi_0))\|_{L_{T_0}^p} \leq \sup_{(x_0, \xi_0) \in \mathbb{R}^6} \|E_k^2(s, X(s, x_0, \xi_0))\|_{L_{T_0}^\infty}^{\theta_0} \sup_{(x_0, \xi_0) \in \mathbb{R}^6} \|E_k^2(s, X(s, x_0, \xi_0))\|_{L_{T_0}^3}^{1-\theta_0}$$

where $\frac{1-\theta_0}{3} = \frac{1}{p}$. We deduce that

$$\sup_{(x_0, \xi_0) \in \mathbb{R}^6} \|E_k^2(s, X(s, x_0, \xi_0))\|_{L_{T_0}^p} \leq C 2^{k[(2\alpha)\theta_0 + (2+0)(1-\theta_0)]} \|\tilde{T}_k f^{in}\|_{L_{x, \xi}^{3+0}} 2^{-(0+0)k}. \quad (42)$$

We have

$$\sup_{(x_0, \xi_0) \in \mathbb{R}^6} \|E^2(s, X(s, x_0, \xi_0))\|_{L_{T_0}^p} \leq \sum_{k=0}^{+\infty} \sup_{(x_0, \xi_0) \in \mathbb{R}^6} \|E_k^2(s, X(s, x_0, \xi_0))\|_{L_{T_0}^p}.$$

Using estimate (42) and applying Hölder inequalities, we deduce that

$$\sup_{(x_0, \xi_0) \in \mathbb{R}^6} \|E^2(s, X(s, x_0, \xi_0))\|_{L_{T_0}^p} \leq C \left(\sum_{k=0}^{+\infty} 2^{(3+0)k[(2\alpha)\theta_0 + (2+0)(1-\theta_0)]} \|\tilde{T}_k f^{in}\|_{L_{x, \xi}^{3+0}} \right)^{\frac{1}{3+0}}.$$

Using the fact that on the one hand $N > 0$ exists such that for all $j, k \in \mathbb{N}$ such that $|j - k| \geq N$, for all functions f

$$\text{supp}_\xi \left(\tilde{T}_k f \cap \tilde{T}_j f \right) = \emptyset,$$

and that on the other hand $(3+0)[(2\alpha)\theta_0 + (2+0)(1-\theta_0)] < 6$, we deduce Proposition 3 in the general case. \square

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