



Some examples of ‘second order elliptic integrable systems associated to a 4-symmetric space’

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1 Hamiltonian Stationary Lagrangian (HSL) surfaces

1.1 A variational problem in \mathbb{R}^4

\mathbb{R}^4 has the canonical Euclidean structure $\langle \cdot, \cdot \rangle$ and the symplectic form $\omega := dx^1 \wedge dx^2 + dx^3 \wedge dx^4$. An immersed surface $\Sigma \subset \mathbb{R}^4$ is

- (i) **Lagrangian** iff $\omega|_{\Sigma} = 0$
- (ii) **Hamiltonian Stationary Lagrangian (HSL)** iff $\omega|_{\Sigma} = 0$ and Σ is a critical point of the area functional \mathcal{A} with respect to all *Hamiltonian vector fields* ξ_h s.t.:
 - $\exists h \in \mathcal{C}_c^\infty(\mathbb{R}^4)$, $\xi_h \lrcorner \omega + dh = 0$
 - equivalently, if J is the complex structure s.t. $\omega = \langle J\cdot, \cdot \rangle$, $\xi_h = J\nabla h$.

It means that $\delta\mathcal{A}_\Sigma(\xi_h) = 0, \forall h \in \mathcal{C}_c^\infty(\mathbb{R}^4)$.

What is the Euler equation ?

The Gauss map is:

$$\gamma : \Sigma \longrightarrow \begin{array}{ccc} Gr_{Lag}(\mathbb{R}^4) & \subset & Gr_2(\mathbb{R}^4) \\ S^1 \times S^2 & \subset & S^2 \times S^2 \end{array}$$

Denote $\gamma = (\rho_\Sigma, \sigma_\Sigma)$ the two components of γ . For a Lagrangian immersion $\rho_\Sigma \simeq e^{i\beta}$. Then the *mean curvature vector* is

$$\vec{H} = J\nabla\beta.$$

Lemma 1.1 Σ is HSL iff

$$\begin{cases} \omega|_{\Sigma} = 0 \\ \Delta_{\Sigma}\beta = 0. \end{cases}$$

Remark: Σ is special Lagrangian iff $\begin{cases} \omega|_{\Sigma} = 0 \\ \beta = \text{Constant}. \end{cases} \iff \begin{cases} \omega|_{\Sigma} = 0 \\ \Sigma \text{ is minimal} \end{cases}$.

An analytic study was done by R. SCHOEN and J. WOLFSON [6] (in a 4-dimensional Calabi–Yau manifold).

1.2 It is a completely integrable system (F.H.–P. ROMON [1, 2])

Let $\Omega \subset \mathbb{C}$ be an open subset and $X : \Omega \rightarrow \mathbb{R}^4$ a (local) conformal parametrization of Σ . Set

$$\rho_X := \rho_{\Sigma} \circ X,$$

the left Gauss map.

Idea: to lift the pair (X, ρ_X) to a map $F : \Omega \rightarrow \mathfrak{G}$, where \mathfrak{G} is a local symmetry group of the problem. The more naive choice is $\mathfrak{G} = SO(4) \times \mathbb{R}^4$, the group of isometries of \mathbb{R}^4 . Then

$$F = \begin{pmatrix} R & X \\ 0 & 1 \end{pmatrix} \simeq (R, X),$$

where $R : \Omega \rightarrow SO(4)$ encodes $\rho_X \simeq e^{i\beta}$. (Alternatively one can choose $\mathfrak{G} = U(2) \times \mathbb{C}^2$, with the identification $\mathbb{C}^2 \simeq (\mathbb{R}^4, J)$ and $U(2)$: subgroup of $SO(4)$. Then the way $R \in U(2)$ encodes β is simply through the relation $\det_{\mathbb{C}} R = e^{i\beta}$).

In all cases there exists an automorphism $\tau : \mathfrak{G} \rightarrow \mathfrak{G}$ s.t. $\tau^4 = Id$. This automorphism acts on the Lie algebra \mathfrak{g} and can be diagonalized with the eigenvalues $i, 1, -i$ and -1 . Hence the vector space decomposition

$$\begin{array}{l} \mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0^{\mathbb{C}} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2^{\mathbb{C}} \\ \text{eigenvalues: } \quad -i \quad \quad 1 \quad \quad i \quad \quad -1 \end{array}$$

Then consider the (pull-back of the) Maurer–Cartan form

$$\alpha := F^{-1}dF$$

and split $\alpha = \alpha_{-1} + \alpha_0 + \alpha_1 + \alpha_2$ according to this decomposition. Then do the further splitting $\alpha_2 = \alpha'_2 + \alpha''_2$, where $\alpha'_2 = \alpha(\frac{\partial}{\partial z})dz$ and $\alpha''_2 = \overline{\alpha'_2}$. And consider the family of deformations

$$\alpha_{\lambda} := \lambda^{-2}\alpha'_2 + \lambda^{-1}\alpha_{-1} + \alpha_0 + \lambda\alpha_1 + \lambda^2\alpha''_2, \quad \lambda \in \mathbb{C}^*.$$

Then:

Theorem 1.1 (i) X is Lagrangian iff $\alpha''_{-1} = 0$

(ii) X is HSL iff $\alpha''_{-1} = 0$ and, $\forall \lambda \in \mathbb{C}^*$, $d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0$.

Using this characterisation one see easily that HSL surfaces are solutions of a completely integrable system.

Note that analogous formulations work for HSL surfaces in $\mathbb{C}P^2 = SU(3)/S(U(2) \times U(1)) = U(3)/U(2) \times U(1)$, $\mathbb{C}D^2 = SU(2, 1)/S(U(2) \times U(1))$, $\mathbb{C}P^1 \times \mathbb{C}P^1$, $\mathbb{C}D^1 \times \mathbb{C}D^1$ [3].

2 Generalizations in \mathbb{R}^4 (after I. KHEMAR [4])

Again \mathbb{R}^4 is endowed with its canonical Euclidean structure. We will also use an identification of \mathbb{R}^4 with the quaternions \mathbb{H} . We recall that this allows to represent rotations $R \in SO(4)$ by a pair $(p, q) \in S^3 \times S^3 \subset \mathbb{H} \times \mathbb{H}$ of unit quaternions such that $\forall z \in \mathbb{H}$, $R(z) = pz\bar{q}$. In other words, denoting by $L_p : z \mapsto pz$ and $R_{\bar{q}} : z \mapsto z\bar{q}$, we have $R = L_p R_{\bar{q}} = R_{\bar{q}} L_p$. The pair (p, q) is unique up to sign, hence the identification $SO(4) \simeq S^3 \times S^3 / \{\pm\}$.

Moreover we can also precise the identification $Gr_2(\mathbb{R}^4) \simeq S^2 \times S^2$. Let

$$Stiefel_2(\mathbb{H}) := \{(e_1, e_2) \in \mathbb{H} \times \mathbb{H} \mid |e_1| = |e_2| = 1, \langle e_1, e_2 \rangle = 0\}.$$

Observe that $\forall (e_1, e_2) \in Stiefel_2(\mathbb{H})$, $e_2\bar{e}_1$ (resp. \bar{e}_1e_2) is unitary (because e_1 and e_2 are so) and imaginary (because $\langle e_1, e_2 \rangle = 0$). Hence this defines two maps

$$\begin{array}{ccc} Stiefel_2(\mathbb{H}) & \longrightarrow & S^2 \\ (e_1, e_2) & \longmapsto & e_2\bar{e}_1 \end{array}, \quad \begin{array}{ccc} Stiefel_2(\mathbb{H}) & \longrightarrow & S^2 \\ (e_1, e_2) & \longmapsto & \bar{e}_1e_2 \end{array}.$$

These maps factor through the natural map $P : (e_1, e_2) \mapsto \text{Span}\{e_1, e_2\}$ from $Stiefel_2(\mathbb{H})$ to the oriented Grassmannian $Gr_2(\mathbb{H})$: let

$$\begin{array}{ccc} \rho : Gr_2(\mathbb{H}) & \longrightarrow & S^2 \\ \text{s. t. } \rho \circ P(e_1, e_2) & = & e_2\bar{e}_1 \end{array}, \quad \begin{array}{ccc} \sigma : Gr_2(\mathbb{H}) & \longrightarrow & S^2 \\ \text{s. t. } \sigma \circ P(e_1, e_2) & = & \bar{e}_1e_2. \end{array}$$

Then $(\rho, \sigma) : Gr_2(\mathbb{H}) \longrightarrow S^2 \times S^2$ is a diffeomorphism.

2.1 Immersions of a surface in \mathbb{H} with a harmonic ‘left Gauss map’

Let $X : \Omega \longrightarrow \mathbb{H}$ be a conformal immersion and $\rho_X : \Omega \longrightarrow S^2$ its left Gauss map, i.e. $\forall z \in \Omega$, $\rho_X(z)$ is the image of $\text{Span}(\frac{\partial X}{\partial x}(z), \frac{\partial X}{\partial y}(z))$ by ρ . It is characterised by

$$\frac{\partial X}{\partial y} = \rho_X \frac{\partial X}{\partial x} \iff i \frac{\partial X}{\partial z} = \rho_X \frac{\partial X}{\partial z}.$$

(In the second equation the i on the l.h.s. is the complex structure on $\Omega \subset \mathbb{C}$, whereas the ρ_X on the r.h.s. denotes the left multiplication in \mathbb{H} .)

Remark: instead of viewing ρ_X as the left component of the Gauss map in $Gr_2(\mathbb{H}) \simeq S^2 \times S^2$, an alternative interpretation is that ρ_X is a map into the ‘left’ connected component of the manifold of compatible complex structures $\mathcal{J}_{\mathbb{H}} \simeq S^2 \cup S^2$ on \mathbb{H} (cf. the work of F. BURSTALL).

Idea: to lift the pair (X, ρ_X) by a framing $F : \Omega \longrightarrow \mathfrak{G}$, \mathfrak{G} is a subgroup of $SO(4) \times \mathbb{R}^4$.

How ? We fix some constant imaginary unit vector $u \in S^2 \subset \text{Im}\mathbb{H}$.

- *First method:* we lift X **and** its full Gauss map $T_X \Sigma \simeq (\rho_X, \sigma_X)$: we let (e_1, e_2) be any moving frame which is an orthonormal basis of $T_{X(z)} \Sigma$ (e.g. $e_1 = \frac{\partial X}{\partial x} / |\frac{\partial X}{\partial x}|$, $e_2 = \frac{\partial X}{\partial y} / |\frac{\partial X}{\partial y}|$) and we choose $F = (R, X)$ s.t. R satisfies:

$$R(1) = e_1, \quad R(u) = e_2.$$

Decompose $R = L_p R_{\bar{q}}$, then

$$R(1) = p\bar{q}, \quad R(u) = pu\bar{q}, \quad \text{so that} \quad \rho_X = e_2 \bar{e}_1 = pu\bar{p}.$$

Note: In this case we must choose $\mathfrak{G} = SO(4) \times \mathbb{R}^4$ (which acts transitively on $Stiefel_2(\mathbb{H})$).

- *Second method:* we lift **only** X and ρ_X . Then it means that we choose $F = (R, X)$, where $R = L_p R_{\bar{q}}$ is s.t.

$$\rho_X = pu\bar{p}.$$

Hence the choice of q is not relevant. In other words introducing the (*left*) Hopf fibration

$$\begin{aligned} \mathcal{H}_L^u : SO(4) &\longrightarrow S^2 \\ L_p R_{\bar{q}} &\longmapsto pu\bar{p}, \end{aligned}$$

we choose the lift $F = (R, X)$ in such a way that $\mathcal{H}_L^u \circ R = \rho_X$.

We observe that in this case one may choose $q = 1$ and assume that $R \in \{L_p | p \in S^3\} \simeq Spin3$, i.e. work with $\mathfrak{G} = Spin3 \times \mathbb{H}$. The restriction of \mathcal{H}_L^u to $Spin3$ (viewed as a subgroup of $SO(4)$) is just the Hopf fibration $\mathcal{H}^u : S^3 \longrightarrow S^2$.

Actually the second point of view is more general and leads to a simpler theory.

Now let $\tau : (R, X) \longmapsto (L_u R L_u^{-1}, -L_u X)$, a 4th order automorphism of \mathfrak{G} (i.e. $\tau^4 = Id$). It induces a 4th order automorphism on its Lie algebra \mathfrak{g} . Let

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0^{\mathbb{C}} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2^{\mathbb{C}}$$

be its associated eigenspace decomposition. Split the Maurer–Cartan form $\alpha = F^{-1}dF$ according to this decomposition: $\alpha = \alpha_{-1} + \alpha_0 + \alpha_1 + \alpha_2$ and let

$$\begin{aligned}\beta_{\lambda^2} &:= \lambda^{-2}\alpha'_2 + \alpha_0 + \lambda^2\alpha''_2, \\ \alpha_\lambda &:= \lambda^{-2}\alpha'_2 + \lambda^{-1}\alpha_{-1} + \alpha_0 + \lambda\alpha_1 + \lambda^2\alpha''_2 = \beta_{\lambda^2} + \lambda^{-1}\alpha_{-1} + \lambda\alpha_1.\end{aligned}$$

Then:

Lemma 2.1 *If $X : \Omega \longrightarrow \mathbb{R}^4$ is a conformal immersion and if $R : \Omega \longrightarrow SO(4)$ is an arbitrary smooth map, then*

$$\mathcal{H}_L^u \circ R = \rho_X \iff \alpha''_{-1} = 0.$$

In other words $F = (R, X) : \Omega \longrightarrow SO(4) \times \mathbb{R}^4$ lifts (X, ρ_X) iff $\alpha''_{-1} = 0$.

Remark: α_1 is the complex conjugate of α_{-1} , so that $\alpha''_{-1} = 0$ iff $\alpha'_1 = 0$.

Lemma 2.2 *We have:*

$$d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = d\beta_{\lambda^2} + \frac{1}{2}[\beta_{\lambda^2} \wedge \beta_{\lambda^2}] + (\lambda^{-3} - \lambda)[\alpha'_2 \wedge \alpha''_{-1}] + (\lambda^3 - \lambda^{-1})[\alpha''_2 \wedge \alpha'_1]. \quad (1)$$

Hence in particular, if F lifts (X, ρ_X) , then $d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = d\beta_{\lambda^2} + \frac{1}{2}[\beta_{\lambda^2} \wedge \beta_{\lambda^2}]$.

In order to interpret (1) we further observe that

- (i) \mathfrak{G}^τ , the fixed subset of $\tau : \mathfrak{G} \longrightarrow \mathfrak{G}$, is a subgroup of \mathfrak{G} with Lie algebra \mathfrak{g}_0
- (ii) $\mathfrak{G}^{\tau^2} = \{(R, 0) \in \mathfrak{G}\}$, the fixed subset of $\tau^2 : \mathfrak{G} \longrightarrow \mathfrak{G}$, is a subgroup of \mathfrak{G} with Lie algebra $\mathfrak{g}_0 \oplus \mathfrak{g}_2$,

with the inclusions

$$\mathfrak{G}^\tau \subset \mathfrak{G}^{\tau^2} \subset \mathfrak{G}.$$

Moreover $\mathfrak{G}/\mathfrak{G}^{\tau^2} \simeq \mathbb{H}$ and $\mathfrak{G}^{\tau^2}/\mathfrak{G}^\tau \simeq S^2$ and the projection map

$$\begin{aligned}\mathfrak{G}^{\tau^2} &\longrightarrow \mathfrak{G}^{\tau^2}/\mathfrak{G}^\tau \simeq S^2 \\ R \simeq (R, 0) &\longmapsto R \bmod \mathfrak{G}^\tau\end{aligned}$$

coincides with the Hopf fibration \mathcal{H}_L^u . Hence, by applying the standard theory of harmonic maps into symmetric spaces, we deduce that:

$$d\beta_{\lambda^2} + \frac{1}{2}[\beta_{\lambda^2} \wedge \beta_{\lambda^2}] = 0 \iff \mathcal{H}_L^u \circ R : \Omega \longrightarrow S^2 \text{ is harmonic.}$$

Putting Lemmas 2.1 and 2.2 and these observations together we conclude with the following:

Theorem 2.1 Let $X : \Omega \longrightarrow \mathbb{H}$ be a conformal immersion and $\rho_X : \Omega \longrightarrow S^2$ its left Gauss map. Let $F = (R, X) : \Omega \longrightarrow \mathfrak{G}$ be any smooth map. Then

(i) $\mathcal{H}_L^u \circ R = \rho_X$ (i.e. F is a lift of (X, ρ_X)) iff $\alpha''_{-1} = 0$

(ii) If so, i.e. if F is a lift of (X, ρ_X) , then ρ_X is harmonic iff

$$d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0.$$

2.2 Examples

2.2.1 HSL surfaces revisited

Let us introduce again the symplectic form $\omega = dx^1 \wedge dx^2 + dx^3 \wedge dx^4$. Note that $\omega = \omega_1 := \langle L_i \cdot, \cdot \rangle$. Let us introduce also $\omega_2 := \langle L_j \cdot, \cdot \rangle = dx^1 \wedge dx^3 + dx^4 \wedge dx^2$ and $\omega_3 := \langle L_k \cdot, \cdot \rangle = dx^1 \wedge dx^4 + dx^2 \wedge dx^3$. Then

$$e_2 \bar{e}_1 = \rho(e_1, e_2) = i\omega_1(e_1, e_2) + j\omega_2(e_1, e_2) + k\omega_3(e_1, e_2).$$

So X is a conformal *Lagrangian* immersion iff $X^*\omega_1 = 0$, i.e. iff ρ_X takes values in

$$S^1 = \{j \cos \beta + k \sin \beta = e^{i\beta} j \mid \beta \in \mathbb{R}\}.$$

Hence a lift of (X, ρ_X) is characterized by

$$pu\bar{p} = \mathcal{H}_L^u \circ R = \rho_X = e^{i\beta} j.$$

A convenient choice for u is to assume that $u \perp i$, e.g. $u = j$. In that case

$$\{p \in S^3 \mid pu\bar{p} = e^{i\beta} j\} = \{e^{i\beta/2} e^{j\theta} \mid \theta \in \mathbb{R}\}$$

and the simplest choices are $p = \pm e^{i\beta/2}$.

With this choice:

- if we start with the group $\mathfrak{G} = SO(4) \times \mathbb{R}^4$, our lift satisfies $R = L_{e^{i\beta/2}} R_{\bar{q}}$, i.e. we can reduce $SO(4) \times \mathbb{R}^4$ to $U(2) \times \mathbb{C}^2$
- if we start with the group $\mathfrak{G} = Spin3 \times \mathbb{H}$, our lift satisfies $R = L_{e^{i\beta/2}}$, i.e. we can reduce $Spin3 \times \mathbb{R}^4$ to $U(1) \times \mathbb{C}^2$ (cf. spinor lifts, related to the KONOPELCHENKO–TAIMANOV representation formula).

2.2.2 Constant mean curvature surfaces in \mathbb{R}^3

Consider an immersed surface Σ in \mathbb{H} with a harmonic left Gauss map. If we assume further that this surface is contained in $\text{Im}\mathbb{H}$, then any orthonormal basis (e_1, e_2) of $T_{X(z)}\Sigma$ is composed of imaginary vectors. Hence

$$\rho_X = e_2 \bar{e}_1 = -\bar{e}_1 e_2 = -\sigma_X,$$

so that ρ_X is harmonic iff σ_X is so. Actually ρ_X is nothing but the Gauss map of Σ in $\text{Im}\mathbb{H} \simeq \mathbb{R}^3$. Hence by Ruh–Vilms theorem we know that Σ is a *constant mean curvature surface* in \mathbb{R}^3 . Conversely any constant mean curvature surface in \mathbb{R}^3 arises that way.

2.3 Other generalizations in dimension 4

This theory can be generalized to surfaces in S^4 or $\mathbb{C}P^2$: then (X, ρ_X) is replaced by a lift of the immersion X in the four dimensional manifold into the twistor bundle of complex structures. The condition of ρ_X being harmonic is replaced by the fact this lift is vertically harmonic (the fiber being the set of (left) compatible complex structures, diffeomorphic to S^2). This follows from independant works by F. BURSTALL and I. KHEMAR.

3 A generalization for surfaces in \mathbb{R}^8 (I. KHEMAR [4])

The following theory is based on the identification of \mathbb{R}^8 with octonions \mathbb{O} . Again the map

$$\begin{aligned} Stiefel_2(\mathbb{O}) &\longrightarrow S^6 \\ (e_1, e_2) &\longmapsto e_2 \bar{e}_1, \end{aligned}$$

where $S^6 \in \text{Im}\mathbb{O} \subset \mathbb{O}$, can be factorized through the map $P : Stiefel_2(\mathbb{O}) \longrightarrow Gr_2(\mathbb{O})$, $(e_1, e_2) \longmapsto \text{Span}\{e_1, e_2\}$ by introducing

$$\begin{aligned} \rho : Gr_2(\mathbb{O}) &\longrightarrow S^6 \\ \text{s.t. } \rho \circ P(e_1, e_2) &= e_2 \bar{e}_1. \end{aligned}$$

Let Σ be an immersed surface in \mathbb{O} we say that Σ is ρ -harmonic iff the composition of the Gauss map $\Sigma \longrightarrow Gr_2(\mathbb{O})$ with ρ is harmonic.

This theory is completely similar with the theory of surfaces in quaternions \mathbb{H} which used the group $\mathfrak{G} = Spin3 \times \mathbb{H}$, where $Spin3$ can be seen as the subgroup of $SO(4)$ generated by L_i, L_j and L_k and the induced representation of $Spin3$ was the spinor representation \mathbb{H} . Here we will use $\mathfrak{G} = Spin7 \times \mathbb{O}$, where $Spin7$ can be identified with the subgroup of $SO(8)$ generated by $\{L_v | v \in S^6 \subset \text{Im}\mathbb{O}\}$ and the induced representation on \mathbb{R}^8 coincides with the spinor representation of $Spin7$ on \mathbb{O} . A difference however is that $Spin7$ is "bigger" than $Spin3$ and in particular acts transitively on $Stiefel_2(\mathbb{O})$ (with isotropy $SU(3)$) and $Gr_2(\mathbb{O})$ (with isotropy G_2), whereas $Spin3$ do not act transitively on $Gr_2(\mathbb{H})$. After fixing an imaginary unit octonion $u \in \mathbb{O}$, a 'Hopf' fibration

$$\begin{aligned} \mathcal{H}^u : Spin7 &\longrightarrow S^6 \\ p &\longmapsto \mathcal{H}^u(p), \text{ s.t. } pL_u p^{-1} = L_{\mathcal{H}^u(p)} \end{aligned}$$

can be defined.

Now let $X : \mathbb{C} \supset \Omega \longrightarrow \mathbb{O}$ be a conformal immersion and denote $\rho_X := \rho \circ T_X \Sigma$ the composition of the Gauss map $T_X \Sigma$ of X with ρ . After having fixed $u \in S^6 \subset \mathbb{O}$ we let

$$F = \begin{pmatrix} R & X \\ 0 & 1 \end{pmatrix} \simeq (R, X) : \Omega \longrightarrow Spin7 \times \mathbb{O},$$

be a smooth map. We say that F lifts (X, ρ_X) iff $\mathcal{H}^u \circ R = \rho_X$. Using the 4th order automorphism $\tau : \mathfrak{G} \longrightarrow \mathfrak{G}$ defined by

$$\tau(R, X) = (L_u R L_u^{-1}, -L_u X),$$

we can characterize among all maps $F = (R, X)$ those which lift ρ_X by the condition $\alpha''_{-1} = 0$ (after a decomposition of the Maurer–Cartan form $\alpha := F^{-1}dF$ along the eigenspaces of the action of τ on the Lie algebra \mathfrak{g} of \mathfrak{G}). Then the ρ -harmonic immersions satisfy a zero curvature equation $d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0$ similar to the previous case.

Again ρ_X can be interpreted as a map into the manifold $\mathcal{J}_\mathbb{O}$ of compatible complex structures on \mathbb{O} , because of the relation $\rho_X \frac{\partial X}{\partial z} = i \frac{\partial X}{\partial \bar{z}}$. However the embedding $S^6 \subset \mathcal{J}_\mathbb{O}$ is much less clear than the inclusion $S^2 \subset \mathcal{J}_\mathbb{H}$ that we used previously: we recall indeed that $\mathcal{J}_\mathbb{H} \simeq S^2_L \cup S^2_R$ and hence that our S^2 was just the (left) connected component of $\mathcal{J}_\mathbb{H}$. However $\mathcal{J}_\mathbb{O} \simeq SO(8)/U(4)$ is 12 dimensional, so that our S^6 is now a particular submanifold of $\mathcal{J}_\mathbb{O}$. Hence a twistor interpretation of the theory in \mathbb{O} seems less clear.

4 Towards a supersymmetric interpretation

Observation : the coefficients of α_{-1} and α_1 actually behave like spinors (they turn half less than those of α_2 when λ run over S^1 and they satisfy a kind of Dirac equation). This motivates the following results by I. KHEMAR [5].

4.1 Superharmonic maps into a symmetric space

For simplicity we restrict ourself to maps into the sphere $S^n \subset \mathbb{R}^{n+1}$. It can be seen as a system of PDE's on a map $u : \Omega \longrightarrow S^n$ (where $\Omega \subset \mathbb{C}$) and *odd* sections ψ_1, ψ_2 of u^*TS^n . This system is

$$\begin{cases} \nabla_{\bar{z}} \frac{\partial u}{\partial z} &= \frac{1}{4} \left(\psi \langle \psi, \frac{\partial u}{\partial \bar{z}} \rangle - \bar{\psi} \langle \bar{\psi}, \frac{\partial u}{\partial z} \rangle \right) \\ \nabla_{\bar{z}} \psi &= \frac{1}{4} \langle \bar{\psi}, \psi \rangle \bar{\psi}, \end{cases} \quad (2)$$

where $\psi = \psi_1 - i\psi_2$. By “odd” we mean that the components ψ_1 and ψ_2 are anticommuting (Grassmann) variables. An alternative elegant reformulation of this system can be obtained by adding the extra field $F : \Omega \longrightarrow \mathbb{R}^{n+1}$, which satisfies the 0th order PDE's

$$F = \frac{1}{2i} \langle \psi, \bar{\psi} \rangle u \quad (3)$$

and by setting

$$\Phi := u + \theta^1 \psi_1 + \theta^2 \psi_2 + \theta^1 \theta^2 F,$$

where θ^1 and θ^2 are anticommuting coordinates, so that $(x, y, \theta^1, \theta^2)$ forms a complete system of coordinates on the *superplane* $\mathbb{R}^{2|2}$. Then (2) and (3) are equivalent to

$$\bar{D}D\Phi + \langle \bar{D}\Phi, D\Phi \rangle \Phi = 0, \quad (4)$$

where $D = \frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial z}$, $\bar{D} = \frac{\partial}{\partial \bar{\theta}} - \bar{\theta} \frac{\partial}{\partial \bar{z}}$.

Actually, from (2) and (3) to (4), we have used the fact that u , ψ_1 , ψ_2 and F are the components (supermultiplet) of a single map Φ from $\mathbb{R}^{2|2}$ to $S^n \subset \mathbb{R}^{n+1}$, which satisfies the superharmonic map equation (4).

Now we lift Φ to a framing supermap $\mathcal{F} : \mathbb{R}^{2|2} \longrightarrow SO(n+1)$ such that the composition of \mathcal{F} with the projection $SO(n+1) \longrightarrow SO(n+1)/SO(n) \simeq S^n$ is Φ . Set $\alpha := \mathcal{F}^{-1}d\mathcal{F}$ and decompose $\alpha = \alpha_0 + \alpha_1$, according to the splitting of the Lie algebra $so(n+1)$ by the Cartan involution.

Before giving a characterization of the superharmonic equation, it is useful to present a technical result concerning the exterior calculus of 1-forms on $\mathbb{R}^{2|2}$.

Lemma 4.1 *For a 1-form α on $\mathbb{R}^{2|2}$ with coefficients in a Lie algebra \mathfrak{g} , we have the equivalence*

$$d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0 \iff \bar{D}\alpha(D) + D\alpha(\bar{D}) + [\alpha(\bar{D}), \alpha(D)] = 0.$$

Remark: $\Lambda^1(\mathbb{R}^{2|2})^*$ is spanned by $(d\theta, d\bar{\theta}, dz + (d\theta)\theta, d\bar{z} + (d\bar{\theta})\bar{\theta})$, the dual basis of $(D, \bar{D}, \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}})$. Hence in particular $\Lambda^2(\mathbb{R}^{2|2})^*$ is 6 dimensional. So the expansion of the l.h.s. of $d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0$ leads to 6 equations which are a priori independant. The content of this lemma is that these 6 terms vanish as soon as one of these 6 coefficients (namely the coefficient of $d\theta \wedge d\bar{\theta}$) vanishes.

Now the supermap \mathcal{F} is superharmonic iff

$$\bar{D}\alpha_1(D) + [\alpha_0(\bar{D}), \alpha_1(D)] = 0.$$

We hence deduce:

Theorem 4.1 *\mathcal{F} is superharmonic iff*

$$\forall \lambda \in \mathbb{C}^*, \quad \bar{D}\alpha(D)_\lambda + D\alpha(\bar{D})_\lambda + [\alpha(\bar{D})_\lambda, \alpha(D)_\lambda] = 0,$$

where $\alpha(D)_\lambda := \alpha_0(D) + \lambda^{-1}\alpha_1(D)$ and $\alpha(\bar{D})_\lambda := \alpha_0(\bar{D}) + \lambda\alpha_1(\bar{D})$.

It results that this problem has the structure of a completely integrable system (F. O'DEA, I. KHEMAR). In particular the DPW algorithm for harmonic maps works.

The DPW potential is a $\Lambda \mathfrak{g}_r^{\mathbb{C}}$ -valued holomorphic 1-form μ on $\mathbb{R}^{2|2}$ s.t.

$$\mu(D) = \mu_0(D) + \theta\mu_\theta(D) = \lambda^{-1}(\cdot) + \lambda^0(\cdot) + \dots$$

One integrates the equation

$$Dg = g\mu(D)$$

to get a holomorphic map $g = g_0 + \theta g_\theta : \mathbb{R}^{2|2} \longrightarrow \Lambda \mathfrak{G}_\tau^{\mathbb{C}}$. This implies in particular that

$$g_0^{-1} \frac{\partial g_0}{\partial z} = -((\mu_0(D))^2 + \mu_\theta(D)) = \lambda^{-2}(\cdot) + \lambda^{-1}(\cdot) + \lambda^0(\cdot) + \dots$$

Similarly, if $\mathcal{F} = \mathcal{F}_0 + \theta \mathcal{F}_\theta + \bar{\theta} \mathcal{F}_{\bar{\theta}} + \theta \bar{\theta} \mathcal{F}_{\theta \bar{\theta}}$, it turns out that $\mathcal{F}_0^{-1} d\mathcal{F}_0 = \lambda^{-2}(\cdot) + \lambda^{-1}(\cdot) + \lambda^0(\cdot) + \lambda^1(\cdot) + \lambda^2(\cdot)$. Hence we recover (for \mathcal{F}_0) something similar to a second order elliptic integrable system.

4.2 Superprimitive maps [5]

More precisely we can recover a second order elliptic integrable system close to the HSL surface theory in \mathbb{R}^4 by looking at *superprimitive maps* from $\mathbb{R}^{2|2}$ to the 4-symmetric space $SU(3)/SU(2)$: if $\Phi : \mathbb{R}^{2|2} \longrightarrow SU(3)/SU(2)$ is a superprimitive map then the first component u in the decomposition $\Phi = u + \theta^1 \psi_1 + \theta^2 \psi_2 + \theta^1 \theta^2 F$ is a conformal HSL immersion (with the restriction that the Lagrangian angle β is equal to a *real* constant plus a harmonic non constant *nilpotent* function).

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