Multiplicative formulas in Cohomology of G/P and in quiver representations

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December 11, 2008

Abstract

1 Introduction

Consider a partial flag variety X which is not a grassmaninan. Consider also its cohomology ring $\mathrm{H}^*(X,\mathbb{Z})$ endowed with the base formed by the Poincaré dual classes of the Schubert varieties. In [Ricar], E. Richmond showed that some coefficient structure of the product in $\mathrm{H}^*(X,\mathbb{Z})$ are products of two such coefficients for smaller flag varieties.

Consider now a quiver without oriented cycle. If α and β denote two dimension-vectors, $\alpha\circ\beta$ denotes the number of α-dimensional subrepresentations of a general $\alpha+\beta$-dimensional representation. In [DW06], H. Derksen and J. Weyman expressed some numbers $\alpha\circ\beta$ as products of two smaller such numbers.

The aim of this work is to prove two generalisations of the two above results by the same way.

We now explain our result about cohomology of the generalized flag varieties. Let G be a semi-simple group, $T\subset B\subset Q\subset G$ be a maximal torus, a Borel subgroup and a parabolic subgroup respectively. In [BK06], P. Belkale and S. Kumar defined a new product (associative and commutative) on the cohomology group $\mathrm{H}^*(G/Q,\mathbb{Z})$ denoted by $\odot_0$. Any coefficient structure of $\odot_0$ in the base of Schubert classes is either zero or the corresponding coefficient structure for the cup product.

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Let now $P \subset Q$ be second parabolic subgroups of G and L denote the Levi subgroup of P containing T. We obtain the following:

Theorem A *Any coefficient structure of* $(\mathrm{H}^*(G/Q, \mathbb{Z}), \odot_0)$ *in the base of Schubert classes is the product of such two coefficients for* $(\mathrm{H}^*(G/P, \mathbb{Z}), \odot_0)$ *and* $(\mathrm{H}^*(L/(L \cap Q), \mathbb{Z}), \odot_0)$ *respectively.*

Actualy, Theorem 4 below is more precise and explicit than Theorem A. This result was already obtained in [Ricar] when $G = \mathrm{SL}_n$, Q is any parabolic subgroup and P is the maximal parabolic subgroup corresponding to the linear subspace in G/Q of minimal dimension.

Let Q be a quiver. The Ringle form (see Section 4.1) is denoted by $\langle \cdot, \cdot \rangle$. We prove the following:

Theorem B *Let* α, β *and* γ *be three dimension-vectors. We assume that* $\langle \alpha, \beta \rangle = \langle \alpha, \gamma \rangle = \langle \beta, \gamma \rangle = 0$. *Then,*

$$(\alpha + \beta \circ \gamma).(\alpha \circ \beta) = (\alpha \circ \beta + \gamma).(\beta \circ \gamma).$$

Note that, Theorem 5 is more general than Theorem B, since s dimension-vectors occur. Moreover, we give in Theorem B a geometric interpretation of the product.

If Q has no oriented cycle, we obtain the following corollary:

Theorem C *We assume that* Q *has no oriented cycle. Let* α, β *and* γ *be three dimension-vectors. We assume that* $\langle \alpha, \beta \rangle = \langle \alpha, \gamma \rangle = 0$ *and* $\beta \circ \gamma = 1$.

Then, $\alpha \circ \beta + \gamma = (\alpha \circ \beta).(\alpha \circ \gamma)$.

This result is not really stated in [DW06]. However, the proof of [DW06, Theorem 7.14] implies it. Note that the proof of Theorem B is really diffenrent from those of [DW06, Theorem 7.14]. Indeed, the numbers $\alpha \circ \beta$ have two non trivially equivalent interpretations (see [DSW07]): as a number of points in a generic fiber of a morphism or as a dimension of the subspace of invariant vectors in a representation. Here we use the first characterisation. Derksen and Weyman used the second one.

In Section 2, we consider more generally a semi-simple group G acting on a variety X.

2 Degree of dominant pairs

2.1 Definitions

Let G be a reductive group acting on a smooth variety X. Let λ be a one parameter subgroup of G. Let G^λ or L denote the centralizer of λ in G. We consider the usual parabolic subgroup associated to λ with Levi subgroup L:

$$P(\lambda) = \left\{ g \in G \ :\ \lim_{t \to 0} \lambda(t).g.\lambda(t)^{-1} \text{ exists in } G \right\}.$$

Let C be an irreducible component of the fix point set X^λ of λ in G. We also consider the Bialinicky-Birula cell C^+ associated to C:

$$C^+ = \{x \in X \mid \lim_{t \to 0} \lambda(t)x \in C\}.$$

Then, C is stable by the action of L and C^+ by the action of $P(\lambda)$.

Consider over $G \times C^+$ the action of $G \times P(\lambda)$ given by the formula (with obvious notation): $(g, p).(g', y) = (gg'p^{-1}, py)$. Consider the quotient $G \times_{P(\lambda)} C^+$ of $G \times C^+$ by the action of $\{e\} \times P(\lambda)$. The class of a pair $(g, y) \in G \times C^+$ in $G \times_{P(\lambda)} C^+$ is denoted by $[g : y]$.

The action of $G \times \{e\}$ induces an action of G on $G \times_{P(\lambda)} C^+$. Moreover, the first projection $G \times C^+ \longrightarrow G$ induces a G-equivariant map $\pi : G \times_{P(\lambda)} C^+ \longrightarrow G/P(\lambda)$ which is a locally trivial fibration with fiber C^+. In particular, we have

$$\dim(G \times_{P(\lambda)} C^+) = \dim(G/P(\lambda)) + \dim(C^+).$$

Consider also the G-equivariant map $\eta : G \times_{P(\lambda)} C^+ \longrightarrow X$, $[g : y] \mapsto gy$. We finally obtain:

$$\begin{tikzcd}
G \times_{P(\lambda)} C^+ \arrow[r, "\eta"] \arrow[d, "\pi"'] & X. \\
G/P(\lambda) &
\end{tikzcd}$$

It is well known that the map

$$\begin{array}{ccc} G \times_{P(\lambda)} C^+ & \longrightarrow & G/P(\lambda) \times X \\ [g : y] & \longmapsto & (gP(\lambda), gy) \end{array} \qquad (1)$$

is an immersion; its image is the set of the $(gP(\lambda), x) \in G/P(\lambda) \times X$ such that $g^{-1}x \in Y$. Note that this fact can be used to prove that $G \times_{P(\lambda)} C^+$ actually exists.

Definition. We set

$$\begin{aligned} \delta(G, X, C, \lambda) &= \dim(X) - \dim(G/P(\lambda)) - \dim(C^+) \\ &= \operatorname{codim}(C^+, X) - \operatorname{codim}(P(\lambda), G). \end{aligned}$$

If $\delta(G, X, C, \lambda) = 0$ and η is dominant, it induces a finite field extension: $k(X) \subset k(G \times_{P(\lambda)} C^+)$. We denote by $d(G, X, C, \lambda)$ the degree of this extension. If $\delta(G, X, C, \lambda) \neq 0$ or η is not dominant, we set $d(G, X, C, \lambda) = 0$.

More generally, we define the degree of any morphism to be the degree of the induced extension if it is finite and zero otherwise.

2.2 A product formula for $d(G, X, C, \lambda)$

Let T be a maximal torus of G and x_0 be a fixed point of T in X. We keep notation of Section 2.1 and assume that the image of λ is contained in T and $x_0 \in C$. We set $P = P(\lambda)$.

Let λ_ε be another one parameter subgroup of T. Set $P_\varepsilon = P(\lambda_\varepsilon)$. Consider the irreducible component C_ε of X^{λ_ε} which contains x_0 and $C_\varepsilon^+ = \{x \in X \,:\, \lim_{t \to 0} \lambda_\varepsilon(t)x \in C\}$. We assume that:

(i) $P_\varepsilon \subset P$,

(ii) $C_\varepsilon^+ \subset C^+$, and

(iii) $C_\varepsilon \subset C$.

Remark. Notice that the set of the λ_ε which satisfy these three assumptions generated an open convex cone in the vector space containing the one parameters subgroups of T as a lattice.

Now, we want to compare η and η_ε. We introduce the natural morphism:

$$\eta_L \,:\, L \times_{P_\varepsilon \cap L} (C_\varepsilon^+ \cap C) \longrightarrow C.$$

This map is a map η as in Section 2.1 with $G = L$, $X = C$, $C = C_\varepsilon$ and $\lambda = \lambda_\varepsilon$. In particular, we have defined $\delta(L, C, C_\varepsilon^+ \cap C, \lambda_\varepsilon)$ and $d(L, C, C_\varepsilon^+ \cap C, \lambda_\varepsilon)$.

We can now state our main result

Theorem 1 *With above notation, we have:*

(i) $\delta(G, X, C_\varepsilon, \lambda_\varepsilon) = \delta(L, C, C_\varepsilon, \lambda_\varepsilon) + \delta(G, X, C, \lambda)$, *and*

(ii) *If* $\delta(L, C, C_\varepsilon, \lambda_\varepsilon) = \delta(G, X, C, \lambda) = 0$, *then*

$$d(G, X, C_\varepsilon, \lambda_\varepsilon) = d(L, C, C_\varepsilon, \lambda_\varepsilon) \cdot d(G, X, C, \lambda).$$

2.3 Proof of Theorem 1

2.3.1 — We set

$$Y_L = L \times_{P_\varepsilon \cap L} (C_\varepsilon^+ \cap C) \quad \text{and} \quad Y_P = P \times_{P_\varepsilon} C_\varepsilon^+.$$

We consider the natural morphism

$$\eta_P \, : \, Y_P \longrightarrow C^+,$$

and

$$[\mathrm{Id} : \eta_P] \, : \, G \times_P Y_P \longrightarrow G \times_P C^+, \; [g : [p : x]] \longmapsto [g : px].$$

Lemma 1 *With above notation, we have:*

(i) *the map* $G \times_P Y_P \longrightarrow G \times_{P_\varepsilon} C_\varepsilon^+$, $[g : [p : x]] \longmapsto [gp : x]$ *is an isomorphism denoted by* ι*; moreover,*

(ii) $\eta_\varepsilon \circ \iota = \eta \circ ([\mathrm{Id} : \eta_P])$.

Proof. The morphism ι commutes with the two projections on G/P. Moreover, the restriction of ι over P/P is the closed immersion $P \times_{P_\varepsilon} C_\varepsilon^+ \longrightarrow G \times_{P_\varepsilon} C_\varepsilon^+$. It follows (see for example [Res04, Appendice]) that ι is an isomorphism.

The morphisms $\eta_\varepsilon \circ \iota$ and $\eta \circ ([\mathrm{Id} : \eta_P])$ are G-equivariant and extend the immersion of C_ε^+ in X. They have to be equal. □

2.3.2 — We are now interested in η_P. Consider the two following limit morphisms:

$$\begin{array}{llll} \Lambda_P : & P & \longrightarrow & L \\ & p & \mapsto & \lim_{t \to 0} \lambda(t) p \lambda(t^{-1}) \end{array} \qquad \text{and} \qquad \begin{array}{llll} \Lambda^+ : & C^+ & \longrightarrow & C \\ & y & \mapsto & \lim_{t \to 0} \lambda(t) y. \end{array}$$

The computation $\lambda(t)px = \lambda(t)p\lambda(t^{-1})\lambda(t)x$ implies the easy

Lemma 2 *We have:* $\Lambda^+(px) = \Lambda_P(p)\Lambda^+(x)$.

2.3.3 — Recall that $\Lambda^+ : C^+ \longrightarrow C$ is a vector bundle. The pullback of this vector bundle by η_L is

$$\eta_L^*(C^+) = \{([l:x], y) \in Y_L \times C^+ \mid lx = \Lambda^+(y)\},$$

endowed with the first projection p_1 on Y_L. Consider the following diagram:

$$\begin{tikzcd}
Y_P \arrow[rr, "\Theta\ :\ {[p:x]} \mapsto ({[\Lambda_P(p):\Lambda^+(x)]}, px)"] \arrow[dr, swap, "{[p:x]} \mapsto {[\Lambda_P(p):\Lambda^+(x)]}"] & & \eta_L^*(C^+) \arrow[dl, "p_1"] \\
& Y_L \arrow[d] & \\
& L/(P_\varepsilon \cap L). &
\end{tikzcd} \tag{2}$$

Lemma 3 *The above diagram is commutative, and the top horizontal map* Θ *is an isomorphism.*

Proof. First, note that the map $Y_P \longrightarrow Y_L$ in Diagram 2 is well defined by Lemma 2. Diagram 2 is obviously commutative.

Since all the morphisms in Diagram 2 are L-equivariant, [Res04, Appendice] implies that it is sufficient to prove that Θ is an isomorphism when restricted over the class of e in $L/(P_\varepsilon \cap L)$. The fiber in Y_L over this point if $C \cap C_\varepsilon^+$. Since $P^u \subset P_\varepsilon^u$, the fiber in Y_P identify with C_ε^+, by $x \in C_\varepsilon^+ \mapsto [e:x]$. The fiber in $\eta_L^*(C^+)$ also identify with C_ε^+ in such a way the restriction of Θ becomes the identity. It follows that Θ is an isomorphism. $\square$

2.3.4 — We can now prove Theorem 1.

Proof. By Lemma 3, we have the following commutative diagram:

$$\begin{tikzcd}
Y_P \arrow[rr, "\eta_P"] \arrow[dr, "\Theta", "\sim"'] \arrow[ddr] & & C^+ \arrow[dd, "\Lambda^+"] \\
& \eta_L^*(C^+) \arrow[ur] \arrow[d] & \\
& Y_L \arrow[r, "\eta_L"] & C.
\end{tikzcd}$$

It follows that $\dim(C^+) - \dim(Y_P) = \delta(L, C, C_\varepsilon, \lambda_\varepsilon)$ and $d(L, C, C_\varepsilon, \lambda_\varepsilon)$ equals the degree of η_P.

Moreover, by Lemma 1, we have the following commutative diagram:

$$\begin{tikzcd}
G \times_{P_\varepsilon} C_\varepsilon^+ \arrow[r, "\sim"'] \arrow[ddr, "\eta_\varepsilon"'] & G \times_P Y_P \arrow[d, "{[id : \eta_P]}"] \\
& G \times_P C^+ \arrow[d, "\eta"] \\
& X.
\end{tikzcd}$$

The first assertion follows immediately. Let d denote the degree of $[id : \eta_P]$ that is the degree of η_P. Since $d = d(L, C, C_\varepsilon, \lambda_\varepsilon)$, we have to prove that $d(G, X, C_\varepsilon, \lambda_\varepsilon) = d.d(G, X, C, \lambda)$. We firstly assume that $d(G, X, C_\varepsilon, \lambda_\varepsilon) = 0$. Since $\delta((G, X, C_\varepsilon, \lambda_\varepsilon) = 0$, η_ε is not dominant. So, η or $[id : \eta_P]$ is not dominant. It follows that either $d(G, X, C, \lambda)$ or d is zero. The assertion follows.

We now assume that $d(G, X, C_\varepsilon, \lambda_\varepsilon) \neq 0$, that is that η_ε is dominant. Since the image of η_ε is contained in the image of η, η is dominant. Since η_ε is dominant, the dimension of the closure of the image of $[id : \eta_P]$ at least those of X. Since $\delta(L, C, C_\varepsilon, \lambda_\varepsilon) = \delta(G, X, C, \lambda) = 0$, this implies that η_P is dominant. Now, the second assertion is simply the multiplicative formula for the degree of a double extension field. $\square$

2.4 Well genericaly finite pairs

2.4.1 — If Y is a smooth variety of dimension n, $\mathcal{T}Y$ denotes its tangent bundle. The line bundle $\bigwedge^n \mathcal{T}Y$ over Y will be called the *determinant bundle* and denoted by $\mathcal{D}etY$. If $\varphi : Y \longrightarrow Y'$ is a morphism between smooth variety, we denote by $T\varphi : \mathcal{T}Y \longrightarrow \mathcal{T}Y'$ its tangent map, and by $\mathcal{D}et\varphi : \mathcal{D}etY \longrightarrow \mathcal{D}etY'$ its determinant.

2.4.2 — Consider $\eta : G \times_{P(\lambda)} C^+ \longrightarrow X$ as in Section 2.1.

Definition. We say that (G, X, C, λ) is *genericaly finite* if $d(G, X, C, \lambda) \neq 0$. We say that (G, X, C, λ) is *well genericaly finite* if it is genericaly finite and there exists $x \in C$ such that $T\eta_{[e:x]}$ is invertible.

2.4.3 — Consider the restriction of $T\eta$ to C:

$$T\eta_{|C^+} : \mathcal{T}(G \times_P C^+)_{|C^+} \longrightarrow \mathcal{T}(X)_{|C^+},$$

and the restriction of $\mathcal{D}et\eta$ to C^+:

$$\mathcal{D}et\eta_{|C^+} : \mathcal{D}et(G \times_P C^+)_{|C^+} \longrightarrow \mathcal{D}et(X)_{|C^+}.$$

Since η is G-equivariant, the morphism $\mathcal{D}et\eta_{|C^+}$ is P-equivariant; it can be thought as a P-invariant section of the line bundle $\mathcal{D} := \mathcal{D}et(G \times_P C^+)^*_{|C^+} \otimes \mathcal{D}et(X)_{|C^+}$ over C^+. For any $x \in C$, $\mathbb{K}^*$ acts linearly via λ on the fiber $\mathcal{D}_x$ over x in $\mathcal{D}$: this action is given by a character of $\mathbb{K}^*$, that is an interger m. Moreover, this integer does not depends on x in C: we denote by $\mu^{\mathcal{D}}(C, \lambda)$ this interger.

Lemma 4 *We assuma that X is smooth. The, the following are equivalent:*

(i) (G, X, C, λ) *is well genericaly finite;*

(ii) (G, X, C, λ) *is genericaly finite and* $\mu^{\mathcal{D}}(C, \lambda) = 0$.

Proof. Let us assume that (G, X, C, λ) is well genericaly finite and let $x \in C$ be such that $T\eta_x$ is invertible. Then, $\mathcal{D}et\eta_x$ is a non zero $\mathbb{K}^*$-fixed point in $\mathcal{D}_x$: the action of $\mathbb{K}^*$ on the line $\mathcal{D}_x$ must be trivial.

Let us now assume that (G, X, C, λ) is genericaly finite and $\mu^{\mathcal{D}}(C, \lambda) = 0$. Since the base field is assumed to be of characteristic zero, the exists a point in $G \times_{P(\lambda)} C^+$ where the $T\eta$ is invertible. Since η is G-equivariant, one can find such a point y in C^+. In particular, $\mathcal{D}et\eta_{|C^+}$ is a non zero $P(\lambda)$-invariant section of $\mathcal{D}$. Since $\mu^{\mathcal{D}}(C, \lambda) = 0$, [Res07, Proposition 5] implies that $\mathcal{D}et\eta_{|C}$ is non identacaly zero. □

2.4.4 — The well genericaly finite pairs provide a nice standing to apply Theorem 1:

Theorem 2 *We use notation of Theorem 1 and assume that X is smooth. Let us also assume that $(G, X, C_\varepsilon, \lambda_\varepsilon)$ is well genericaly finite.*

Then, (G, X, C, λ) and $(L, C, C_\varepsilon, \lambda_\varepsilon)$ are well genericaly finite.

Proof. If V is a vector space endowed with a linear action of a one parameter subgroup λ we denote by $V^\lambda_{<0}$ the set of $v \in V$ such that $\lim_{t\to 0} \lambda(t^{-1})v = 0$.

Let $x \in C_\varepsilon$ be such that T_{η_ε} is invertible at $[e : x]$. Consider the subtorus S of dimension two containing the images of λ and λ_ε. It fixes x. The tangent map of η_ε at the point $[e : x]$ induces a S-equivariant linear isomoprhim: $\theta : \mathfrak{g}/\mathfrak{p}_\varepsilon \simeq \mathfrak{g}^{\lambda_\varepsilon}_{<0} \longrightarrow (T_xX)^{\lambda_\varepsilon}_{<0}$. By assumption, $\mathfrak{g}^\lambda_{<0} \subset \mathfrak{g}^{\lambda_\varepsilon}_{<0}$ and $(T_xX)^\lambda_{<0} \subset (T_xX)^{\lambda_\varepsilon}_{<0}$. Since θ is S-equivariant, it follows that it induces an isomorphism between $\mathfrak{g}^\lambda_{<0}$ and $(T_xX)^\lambda_{<0}$. In particular, $\delta(G, X, C, \lambda) = 0$.

Now, the second assertion of Lemma 1 implies that $T_{[}e : x]\eta$ is invertible. It follows that (G, X, C, λ) is well genericaly finite.

Since $\delta(G, X, C_\varepsilon, \lambda_\varepsilon) = 0$, Theorem 1 implies that $\delta(L, C, C_\varepsilon, \lambda_\varepsilon) = 0$. Now, Lemma 1 implies that $T_{[}e : x]\eta_P$ is invertible. By Lemma 3, it follows that $T_{[}e : x]\eta_L$ is invertible. Then, $(L, C, C_\varepsilon, \lambda_\varepsilon)$ is well genericaly finite.

□

3 Application to Belkale-Kumar's product

3.1 An interpretation of coefficient structures

3.1.1 — Let P be a parabolic subgroup of the semisimple group G. Let $T \subset B \subset P$ be a maximal torus and a Borel subgroup of G. Let W denote the Weyl group of T and G. For $w \in W$, we set $X(w) = \overline{BwP/P}$, $X^\circ(w) = BwP/P$ and denote by $[X(w)] \in \mathrm{H}^*(G/P, \mathbb{Z})$ the Poincaré dual class of $X(w)$ in cohomology. Let $w_1, \cdots, w_s \in W$ be such that $\sum_i \mathrm{codim} X(w_i) = \dim G/P$. Let c be the non negative integer such that

$$[X(w_1)].\cdots.[X(w_s)] = c[\mathrm{pt}].$$

Let λ be a one parameter subgroup of T such that $P = P(\lambda)$. Consider $X = (G/B)^s$ and the following T-fixed point $x = (w_1^{-1}B/B, \cdots, w_s^{-1}B/B)$. Let C denote the irreducible component of X^λ containing x. An easy consequence of Kleiman's transversality Theorem (see [Kle76]) is the following lemma which express c has a degree.

Lemma 5 *We have: $\delta(G, X, C, \lambda) = 0$ and $c = d(G, X, C, \lambda)$.*

Proof. See [Res07, proof of Lemma 14]. ☐

3.1.2 — Lemma 5 explains how to express the structure coefficients of $\mathrm{H}^*(G/P,\mathbb{Z})$ in the basis of Schubert classes in terms of maps η's as in Section 2. We are now going to discuss Levi-movability, a notion introduced in [BK06]:

Definition. Let $w_i \in W$ such that $\sum_i \mathrm{codim}(X(w_i), X) = \dim(X)$. Then, $(X(w_1), \cdots, X(w_s))$ is said to be *Levi-movable* if for generic $l_1, \cdots, l_s \in L$ the intersection $l_1X^\circ(w_1) \cap \cdots \cap l_sX^\circ(w_s)$ is transverse at P/P.

Lemma 6 *The following are equivalent:*

(i) $(X(w_1), \cdots, X(w_s))$ *is Levi-movable;*

(ii) there exists $y \in C$ *such that the tangent map* $T_{[e:y]}\eta$ *of* η *at* $[e : y]$ *is invertible.*

Proof. Let $y \in C$ and $l_1, \cdots, l_s \in L$ such that $y = (l_1w_1^{-1}B/B, \cdots, l_sw_s^{-1}B/B)$. Since η extends the immersion of C^+ in C^+; the tangent map $T\eta_[e : y]$ restricts to the identity on $T_[e : y]C^+$. In particular, it induces a linear map:

$$\overline{T}\eta_{[e:y]}\ :\ N_{[e:y]}(C^+, G\times_P C^+) \longrightarrow N_y(C^+, X)$$

such that $T_{[e:y]}\eta$ is an isomorphism if and only if $\overline{T}\eta_{[e:y]}$ is. By π, $N_{[e:y]}(C^+, G\times_P C^+)$ identifies with T_eG/P that is with $\mathfrak{g}/\mathfrak{p}$. Moreover, $N_y(C^+, X)$ equals $\bigoplus_i N_{l_iw_i^{-1}B/B}(Pl_iw_i^{-1}B/B, G/B)$ which identifies with $\oplus_i\mathfrak{g}/(\mathfrak{p}+l_iw_i^{-1}\mathfrak{b}w_il_i)$. Moreover, after composing by these isomorphisms $\overline{T}\eta_{[e:y]}$ is the canonical map $\mathfrak{g}/\mathfrak{p} \longrightarrow \oplus_i\mathfrak{g}/(\mathfrak{p}+l_iw_i^{-1}\mathfrak{b}w_il_i)$. The lemma follows immediately. ☐

3.2 Azad-Barry-Seitz's Theorem

For later use, we recall in this section the main result of [ABS90]. Let G be a semisimple group and P be a parabolic subgroup of G. We choose a Levi subgroup L of P and denote by U its unipotent radical. We are interested in the action of L on the Lie algebra $\mathfrak{u}$ of U.

Let T be a maximal torus of L and B be a Borel subgroup of G containing T. Let $\mathfrak{g}$ denote the Lie algebra of G. Let $\Delta \subset \Phi^+ \subset \Phi$ (resp. $\Delta_L \subset \Phi_L^+ \subset \Phi_L$) be the set of simple roots, positive roots and roots of G (resp. L) for T corresponding of B (resp. $B \cap L$). For any $\alpha \in \Phi$, we denote by $\mathfrak{u}_\alpha$ the line generated by the eigenvectors in $\mathfrak{g}$ of weight α.

Since $\mathfrak{u}$ has no multiplicity for the action of T, it has no multiplicity for the action of L: we have a canonical decomposition of $\mathfrak{u}$ as a sum $\oplus_i V_i$ of irreducible L-modules. Since $T \subset L$, each V_i is a sum of $\mathfrak{u}_\alpha$ for some $\alpha \in \Phi^+ - \Phi^+_L$: the decomposition $\mathfrak{u} = \oplus_i V_i$ corresponds to a partition $\Phi^+ - \Phi^+_L = \bigsqcup_i \Phi_i$.

Let β and β' be two positive roots. We write

$$\beta = \sum_{\alpha\in\Delta_L} c_\alpha \alpha + \sum_{\alpha\in\Delta-\Delta_L} d_\alpha \alpha, \tag{3}$$

with c_α and d_α in $\mathbb{N}$. We also write β' in the same way with some c'_α and d'_α. We write $\beta \equiv \beta'$ if and only if $\sum_{\alpha\in\Delta-\Delta_L} d_\alpha\alpha = \sum_{\alpha\in\Delta-\Delta_L} d'_\alpha\alpha$. The relation $\equiv$ is obviously an equivalence relation. Let S denote the set of equivalence classes in $\Phi^+ - \Phi^+_L$ for $\equiv$. We can now rephrase the main result of [ABS90]:

Theorem 3 (Azad-Barry-Seitz) *For any $s \in S$, $V_s := \oplus\alpha \in s\mathfrak{u}_\alpha$ is an irreducible L-module. In particular, $\bigsqcup_i \Phi_i$ is the partition in equivalence classes for $\equiv$.*

For any $\alpha \in \Phi$, we denote by $\mathrm{Ker}\alpha$ the Kernel of the character α of T. Let Z be the center of L. Let Z° denote the neutral component of Z and $X(Z^\circ)$ denote the group of characters of Z°. Under the action of Z°, $\mathfrak{u}$ decompose as

$$V = \oplus_{\chi\in X(Z^\circ)} V_\chi,$$

where V_χ is the vector subspace of weight χ. Since Z° is central in L, each V_χ is L-stable.

Note that $Z^\circ \subset Z \subset T$; and more precisely

$$Z = \bigcup_{\alpha\in\Delta_L} \mathrm{Ker}\alpha.$$

It follows that for β as in Equation 3, the restriction $\beta_{|Z^\circ}$ of β to Z° equals $\sum_{\alpha\in\Delta-\Delta_L} d_\alpha \alpha_{|Z^\circ}$. Moreover, the family $(\alpha_{|Z^\circ})_{\alpha\in\Delta-\Delta_L}$ is free in the rational vector vector space containing the characters of the torus Z°. We obtain that

$$\beta \equiv \beta' \iff \beta_{|Z^\circ} = \beta'_{|Z^\circ}.$$

In particular, each V_s is one V_χ with above notation. In particular, we have:

Corollary 1 *Each V_χ (with $\chi \in X(Z^\circ)$) is an irreducible L-module.*

3.3 A multiplicative formula for structure coefficients of $\odot_0$

3.3.1 — Let now $Q\subset P$ be two parabolic subgroups of the semisimple group G. Let $T\subset B\subset Q$ be a maximal torus and a Borel subgroup of G.

Let L denote the Levi subgroup of P containing T. Let W (resp. W_P) denote the Weyl group of T and G (resp. L).

For any $w\in W$, $w^{-1}Bw\cap L$ is a Borel subgroup of L containing T. So, there exists a unique $\overline{w}\in W_P$ such that

$$\overline{w}^{-1}(B\cap L)\overline{w}=w^{-1}Bw\cap L. \tag{4}$$

To any $w\in W$, we now associated three Schubert varieties in G/P, G/Q and $L/L\cap Q$ respectively:

$$X^{G/P}(w)=\overline{BwP/P},\quad X^{G/Q}(w)=\overline{BwQ/Q}$$

and

$$X^{L/L\cap Q}(w)=\overline{(L\cap B)\overline{w}(L\cap Q/L\cap Q)}.$$

Theorem 4 *Let $w_1,\cdots,w_s\in W$. We assume that $\sum_i \operatorname{codim} X^{G/Q}(w_i)=\dim G/Q$ and $(X^{G/Q}(w_1),\cdots,X^{G/Q}(w_s))$ is Levi-movable. Then, we have:*

(i) $\sum_i \operatorname{codim} X^{G/P}(w_i)=\dim G/P$ *and* $\sum_i \operatorname{codim} X^{L/L\cap Q}(w_i)=\dim L/(L\cap Q)$*;*

(ii) $(X^{G/P}(w_1),\cdots,X^{G/P}(w_s))$ *and* $(X^{L/L\cap Q}(w_1),\cdots,X^{L/L\cap Q}(w_s))$ *are Levi-movable.*

Moreover, by Assertion (i) we can define three integers by the formulas:

$$\begin{array}{rcl}[X^{G/Q}(w_1)].\cdots.[X^{G/Q}(w_s)] &=& c^{G/Q}_{w_1,\cdots,w_s}[\mathrm{pt}],\\ [X^{G/P}(w_1)].\cdots.[X^{G/P}(w_s)] &=& c^{G/P}_{w_1,\cdots,w_s}[\mathrm{pt}]\text{ and}\\ [X^{L/L\cap Q}(w_1)].\cdots.[X^{L/L\cap Q}(w_s)] &=& c^{L/L\cap Q}_{w_1,\cdots,w_s}[\mathrm{pt}].\end{array}$$

Then, we have:

$$c^{G/Q}_{w_1,\cdots,w_s}=c^{G/P}_{w_1,\cdots,w_s}.c^{L/L\cap Q}_{w_1,\cdots,w_s}.$$

Proof. We begin the proof by making some remarks about the tangent space $T_{Q/Q}G/Q$ of G/Q at Q/Q. Let L_Q denote the Levi subgroup of Q containing T and Z° denote its connected center. We decompose $T_{Q/Q}G/Q$ as a sum $\oplus_{\chi\in X(Z^\circ)}V_\chi$ of eigenvector spaces for the action of the torus Z°. Note that

$T_{Q/Q}P/Q \subset T_{Q/Q}G/Q$ is stable by the action of L_Q. Now, Corollary 1 implies that there exists $S \subset X(Z^\circ)$ such that

$$T_{Q/Q}P/Q = \oplus_{\chi\in S}V_\chi. \tag{5}$$

Let $l \in L_Q$ and $w \in W$. We set $Y^\circ(w) = w^{-1}BwQ/Q$. One easily checks that $lY^\circ(w)$ is stable by the action of Z°. Since $Q/Q \in lY^\circ(w)$ is a point fixed by Z°, Z° acts on $T_{Q/Q}lY^\circ(w)$. In particular,

$$T_{Q/Q}lY^\circ(w) = \oplus_{\chi\in X(Z^\circ)}V_\chi \cap T_{Q/Q}lY^\circ(w). \tag{6}$$

Since $(X^{G/Q}(w_1), \cdots, X^{G/Q}(w_s))$ is Levi-movable, there exist $l_1, \cdots, l_s \in L_Q$ such that

$$T_{Q/Q}l_1Y^\circ(w_1^\circ) \oplus \cdots \oplus T_{Q/Q}l_sY^\circ(w_s^\circ) = T_{Q/Q}G/Q. \tag{7}$$

Consider now the G-equivariant projection $\pi : G/Q \longrightarrow G/P$. Note that the Kernel of the tangent map $T_{Q/Q}\pi$ of π at Q/Q is $T_{Q/Q}P/Q$. So, Equations 5 and 6 imply that for any $i = 1, \cdots, s$, $T_{Q/Q}$ induces an isomorphism from $T_{Q/Q}l_iY^\circ(w_i) \cap \oplus_{\chi\notin S}V_\chi$ onto $T_{Q/Q}l_i\pi(Y^\circ(w_i))$. Now, Equation 7 imply that $\oplus_i T_{P/P}l_i\pi(Y^\circ(w_i)) = T_{P/P}G/P$. Moreover, L_Q is contained in L; Assertions (i) and (ii) of the theorem follows for G/P.

Recall that X is the variety $(G/B)^s$ and $x = (w_1^{-1}B/B, \cdots, w_s^{-1}B/B)$. Let λ (resp. λ_ε) be a one parameter subgroup of T such that $P(\lambda)$ (resp. $P(\lambda_\varepsilon)$) equals P and Q. Let C (resp. C_ε) denote the irreducible component of X^λ (resp. X^{λ_ε}) containing x. With notation of Section 2, Lemma 5 implies that $\delta(G, X, C_\varepsilon, \lambda_\varepsilon)$ and $\delta(G, X, C, \lambda)$ equal zero. Theorem 1 implies that $\delta(L, C, C_\varepsilon, \lambda_\varepsilon) = 0$. Assertion (i) for $L/L \cap Q$ follows. Now, the second assertion of Theorem 1 with Lemma 5 imply the last formula of the theorem.

It remains to prove that $(X^{L/L\cap Q}(w_1), \cdots, X^{L/L\cap Q}(w_s))$ is Levi-movable. Since $(X^{G/Q}(w_1), \cdots, X^{G/Q}(w_s))$ is Levi-movable, Lemma 6 shows that there exists $y \in C_\varepsilon$ such that $T_{[e:y]}\eta_\varepsilon$ is invertible. Now, Lemmas 1 and 3 imply that $T_{[e:y]}\eta_L$ is invertible. So, Lemma 6 allows to conclude. ☐

Remark. In the case when $G = \mathrm{SL}_n$, Theorem 4 was already obtained in [Ricar] for a lot of pairs $Q \subset P$.

3.3.2 — If one know how to compute in $(\mathrm{H}^*(G/P, \mathbb{Z}), \odot_0)$ for any maximal P and any G, then Theorem 4 can be used to compute the structure coefficients of $(\mathrm{H}^*(G/Q, \mathbb{Z}), \odot_0)$ for any parabolic subgroup Q. To illustrate this principle, we state an analogous to [Ricar, Corollary 23]:

Corollary 2 *Let* $G = \mathrm{Sp}_{2n}$*. The non-zero coefficients structures of the ring* $(\mathrm{H}^*(G/B, \mathbb{Z}), \odot_0)$ *are all equal to* 1*.*

Proof. The proof proceeds by induction on n. Let c be a non-zero coefficient structure of $(\mathrm{H}^*(G/B, \mathbb{Z}), \odot_0)$. Let (w_1, w_2, w_3) be such that $[X(w_1)].[X(w_2)].[X(w_3)] = c[\mathrm{pt}]$. Since c is non-zero, $(X(w_1), X(w_2), X(w_3))$ is Levi-movable.

Consider the stabilizer P in G of a line in $\mathbb{K}^{2n}$. Theorem 4 applied with $B \subset P$ shows that c is the product of coefficient structure of $(\mathrm{H}^*(G/P, \mathbb{Z}), \odot_0)$ and one of $(\mathrm{H}^*(\mathrm{Sp}_{2n-2}/B, \mathbb{Z}), \odot_0)$. The fact that G/P is a projective space and the induction allow to conclude. ☐

4 Application to quiver representations

4.1 Definitions

In this section, we fix some classical notation about quiver representations.

Let Q be a quiver (that is, a finite oriented graph) with vertexes Q_0 and arrows Q_1. An arrow $a \in Q_1$ has initial vertex ia and terminal one ta. A representation R of Q is a family $(V(s))_{s\in Q_0}$ of finite dimensional vector spaces and a family of linear maps $u(a) \in \mathrm{Hom}(V(ia), V(ta))$ indexed by $a \in Q_1$. The dimension vector of R is the family $(\dim(V(s)))_{s\in Q_0} \in \mathbb{N}^{Q_0}$.

Let us fix $\alpha \in \mathbb{N}^{Q_0}$ and a vector space $V(s)$ of dimension $\alpha(s)$ for each $\alpha \in Q_0$. Set

$$\mathrm{Rep}(Q, \alpha) = \bigoplus_{a\in Q_1} \mathrm{Hom}(V(ia), V(ta)).$$

Consider also the groups:

$$\mathrm{GL}(\alpha) \prod_{s\in Q_0} \mathrm{GL}(V(s)) \text{ and } \mathrm{SL}(\alpha) \prod_{s\in Q_0} \mathrm{SL}(V(s).$$

Let $\alpha, \beta \in \mathbb{Z}^{Q_0}$. The Ringle form is defined by

$$\langle \alpha, \beta \rangle = \sum_{s\in Q_0} \alpha(s)\beta(s) - \sum_{a\in Q_1} \alpha(ia)\beta(ta).$$

Assume now that $\alpha, \beta \in \mathbb{N}^{Q_0}$. Following Derksen-Schofield-Weyman (see [DSW07]), we define $\alpha \circ \beta$ to be the number of α-dimensional subrepresentation of a general representation of dimension $\alpha + \beta$ if it is finite, and 0 otherwise.

4.2 Dominant pairs

4.2.1 — Let λ be a one parameter subgroups of $\mathrm{GL}(\alpha)$. For any $i \in \mathbb{Z}$ and $s \in Q_0$, we set $V_i(s) = \{v \in V(s) \mid \lambda(t)v = t^i v\}$ and $\alpha_i(s) = \dim V_i(s)$. Obviously, almost all α_i are zero; and, $\alpha = \sum_{i\in\mathbb{Z}} \alpha_i$. Moreover, λ is determined up to conjugacy by the α_i's.

The parabolic subgroup $P(\lambda)$ of $\mathrm{GL}(\alpha)$ associated to λ is the set of $(g(s))_{s\in Q_0}$ such that for all $i \in \mathbb{Z}$ $g(s)(V_i(s)) \subset \oplus_{j\leq i} V_j(s)$.

Now, $\mathrm{Rep}(Q,\alpha)^\lambda$ is the set of the $(u(a))_{a\in Q_1}$'s such that for any $a \in Q_1$ and for any $i \in \mathbb{Z}$, $u(a)(V_i(ia)) \subset V_i(ta)$. It is isomorphic to $\prod_i \mathrm{Rep}(Q,\alpha_i)$. In particular, it is irreducible and denoted by C from now on.

Moreover, C^+ is the set of the $(u(a))_{a\in Q_1}$'s such that for any $a \in Q_1$ and for any $i \in \mathbb{Z}$, $u(a)(V_i(ia)) \subset \oplus_{j\leq i} V_j(ta)$.

Consider the morphism $\eta_\lambda \,:\, G \times_{P(\lambda)} C^+ \longrightarrow \mathrm{Rep}(Q,\alpha)$. Note that, $P(\lambda)$, C and C^+ only depend on the list (ordered by the index i) of non-zero α_i's.

4.2.2 — The last observation allows the following

Definition. A *decomposition* of the vector dimension α, is a family $(\beta_1, \cdots, \beta_s)$ of non-zero vector dimensions such that $\alpha = \beta_1 + \cdots + \beta_s$. We denote the decomposition by $\alpha = \beta_1 \tilde{+} \cdots \tilde{+} \beta_s$.

Any one parameter subgroup λ induces a decomposition of $\alpha = \beta_1 \tilde{+} \cdots \tilde{+} \beta_s$ where the β_j's are the non-zero α_i's ordered by the index i. Note that, up to conjugacy, $P(\lambda)$, C and C^+ only depend on this decomposition. In particular, one can define (up to conjugacy) the *map* $\eta_{\beta_1 \tilde{+} \cdots \tilde{+} \beta_s}$ *associated to a decomposition of* α.

4.2.3 — Consider a decomposition $\alpha = \beta_1 \tilde{+} \beta_2$ with two dimension-vectors and the associated morphism η. In this section, we collect some easy properties of η. Let $(u,v) \in \mathrm{Rep}(Q,\beta_1) \times \mathrm{Rep}(Q,\beta_2) = C \subset \mathrm{Rep}(Q,\alpha) = X$. Since η extends the immersion of C^+ in X, the tangent map $T_{(u,v)}\eta$ induces the identity on $T_{(u,v)}C^+$. In particular, it induces a linear map

$$\overline{T}\eta_{[e:(u,v)]} \,:\, N_{[e:(u,v)]}(C^+, G \times_P C^+) \longrightarrow N_{(u,v)}(C^+, \mathrm{Rep}(Q,\alpha)).$$

Moreover, $N_{[e:(u,v)]}(C^+, G \times_P C^+)$ identifies with $\oplus_{s\in Q_0}\mathrm{Hom}(V(s), W(s))$ and $N_{(u,v)}(C^+, \mathrm{Rep}(Q,\alpha))$ with $\oplus_{a\in Q_1}\mathrm{Hom}(V(ia), W(ta))$. A direct computation gives the following

Lemma 7 *With the above identification, we have:*

$$\overline{T}\eta_{[e:(u,v)]}(\sum_{s\in Q_0}\varphi(s)) = \sum_{a\in Q_1} v(a)\varphi(ta) - \varphi(ha)u(a).$$

In particular, the Kernel of $\overline{T}\eta_{(u,v)}$ *is* $\mathrm{Hom}(u,v)$ *and its Image is* $\mathrm{Ext}(u,v)$.

The quantities $\delta(\eta)$ and $d(\eta)$ are also particularly interesting:

Lemma 8 *Consider a decomposition* $\alpha = \beta_1 \tilde{+} \beta_2$ *and the associated map* η. *Then,*

(i) $\delta(\eta) = -\langle \beta_1, \beta_2 \rangle$, *and*

(ii) $d(\eta) = \beta_1 \circ \beta_2$.

Proof. By the discussion preceding Lemma 7, $\delta(\eta)$ equals the difference between the dimension of $\oplus_{a\in Q_1}\mathrm{Hom}(V(ia), W(ta))$ and of $\oplus_{s\in Q_0}\mathrm{Hom}(V(s), W(s))$. The first assertion follows.

Let $u \in \mathrm{Rep}(Q,\alpha)$. Using Immersion 1, one identifies the fiber $\eta^{-1}(u)$ with the set u-stable subspaces of V of dimension β_1. In particular, $\eta^{-1}(u)$ identifies with the set of β_1-dimensional subrepresentations of u. Since the characteristic of k is assumed to be zero, when u is generic this numbers equals $d(\eta)$ on one hand and $\beta_1 \circ \beta_2$ on the other one. $\square$

If Y is a smooth variety of dimension n, $\mathcal{T}Y$ denotes its tangent bundle. The line bundle $\bigwedge^n \mathcal{T}Y$ over Y will be called the *determinant bundle* and denoted by $\mathcal{D}etY$. If $\varphi : Y \longrightarrow Y'$ is a morphism between smooth variety, we denote by $\mathcal{D}et\varphi : \mathcal{D}etY \longrightarrow \mathcal{D}etY'$ the determinant of its tangent map $T\varphi$. We consider now the restriction of $\mathcal{D}et\eta$ to C^+: it is a P-invariant section of the P-linearized line bundle $\mathcal{D}et$ over C^+ defined by $\mathcal{D}et = \mathcal{D}et(G \times_P C^+)^*_{|C^+} \otimes \mathcal{D}et(X)_{|C^+}$.

Recall that for any $s \in Q_0$, we have fixed a vector space $V(s)$ of dimension $\alpha(s)$. Let us fix, for any $s \in Q_0$ a decomposition $V(s) = V_1(s) \oplus V_2(s)$ such that $\dim V_i(s) = \beta_i(s)$ for $i = 1, 2$. Consider the one parameter subgroup λ of $\mathrm{GL}(\alpha)$ defined by $\lambda(s)(t)$ stabilizes the decompostion $V_1(s) \oplus V_2(s)$, equals to Id when restricted to $V_1(s)$ and tId when restrected to $V_2(s)$. It satisfies $P(\lambda) = P$, $\mathrm{Rep}(Q,\alpha)^\lambda = C$ and $C^+(\lambda) = C^+$.

Lemma 9 *We assume that* $\langle \beta_1, \beta_2 \rangle = 0$. *The one parameter subgroup* λ *acts trivially on* $\mathcal{D}et_{|C}$.

Proof. Since C is an affine space, λ acts by the same character on each fiber of $\mathcal{D}et_{|C}$. Since η extend the identity on C^+, its character is the difference between the weights of λ in

$$N_0(C^+, X) \simeq \oplus_{a\in Q_1}\mathrm{Hom}(V_1(ia), V_2(ta))$$

and in

$$N_0(C^+, G\times_P C^+) \simeq T_eG/P \simeq \oplus_{s\in Q_0}\mathrm{Hom}(V_1(s), V_2(s)).$$

So, this character equals:

$$\sum_{a\in Q_1}\beta_1(ia)\beta_2(ta) - \sum_{s\in Q_0}\beta_1(s)\beta_2(s);$$

that is, $-\langle\beta_1, \beta_2\rangle$. The lemma follows. $\square$

4.3 Two formulas for $d(\eta_{\beta_1\tilde{+}\cdots\tilde{+}\beta_s})$

Here comes the main result of this section:

Theorem 5 *Let $\alpha = \beta_1\tilde{+}\cdots\tilde{+}\beta_s$ be a decomposition of α such that for all $i < j$, $\langle\beta_i, \beta_j\rangle = 0$.*

Then, $\delta(\eta_{\beta_1\tilde{+}\cdots\tilde{+}\beta_s}) = 0$ and

$$\begin{aligned} d(\eta_{\beta_1\tilde{+}\cdots\tilde{+}\beta_s}) &= (\beta_1\circ\alpha - \beta_1).(\beta_2\circ\alpha - \beta_1 - \beta_2).\cdots.(\beta_{s-1}\circ\beta_s) \\ &= (\alpha - \beta_s\circ\beta_s).(\beta - \beta_s - \beta_{s-1}\circ\beta_{s-1}).\cdots.(\beta_1\circ\beta_2). \end{aligned}$$

Proof. By Section 4.2.1, the codimension of C^+ in $G\times_P C^+$ is

$$\sum_{i<j}\sum_{s\in Q_0}\beta_i(s)\beta_j(s);$$

and, the codimension of C^+ in $\mathrm{Rep}(Q, \alpha)$ is

$$\sum_{i<j}\sum_{a\in Q_1}\beta_i(ia)\beta_j(ta).$$

Since $\forall i < j$ $\langle\beta_i, \beta_j\rangle = 0$, this implies that $\delta(\eta_{\beta_1\tilde{+}\cdots\tilde{+}\beta_s}) = 0$.

We will just prove the first formula for $d(\eta_{\beta_1\tilde{+}\cdots\tilde{+}\beta_s})$. The second one can be proved in a similar way. When $s = 2$, the theorem follows from Lemma 8.

Assume that $s = 3$. A direct application of Theorem 1 with $\eta_\varepsilon = \eta_{\beta_1 \tilde{+} \beta_2 \tilde{+} \beta_3}$ and $\eta = \eta_{\beta_1 \tilde{+} (\alpha - \beta_1)}$ gives

$$\begin{aligned} d(\eta_{\beta_1 \tilde{+} \beta_2 \tilde{+} \beta_3}) &= (\beta_1 \circ \alpha - \beta_1).d(\eta_{\beta_2 \tilde{+} \beta_3}) \\ &= (\beta_1 \circ \alpha - \beta_1).(\beta_2 \circ \beta_3). \end{aligned}$$

One can easily ends the proof by an induction on s. ☐

Remark. In the proof of Theorem 5, the induction was made by the paranthésages $\beta_1 \tilde{+} \cdots \tilde{+} \beta_s = \beta_1 \tilde{+} (\beta_2 \tilde{+} \cdots \tilde{+} \beta_s)$ and $\beta_1 \tilde{+} \cdots \tilde{+} \beta_s = (\beta_1 \tilde{+} (\beta_2 \cdots \tilde{+} \beta_s)$. All other paranthésage gives a similar formula.

4.3.1 — We now want to discuss the assumption "$\forall i < j \ \langle \beta_i, \beta_j \rangle = 0$". This assumption is actually similar to Levi-movability. Indeed, we have the equivalent of Lemma 6:

Lemma 10 *Let $\alpha = \beta_1 \tilde{+} \cdots \tilde{+} \beta_s$ be a decomposition of α such that $\delta(\eta_{\beta_1 \tilde{+} \cdots \tilde{+} \beta_s}) = 0$. Then, the following are equivalent:*

(i) for all $i < j$, $\langle \beta_i, \beta_j \rangle = 0$ and $d(\eta_{\beta_1 \tilde{+} \cdots \tilde{+} \beta_s}) \neq 0$;

(ii) there exists $y \in C$ such that the tangent map of $\eta_{\beta_1 \tilde{+} \cdots \tilde{+} \beta_s}$ at $[e : y]$ is invertible.

Proof. Let $\underline{V} = \oplus_i \underline{V}_i$ be a decomposition of $\underline{V}$ such that $\dim \underline{V}_i = \beta_i$. Consider the linear action of the torus $Z = \mathbb{G}_m^s$ on $\underline{V}$ such that $(t_1, \cdots, t_s).v = t_i v$ for all $t_i \in \mathbb{G}_m$ and $v \in \underline{V}_i(s)$ for any $s \in Q_0$. Since Z is embedded in $\mathrm{GL}(\alpha)$ it also acts on $G \times_P C^+$.

Let y be a point in C satisfying Assertion (ii). Since, Z fixes $[e : y]$ and η is G-equivariant, $T\eta_{\beta_1 \tilde{+} \cdots \tilde{+} \beta_s}$ is Z-equivariant for the tangent action of Z. It follows that for all $i < j$, $T\eta_{\beta_1 \tilde{+} \cdots \tilde{+} \beta_s}$ induces an isomorphism from the eigensubspaces of $T_{[e:y]} G \times_P C^+$ and $T_y \mathrm{Rep}(Q, \alpha)$ of weight $t_j t_i^{-1}$. In particular, these two eigensubspaces have the same dimension. But, a direct compution shows that the difference between these two dimension is precisely $\langle \beta_i, \beta_j \rangle$. Assertion (i) follows.

Conversely, let us assume that Assertion (i) follows. Since $d(\eta_{\beta_1 \tilde{+} \cdots \tilde{+} \beta_s}) \neq 0$, there exists a point $G \times_P C^+$ where the tangent map of $\eta_{\beta_1 \tilde{+} \cdots \tilde{+} \beta_s}$ is invertible. Since η is G-equivariant, its determinant is not identicaly zero on C^+. Using the fact for all $i < j$ $\langle \beta_i, \beta_j \rangle = 0$, a direct computation (like in the proof of Lemma 9) shows that Z acts trivialy on $\mathcal{D}et_{|C}$. By [Res07, Proposition 5], the determinant of η is not identicaly zero on C. Assertion (ii) follows. ☐

4.3.2 — The dimension of $\mathrm{Ext}(\mathrm{u},\mathrm{v})$ for generic α and β dimensional representations u and v is denoted by $\mathrm{ext}(\alpha,\beta)$.

Corollary 3 *We assume that Q has no oriented cycle. Let α, β and γ be three dimension-vectors. We assume that $\langle\alpha,\beta\rangle=\langle\alpha,\gamma\rangle=0$ and $\beta\circ\gamma=1$.*

Then, $\alpha\circ\beta+\gamma=(\alpha\circ\beta).(\alpha\circ\gamma)$.

Proof. Theorem 5 applied to $\alpha\tilde{+}\beta\tilde{+}\gamma$ gives:

$$(\alpha+\beta\circ\gamma).(\alpha\circ\beta)=(\alpha\circ\beta+\gamma).(\beta\circ\gamma)=(\alpha\circ\beta+\gamma),$$

since $(\beta\circ\gamma)=1$. If $\alpha\circ\beta=0$, the corollary follows. Now assume that $\alpha\circ\beta\neq 0$. Lemma 10 implies that the determinant of $\eta_{\alpha\tilde{+}\beta}$ is not identically zero on C. But, Lemma 7 implies that $\mathrm{ext}(\alpha,\beta)=0$. Now, the corollary is a direct consequence of Lemma 11 below. □

Lemma 11 *We assume that Q has no oriented cycle. Let α, β and γ be three vector dimensions. We assume that $\beta\circ\gamma=1$ and $\mathrm{ext}(\alpha,\beta)=0$.*

Then, $\alpha+\beta\circ\gamma=\alpha\circ\gamma$.

Proof. In [DSW07], Derksen-Schofield-Weyman prove that $\alpha\circ\gamma$ equals the dimension of $\mathbb{K}[\mathrm{Rep}(Q,\gamma)]_{\langle\alpha,\cdot\rangle}$. Consider the multiplication morphism:

$$m\,:\,\mathbb{K}[\mathrm{Rep}(Q,\gamma)]_{\langle\alpha,\cdot\rangle}\otimes\mathbb{K}[\mathrm{Rep}(Q,\gamma)]_{\langle\beta,\cdot\rangle}\longrightarrow\mathbb{K}[\mathrm{Rep}(Q,\gamma)]_{\langle\alpha+\beta,\cdot\rangle}.$$

We claim that m is an isomorphism. The lemma will follow directly. Since $\dim(\mathbb{K}[\mathrm{Rep}(Q,\gamma)]_{\langle\beta,\cdot\rangle})=1$ and $\mathbb{K}[\mathrm{Rep}(Q,\gamma)]$ has no zero-divisors, m is injective.

Since $\mathrm{ext}(\alpha,\beta)=0$, $\eta_{\alpha\tilde{+}\beta}$ is dominant. But, it is proper; so, it is surjective.

In [DW00], Derksen-Weyman prove that $\mathbb{K}[\mathrm{Rep}(Q,\gamma)]_{\langle\alpha+\beta,\cdot\rangle}$ is generated by functions c^V associated to various $\alpha+\beta$-dimensional representation V (see also [DZ01]). Since $\eta_{\alpha\tilde{+}\beta}$ is surjective, there exists an α-dimensional subrepresentation V' of V. By [DW00, Lemma 1], $c^V=c^{V'}.c^{V/V'}$. It follows that m is surjective. □

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