

Quantum limits in the measurement of very small displacements in optical images

C. Fabre, J.B. Fouet, A. Maître

*Laboratoire Kastler Brossel, Université Pierre et Marie Curie, Case 74,
75252 Paris cedex 05, France*

Abstract

We consider the problem of the measurement of very small displacements in the transverse plane of an optical image using a split-photodetector. We show that the standard quantum limit for such a measurement, equal to the diffraction limit divided by the square root of the number of photons used in the measurement, cannot be overcome by using "ordinary" single mode squeezed light. We give the form of possible multimode nonclassical states of light enabling us to enhance by orders of magnitude the resolution of such a measurement beyond the standard quantum limit.

PACS numbers: 42.50 Dv, 42.50 Lc, 42.30 Yc

As CCD cameras and photodetector arrays are now widely used to record optical images, it is important to know to which extent does the quantum nature of light impose fundamental limits to the quality and resolution in this kind of optical measurement. Quantum limits in optics have been already studied in many configurations, but only in the case of light monitored by large area photodetectors that give an information integrated over all the transverse plane. On the other hand, subtle quantum effects occurring in images are now actively studied [1], but not, to the best of our knowledge, for the determination of quantum limits in transverse measurements. In order to give precise assessments of such quantum limits, we will focus our attention to a precise problem, namely the measurement of very small displacements in an optical image. After recalling the standard quantum limit for the

optical resolution in such a configuration, we will show that one must use multi-transverse modes nonclassical states of light to improve the resolution beyond the standard quantum limit, and give examples of such multimode nonclassical states.

According to the Rayleigh criterion [2,3], the resolution in optical images is limited by diffraction. This classical criterion, based on the capabilities of the human visual system [4], can be violated when one uses modern photodetectors which are able to resolve image details much smaller than the size of a diffraction spot. For such measurements the resolution is limited by the quantum fluctuations of the light affecting each pixel [5,6].

Let us take an orthonormal basis $u_i(x, y)$ of transverse modes (where x and y are the coordinates in the transverse plane), and call a_i the corresponding annihilation operators. The photocurrent N_S measured on a pixel of area S , expressed in units of number of photons recorded during the measurement time, is equal to the integral over S of the quantity $\langle (E^{(+)})^\dagger E^{(+)} \rangle$, where $E^{(+)}(x, y) = \sum_j a_j u_j(x, y)$. Let us first assume that the input beam is described by a *single mode quantum state* in the transverse mode u_1 . A straightforward calculation based on standard photodetection theory shows that

$$\begin{aligned} C_{N_A N_B} &= \langle N_A N_B \rangle - \langle N_A \rangle \langle N_B \rangle \\ &= N_{S_A \cap S_B} + \frac{\langle N_A \rangle \langle N_B \rangle}{\langle N_{tot} \rangle^2} (\Delta N_{tot}^2 - \langle N_{tot} \rangle) \end{aligned} \quad (1)$$

where N_A , N_B and $N_{S_A \cap S_B}$ are the photocurrents measured by photodetectors of areas S_A , S_B and $S_A \cap S_B$. $\langle N_{tot} \rangle$ and ΔN_{tot}^2 are respectively the mean and the variance of the total photocurrent, measured by a detector covering the entire transverse plane. If the quantum state describing the single mode beam is a coherent state, then $C_{N_A N_B} = N_{S_A \cap S_B}$: the photocurrent fluctuations are at the shot noise level on all the pixels, whatever their size and position, and the fluctuations at two different pixels are uncorrelated. This result is consistent with the simple picture that a coherent state is 'composed' of photons which are randomly distributed not only in time, but also in the transverse plane inside the beam area.

Let us consider as an example a configuration which is very often used for the measurement of small displacements, for example in atomic force microscopy [8], ultra-weak

absorption measurement [9], or in single molecule tracking in biology [10] : the photodetector array is a two-pixel photodetector ("split detector"), delivering two photocurrents N_A and N_B proportional to the light intensities integrated over two halves of the transverse plane (A : $x > 0$; B : $x < 0$). A light beam of intensity $I(x, y)$ (expressed in photons per unit area), assumed symmetrical with respect to the pixel boundary, is incident on the split photodetector. If the beam is initially centered on the detector, the mean value of the photocurrent difference $N_- = N_A - N_B$ is directly proportional to the relative lateral displacement D of the whole beam with respect to the initial symmetrical configuration, at least for displacements small compared to the beam size. The noise affecting this quantity, sometimes called "position noise", limits the accuracy in the measurement of D . It has been studied experimentally and theoretically for various laser beams [11,12]. When the light beam is in a coherent state, the displacement D_{sql} providing a value of $\langle N_- \rangle$ equal to this noise is the standard quantum limit in the measurement of a small transverse displacement. One finds from Eq(1)

$$D_{sql} = \frac{\Delta}{\sqrt{\langle N_{tot} \rangle}} \quad (2)$$

where Δ is the beam effective width, which is defined by

$$\Delta = \frac{\langle N_{tot} \rangle}{\left(\frac{\partial \langle N_- \rangle}{\partial D} \right)_{D=0}} = \frac{\langle N_{tot} \rangle}{2 \int_{-\infty}^{+\infty} dy I(0, y)} \quad (3)$$

and which depends on the exact beam shape. For example, $\Delta = 0.63w_0$ for a TEM_{00} Gaussian beam of waist w_0 . Δ is, within some numerical factor of order one, the Rayleigh, or diffraction, limit for the optical resolution in this specific measurement. Expressions analog to (2) have been already obtained [8]. D_{sql} can be much smaller than the optical wavelength even with light beams of moderate intensities.

Let us now replace the single mode coherent beam by a single mode nonclassical beam, and more precisely by a sub-Poissonian beam, for which $\Delta N_{tot}^2 < \langle N_{tot} \rangle$, such as those produced by some semiconductor lasers [13]. Formula (1) implies that for any pixel of area S_A

$$\frac{\Delta N_A^2}{\langle N_A \rangle} - 1 = \frac{\langle N_A \rangle}{\langle N_{tot} \rangle} \left(\frac{\Delta N_{tot}^2}{\langle N_{tot} \rangle} - 1 \right) \quad (4)$$

As $\langle N_A \rangle \leq \langle N_{tot} \rangle$, this expression shows that the relative noise reduction with respect to shot noise is smaller when one measures a part of a single mode beam than on the total beam. The degradation in the measured intensity squeezing is the proportion $\frac{\langle N_A \rangle}{\langle N_{tot} \rangle}$ of the intensity measured in the partial detection. One gets an expression similar to formula (4) when one determines the effect on squeezing of a lossy medium of intensity transmission coefficient T (replacing $\frac{\langle N_{SA} \rangle}{\langle N_{tot} \rangle}$ by T). For a single mode beam, a partial detection is thus equivalent to a loss mechanism. This property can be simply pictured by asserting that a single mode sub-Poissonian beam is composed of photons which are somehow antibunched in time, but still completely randomly distributed in the transverse plane, like in a coherent state : transverse randomness is therefore associated to the single mode nature of the field, and not to its coherent or quasi-classical nature. From (1), it is easy to show that the noise on $N_- = N_A - N_B$ does not depend on the quantum state of the single mode beam used in the experiment, and therefore that the minimum displacement measurable with a split detector is still D_{sql} : we are led to the conclusion that a *single mode nonclassical state cannot improve the transverse resolution beyond the standard quantum limit*.

One must therefore use a *transverse multimode state of light*. Let us first consider a two-mode nonclassical state, spanned on the first two transverse modes u_1 and u_2 . In the small fluctuation limit (i.e. neglecting terms quadratic in the fluctuations), which is valid when one uses intense beams, assuming that the modes u_1 and u_2 are real, and using the orthonormality and closure relations for the transverse modes, the photocurrent fluctuation measured on a pixel of area S_A can be shown to be

$$\begin{aligned} \Delta N_A^2 = \langle N_A \rangle + & \left[\sum_{i=1,2} (C_{a_i^+ a_i} |A_i|^2 + C_{a_i a_i} (A_i^*)^2) \right. \\ & + 2C_{a_1 a_2^+} A_2 A_1^* + 2C_{a_1 a_2} A_1^* A_2^* \\ & \left. + \text{complex conjugate} \right] \end{aligned} \quad (5)$$

where $C_{a_i^+ a_i}$ (and analogous quantities) are the correlation functions of operators a_i^+ and a_i ,

and A_i ($i = 1, 2$) is the overlap integral on the pixel surface

$$A_i = \int \int_{S_A} u_i(x, y) \langle E^{(+)}(x, y) \rangle dx dy \quad (6)$$

If the system is in a two-mode coherent state, all the quantum mean values are zero in (5), except $\langle N_A \rangle$: one finds again the shot noise, like in the single mode case. If the system is not in a coherent state, it is no longer possible to write expression (5) in a form analogous to (4), reminiscent of a loss mechanism. One finds therefore that considering a partial photodetection as equivalent to a loss is not true in general. One can find in [11] and [12] examples of nontrivial noise variations in partial photodetection, in the case of multimode laser beams with excess noise, providing useful information on the laser used in the experiment.

Let us now use a two-mode nonclassical state. It is easy to show that the noise on the intensity difference $N_A - N_B$ between the two zones is given by an expression similar to (5), where $\langle N_A \rangle$ is replaced by the total number of photons $\langle N_{tot} \rangle$ in the light beam, and where the A_i coefficients are replaced by A'_i given by

$$A'_i = \int \int_{x>0} u_i \langle E^{(+)} \rangle dx dy - \int \int_{x<0} u_i \langle E^{(+)} \rangle dx dy \quad (7)$$

Let us give now an example of a two-mode state allowing us to reduce by a large amount the variance in the measurement of the intensity difference between the two half planes : we consider an even mode u_e , and an odd mode u_o , with respect to the coordinate x , and we assume that the light state consists of a tensor product of a coherent state having a non zero mean value in mode u_e , and of a squeezed vacuum in mode u_o . The mean value of the field $\langle E^{(+)}(x, y) \rangle$ is then an even function of x , which gives a zero mean value for the measured signal $N_A - N_B$ when the displacement is zero. Using expressions (5) and (7), and assuming that $\langle E^{(+)}(x, y) \rangle$ is real, one gets

$$\begin{aligned} & \Delta (N_A - N_B)^2 \\ &= \langle N_{tot} \rangle \left[1 + 4 (\Delta p_o^2 - 1) \left(\int \int_{x>0} u_o u_e dx dy \right)^2 \right] \end{aligned} \quad (8)$$

where Δp_o^2 represents the variance of the real part of the odd mode field fluctuations (normalized so that its value is 1 in vacuum state). The minimum noise, and therefore the smallest measurable displacement D_{\min} , is obtained when one uses a perfectly squeezed vacuum state ($\Delta p_o^2 = 0$). D_{\min} is then equal to zero if $\int \int_{x>0} u_o u_e dx dy = 0.5$ [14]. Given that the two modes are normalized, one can show that this occurs when u_e and u_o coincide exactly at all points of one of the two zones, and are opposite in the other. One notices that in this configuration u_e must vary abruptly from $u_o(0, y)$ to $-u_o(0, y)$ at the edge between the two pixels. Such an unusual mode can be experimentally approximated by inserting a π dephasing plate with a sharp edge at $x = 0$ in one half of the transverse plane. It must be detected within a very short distance, as it will diffract quickly when it propagates. It can be also imaged using lenses, with some high spatial frequency filtering because of the finite size of the optical system. For all these reasons D_{\min} will not be exactly zero, even with a perfectly squeezed vacuum, but it will be much smaller than the standard quantum limit.

A simpler configuration, less sensitive to propagation effects, can be achieved, which consists of a TEM₀₀ mode as u_e , and a perfectly squeezed vacuum TEM_{q0} mode (q odd, same waist) as u_o . D_{\min} is equal in this case to $0.60D_{sql}$ if $q = 1$ and $0.94D_{sql}$ if $q = 3$. This modest improvement with respect to the standard quantum limit is due to the too slow variation of the odd squeezed mode amplitude when one crosses the edge $x = 0$. Much abrupt changes can be obtained by using linear superpositions of many different transverse TEM_{pq} modes in nonclassical states. In this respect parametric interaction in an optical cavity with a great number of degenerate transverse modes seems a very promising source of nonclassical transverse states of light with noise properties varying abruptly at $x = 0$. The quantum properties of the fields generated in cavities containing thin parametric media, pumped by a plane wave, in the planar, confocal, or concentric configurations have been recently studied in great detail, especially below the oscillation threshold. In particular the quasi-planar below threshold configuration has been shown in [15] to produce a field with almost identical quantum intensity fluctuations at points symmetrical with respect to the cavity axis. This device would certainly be a good candidate to increase by a large factor

the ultimate sensitivity in the measurement of a small displacement.

In conclusion, we have demonstrated that the standard quantum limit for the measurement of a small transverse displacement cannot be broken by using single mode nonclassical states of light, such as those produced in many experiments performed so far. We have shown that different kinds of multimode non-classical states of light can be used to increase by a large factor the sensitivity in such a measurement.

We thank L. Lugiato, A. Gatti and M. Kolobov for many enlightening discussions. This work has been partially funded by an E.C. contract (ESPRIT IV ACQUIRE 20029). Laboratoire Kastler Brossel, from Ecole Normale Supérieure and Université P.M. Curie, is associated to CNRS.

REFERENCES

- [1] for a review, see for example : L. Lugiato, A. Gatti, H. Wiedemann, Les Houches Session LXIII, p. 431, North-Holland (1997), and M. Kolobov, "the spatial behavior of quantum noise", to be published in Rev. Mod. Physics (1999)
- [2] J.W. Rayleigh, Phil. Mag. 261,403,477 (1979), and : Collected Optics Papers of Lord Rayleigh, part A, p.117 (Opt. Soc. Am., Washington 1994)
- [3] C.K. Rushforth, R.W. Harris J. Opt. Soc. Am., **58**, 539 (1968)
- [4] for a survey, see for example : A.J. den Dekker, A. van den Bos, J. Opt. Soc. Am. A **14**, 547 (1997)
- [5] J. Goodman, *Statistical Optics*, p. 85 (Wiley, New York 1985)
- [6] M. Kolobov, I. Sokolov, Sov Phys JETP, **63**, 1105 (1986)
- [7] M. Kolobov, I. Sokolov, Phys. Lett. A**140**, 101 (1989)
- [8] C. Putman, B. De Grooth, N. Van Hulst, J. Greve J. Appl. Phys. **72**, 6 (1992)
- [9] D. Fournier, A. Boccara, N. Amer, R. Gerlach, Appl. Phys. Lett.**37**, 519 (1980)
- [10] S. Kamimura, Applied Opt. **26**, 3425 (1987)
- [11] M. Levenson, W. Richardson, S. Perlmutter, Opt. Lett. **14**, 779 (1990)
- [12] J-P. Poizat, T-J. Chang, O. Ripoll, P. Grangier, J. Opt. Soc. Am.B **15**, 1757 (1998)
- [13] S. Machida, Y. Yamamoto, Y. Itaya, Phys. Rev. Lett.**58**, 1000 (1987); F. Marin, A. Bramati, E. Giacobino, T-C. Zhang, J-P. Poizat, J-F. Roch, P. Grangier, Phys. Rev. Lett.**75**, 4606 (1995)
- [14] Actually a perfectly squeezed vacuum is excluded by our linearization procedure, as it has an infinite noise on the other quadrature. A simple reasoning shows that D_{\min} is not zero in such a case, but on the order of $\frac{\Delta}{\langle N_{tot} \rangle}$.

[15] I. Marzoli, A. Gatti, L. Lugiato, Phys. Rev. Lett.**78**, 2092 (1997)