

Covering a ball with smaller equal balls in \mathbb{R}^n

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Abstract

We give an explicit upper bound of the minimal number $v_{T,n}$ of balls of radius $1/2$ which form a covering of a ball of radius $T > 1/2$ in \mathbb{R}^n , $n \geq 2$. The asymptotic estimates of $v_{T,n}$ we deduce when n is large are improved further by recent results of Böröczky Jr. and Wintsche on the asymptotic estimates of the minimal number of equal balls of \mathbb{R}^n covering the sphere S^{n-1} . The optimality of the asymptotic estimates is discussed.

1. Introduction

Let $T > 1/2$ and $v_{T,n}$ the minimal number of (closed) balls of radius $1/2$ which can cover a (closed) ball of radius T in \mathbb{R}^n , $n \geq 2$. In [R1] (pp 163-164 and theorem 2) Rogers has obtained the following result.

THEOREM 1.1. — (i) If $n \geq 3$, with $\vartheta_n = n \ln n + n \ln(\ln n) + 5n$, we have

$$1 < v_{T,n} \leq \begin{cases} e\vartheta_n(2T)^n & \text{if } T \geq n/2, \\ n\vartheta_n(2T)^n & \text{if } \frac{n}{2\ln n} \leq T < \frac{n}{2}. \end{cases} \quad (1)$$

(ii) If $n \geq 9$ we have

$$1 < v_{T,n} \leq \frac{4e(2T)^n n \sqrt{n}}{\ln n - 2} (n \ln n + n \ln(\ln n) + n \ln(2T) + \frac{1}{2} \ln(144n)) \quad (2)$$

for all $1/2 < T < \frac{n}{2\ln n}$.

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The assertion (i) can easily be extended to the case $n = 2$ by invoking Rogers [R] (p 47) so that the strict upper bound $\vartheta_n = n \ln n + n \ln(\ln n) + 5n$ of the covering density of equal balls in \mathbb{R}^n is still a valid one in this case. Thus the inequalities (1) are still true for $n = 2$. In the case $n = 2$, see also Kershner [K]. On the other hand, the result (ii) does not seem to have been improved since then, see for instance [Hand], Fejes-Toth [Ft], Schramm [Sc], Raigorodski [Ra] or Bourgain et al [BL]. This problem is linked to the existence of explicit lower bounds of the packing constant of equal spheres in \mathbb{R}^n [MVG] and to various problems [MR] [IM] [FF] [Ma].

In this contribution we give an improvement of the upper bound of $\nu_{T,n}$ given by the assertion (ii), i.e. when the radius T is less than $n/(2 \ln n)$. Namely, we will prove

THEOREM 1.2. — *Let $n \geq 2$. The following inequalities hold:*

(i) $n < \nu_{T,n} \leq$

$$\frac{7^{4(\ln 7)/7}}{4} \sqrt{\frac{\pi}{2}} \frac{n\sqrt{n} \left[(n-1) \ln(2Tn) + (n-1) \ln(\ln n) + \frac{1}{2} \ln n + \ln \left(\frac{\pi\sqrt{2n}}{\sqrt{\pi n-2}} \right) \right]}{T \left(1 - \frac{2}{\ln n}\right) \left(1 - \frac{2}{\sqrt{\pi n}}\right) (\ln n)^2} (2T)^n$$

if $1 < T < \frac{n}{2 \ln n}$, (3)

(ii) $n < \nu_{T,n} \leq$

$$\sqrt{\frac{\pi}{2}} \frac{\sqrt{n} \left[(n-1) \ln(2Tn) + (n-1) \ln(\ln n) + \frac{1}{2} \ln n + \ln \left(\frac{\pi\sqrt{2n}}{\sqrt{\pi n-2}} \right) \right]}{T \left(1 - \frac{2}{\ln n}\right) \left(1 - \frac{2}{\sqrt{\pi n}}\right)} (2T)^n$$

if $1/2 < T \leq 1$. (4)

The following question seems fundamental: what are the integers $\nu_{T,n}$ when $1/2 < T$, $2 \leq n$ and the corresponding configurations of balls of radius $1/2$ when they form the most economical covering of the closed ball $B(0, T)$ of radius T centred at the origin?

In section 3 we will recall the recent results of Böröczky Jr. and Wintsche [BW] on the asymptotic estimates in the sphere covering problem by smaller equal balls when n is large. These estimates will allow to make further improvements on the upper bounds of $\nu_{T,n}$ (theorem 3.1), to appreciate the optimality of these upper bounds with respect to known lower bounds and to state some conjectures.

2. Proof of the theorem 1.2

The idea of the proof is simple: (i) when T is small enough, it amounts to show that the sphere $S(0, T)$ can be covered by a collection of N balls of radius $1/2$ suitably placed at equidistance from the origin, and that this covering to which we add the central ball $B(0, 1/2)$ actually covers the ball $B(0, T)$ itself; in the subsection 2, an upper bound of the minimal value of N is calculated from the results given by the lemmas of subsection 1; (ii) when T is

larger, we proceed recursively using (i) to give an upper bound of N . The configuration of balls of radius $1/2$ covering $B(0, T)$ is then ordered by layers, the last layer of balls of radius $1/2$ being at an optimal distance from the origin so as to cover the sphere $S(0, T)$.

1. *Caps and sectors.*—Let $T > 1/2$ and $n \geq 2$ in the following. If the closed ball $B(0, T)$ is covered by N smaller balls of radius $1/2$, the smaller balls will intersect the sphere $S(0, T)$, for a certain proportion of them. The intersection of a closed ball of radius $1/2$ and the sphere $S(0, T)$, if it is not empty, is called a (spherical) cap. For fixing the notations let us define properly what is a cap and the sector it generates in $B(0, T)$.

Let $h \geq 0$ and u be a unit vector of \mathbb{R}^n . Let us denote by $H_{h,u}$ the affine hyperplane $\{z + hu \mid z \in \mathbb{R}^n, z \cdot u = 0\}$ of \mathbb{R}^n . Assume that $H_{h,u}$ intersects the ball $B(0, T)$, i.e. $h \leq T$. We will denote

$$C_{T,h,u} := \left\{ z \in S(0, T) \mid \frac{z \cdot u}{\|z\|} \geq \frac{h}{T} \right\}$$

The $n-2$ -dimensional sphere $H_{h,u} \cap C_{T,h,u}$ admits $x = \sqrt{T^2 - h^2}$ for radius. The correspondence between $x \in [0, T]$ and $h \in [0, T]$ is one-to-one. We will say that $C_{T,h,u}$ is the cap of chord $2x$ and of centre Tu . If a subset Y of $S(0, T)$ is such that there exists $h \geq 0$ and u a unit vector of \mathbb{R}^n such that $Y = C_{T,h,u}$, then we will say that Y is a cap of chord $2x$ of $S(0, T)$.

Every cap $C_{T,h,u}$ of chord $2x$ of $S(0, T)$ generates a sector in $B(0, T)$. We will denote it by

$$\mathcal{S}(T, h, u) := \left\{ z \in B(0, T) \mid \frac{z \cdot u}{\|z\|} \geq \frac{h}{T} \right\}$$

We will denote by $V_{(T,x)}$ (indexing with x instead of h) the volume of a sector generated by a cap of chord $2x$ in $S(0, T)$ with $x \leq T$. Let $\omega_n := \pi^{n/2}/\Gamma(1+n/2)$ so that the (n -dimensional) volume of a ball of radius T in \mathbb{R}^n is $\omega_n T^n$.

LEMMA 2.1. — *We have*

$$\frac{\omega_{n-1}}{\omega_n} \geq \frac{1}{\sqrt{2\pi}} \sqrt{n} \left(1 - \frac{2}{\sqrt{\pi n}}\right). \quad (1)$$

Proof. — The following inequalities are classical ([Va] p 171):

$$\frac{\omega_{n-1}}{\omega_n} \geq \begin{cases} \frac{1}{\sqrt{2\pi}} \sqrt{n} \left(1 + \left(\frac{n}{2e}\right)^{n/2} \frac{1}{(n/2)!}\right)^{-1} & \text{if } n \text{ is even,} \\ \frac{1}{\sqrt{2\pi}} \sqrt{n+1} \left(1 - \left(\frac{n+1}{2e}\right)^{(n+1)/2} \frac{1}{((n+1)/2)!}\right) & \text{if } n \text{ is odd.} \end{cases} \quad (2)$$

By Stirling's formula we deduce the result. □

LEMMA 2.2. — *Let $0 < x < T$. Let n be odd and put $y = (n-1)/2$. The volume $V_{(T,x)}$ of a sector in $B(0, T) \subset \mathbb{R}^n$ generated by a cap of chord $2x$ in $S(0, T)$ is equal to*

$$\omega_{n-1} x^{n-1} \left[\frac{\sqrt{T^2 - x^2}}{n} + \frac{2(T - \sqrt{T^2 - x^2})}{n+1} \sum_{j=0}^y \frac{y!(y+1)!}{(y+1+j)!(y-j)!} \left(\frac{T - \sqrt{T^2 - x^2}}{T + \sqrt{T^2 - x^2}} \right)^j \right]. \quad (3)$$

It satisfies the relations

$$(i) \quad V_{(T,x)} = x^n V_{(T/x,1)}, \quad (4)$$

$$(ii) \quad \frac{T}{nx} \leq \frac{2n(T/x) + (1-n)\sqrt{(T/x)^2 - 1}}{n(n+1)} \leq \frac{1}{\omega_{n-1}} V_{(T/x,1)}. \quad (5)$$

Proof. — Let us show (3). The first term $\omega_{n-1}x^{n-1}\frac{\sqrt{T^2-x^2}}{n}$ is the volume of the truncated cone $\{z \in \mathcal{P}(T, h, u) \mid z \cdot u \leq h\}$ with $h = \sqrt{T^2 - x^2}$. The second term in (3) is the volume of $\{z \in \mathcal{P}(T, h, u) \mid z \cdot u \geq h\}$: any point of $C_{T, \sqrt{T^2-x^2}, u}$ which is at distance t from $H_{\sqrt{T^2-x^2}, u}$ is at distance $(x^2 - t^2 - 2t\sqrt{T^2 - x^2})^{1/2}$ from the line $\mathbb{R}u$. Hence, this volume equals to

$$\int_0^{T-\sqrt{T^2-x^2}} \omega_{n-1} \left[x^2 - t^2 - 2t\sqrt{T^2 - x^2} \right]^{(n-1)/2} dt.$$

It is obtained by integration by parts, y times, of the integral

$$\omega_{n-1} \int_0^\alpha (\alpha - t)^y (t - \beta)^y dt \quad (6)$$

with $\alpha = T - \sqrt{T^2 - x^2}$ and $\beta = -T - \sqrt{T^2 - x^2}$.

The relation (4) is obvious. Let us show (5). We deduce it from the fact that the summation in (3) has positive terms and is greater than its first term which is 1. \square

LEMMA 2.3. — Assume $n \geq 2$ even and $0 < x < 1$. The volume $V_{(T,x)}$ of a sector in $B(0, T) \subset \mathbb{R}^n$ generated by a cap of chord $2x$ in $S(0, T)$ satisfies the relations:

$$(i) \quad V_{(T,x)} = x^n V_{(T/x,1)}, \quad (7)$$

$$(ii) \quad \frac{T}{nx} \leq \frac{2n(T/x) + (2-n)\sqrt{(T/x)^2 - 1}}{n(n+2)} \leq \frac{1}{\omega_{n-1}} V_{(T/x,1)}. \quad (8)$$

Proof. — The equality (7) is obvious. In order to prove (8), let us observe that the function $t \rightarrow (\alpha - t)(t - \beta)$ defined on the interval $[0, \alpha]$ is valued in the interval $[0, 1]$ since it lies below the horizontal line of y -coordinate $-\alpha\beta = x^2 < 1$. We deduce the following inequalities

$$(\alpha - t)^{\frac{n+1}{2}} (t - \beta)^{\frac{n+1}{2}} \leq (\alpha - t)^{\frac{n}{2}} (t - \beta)^{\frac{n}{2}} \leq (\alpha - t)^{\frac{n-1}{2}} (t - \beta)^{\frac{n-1}{2}}$$

for all $t \in [0, \alpha]$. From (6) in the proof of lemma 2.2 we deduce a lower bound of the volume of the convex hull of $C_{T, \sqrt{T^2-x^2}, u}$ for n even using the preceding n odd case of lemma 2.2: changing n to $n+1$ now odd in the computation of the lower bound of the summation in (3). Let us note that the computation of the volume of $\{z \in \mathcal{P}(T, h, u) \mid z \cdot u \leq h\}$ with $h = \sqrt{T^2 - x^2}$ still gives $\omega_{n-1}x^{n-1}\frac{\sqrt{T^2-x^2}}{n}$ for n even so that the first term of $V_{(T,x)}$ remains the same as in the n odd case. We deduce the inequality (8). \square

LEMMA 2.4. — Let $0 < x \leq 1/2$. Let D be a point of the cap $C_{T, \sqrt{T^2-1/4}, u} \subset S(0, T) \subset \mathbb{R}^n$ at a distance x from the line $\mathbb{R}u$. Let B denote the unique point which lies in the intersection of $C_{T, \sqrt{T^2-1/4}, u} \cap H_{\sqrt{T^2-1/4}, u}$ with the plane $(0, D, Tu)$ with the property that it is the closest to D . If η denotes the distance between D and the line OB , we have the following relation between x, T and η :

$$x = \frac{1}{2}\sqrt{1 - \left(\frac{\eta}{T}\right)^2} - \frac{\eta}{2}\sqrt{4 - \frac{1}{T^2}} \quad \text{equivalently} \quad \eta = \frac{1}{2}\sqrt{1 - \left(\frac{x}{T}\right)^2} - \frac{x}{2}\sqrt{4 - \frac{1}{T^2}}. \quad (9)$$

Proof. — Let ψ be the angle between the lines OB and OD , ψ' the angle between the lines OD and $\mathbb{R}u$, so that $\sin(\psi) = \eta/T$ and $\sin(\psi') = x/T$. Since $\sin(\psi + \psi') = 1/(2T)$ we obtain

$$1 = 2x\sqrt{1 - (\eta/T)^2} + 2\eta\sqrt{1 - (x/T)^2}.$$

This expression is symmetrical in x and η . It is now easy, from it, to deduce the expression of x as a function of η , as stated by the eq.(9). \square

LEMMA 2.5. — *Let us assume that a collection of N balls $(B(c_j, 1/2))_{j=1,2,\dots,N}$ of \mathbb{R}^n is such that (i) for all $j = 1, 2, \dots, N$, $B(c_j, 1/2) \cap S(0, T)$ is a cap of chord 1 in $S(0, T)$ and (ii) these N caps form a covering of $S(0, T)$. Then (i) if $T > \sqrt{2}/2$, the union*

$$\bigcup_{j=1}^N B(c_j, 1/2) \text{ covers the annulus } \{z \in \mathbb{R}^n \mid T - \frac{1}{2T} \leq \|z\| \leq T\}$$

of the ball $B(0, T)$; (ii) if $1/2 < T \leq \sqrt{2}/2$ this union covers $B(0, T)$.

Proof. — Any such ball $B(c_j, 1/2)$ covers the part of the sector

$$\{z \in \mathcal{S}(T, \sqrt{T^2 - 1/4}, Oc_j / \|Oc_j\|) \mid \alpha T \leq \|z\|\}$$

with α to be determined. To compute α , let us consider two adjacent balls, say $B(c_1, 1/2)$ and $B(c_2, 1/2)$, such that the intersection of the respective caps $B(c_1, 1/2) \cap S(0, T)$ and $B(c_2, 1/2) \cap S(0, T)$ is reduced to one point. Then, on the line $O\frac{c_1+c_2}{2}$, it is easy to check that all the points z such that $T - 1/2T \leq \|z\| \leq T$ are covered. This gives $\alpha = 1 - 1/2T^2$. Now, since the caps $B(c_j, 1/2) \cap S(0, T)$ form a covering of $S(0, T)$, the balls $B(c_j, 1/2)$ form a covering of the annulus $\{z \in \mathbb{R}^n \mid \alpha T \leq \|z\| \leq T\}$. The last assertion is obvious. \square

Let us consider $N(\geq 1)$ distinct points M_1, M_2, \dots, M_N of $S(0, T) \subset \mathbb{R}^n$. We will consider that they are the respective centres of caps of chord $2x$ of $S(0, T)$. We will denote by $\theta_{(T,x)}(M_1, M_2, \dots, M_N)$ the proportion of $S(0, T)$ occupied by these caps. In other terms, with $u_i := \frac{OM_i}{\|OM_i\|}$ for all $i = 1, 2, \dots, N$, we have

$$\theta_{(T,x)}(M_1, M_2, \dots, M_N) := \frac{\text{Vol}_{n-1}(\bigcup_{i=1}^N C_{T, \sqrt{T^2 - x^2}, u_i})}{\text{Vol}_{n-1}(S(0, T))}.$$

LEMMA 2.6. — *Let $N \geq 1$, $x \in (0, 1/2]$. The mean $E\theta(N, T, x)$ of $\theta_{(T,x)}(M_1, M_2, \dots, M_N)$ over all possibilities of collections of N distinct points (M_1, M_2, \dots, M_N) of $S(0, T)$ is equal to*

$$E\theta(N, T, x) = 1 - \left(1 - \frac{V_{(T,x)}}{\omega_n T^n}\right)^N.$$

Proof. — Let M_1, M_2, \dots, M_N be N points of $S(0, T)$. We define

$$p_i = \frac{\text{Vol}_{n-1}(C_{T, \sqrt{T^2 - x^2}, u_i})}{\text{Vol}_{n-1}(S(0, T))}, \quad i = 1, 2, \dots, N,$$

the probability that a point $M \in S(0, T)$ belongs to the cap of chord $2x$ of centre M_i . It is the probability, hence independent of i , that M_i belongs to the cap of chord $2x$ of centre M .

We have $p_i = \frac{V_{(T,x)}}{\omega_n T^n}$. Therefore, the probability that M belongs to none of the caps of chord $2x$ of centre M_i for all $i = 1, 2, \dots, N$ is, by the independence of the points, the product of the probabilities that none of the M_i 's belongs to the cap of chord $2x$ of centre M , that is the product

$$\left(1 - \frac{V_{(T,x)}}{\omega_n T^n}\right)^N.$$

This value is independent of the collection of points $\{M_i\}$. We deduce the mean $E\theta(N, T, x)$ by complementarity. \square

2. *Proof of the theorem 1.2.*—

PROPOSITION 2.7. — Let $0 < x < 1/2$. With $\eta(x) = \frac{1}{2}\sqrt{1 - \left(\frac{x}{T}\right)^2} - \frac{x}{2}\sqrt{4 - \frac{1}{T^2}}$, if

$$N \geq \frac{\omega_n T^n}{V_{(T,x)}} \ln \left(\frac{\omega_n T^n}{V_{(T,\eta(x))}} \right) \quad (10)$$

then there exists a collection of N distinct caps of centres M_1, M_2, \dots, M_N of chord 1 of $S(0, T) \subset \mathbb{R}^n$ satisfying

$$\ln \left(\frac{1}{1 - \theta_{(T,x)}(M_1, M_2, \dots, M_N)} \right) > N \frac{V_{(T,x)}}{\omega_n T^n} \quad (11)$$

which covers $S(0, T)$.

Proof. — Given $x \in (0, 1/2]$ there exists at least one collection of caps $\{C_{T, \sqrt{T^2 - x^2}, u_i} \mid i = 1, 2, \dots, N\}$ of centres M_1, M_2, \dots, M_N , where the unit vectors $u_i := OM_i / \|OM_i\|$ are all distinct, such that the relation (11) is true since, after lemma 2.6, the mean $E\theta(N, T, x)$ is equal to $1 - \left(1 - \frac{V_{(T,x)}}{\omega_n T^n}\right)^N$ and that

$$\ln \left(\frac{1}{1 - E\theta(N, T, x)} \right) = -N \ln \left(1 - \frac{V_{(T,x)}}{\omega_n T^n} \right) > N \frac{V_{(T,x)}}{\omega_n T^n}. \quad (12)$$

Let us note that the points M_1, M_2, \dots, M_N depend upon x . Keeping fixed the centres M_1, M_2, \dots, M_N and putting around them caps of chord 1 instead of $2x$, we obtain a new collection of caps. Let us show that this new collection of caps of chords 1 of $S(0, T)$ forms a covering. We will assume that it does not and will show the contradiction.

Then there exists a point $M \in S(0, T)$ such that

$$M \notin \bigcup_{i=1}^N C_{T, \sqrt{T^2 - 1/4}, u_i}.$$

Let us write $u := OM / \|OM\|$ for the unit vector on the line OM . At worse, M lies close to the boundary of the domain $\bigcup_{i=1}^N C_{T, \sqrt{T^2 - 1/4}, u_i}$, hence close to the boundary of one of the caps $C_{T, \sqrt{T^2 - 1/4}, u_i}$ of chord 1. We can now apply lemma 2.4 as if M were on this boundary: $\eta = \eta(x)$ is strictly positive since $x < 1/2$ by the eq.(9). Therefore the cap $C_{T, \sqrt{T^2 - \eta(x)^2}, u}$ is not trivial and is disjoint from the union

$$\bigcup_{i=1}^N C_{T, \sqrt{T^2 - x^2}, u_i}.$$

This means that

$$1 - \theta_{(T,x)}(M_1, M_2, \dots, M_N) > \theta_{(T,\eta(x))}(M) > 0.$$

Therefore

$$\ln \left(\frac{1}{1 - \theta_{(T,x)}(M_1, M_2, \dots, M_N)} \right) < \ln \left(\frac{1}{\theta_{(T,\eta(x))}(M)} \right).$$

From the eq.(12) we deduce the relation

$$N \frac{V_{(T,x)}}{\omega_n T^n} < \ln \left(\frac{\omega_n T^n}{V_{(T,\eta(x))}} \right).$$

Hence the contradiction. \square

By lemma 2.1 and the eq.(4), (5), (7), (8), we deduce

$$\begin{aligned} \frac{\omega_n T^n}{V_{(T,x)}} \ln \left(\frac{\omega_n T^n}{V_{(T,\eta(x))}} \right) &= \frac{\omega_n}{\omega_{n-1}} \frac{n T^n}{x^n} \frac{\omega_{n-1}}{n V_{(T/x,1)}} \ln \left(\frac{\omega_n}{\omega_{n-1}} \frac{n T^n}{(\eta(x))^n} \frac{\omega_{n-1}}{n V_{(T/(\eta(x)),1)}} \right) \\ &\leq \sqrt{\frac{\pi}{2}} \frac{\sqrt{n} (2T)^n (1 - 4\eta(x))^{-n/2}}{T \left(1 - \frac{2}{\sqrt{\pi n}}\right)} \left[-(n-1) \ln(\eta(x)) + (n-1) \ln T + \ln \left(\frac{\sqrt{2\pi n}}{\sqrt{\pi n} - 2} \right) \right]. \end{aligned} \quad (13)$$

In the proposition 2.7, we can take any x , hence any η , in the open interval $(0, 1/2)$ such that the condition (11) is satisfied. We will chose η and $x = x(\eta)$ as functions of n only with η tending monotonically to zero when n goes to infinity, hence x tending to $1/2$. This will give a minimal integer

$$\left\lfloor \frac{\omega_n T^n}{V_{(T,x)}} \ln \left(\frac{\omega_n T^n}{V_{(T,\eta(x))}} \right) \right\rfloor + 1$$

for obtaining the covering property of $S(0, T)$ as a function of n and T only.

Let us now state the central problem (P).

(P) The problem consists now in finding, in the set of strictly positive monotone decreasing functions $f(x)$ defined on $(1/4, +\infty)$ such that $\lim_{x \rightarrow +\infty} f(x) = 0$, one function for which $-(1 - 4f(x))^{-x/2} \ln(f(x))$ goes the slowest to $+\infty$ when x tends to $+\infty$. We will not solve this problem here. We will simply take $f(x) = 1/(2xu(x))$ with $u(x)$ an increasing monotone continuous function such that $\lim_{x \rightarrow +\infty} u(x) = +\infty$, in particular $u(x) = \ln x$. By reporting this function in the eq.(13) we take $\eta = 1/(2n \ln n)$, $n \geq 3$. This gives an expression of x as a function of n from the eq.(9). This function represents a fairly good compromise.

The second member of the inequality (10) appears as a configurational entropy which has to be exceeded for the existence of (at least one) a certain configuration of equal caps of chord 1 for covering $S(0, T)$. But the condition (11) is non-constructive.

We will now explicit the second member of the inequality (13) with $\eta = 1/(2n \ln n)$. Thus, for all $n \geq 2$, since $(1 - 2/(n \ln n))^{-n/2} < (1 - 2/(\ln n))^{-1}$, we obtain

$$\sqrt{\frac{\pi}{2}} \frac{\sqrt{n}}{T} \left(\frac{(2T)^n}{\left(1 - \frac{2}{\ln n}\right) \left(1 - \frac{2}{\sqrt{\pi n}}\right)} \right) \left[(n-1) \ln(2T n \ln n) + \frac{1}{2} \ln n + \ln \left(\frac{\pi \sqrt{2n}}{\sqrt{\pi n} - 2} \right) \right] \quad (14)$$

for $1/2 < T \leq 1$.

By lemma 2.5, if $1/2 < T \leq 1$, then, in order to cover the ball $B(0, T)$ by balls of radius $1/2$, it suffices to put a ball of radius $1/2$ centred at the origin (not necessary if $1/2 < T \leq \sqrt{2}/2$) and to put a collection of N balls (with N chosen minimal) given by the proposition 2.7 around such that their intersections with $S(0, T)$ are caps of chord 1 which cover $S(0, T)$. This total number of balls, $N + 1$, is certainly exceeded by (14). This proves the assertion (i) in the theorem 1.2.

Let us prove the assertion (ii) in the theorem 1.2. If $T > 1$, we proceed inductively using the lemma 2.5. We will cover $B(0, T)$ as follows. We put a ball of radius $1/2$ centred at the origin. Then we put balls of radius $1/2$ in such a way that their intersections with the spheres $S(0, T_m)$ are caps of chord 1 which cover $S(0, T_m)$, where the decreasing sequence $\{T_m\}$ is defined by $T_0 = T, T_1 = T_0 - \frac{1}{2T_0}, \dots, T_m = T_{m-1} - \frac{1}{2T_{m-1}}, \dots$ with $m \in \{0, 1, \dots, m_0\}$ and m_0 defined by the condition that $T_{m_0} \leq 1$ and $T_{m_0-1} > 1$. Since, for all integer $m \in \{0, 1, \dots, m_0\}$, we have

$$T - \frac{m}{2T} \geq T_m$$

the total number of balls of radius $1/2$ disposed in such a configuration required for covering $B(0, T)$ is certainly less than

$$\begin{aligned} & \sum_{m=0}^{m_0} \left(2 \left(T - \frac{m}{2T}\right)\right)^n \frac{\sqrt{\pi n} \left(1 - \frac{2}{\ln n}\right)^{-1}}{T \sqrt{2} \left(1 - \frac{2}{\sqrt{\pi n}}\right)} \left[(n-1) \ln(2(T - \frac{m}{2T})n \ln n) + \frac{\ln n}{2} + \ln \left(\frac{\pi \sqrt{2n}}{\sqrt{\pi n} - 2}\right) \right] \\ & \leq \frac{\sqrt{\pi n} \left(1 - \frac{2}{\ln n}\right)^{-1}}{T \sqrt{2} \left(1 - \frac{2}{\sqrt{\pi n}}\right)} \left[(n-1) \ln(2Tn \ln n) + \frac{\ln n}{2} + \ln \left(\frac{\pi \sqrt{2n}}{\sqrt{\pi n} - 2}\right) \right] \sum_{m=0}^{m_0} \left(2(T - \frac{m}{2T})\right)^n. \end{aligned}$$

But

$$\sum_{m=0}^{m_0} \left(2(T - \frac{m}{2T})\right)^n \leq (2T)^n \sum_{m=0}^{m_0} e^{-\frac{nm}{T^2}} \leq (2T)^n \sum_{m=0}^{+\infty} e^{-\frac{nm}{T^2}} = \frac{(2T)^n}{1 - e^{-n/T^2}}.$$

Since $T < n/(2 \ln n)$, we have

$$\frac{e^{n/T^2}}{e^{n/T^2} - 1} < \frac{e^{4(\ln n)^2/n}}{e^{4(\ln n)^2/n} - 1} < \frac{n}{4(\ln n)^2} e^{4(\ln n)^2/n}.$$

The function $t \rightarrow (\ln t)^2/t$ reaches its maximum on $[2, +\infty)$ at $t = e^2$. Hence, for all integer $n \geq 2$, we have $(\ln n)^2/n \leq (\ln 7)^2/7$. We deduce that

$$\sum_{m=0}^{m_0} \left(2(T - \frac{m}{2T})\right)^n \leq \frac{e^{4(\ln 7)^2/7} n (2T)^n}{4 (\ln n)^2}$$

with a constant $e^{4(\ln 7)^2/7}/4 = 2.176\dots$ This gives the assertion (ii).

As for the strict lower bound n in the eq.(3) and (4), it obviously comes from the dimension of the ambient space: n balls being placed along the n coordinates axis of any basis of \mathbb{R}^n never cover $B(0, T)$ when $T > 1/2$.

3. Asymptotic estimates: results and conjectures

Rogers [R1] constructed certain economic coverings of a larger Euclidean ball by equal smaller balls. When T is large ($T > \frac{n}{2}$) he has computed an upper bound on the quantity $\nu_{T,n}$ that is close to being optimal up to a $\ln n$ factor. On the other hand, his upper bounds are of higher order when $T < \frac{n}{2}$, and it is the object of theorem 1.2 to improve them in the case $T < \frac{n}{2 \ln(n)}$.

In this section we will reformulate theorem 1.2 in terms of asymptotic estimates. Then we will state further improved upper bounds on the quantity $\nu_{T,n}$ for $T < \frac{n}{2}$ that are most probably close to being optimal up to a $\ln n$ factor. In addition lower bounds on the quantity $\nu_{T,n}$ are discussed. The arguments for further improvements will use recent results of [BW].

This sequel only discusses the order of the bounds, hence let us introduce the corresponding notation: given non-negative functions f and g , if $f(n) < c \cdot g(n)$ for a positive absolute constant c then we will write $f(n) \ll g(n)$, or $g(n) \gg f(n)$, or $f(n) = O(g(n))$.

The starting point is the following list of estimates by Rogers [R1]:

$$\nu_{T,n} \ll n \ln n \cdot (2T)^n \quad \text{if } T \geq \frac{n}{2}; \quad (1)$$

$$\nu_{T,n} \ll n^2 \ln n \cdot (2T)^n \quad \text{if } \frac{n}{2 \ln n} \leq T < \frac{n}{2}; \quad (2)$$

$$\nu_{T,n} \ll n^2 \sqrt{n} \cdot (2T)^n \quad \text{if } \frac{1}{2} < T < \frac{n}{2 \ln n}. \quad (3)$$

Most probably (1) can not be improved with the present methods, and it is actually optimal up to a $\ln n$ factor (see (10)). The theorem 1.2 improves (3) into the following estimates:

$$\nu_{T,n} \ll \frac{n^2 \sqrt{n}}{T \ln n} \cdot (2T)^n \quad \text{if } 1 < T \leq \frac{n}{2 \ln n} \quad (4)$$

$$\text{where } \frac{n^2 \sqrt{n}}{T \ln n} > n \sqrt{n};$$

$$\nu_{T,n} \ll n \sqrt{n} \ln n \cdot (2T)^{n-1} \quad \text{if } \frac{1}{2} < T \leq 1. \quad (5)$$

Using some bounds of [BW] (subsection 3.2) the estimates (2) (3) of Rogers and the present estimates (4) and (5) can be further improved as follows.

THEOREM 3.1. — *The following asymptotic estimates hold:*

$$\nu_{T,n} \ll n \ln n \cdot (2T)^n \quad \text{if } T \geq \frac{\sqrt{n}}{2}; \quad (6)$$

$$\nu_{T,n} \ll \frac{n \sqrt{n} \ln n}{T} \cdot (2T)^n \quad \text{if } 1 \leq T \leq \frac{\sqrt{n}}{2}; \quad (7)$$

$$\frac{\nu_{T,n}}{(2T)^{n-1}} \ll n \sqrt{n} \cdot \sqrt{T - \frac{1}{2}} \cdot \ln 8(T - \frac{1}{2})n \quad \text{if } \frac{1}{2} + \frac{1}{4n} \leq T \leq 1; \quad (8)$$

$$\nu_{T,n} < 2n \quad \text{if } \frac{1}{2} < T \leq \frac{1}{2} + \frac{1}{4n}. \quad (9)$$

Let us observe that the estimates in the list change continuously as T increases (up to absolute constant factors).

In order to provide a feeling about the optimality of the estimates above, let us list the corresponding known and conjectured lower bounds:

$$\nu_{T,n} \gg n \cdot (2T)^n \quad \text{if } T \geq \frac{n}{2} \text{ or } T = \frac{\sqrt{n}}{2}; \quad (10)$$

$$\nu_{T,n} \gg \frac{n\sqrt{n}}{T} \cdot (2T)^n \quad \text{if } 1 \leq T \leq \frac{\sqrt{n}}{2}, \text{ conjectured}; \quad (11)$$

$$\frac{\nu_{T,n}}{(2T)^n} \gg n\sqrt{n} \cdot \sqrt{T - \frac{1}{2}} \quad \text{if } \frac{1}{2} + \frac{1}{4n} \leq T \leq 1, \text{ conjectured}; \quad (12)$$

$$\nu_{T,n} > n \quad \text{if } \frac{1}{2} < T \leq \frac{1}{2} + \frac{1}{4n}. \quad (13)$$

It is conjectured [BW] that (10) holds for any $T \geq \frac{\sqrt{n}}{2}$ but Böröczky Jr. and Wintsche only verified that

$$\nu_{T,n} \gg \frac{n\sqrt{n}}{T} \cdot (2T)^n \quad \text{if } \frac{\sqrt{n}}{2} \leq T \leq \frac{n}{2}. \quad (14)$$

The quantity $\frac{\nu_{T,n}}{(2T)^n}$ is the minimal density of a covering of a ball of radius T by balls of radius $\frac{1}{2}$. Since $\nu_{T,n}$ balls of radius $\frac{1}{2}$ cover the ball of radius T , readily

$$\nu_{T,n} > (2T)^n.$$

3.1 Covering a sphere

The arguments for (6)–(14) depend on estimates on the minimal number of equal balls covering a sphere. Let $\tilde{\nu}_n(T, \varrho)$ denote the minimal number of balls of radius ϱ in \mathbb{R}^n that cover the sphere $S(o, T)$ of radius T . The number $\tilde{\nu}_n(T, \varrho)$ corresponds to an optimal function in the problem (P) (in section 2). A better upper estimate of $\tilde{\nu}_n(T, \varrho)$ is given by corollary 1.2 of Böröczky, Jr. and Wintsche [BW]:

$$\tilde{\nu}_n(T, \varrho) \ll n\sqrt{n} \ln n \cdot \left(\frac{T}{\varrho}\right)^{n-1} \quad \text{if } T \geq 2\varrho; \quad (15)$$

$$\frac{\tilde{\nu}_n(T, \frac{1}{2})}{(2T)^{n-1}} \ll n\sqrt{n} \cdot \sqrt{T - \frac{1}{2}} \cdot \ln 8(T - \frac{1}{2})n \quad \text{if } \frac{1}{2} + \frac{1}{4n} \leq T \leq 1. \quad (16)$$

Concerning lower bounds, Example 6.3 of Böröczky, Jr. and Wintsche [BW] says

$$\tilde{\nu}_n(T, \frac{1}{2}) \gg n\sqrt{n} \cdot (2T)^{n-1} \quad \text{if } T \geq \frac{\sqrt{n}}{2}. \quad (17)$$

It is conjectured [BW] that

$$\tilde{\nu}_n(T, \frac{1}{2}) \gg n\sqrt{n} \cdot (2T)^{n-1} \quad \text{if } 1 \leq T < \frac{\sqrt{n}}{2}; \quad (18)$$

$$\frac{\tilde{\nu}_n(T, \frac{1}{2})}{(2T)^{n-1}} \gg n\sqrt{n} \cdot \sqrt{T - \frac{1}{2}} \quad \text{if } \frac{1}{2} + \frac{1}{4n} \leq T \leq 1. \quad (19)$$

3.2 Proofs of the improved upper bounds in the theorem 3.1

If $T \geq \frac{\sqrt{n}}{2}$ then let $\varrho = (1 - \frac{1}{n})\frac{1}{2}$. Given any $R \geq \frac{1}{2}$, let us cover $S(o, R)$ with $\tilde{\nu}_n(T, \varrho)$ balls of radius ϱ in a way that each ball intersects $S(o, R)$ in an $(n-2)$ -sphere of radius ϱ . Since $\sqrt{\frac{1}{4} - \varrho^2} > \frac{1}{2\sqrt{n}}$, the balls of the same centre and of radius $\frac{1}{2}$ cover the annulus between $S(o, R)$

and $S(o, R - \frac{1}{2\sqrt{n}})$. Writing m to denote the maximal integer such that $T - \frac{m}{2\sqrt{n}} \geq \frac{1}{2}$, it follows by (15)

$$\begin{aligned} v_{T,n} &\leq 1 + \sum_{i=0}^m \tilde{v}_n(T - \frac{i}{2\sqrt{n}}, \varrho) \\ &\ll \varrho^{-(n-1)} n\sqrt{n} \ln n \sum_{i=0}^m (T - \frac{i}{2\sqrt{n}})^{n-1} \\ &\ll 2^{n-1} n\sqrt{n} \ln n \cdot 2\sqrt{n} \int_0^{T+\frac{1}{2\sqrt{n}}} x^{n-1} dx \\ &\ll n \ln n \cdot (2T)^n. \end{aligned}$$

If $1 \leq T \leq \frac{\sqrt{n}}{2}$ then the argument is based on lemma 2.5, which actually holds for any $R \geq 1$: let us cover $S(o, R)$ with $\tilde{v}_n(T, \frac{1}{2})$ balls of radius $\frac{1}{2}$ in a way that each ball intersects $S(o, R)$ in an $(n-2)$ -sphere of radius $\frac{1}{2}$. Then the balls cover the annulus between $S(o, R)$ and $S(o, R - \frac{1}{2R})$. Writing m to denote the maximal integer such that $T - \frac{m}{2T} \geq \frac{1}{2}$, it follows by (15)

$$\begin{aligned} v_{T,n} &\leq 1 + \sum_{i=0}^m \tilde{v}_n(T - \frac{i}{2T}, \frac{1}{2}) \\ &\ll 2^{n-1} n\sqrt{n} \ln n \sum_{i=0}^m (T - \frac{i}{2T})^{n-1} \\ &< 2^{n-1} n\sqrt{n} \ln n \cdot \frac{T^{n-1}}{1 - e^{-\frac{n-1}{2T^2}}} \\ &\ll \frac{n\sqrt{n} \ln n}{T} \cdot (2T)^n. \end{aligned}$$

using $\frac{n-1}{2T^2} > 1$ in the last step.

If $\frac{1}{2} + \frac{1}{4n} \leq T \leq 1$ then it is essentially sufficient to cover $S(o, T)$, hence (16) yields (8).

Finally if $\frac{1}{2} < T \leq \frac{1}{2} + \frac{1}{4n}$ then the balls centred at the vertices of the inscribe regular crosspolytope show (9) [BW].

3.3 About the lower bounds

The lower bound (10) for $T \geq \frac{n}{2}$ follows essentially directly from the celebrated lower bound of order n on the covering density of a ball, which bound is due to Coxeter, Few and Rogers [CFR]. In addition, (14) is a consequence of (17) because the balls covering $B(o, T)$ cover $S(o, T)$, as well. Now (10) for $T = \frac{\sqrt{n}}{2}$ is a consequence of (14).

If the conjectures (18) and (19) hold then they yield (11) and (12).

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