Control of $(max, +)$-linear systems minimizing delays

L. Houssin[*], S. Lahaye and J.-L. Boimond[†]

December 10, 2008

[*]L. Houssin is with the Laboratoire d'Analyse et d'Architecture des Systèmes (LAAS-CNRS), 31077 Toulouse Cedex 4, France (e-mail: laurent.houssin@laas.fr).

[†]S. Lahaye and J.-L. Boimond are with the Laboratoire d'Ingenierie des Systèmes Automatisés (LISA), Institut des Sciences et Techniques de l'Ingénieur d'Angers (ISTIA), Université d'Angers, 49000 Angers, France (e-mail: sebastien.lahaye@istia.univ-angers.fr and jean-louis.boimond@istia.univ-angers.fr).

1 Introduction

In this paper, we study Discrete Events Dynamic Systems (DEDS) that can be modeled by a linear representation on $(max, +)$ algebra. This class corresponds to Timed Event Graphs (TEG). A linear system theory has been developed for these particular systems in [1]. Strong analogies then appear between the classical linear system theory and the $(max, +)$-linear system theory. Some control problems for these systems have previously been tackled. In all these works, the problematic was to compute an optimal solution in regard to the just-in-time criterion, indeed the proposed control laws satisfy some given control objectives while delaying as much as possible occurrences of input or internal events. In [5], the authors compute a greatest feedback controller (which enables to delay as much as possible occurrences of input or internal events) in order that the controlled system behaves as close as possible to a given reference model.

For our concern, the aim of the control is to delay as less as possible the system while ensuring some given specifications. A first study of this problem is [10]. For example, in a railway network, one can aim at limiting the number of trains in a path while minimizing the induced delays. Another possible application concerns production systems subject to critical time constraints, in which sojourn times of pieces must not exceed a given value at some stages. We may then be interested at bounding the sojourn times while delaying as less as possible the system. We consider two control structures: the control is firstly formalized as a *state feedback on state* and then as a *state feedback on inputs*.

Originalities of this paper lies in the considered criterion (minimization of delays instead of just-in-time), the specification of the control objective (constraints instead of reference model) and the approach for the controller synthesis (new results on fixed points of antitone mappings, instead of residuation theory used in previous papers about control of $(max, +)$-linear systems).

In section 2, we recall some results from the dioid theory and introduce results concerning isotone and antitone mappings. Section 3 is devoted to the modeling of DEDS. The proposed control laws are presented in section 4.

2 Algebraic tools

2.1 Dioid theory

A dioid $(\mathcal{D}, \oplus, \otimes)$ is a semi-ring in which the sum, denoted $\oplus$, is idempotent. The sum (resp. product) admits a neutral element denoted ε (resp. e). A dioid is said to be complete if it is closed for infinite sums and if product distributes over infinite sums too. The sum of all its elements is generally denoted $\top$ (for top).

Example 1 *The set $\mathbb{Z}_{max} = (\mathbb{Z} \cup \{-\infty\})$ endowed with the max operator as sum and the classical sum as product is a (non-complete) dioid. If we add $\top = +\infty$ (with convention $\top \otimes \varepsilon = +\infty + (-\infty) = -\infty = \varepsilon$) to this set, the resulting dioid is complete and is denoted $\overline{\mathbb{Z}}_{max}$.*

Due to the idempotency of the sum, a dioid is endowed with a partial order relation, denoted $\succeq$ and defined by the following equivalence: $a \succeq b \;\Leftrightarrow\; a = a \oplus b$. A complete dioid has a structure of complete lattice [1, §4], *i.e.*, two elements in a complete dioid always have a *least* *upper bound*, namely $a \oplus b$, and a *greatest lower bound* denoted $a \wedge b = \bigoplus_{\{x|x \preceq a,\ x \preceq b\}} x$ in the considered dioid.

Let $\mathcal{D}$ and $\mathcal{C}$ be two complete dioids. A mapping $f : \mathcal{D} \to \mathcal{C}$ is said to be isotone (resp. antitone) if $a, b \in \mathcal{D}$, $a \preceq b \Rightarrow f(a) \preceq f(b)$ (resp. $f(a) \succeq f(b)$).

2.2 Residuation theory

Residuation theory [3] defines "pseudo-inverses" for some isotone mappings defined over ordered sets such as complete dioids [4]. More precisely, if the greatest element of set $\{x \in \mathcal{D} | f(x) \preceq b\}$ exists for all $b \in \mathcal{C}$, then it is denoted $f^\sharp(b)$ and $f^\sharp$ is called *residual* of f. Dually, one may consider the least element satisfying $f(x) \succeq b$, if it exists for all $b \in \mathcal{C}$, it is denoted $f^\flat(b)$ and $f^\flat$ is called *dual residual* of f.

Example 2 *The mapping $T_a : \mathcal{D} \to \mathcal{D}; x \mapsto a \oplus x$ is dually residuated (proof is available in [1, §4.4.4]). The dual residual is denoted $T_a^\flat(b) = b \ominus a$. It should be clear that $a \succeq b \Leftrightarrow T_a^\flat(b) = \varepsilon$. If T_a is defined over $\overline{\mathbb{Z}}_{max}$ then*

$$T_a^\flat(b) = \begin{cases} b & \textit{if } b > a, \\ \varepsilon & \textit{otherwise.} \end{cases}$$

We recall the following property of $T_a^\flat$ which is usefull later :

$$a(x \ominus b) \succeq ax \ominus ab. \tag{1}$$

A relevant remark is that although $T_a^\flat(x) = x \ominus a$ is isotone, the mapping $x \mapsto a \ominus x$ is antitone since

$$x_1 \preceq x_2 \Leftrightarrow a \ominus x_1 \succeq a \ominus x_2 \qquad \forall a.$$

It should be clear that $a \ominus x_1$ is the least solution of $x_1 \oplus x \succeq a$ and $a \ominus x_2$ is the least solution of $x_2 \oplus x \succeq a$ (see [1, 4.4.4] for more details).

2.3 Fixed points of mappings defined over dioids

Because of their lattice structure, properties about fixed points stated for lattices also apply over dioids.

Notation 1 *Let $f : \mathcal{D} \to \mathcal{D}$ with $\mathcal{D}$ a complete dioid, we use the following notations: $\mathcal{F}_f = \{x \in \mathcal{D} | f(x) = x\}$, $\mathcal{P}_f = \{x \in L | f(x) \succeq x\}$, $\mathcal{Q}_f = \{x \in \mathcal{D} | f(x) \preceq x\}$ and f^2 denotes $f \circ f$.*

For an isotone mapping f, [13] and [6] have shown that sets $\mathcal{F}_f$, $\mathcal{P}_f$ and $\mathcal{Q}_f$ are nonempty complete lattices. Moreover, it can be shown that the greatest (resp. least) fixed point coincides with the greatest (resp. least) element of $\mathcal{P}_f$ (resp. $\mathcal{Q}_f$):

$$\begin{array}{lll} Sup\ \mathcal{P}_f = Sup\ \mathcal{F}_f & \text{and} & Sup\ \mathcal{F}_f \in \mathcal{F}_f, \\ Inf\ \mathcal{Q}_f = Inf\ \mathcal{F}_f & \text{and} & Inf\ \mathcal{F}_f \in \mathcal{F}_f. \end{array} \tag{2}$$

A lattice is depicted in fig.1. We represent sets $\mathcal{F}_f$, $\mathcal{P}_f$ and $\mathcal{Q}_f$ for an isotone mapping f.

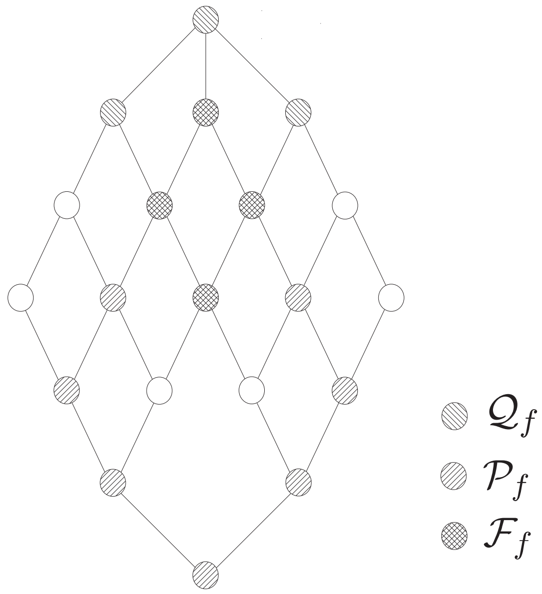

Figure 1: Hasse diagram and sets $\mathcal{F}_f$, $\mathcal{P}_f$ and $\mathcal{Q}_f$ for an isotone mapping f.

In the following proposition given without proof, we specify to dioids a well known method to compute the greatest fixed point of an isotone mapping f.

Proposition 1 *If the following iterative computation*

$$\begin{array}{rcl} y_0 & = & \top \\ y_{k+1} & = & f(y_k) \end{array} \tag{3}$$

converges in a finite number k_e of iterations, then y_{k_e} is the greatest fixed point of f.

Concerning antitone mappings, properties about fixed points are not that well established, and only few works have tackled this problem [2], [7]. To the best of our knowledge, results presented in the sequel are original. However, proposition 6 has been inspired by [7, th. A].

Notice that if f is an antitone mapping then f^2 is isotone. Let us first characterize the structure of $\mathcal{P}_f$ and $\mathcal{Q}_f$.

Proposition 2 *Let $f : \mathcal{D} \to \mathcal{D}$ be an antitone mapping. Set $\mathcal{Q}_f$ (resp. $\mathcal{P}_f$) is a complete upper semi-lattice (resp. complete lower semi-lattice).*

Proof: Let us consider two elements $x, y \in \mathcal{Q}_f$, we have $f(x \oplus y) \preceq f(x) \wedge f(y) \preceq f(x) \oplus f(y) \preceq x \oplus y$, and so $x \oplus y \in \mathcal{Q}_f$. This assertion also applies to infinite sums. Set $\mathcal{P}_f$ is proved to be a complete lower semi-lattice by identical arguments.

□

Proposition 3 *Let $f : \mathcal{D} \to \mathcal{D}$ be an antitone mapping and $x \in \mathcal{D}$. We have*

$$\begin{array}{rcl} x \oplus f(x) & \in & \mathcal{Q}_f, \\ x \wedge f(x) & \in & \mathcal{P}_f. \end{array}$$

Proof:

$$\begin{cases} f(x)\oplus x & \succeq & x \\ f(x)\oplus x & \succeq & f(x) \end{cases}$$

which implies by antitony of f

$$f(f(x)\oplus x) \preceq f(x) \preceq f(x)\oplus x.$$

Respectively, $f(x)\wedge x \in \mathcal{P}_f$ since $f(f(x)\wedge x) \succeq f(x) \succeq f(x)\wedge x$.

□

Proposition 4 *Let $f:\mathcal{D}\to\mathcal{D}$ be an antitone mapping, $y\in\mathcal{P}_f$ and $z\in\mathcal{Q}_f$. For all $x\in\mathcal{D}$ such that $x\preceq y$ (resp. $x'\in\mathcal{D}$ such that $x'\succeq z$), we have $x\in\mathcal{P}_f$ (resp. $x'\in\mathcal{Q}_f$).*

The proof is based on the antitony of f :

$$\begin{array}{ccccccccccc} x & \preceq & y & \Rightarrow & f(x) & \succeq & f(y) & \succeq & y & \succeq & x \\ x' & \succeq & z & \Rightarrow & f(x') & \preceq & f(z) & \preceq & z & \preceq & x' \end{array}$$

Proposition 5 *If x is a fixed point of an antitone mapping $f:\mathcal{D}\to\mathcal{D}$, then x is a minimal (resp. maximal) element of $\mathcal{Q}_f$ (resp. $\mathcal{P}_f$).*

Proof: Let $x\in\mathcal{F}_f$, $y\in\mathcal{P}_f$ and $z\in\mathcal{Q}_f$ such that $y\succeq x\succeq z$. Using antitony of f, we obtain $f(y)\preceq f(x)\preceq f(z) \Rightarrow y\preceq f(y)\preceq x\preceq f(z)\preceq z$ hence $y=x=z$. We conclude that there is no element of $\mathcal{Q}_f$ (resp. $\mathcal{P}_f$) which is less (resp. greater) than x.

□

As a corollary of this proposition, notice that if f admits several distinct fixed points, then they are not comparable. Furthermore, remark that set $\mathcal{F}_f$ can be empty.

Proposition 6 *Let $f:\mathcal{D}\to\mathcal{D}$ be an antitone mapping. Denoting $u = Inf\ \mathcal{F}_{f^2}$ and $v = Sup\ \mathcal{F}_{f^2}$, we have $u\in\mathcal{P}_f$ and $v\in\mathcal{Q}_f$.*

Proof: We show that $f(u)=v$ and $f(v)=u$ (since $u\preceq v$ by definition, this proves $f(u)\succeq u$ and $f(v)\preceq v$). The expression of $f(u)$ leads to

$$f(u) = f(\bigwedge_{x\in\mathcal{F}_{f^2}} x) \succeq \bigoplus_{x\in\mathcal{F}_{f^2}} f(x)$$

(f antitone $\Rightarrow f(a\wedge b)\succeq f(a)\oplus f(b)$).

However, elements of $\{f(x)|x\in\mathcal{F}_{f^2}\}$ are fixed points of f^2 too since $f^2(f(x)) = f(f^2(x)) = f(x)$. So we can deduce that $f_{|\mathcal{F}_{f^2}}$ is a permutation, it can be proved by considering $x,y\in\mathcal{F}_{f^2}$, $x\neq y$ and $f(x)=f(y)$, we would obtain $f^2(x)=f^2(y)$ and so $x=y$ which is a contradiction. So last inequality can be rewritten:

$$f(u) \succeq \bigoplus_{y\in\mathcal{F}_{f^2}} y = v.$$

We previously remark that $f(x)$ with $x \in \mathcal{F}_{f^2}$ is a fixed point of f^2 so does $f(u)$ and it leads to $f(u) = \bigoplus_{y \in \mathcal{F}_{f^2}} y = v$. From last equality, we obtain also $f(f(u)) = u = f(v)$.

□

We illustrate the study of sets $\mathcal{F}_f$, $\mathcal{P}_f$ and $\mathcal{Q}_f$ for an antitone mapping f in fig.2.

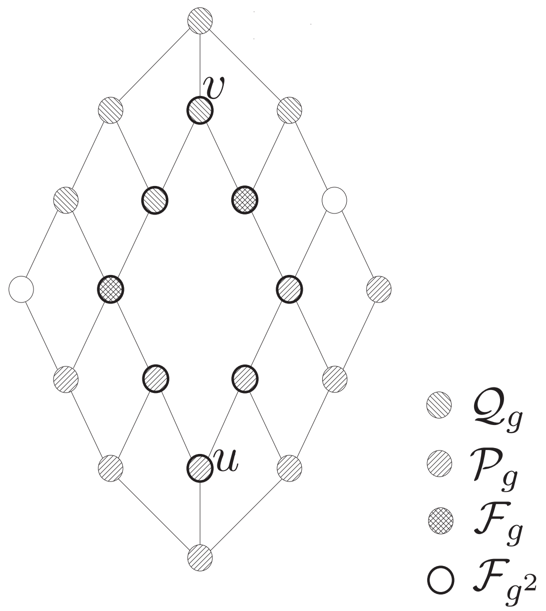

Figure 2: Hasse diagram of sets $\mathcal{Q}_g$, $\mathcal{P}_g$, $\mathcal{F}_g$ and $\mathcal{F}_{g^2}$ for an antitone mapping g.

Remark 1 *For the following control problem, we are interested in the computation of minimal elements of $\mathcal{Q}_f$. The element v, which can be computed using proposition 1, constitutes an interesting approximation since any $x \in \mathcal{F}_f$, minimal element of $\mathcal{Q}_f$ (see proposition 5), is such that $u \preceq x \preceq v$ (since x also belongs to $\mathcal{F}_{f^2}$). As mentioned in the following corollary, v may sometimes be the "best approximation".*

Corollary 1 *If $v = u$, then $\mathcal{F}_f = \{v\}$ and v is a minimal element of $\mathcal{Q}_f$.*

3 Modeling DEDS using dioids

3.1 State and transfer representation

Dioids enable one to obtain linear models for DEDS which involve (only) synchronization and delay phenomena (but not choice phenomena). It corresponds to the class of DEDS which can be modeled by Timed Event Graphs (TEG). The behavior of such systems can be represented by some discrete functions called *dater* functions (see [1]). More precisely, a discrete variable $x(\cdot)$ is associated to an event labeled x (firing times of transition labeled x in the corresponding TEG). This variable represents the occurring dates of event x. Notice that a dual representation for these DEDS is called *counter representation* and it manipulates variables depending on time which represent the cumulated number of firings of transition x.

The considered DEDS can be modeled by a linear *state equation*

$$x(k) = Ax(k-1) \oplus Bu(k), \tag{4}$$

where x and u are the state and the input vectors.

An analogous transform to $\mathcal{Z}$-transform (used to represent discrete-time trajectories in conventional theory) can be introduced for DEDS considered here: the γ,δ-transform. This transform enables to manipulate formal power series, with two commutative variables γ and δ, representing daters trajectories. The set of these formal series is a complete dioid denoted $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ with $e = \gamma^0\delta^0$ as neutral element of product and $\varepsilon = (\gamma^{-1})^*(\delta^1)^*$ as neutral element of sum (the construction of this dioid is detailed in [1]). In the following, we denote x the corresponding element of $\{x(k)\}_{k\in\mathbb{Z}}$ in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ and we assume that each $x \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is represented by its minimum representative (see [1, §5]).

In $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$, state representation (4) becomes

$$x = Ax \oplus Bu, \tag{5}$$

in which entries of matrices A and B are elements of $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$.

Considering the earliest functioning rule (an event occurs as soon as possible), we select the least solution given by $x = A^*Bu$ with $A^* = \bigoplus_{i\in\mathbb{N}} A^i$ [1, Th 4.75], and A^*B corresponds to the *transfer* between u and x.

Assumption 1 *Afterwards, we assume that the input matrix B is a diagonal square matrix with entries equal to e or ε. This assumption is not restrictive since it can always be satisfied by extending the state vector. Remark that the assumed structure of B is such that $B \preceq Id$ and $B^n = B$ for $n \geq 1$.*

3.2 Causality and causal upper approximation

Variables $x \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ used to model TEG have the causality property [1].

Definition 1 *Let $x \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$, x is said to be causal if either $x = \varepsilon$ or all exponents of x are in $\mathbb{N}$. A matrix is said causal if its entries are all causal. The set of causal elements of $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is a complete dioid denoted $\mathcal{M}_{in}^{ax+}[\![\gamma,\delta]\!]$.*

Considering a TEG, a causal transfer means that the system does not require any anticipation (time or event). We now introduce the notion of causal upper approximation (see [9, §2.4]).

Proposition 7 *Let $x \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. The two following assertions are equivalent:*

(i) x has no negative exponent in γ,

(ii) there exists a least $x' \in \mathcal{M}_{in}^{ax+}[\![\gamma,\delta]\!]$ such that $x' \succeq x$. It means that x admits a causal upper approximation.

Proof: If x is causal, the demonstration is obvious and $x' = x$. We now consider x not causal. We can limit the proof to the case of monomials since a series is nothing more than a sum of monomials.

(i)⇒(ii) : Let $x = \gamma^n\delta^t$, with $n \geqslant 0$ and $t < 0$. It is easy to see that the monomials $\gamma^n\delta^0$ is the least element of $\mathcal{M}_{in}^{ax+}[\![\gamma,\delta]\!]$ such that $x' \succeq x$. So, $x' = \gamma^n\delta^0$.

(ii)⇒(i) : If there exists a least $x' \in \mathcal{M}_{in}^{ax+}[\![\gamma,\delta]\!]$ such that $x' \succeq x$ with $x' = \gamma^{n'}\delta^{t'}$ and $x = \gamma^n\delta^t$, we have $n' \leq n$ and $t' \geq t$. However $x' \in \mathcal{M}_{in}^{ax+}[\![\gamma,\delta]\!]$, so $n' \geq 0$ and we obtain $n \geq 0$.

□

We can remark that proposition (7) is also available for matrix.

We now demonstrate that if an element x admits a causal upper approximation then every element less than x admits a causal approximation too.

Corollary 2 *Let x be an element of $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ which admits a causal approximation. Every element y such that $y \preceq x$ admits also a causal approximation.*

Proof: The series x can be rewritten $x = \bigoplus_{i\in I}\gamma^{n_i}\delta^{t_i}$. Since x admits a causal approximation, we have $\forall i \in I$, $n_i \geqslant 0$, in other words, $min_{i\in I}n_i \geqslant 0$. If $y = \bigoplus_{j\in J}\gamma^{n'_j}\delta^{t'_j}$ is such as $x \succeq y$, we obtain

$$min_{j\in J}n'_j \geqslant min_{i\in I}n_i$$

and so $\forall j \in J$, $n'_j \succeq 0$. We conclude that y has no negative exponent in γ and consequently y admits a causal approximation.

□

4 Controllers synthesis for constrained systems

4.1 Problem statement

Considering DEDS modeled by their state equation (5), we are interested here in the design of a *"state feedback"* controller for $(max,+)$-linear systems.

More precisely, if we consider a DEDS modeled by a TEG, then the feedback controller can be realized by another TEG merged on the initial ones. In this controlled TEG, the additional arcs due to the controller authorize or prohibit the firing of the controlled transitions (see figure 5). This control structure is comparable with some Petri nets methods for controlled DEDS [8].

Synthesis of controllers for DEDS in dioids has previously been tackled in many papers ([5], [12] is a non exhaustive list). In all these works, authors focus on just-in-time model reference control, that is, the synthesis of a feedback which delays as much as possible events in the system (*i.e.* the greatest feedback) such that the controlled system is not slower than a reference model. In this paper, the control objective is different :

- we aim at ensuring some given constraints on state (rather than satisfying a reference model matching) for all input u. These constraints are defined by a matrix ϕ and are formulated as the implicit inequality :

$$\phi x \preceq x, \quad \forall u. \tag{6}$$

- we search a feedback which delays as less as possible the functioning of the system (rather than the just-in-time criterion). More precisely, we aim at computing the least feedback such that the state of the controlled system satisfies the constraints given by (6).

In the following, we illustrate three constraints which can be imposed on the controlled systems as an inequality (6). Next, we consider two controller structures. We remind that, for some conditions, a state feedback on state controller is appropriate [10, 9]. We also show that, for some conditions on the specifications, a state feedback on the inputs of the system fits to the problem.

4.2 Constraints specification

We now detail three kinds of constraints for DEDS described by a TEG, that can be formulated as an inequality (6):

- Some inner variables can be subject to a minimum time separation between two firings. For a state variable x_i and a time separation denoted Δ_{min}, we claim that $x_i(k+1) \succeq \Delta_{min}\ x_i(k)$. Then, the counterpart of this constraint in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ is $\gamma\delta^{\Delta_{min}} x_i \preceq x_i$.
- We can also aim at bounding the sojourn times of tokens in given paths of the TEG (critical time constraints). Let us consider a path from transition x_i to transition x_j containing initially α tokens, we denote τ the desired maximum sojourn time in this path. This imposes $x_j(k+\alpha) - x_i(k) \preceq \tau$, which can be formulated in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ by $\gamma^{-\alpha}\delta^{-\tau} x_j \preceq x_i$.
- We may also limit the number of tokens in paths of the TEG. Let us consider a path from x_i to x_j containing initially α tokens, we denote κ the desired maximum number of tokens in this path. This constraint can be specified by: $\kappa \succeq x_i(t) - x_j(t) + \alpha$, which leads in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ to $\gamma^{\kappa-\alpha} x_j \preceq x_i$.

4.3 Synthesis of a state feedback on state controller

4.3.1 Formalization

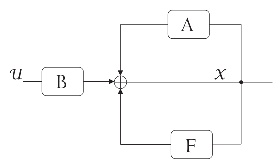

Figure 3: state feedback on state controller

In this structure, a controller denoted F, is added between internal states (see figure 3). This particular structure is not usual in control theory, but has a specific interest for DEDS (*e.g.* the state feedback on state controller has been considered in [11]).

In fact, if we assume that internal events are measurable, controller F uses this measure to possibly delay occurrences of some internal events.

The state evolution is then described by

$$x = Ax \oplus Fx \oplus Bu. \tag{7}$$

The least solution of this equation which corresponds to the earliest functioning, is given by $x = (A \oplus F)^* Bu$.

From (7), the state of controlled system is s.t.

$$x \succeq Ax \oplus Bu.$$

Furthermore, as control objective, we aim at satisfying (6), then

$$x \succeq (A \oplus \phi)x \oplus Bu.$$

Since we aim at delaying as less as possible the system, we seek the least controlled state x, that is the least x greater than the least solution $x \succeq (A \oplus \phi)^* Bu$. The problem can then be formulated as the search for a least feedback F such that

$$(A \oplus F)^* Bu \succeq (A \oplus \phi)^* Bu, \quad \forall u$$
$$\Leftrightarrow (A \oplus F)^* B \succeq (A \oplus \phi)^* B. \tag{8}$$

Some constraints may be unsuitable, that is, there may not exist a causal feedback enabling to satisfy these constraints. The following proposition furnishes a test on given constraints ϕ, stating a necessary and sufficient condition for the existence of a causal feedback.

Proposition 8 *There exists a causal feedback F satisfying (8) if and only if each entry of $(A \oplus \phi)^* B$ has no negative exponent in γ. We then denote G the upper causal approximation of $(A \oplus \phi)^* B$.*

Proof: According to proposition 7, there exists a least causal matrix G such that $G \succeq (A \oplus \phi)^* B$ if and only if each entry of $(A \oplus \phi)^* B$ has no negative exponent in γ. It is always possible to find a causal matrix F such that $(A \oplus F)^* \succeq G$. Furthermore, by isotony of product, we have $GB \succeq (A \oplus \phi)^* B^2 = (A \oplus \phi)^* B$ since $B^2 = B$ (see assumption 1). The product GB is causal (since G and B are causal) and G is the least causal matrix such that $G \succeq (A \oplus \phi)^* B$, so we claim that $GB \succeq G$. However $B \preceq Id$, one has that $GB \preceq G$. Therefore, and since B is causal, there exists a causal F such that $(A \oplus F)^* B \succeq GB = G$.

□

4.3.2 Feedback computation

From proposition 8, we aim at computing the least feedback F such that

$$(A \oplus F)^* B \succeq G \tag{9}$$

in which G is the upper causal approximation of $(A \oplus \phi)^* B$.

Proposition 9 *Solutions of equation (9) are elements of* $\mathcal{Q}_f$ *(see notation 1) with* $f : F \mapsto G \ominus (A \oplus F)^*$.

Proof: We have the following equivalences

$$
\begin{array}{lll}
 & G \preceq (A \oplus F)^* B & \\
\Leftrightarrow & G \preceq (A \oplus F)^* & \text{since } GB = G \text{ (see proof} \\
 & & \text{of prop. 8) and } B \preceq Id \text{ (ass. 1)} \\
\Leftrightarrow & G \preceq (A \oplus F)^* \oplus F & \text{since } (A \oplus F)^* \succeq F \\
\Leftrightarrow & G \ominus (A \oplus F)^* \preceq F & \text{since } T_{(A \oplus F)^*} \text{ is dually} \\
 & & \text{residuated } (\textit{cf.}\ \text{ex.2})
\end{array}
$$

□

Notice that mapping f is antitone. From proposition 6, the element $v = Sup\ \mathcal{F}_{f^2}$ (which can be computed using proposition 1) belongs to $\mathcal{Q}_f$ and is then a feedback satisfying (9). From corollary 1, v is a minimal element of $\mathcal{Q}_f$ (i.e. a minimal feedback satisfying (9)) if $v = u$. Otherwise, as mentioned in remark 1, v can be used to approximate a minimal feedback satisfying (9).

4.4 Synthesis of a state feedback on inputs controller

4.4.1 Formalization

In previous section, the designed controller requires that every state variables are controllable. This assumption is not needed in the case of a state feedback on the inputs of the system. This kind of controller implies that the delayed events are only the inputs one. The concerned variables are the ones belonging to the set $\mathcal{U}_c = \{u_i | B_{ii} = e\}$.

By considering the earliest functioning rule, the transfer relation of such controlled system is

$$x_c = (A \oplus BF)^* Bv = H_c v. \tag{10}$$

The figure 4(b) describes the controlled system.

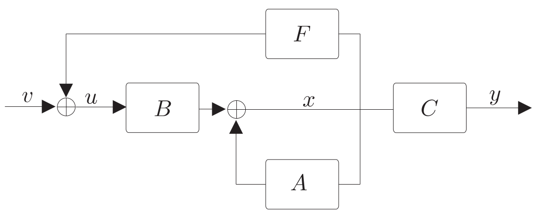

Figure 4: s tate feedback on inputs controller.

Remark 2 *The assumption 1 implies that the feedback on inputs has an effect on the state variables directly controllable, i.e. state variables* x_i *such that* $B_{ii} = e$*. These state variables* x_i

are such that $x_i = u_i$ since there is no shift between them. We denote $\mathcal{X}_c = \{x_i | B_{ii} = e\}$ this set.

From (10), it is obvious that the state of the controlled system is such that

$$x_c \succeq A^* B v \qquad \forall v.$$

Furthermore, x_c should satisfy the control objective (6), *i.e.* $x_c \succeq \phi x_c$, then

$$x_c \succeq A^* B v \oplus \phi x_c \qquad \forall v.$$

We aim at delaying as less as possible the system, therefore we seek the least controlled x_c given by

$$x_c \succeq \phi^* A^* B v \qquad \forall v.$$

Using (10), we then seek the least feedback F such that

$$\begin{aligned} (A \oplus BF)^* B v \succeq \phi^* A^* B v \quad \forall v \\ \Leftrightarrow (A \oplus BF)^* B \succeq \phi^* A^* B. \end{aligned} \tag{11}$$

We can easily prove that (11) is equivalent to

$$(A \oplus BF)^* B \succeq \phi^+ A^* B, \tag{12}$$

since (11) $\Rightarrow$ (12) : $\phi^* A^* B \succeq \phi^+ A^* B$, and (11) $\Leftarrow$ (12) :

$$\begin{array}{rlcl} & (A \oplus BF)^* B & \succeq & \phi^+ A^* B \\ \Rightarrow & (A \oplus BF)^* B \oplus A^* B & \succeq & \phi^+ A^* B \oplus A^* B \\ \Leftrightarrow & (A \oplus BF)^* B & \succeq & \phi^* A^* B. \end{array}$$

Assumption 2 *The matrix of constraints ϕ is supposed to satisfy $B\phi = \phi$. This assumption comes down to formulating all constraints $\phi_{ij} x_j \preceq x_i$ (see §4.2) such that $x_i \in \mathcal{X}_c$.*

The following proposition gives a necessary and sufficient condition on given constraints ϕ for the existence of a causal feedback satisfying (12).

Proposition 10 *There exists a causal feedback F satisfying (12) if, and only if $\phi^+ A^* B$ admits a causal upper approximation. If it exists, the causal upper approximation is denoted G.*

Proof:

- $\Rightarrow$ If a causal feedback F exists, then $(A \oplus BF)^* B$ is also causal (since A and B are causal). We can state from corollary 2 and (12) that $\phi^+ A^* B$ admits a causal upper approximation.
- $\Leftarrow$ If $\phi^+ A^* B$ admits a causal upper approximation, then one can find a causal element X such that $X \succeq \phi^+ A^* B$. Since $B^2 = B$ and $B\phi^+ = B(\phi \oplus \phi\phi \oplus ...) = \phi^+$ (by assumption $B\phi = \phi$), we have

$$\begin{array}{rrcll} & BXB & \succeq & \phi^+A^*B & \\ \Rightarrow & (BX)^*B & \succeq & \phi^+A^*B & \text{since } a^* \succeq a \\ \Rightarrow & (BX)^*B \oplus A^*B & \succeq & \phi^+A^*B & \\ \Rightarrow & ((BX)^* \oplus A^*)B & \succeq & \phi^+A^*B & \\ \Rightarrow & ((BX) \oplus A)^*B & \succeq & \phi^+A^*B & \text{since } (a\oplus b)^* \succeq a^* \oplus b^* \end{array}$$

which proves that a causal feedback (here denoted X) satifying (12) exists.

□

Corollary 3 *The causal upper approximation G, if it exists, is such that $GB = G$ and $BG = G$.*

Proof:

We first demonstrate that $GB = G$. Since $B \preceq Id$, we have $GB \preceq G$. From proposition 10, G is such that $G \succeq \phi^+A^*B$ and we have $GB \succeq \phi^+A^*B$ ($B^2 = B$). GB is causal since G and B are, and as G is the least causal element greater than ϕ^+A^*B, we deduce $GB \succeq G$. By the same reasonning, We can easily prove $BG = G$.

□

4.4.2 Feedback computation

In this section, we tackle how to compute a solution to (12).

Proposition 11 *Suppose that ϕ^+A^*B admits a causal upper approximation denoted G. Solutions of (12) are elements of $\mathcal{Q}_g$ (see notation 1) with $g : F \mapsto B(G \ominus (A \oplus BF)^*)$.*

Proof:

Causal feedbacks we are interested in are such that

$$\begin{array}{rlcll} & (A\oplus BF)^*B & \succeq & G & \\ \Leftrightarrow & (A\oplus BF)^* & \succeq & G & (\text{since } GB = G \text{ and } B \preceq Id) \\ \Leftrightarrow & BF \oplus (A\oplus BF)^* & \succeq & G & (\text{since } (A\oplus BF)^* \succeq BF) \\ \Leftrightarrow & BF & \succeq & G \ominus (A\oplus BF)^* & (T_{(A\oplus BF)^*} \text{ is dually residuated}) \\ \Leftrightarrow & F & \succeq & B(G \ominus (A\oplus BF)^*). & \end{array}$$

For the last equivalence :

$(\Rightarrow)$

$$\begin{array}{rcl} BF \succeq G \ominus (A\oplus BF)^* & \Rightarrow & B^2F \succeq B(G \ominus (A\oplus BF)^*) \\ & \Rightarrow & F \succeq B(G \ominus (A\oplus BF)^*) \\ & & (\text{since } F \succeq BF = B^2F) \end{array}$$

$(\Leftarrow)$

$$\begin{array}{rcl} F \succeq B(G \circ\!\!- (A\oplus BF)^*) & \Rightarrow & BF \succeq B^2(G \circ\!\!- (A\oplus BF)^*) \\ & \Rightarrow & BF \succeq B(G \circ\!\!- (A\oplus BF)^*) \\ & \Rightarrow & BF \succeq BG \circ\!\!- B(A\oplus BF)^* \\ & & (\text{since } a(x \circ\!\!- b) \succeq ax \circ\!\!- ab) \\ & \Rightarrow & BF \succeq BG \circ\!\!- (A\oplus BF)^* \\ & & (x \mapsto a \circ\!\!- x \text{ is antitone}) \\ & \Rightarrow & BF \succeq G \circ\!\!- (A\oplus BF)^* \\ & & (\ BG = G). \end{array}$$

□

The computation of $v = Sup\ \mathcal{F}_{g^2}$ furnishes a feedback ensuring (12) since $v \in \mathcal{Q}_g$ (see propos. 6).

4.5 Example

We consider the DEDS modeled by the TEG of fig. 5, and represented by the following matrices:

$$A = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \delta^3 & \gamma^3\delta & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \delta & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \gamma\delta^5 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \delta^4 & \varepsilon & \varepsilon & \gamma^2\delta^2 & \gamma\delta^2 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta^3 & \varepsilon \end{pmatrix},$$

and B is a diagonal matrix s.t.

$$B_{ii} = \begin{cases} e & \text{if } i \in \{1,3\}, \\ \varepsilon & \text{otherwise.} \end{cases}$$

In a first place, we aim at illustrating that all constraints defined as in §4.2, are not suitable. For example, we can not impose that tokens sojourn less than 2 time units in the place between transitions x_5 and x_6 for all input. In fact, if transition u_1 is not fired, then initial tokens will not be removed from this place, and we can not find any relevant feedback. As stated in proposition 8, the computation of $(A \oplus \phi)^* B$ enables to detect this unsuitable constraint since it contains an entry with a negative exponent in γ.

We now consider suitable constraints:

- tokens must not sojourn more than 4 time units in the place between transitions x_2 and x_6, then $\delta^{-4}x_6 \preceq x_2$,
- the number of tokens between x_5 and x_4 must not exceed 3, so $\gamma^2 x_5 \preceq x_4$.

We have

$$\phi = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta^{-4} & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \gamma^2 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}.$$

4.5.1 The state feedback on state case

According to §4.3.2, we can compute the following state feedback on state

$$F = v = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \gamma\delta^5(\gamma\delta^2)^* & \varepsilon & \gamma^3\delta^4(\gamma\delta^2)^* & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \gamma^3\delta^6(\gamma^3\delta^5)^* & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \gamma^4\delta^{11}(\gamma^3\delta^5)^* & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}$$

which satisfies both constraints. A realization of this controller is represented in thin lines on fig.5. Let us note that, for this example, we have $v \neq u$, and we hence can not argue thanks to corollary 1 that v is a minimal feedback. It can be checked that there exists a relevant feedback F', defined by $F'_{ij} = v_{ij}$ for $(i, j) \neq (5, 3)$ and $F'_{53} = \varepsilon$, which is less than v. Nevertheless, let us point out that the controlled system with F' has the same transfer as the controlled system with v. This observation reinforces our suggestion that v constitutes a good approximated solution for our control problem (see remark 1).

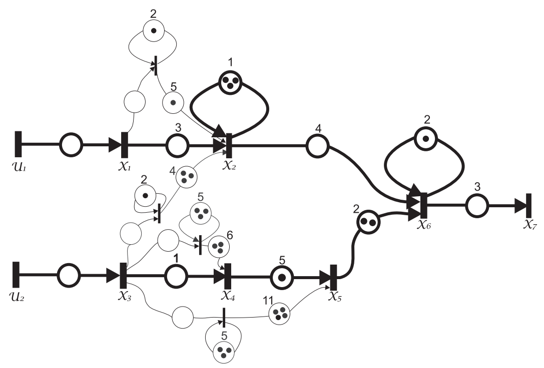

Figure 5: A TEG (thick lines) merged with a realization of its state on state feedback controller (thin lines)

4.5.2 The state feedback on inputs case

In the case of the state feedback on inputs, the new constraint matrix ϕ' must ensure assumption 2. Therefore, we have to adapt constraints from §4.5. With that aim, we consider constraints described as the following.

- Tokens must not sojourn more than 7 time units in the path the path between transition x_1 and transition x_7. Consequently, tokens can't remain more than 4 time units in the place between transitions x_2 and x_6.
- The number of tokens in the path from x_3 to x_5 must not exceed 3. It follows to the limitation of the place between x_4 and x_5 to 3 tokens.

We have

$$\phi' = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta^{-7} & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \gamma^2 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}.$$

The computation of the state feedback on inputs yields

$$F' = \begin{pmatrix} \gamma\delta^2(\gamma\delta^2)^* & \varepsilon & \gamma^3\delta(\gamma\delta^2)^* & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \gamma^3\delta^6 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}.$$

A realization of F' is given in fig.6.

It's worth mentioning that, if we consider x_7 as the output of the system, the transfer of the controlled system is the same for the two controllers since

$$C(A \oplus BF')^*B = C(A \oplus F)^*B = \begin{pmatrix} \delta^{10}(\gamma\delta^2)^* \\ \gamma^3\delta^{11}(\gamma\delta^2)^* \end{pmatrix},$$

with $C = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e \end{pmatrix}$.

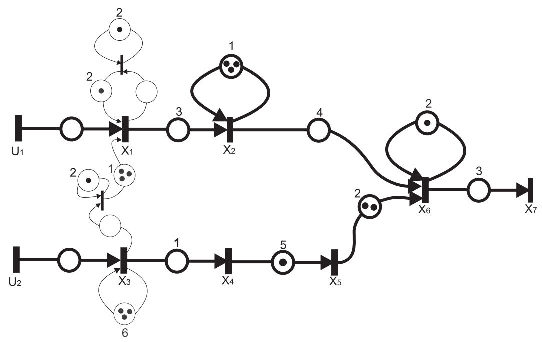

Figure 6: A TEG (thick lines) merged with a realization of its state on inputs feedback controller (thin lines)

5 Conclusion

We have presented a new control problem in $(max, +)$-linear system theory: ensure some given constraints while delaying as less as possible the system. Using results on antitone and isotone mappings, two state feedback structures are proposed. However, it must be noted that the obtained controllers are not necessarily minimal. In the near future, we will focus on improvements of our control approach in that sense. We also plan to consider TEG with uncontrollable transitions and non linear constraints.

References

[1] F. Baccelli, G. Cohen, G. J. Olsder, and J. P. Quadrat. *Synchronization and Linearity.* Wiley, 1992.

[2] K. Baclawski and A. Björner. Fixed points in partially ordered sets. *Advances in Mathematics*, 31(3):263–287, mar. 1979.

[3] T. S. Blyth and M. F. Janowitz. *Residuation Theory.* Pergamon press, 1972.

[4] G. Cohen. Residuation and Applications. In *Special issue on Ecole d'informatique théorique*, Noirmoutier, 1998. INRIA.

[5] B. Cottenceau, L. Hardouin, J. L. Boimond, and J. L. Ferrier. Model reference control for timed event graphs in dioids. *Automatica*, 37:1451–1458, 2001.

[6] P. Cousot and R. Cousot. Constructive versions of Tarski's fixed point theorems. *Pacific Journal of Mathematics*, 81(1):43–57, 1979.

[7] Rade M. Dacić. On fixed edges of antitone self-mappings of complete lattices. *Publications de l'Institut Mathématique*, 34:49–53, 1983.

[8] L. E. Holloway, B. H. Krogh, and A. Giua. A survey of petri net methods for controlled discrete event systems. *DEDS*, 7:151–190, 1997.

[9] L. Houssin. *Contributions à la commande des systèmes (max,+)-linéaires. Applications aux réseaux de transport.* Thèse de doctorat, ISTIA - Université d'Angers, Dec. 2006.

[10] L. Houssin, S. Lahaye, and J. L. Boimond. Control of constrained $(max, +)$-linear systems minimizing delays. In *Proceedings of INCOM'2006*, Saint-Etienne, France, 2006.

[11] R. Lüders and R. Santos-Mendes. Generalized multivariable control of discrete event systems in dioid. In *Proceedings of WODES 2002*, Zaragoza, Spain, 2002.

[12] C.A. Maia, L. Hardouin, R. Santos Mendes, and B. Cottenceau. Optimal closed-loop control for timed event graphs in dioid. *IEEE Trans. on Automatic Control*, 48:2284–2287, 2003.

[13] A. Tarski. A lattice theorical fixed point theorem and its applications. *Pacific Journal of Mathematics*, 5(2):285–309, 1955.