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Jean Fromentin

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THE WELL-ORDERING OF DUAL BRAID MONOIDS

JEAN FROMENTIN

ABSTRACT. We describe the restriction of the Dehornoy ordering of braids to the dual braid monoids introduced by Birman, Ko and Lee: we give an inductive characterization of the ordering of the dual braid monoids and compute the corresponding ordinal type. The proof consists in introducing a new ordering on the dual braid monoid using the rotating normal form of arXiv:math.GR/0811.3902, and then proving that this new ordering coincides with the standard ordering of braids.

It is known since [7] and [15] that the braid group B_n is left-orderable, by an ordering whose restriction to the positive braid monoid is a well-order. Initially introduced by complicated methods of self-distributive algebra, the standard braid ordering has then received a lot of alternative constructions originating from different approaches—see [10]. However, this ordering remains a complicated object, and many questions involving it remain open.

Dual braid monoids have been introduced by Birman, Ko, and Lee in [2]. The dual braid monoid B_n^{+*} is a certain submonoid of the n -strand braid group B_n . It is known that the monoid B_n^{+*} admits a Garside structure, where simple elements correspond to non-crossing partitions of n —see [1]. In particular, there exists a standard normal form associated with this Garside structure, namely the so-called greedy normal form.

The rotating normal form is another normal form on B_n^{+*} that was introduced in [12]. It relies on the existence of a natural embedding of B_{n-1}^{+*} in B_n^{+*} and on the easy observation that each element of B_n^{+*} admits a maximal right divisor that belongs to B_{n-1}^{+*} . The main ingredient in the construction of the rotating normal form is the result that each braid β in B_{n-1}^{+*} admits a unique decomposition

$$\beta = \phi_n^{b-1}(\beta_b) \cdot \dots \cdot \phi_n^2(\beta_3) \cdot \phi_n(\beta_2) \cdot \beta_1$$

with β_b, \dots, β_1 in B_{n-1}^{+*} such that $\beta_b \neq 1$ and such that for each $k \geq 1$, the braid β_k is the maximal right-divisor of $\phi_n^{b-k}(\beta_b) \cdot \dots \cdot \beta_k$ that lies in B_{n-1}^{+*} . The sequence $(\beta_b, \dots, \beta_1)$ is then called the ϕ_n -splitting of β .

The main goal of this paper is to establish the following simple connection between the order on B_n^{+*} and the order on B_{n-1}^{+*} through the notion of ϕ_n -splitting.

Theorem 1. *For all braids β, γ in B_n^{+*} with $n \geq 2$, the relation $\beta < \gamma$ is true if and only if the ϕ_n -splitting $(\beta_b, \dots, \beta_1)$ of β is smaller than the ϕ_n -splitting $(\gamma_c, \dots, \gamma_1)$ of γ with respect to the ShortLex-extension of the ordering of B_{n-1}^{+*} , i.e., we have either $b < c$, or $b = c$ and there exists t such that $\beta_t < \gamma_t$ holds and $\beta_k = \gamma_k$ holds for $b \geq k > t$.*

A direct application of Theorem 1 is:

Corollary. *For $n \geq 2$, the restriction of the braid ordering to B_n^{+*} is a well-ordering of ordinal type $\omega^{\omega^{n-2}}$.*

This refines a former result by Laver stating that the restriction of the braid ordering to B_n^{+*} is a well-ordering without determining its exact type.

Another application of Theorem 1 or, more exactly, of its proof, is a new proof of the existence of the braid ordering. What we precisely obtain is a new proof of the result that every nontrivial braid can be represented by a so-called σ -positive or σ -negative word (“Property **C**”).

The connection between the restrictions of the braid order to B_n^{+*} and B_{n-1}^{+*} via the ϕ_n -splitting is formally similar to the connection between the restrictions of the braid order to the Garside monoids B_n^+ and B_{n-1}^+ via the so-called Φ_n -splitting established in [9] as an application of Burckel’s approach of [3, 4, 5]. However, there is an important difference, namely that, contrary to Burckel’s approach, our construction requires no transfinite induction: although intricate in the general case of 5 strands and above, our proof remains elementary. This is an essential advantage of using the Birman–Ko–Lee generators rather than the Artin generators.

The paper is organized as follows. In Section 1, we briefly recall the definition of the Dehornoy ordering of braids and the definition of the dual braid monoid. In Section 2, we use the ϕ_n -splitting to construct a new linear ordering of B_n^{+*} , called the rotating ordering. In Section 3, we deduce from the results about ϕ_n -splittings established in [12] the result that certain specific braids are σ -positive or trivial. Finally, Theorem 1 is proved in Section 4.

1. THE GENERAL FRAMEWORK

Artin’s braid group B_n is defined for $n \geq 2$ by the presentation

$$\left\langle \sigma_1, \dots, \sigma_{n-1}; \begin{array}{ll} \sigma_i \sigma_j = \sigma_j \sigma_i & \text{for } |i - j| \geq 2 \\ \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j & \text{for } |i - j| = 1 \end{array} \right\rangle. \quad (1.1)$$

The submonoid of B_n generated by $\{\sigma_1, \dots, \sigma_{n-1}\}$ is denoted by B_n^+ .

1.1. The standard braid ordering. We recall the construction of the Dehornoy ordering of braids. By a *braid word* we mean any word on the letters $\sigma_i^{\pm 1}$.

Definition 1.1.

- A braid word w is called σ_i -positive (resp. σ_i -negative) if w contains at least one σ_i (resp. at least one σ_i^{-1}), no σ_i^{-1} (resp. no σ_i), and no letter $\sigma_j^{\pm 1}$ with $j > i$.
- A braid β is said to be σ_i -positive (resp. σ_i -negative) if, among the braid words representing β , at least one is σ_i -positive (resp. σ_i -negative).
- A braid β is said to be σ -positive (resp. σ -negative) if it is σ_i -positive (resp. σ_i -negative) for some i .
- For β, γ braids, we declare that $\beta < \gamma$ is true if the braid $\beta^{-1} \gamma$ is σ -positive.

By definition of the relation $<$, every σ -positive braid β satisfy $1 < \beta$. Then, every braid of B_n^+ except 1 is larger than 1.

Example 1.2. Put $\beta = \sigma_2$ and $\gamma = \sigma_1 \sigma_2$. Let us show that β is $<$ -smaller than γ . The quotient $\beta^{-1} \gamma$ is represented by the word $\sigma_2^{-1} \sigma_1 \sigma_2$. Unfortunately, the latter word is neither σ_2 -positive (since it contains σ_2^{-1}), nor σ_2 -negative (since it contains σ_2), nor σ_1 -positive and σ_1 -negative (since it contains a letter $\sigma_2^{\pm 1}$). However, the word $\sigma_2^{-1} \sigma_1 \sigma_2$ is equivalent to $\sigma_1 \sigma_2 \sigma_1^{-1}$, which is σ_2 -positive. Then, the braid $\beta^{-1} \gamma$ is σ_2 -positive and the relation $\beta < \gamma$ holds.

Theorem 1.3. [7] *For each $n \geq 2$, the relation $<$ is a linear ordering on B_n that is invariant under left multiplication.*

Remark 1.4. In this paper, we use the flipped version of the braid ordering [10], in which one takes into account the generator σ_i with greatest index, and not the original version, in which one considers the generator with lowest index. This choice is necessary here, as we need that B_{n-1}^{+*} is an initial segment of B_n^{+*} . That would not be true if we were considering the lower version of the ordering. We recall that the flip automorphism Φ_n that maps σ_i and σ_{n-i} for each i exchanges the two version, and, therefore, the properties of both orderings are identical.

Following [7], we recall that Theorem 1.3 relies on two results:

Property A. Every σ_i -positive braid is nontrivial.

Property C. Every braid is either trivial or σ -positive or σ -negative.

In the sequel, as we shall prove that Property C is a consequence of Theorem 1, we never use Theorem 1.3, *i.e.*, we never use the fact that the relation $<$ of Definition 1.1 is a linear ordering. The only properties of $<$ we shall use are the following trivial facts—plus, but exclusively for the corollaries, Property A.

Lemma 1.5. *The relation $<$ is transitive, and invariant under left-multiplication.*

The ordering $<$ on B_n admits lots of properties but it is not a well-ordering. For instance, $\sigma_{n-1}^{-1}, \sigma_{n-1}^{-2}, \dots, \sigma_{n-1}^{-k}, \dots$ is an infinite descending sequence. However, R. Laver proved the following result:

Theorem 1.6. [15] *Assume that M is a submonoid of B_∞ generated by a finite number of braids, each of which is a conjugate of some braid σ_i . Then the restriction of $<$ to M is a well-ordering.*

The positive braid monoid B_n^+ satisfies the hypotheses of Theorem 1.6. Therefore, the restriction of the braid ordering to B_n^+ is a well-ordering. However, Laver's proof of Theorem 1.6 leaves the determination of the isomorphism type of $(M, <)$ open. In the case of the monoid B_n^+ , the question was solved by S. Burckel:

Proposition 1.7. [4] *For each $n \geq 2$, the order type of $(B_n^+, <)$ is the ordinal $\omega^{\omega^{n-2}}$.*

1.2. Dual braid monoids. In this paper, we consider another monoid of which B_n is a group of fractions, namely the Birman-Ko-Lee monoid, also called *the dual braid monoid*.

Definition 1.8. For $1 \leq p < q$, put

$$a_{p,q} = \sigma_p \dots \sigma_{q-2} \sigma_{q-1} \sigma_{q-2}^{-1} \dots \sigma_p^{-1}. \quad (1.2)$$

Then the *dual braid monoid* B_n^{+*} is the submonoid of B_n generated by the braids $a_{p,q}$ with $1 \leq p < q \leq n$; the braids $a_{p,q}$ are called the *Birman-Ko-Lee generators*.

Remark 1.9. In [2], the braid $a_{p,q}$ is defined to be $\sigma_{q-1} \dots \sigma_{p+1} \sigma_p \sigma_{p+1}^{-1} \dots \sigma_{q-1}^{-1}$. Both options lead to isomorphic monoids, but our choice is the only one that naturally leads to the suitable embedding of B_{n-1}^{+*} into B_n^{+*} .

By definition, we have $\sigma_p = a_{p,p+1}$, so the dual braid monoid B_n^{+*} includes the positive braid monoid B_n^+ . The inclusion is proper for $n \geq 3$: the braid $a_{1,3}$, which belongs to B_n^{+*} by definition, does not belong to B_n^+ .

The following notational convention will be useful in the sequel. For $p \leq q$, we write $[p, q]$ for the interval $\{p, \dots, q\}$ of \mathbb{N} . We say that $[p, q]$ is *nested* in $[r, s]$ if we have $r < p < q < s$. The following results were proved by Birman, Ko, and Lee.

Proposition 1.10. [2]

(i) In terms of the generators $a_{p,q}$, the monoid B_n^{+*} is presented by the relations

$$a_{p,q} a_{r,s} = a_{r,s} a_{p,q} \quad \text{for } [p, q] \text{ and } [r, s] \text{ disjoint or nested,} \quad (1.3)$$

$$a_{p,q} a_{q,r} = a_{q,r} a_{p,r} = a_{p,r} a_{p,q} \quad \text{for } 1 \leq p < q < r \leq n. \quad (1.4)$$

(ii) The monoid B_n^{+*} is a Garside monoid with Garside element δ_n defined by

$$\delta_n = a_{1,2} a_{2,3} \dots a_{n-1,n} \quad (= \sigma_1 \sigma_2 \dots \sigma_{n-1}). \quad (1.5)$$

Garside monoids are defined for instance in [11] or [8]. In every Garside monoid, conjugating by the Garside element defines an automorphism (\cdot). In the case of B_n^{+*} , the automorphism ϕ_n defined by $\phi_n(\beta) = \delta_n \beta \delta_n^{-1}$ has order n , and its action on the generators $a_{p,q}$ is as follows.

Lemma 1.11. For all p, q with $1 \leq p < q \leq n$, we have

$$\phi_n(a_{p,q}) = \begin{cases} a_{p+1,q+1} & \text{for } q \leq n-1, \\ a_{1,p+1} & \text{for } q = n. \end{cases} \quad (1.6)$$

The relations (1.6) show that the action of ϕ_n is similar to a rotation. Note that the relation $\phi_n(a_{p,q}) = a_{p+1,q+1}$ always holds provided indices are taken mod n and possibly switched, so that, for instance, $a_{p+1,n+1}$ means $a_{1,p+1}$.

For every braid β in B_n^{+*} and every k , the definition of ϕ_n implies the relation

$$\delta_n \phi_n^k(\beta) = \phi_n^{k+1}(\beta) \delta_n, \quad \delta_n^{-1} \phi_n^k(\beta) = \phi_n^{k-1}(\beta) \delta_n^{-1}. \quad (1.7)$$

By definition, the braid $a_{p,q}$ is the conjugate of σ_{q-1} by the braid $\sigma_p \dots \sigma_{q-2}$. Braids of the latter type play an important role in the sequel, and we give them a name.

Definition 1.12. For $p \leq q$, we put

$$\delta_{p,q} = a_{p,p+1} a_{p+1,p+2} \dots a_{q-1,q} \quad (= \sigma_p \sigma_{p+1} \dots \sigma_{q-1}). \quad (1.8)$$

Note that the Garside element δ_n of B_n^{+*} is equal to $\delta_{1,n}$ and that the braid $\delta_{p,p}$ is the trivial one, *i.e.*, is the braid 1.

With this notation, we easily obtain

$$a_{p,q} = \delta_{p,q} \delta_{p,q-1}^{-1} = \delta_{p,q-1} \sigma_{q-1} \delta_{p,q-1}^{-1} \quad \text{for } p < q, \quad (1.9)$$

$$a_{p,q} = \delta_{p+1,q}^{-1} \delta_{p,q} \quad \text{for } p < q, \quad (1.10)$$

$$\delta_{p,r} = \delta_{p,q} \delta_{q,r} \quad \text{for } p \leq q \leq r, \quad (1.11)$$

$$\delta_{p,q} \delta_{r,s} = \delta_{r,s} \delta_{p,q} \quad \text{for } p \leq q < r \leq s. \quad (1.12)$$

Relations (1.9), (1.11), and (1.12) are direct consequences of the definition of $\delta_{p,q}$. Relation (1.10) is obtained by using an induction on p and the equality $\delta_{p+1,q}^{-1} \delta_{p,q} = \sigma_p \delta_{p+2,q}^{-1} \delta_{p+1,q} \sigma_p^{-1}$, which holds for $p < q$.

1.3. The restriction of the braid ordering to the dual braid monoid. The aim of this paper is to describe the restriction of the braid ordering of Definition 1.1 to the dual braid monoid B_n^{+*} . The initial observation is:

Proposition 1.13. [15] *For each $n \geq 2$, the restriction of the braid ordering to the monoid B_n^{+*} is a well-ordering.*

Proof. By definition, the braid $a_{p,q}$ is a conjugate of the braid σ_{q-1} . So B_n^{+*} is generated by finitely many braids, each of which is a conjugate of some σ_i . By Laver's Theorem (Theorem 1.6), the restriction of $<$ to B_n^{+*} is a well-ordering. \square

As in the case of B_n^+ , Laver's Theorem, which is a non-effective result based on the so-called Higman's Lemma [13], leaves the determination of the isomorphism type of $<|_{B_n^{+*}}$ open. This is the question we shall address in the sequel.

Before introducing our specific methods, let us begin with some easy observations.

Lemma 1.14. *Every braid β in B_n^{+*} except 1 satisfies $\beta > 1$.*

Proof. By definition, the braid $a_{p,q}$ is σ -positive—actually it is σ_{q-1} -positive in the sense of Definition 1.1. \square

Lemma 1.15. *For each $n \geq 2$, we have*

$$1 < a_{1,2} < a_{2,3} < a_{1,3} < \dots < a_{1,n-1} < a_{n-1,n} < a_{n-2,n} < \dots < a_{1,n}. \quad (1.13)$$

Proof. We claim that $a_{p,q} < a_{r,s}$ holds if and only if we have either $q < s$, or $q = s$ and $p > r$. Assume first $q < s$. Then, the braid $a_{p,q}^{-1}$ is σ_{q-1} -negative while $a_{r,s}$ is σ_{s-1} -positive with $q < s$, hence the quotient $a_{p,q}^{-1} a_{r,s}$ is σ_{s-1} -positive, which implies $a_{p,q} < a_{r,s}$. Assume now $q = s$ and $p > r$. Then, by relation (1.9), the quotient $a_{p,q}^{-1} a_{r,s}$ is equal to $\delta_{p,s-1}^{-1} \delta_{p,s}^{-1} \delta_{r,s} \delta_{r,s-1}^{-1}$. Applying Relation (1.11) on $\delta_{r,s}$, we obtain

$$a_{p,q}^{-1} a_{r,s} = \delta_{p,s-1} \delta_{p,s}^{-1} \delta_{r,p-1} \delta_{p-1,s} \delta_{r,s-1}^{-1}.$$

Then, by applying Relation (1.12) on $\delta_{p,s}^{-1} \delta_{r,p-1}$, we obtain

$$a_{p,q}^{-1} a_{r,s} = \delta_{p,s-1} \delta_{r,p-1} \delta_{p,s}^{-1} \delta_{p-1,s} \delta_{r,s-1}^{-1}.$$

Finally, Relation (1.10) on $\delta_{p,s}^{-1} \delta_{p-1,s}$ implies

$$a_{p,q}^{-1} a_{r,s} = \delta_{p,s-1} \delta_{r,p-1} a_{p-1,s} \delta_{r,s-1}^{-1}.$$

The braid $a_{p-1,s}$ is σ_{s-1} -positive, while the braids $\delta_{p,s-1}$, $\delta_{r,p-1}$ and $\delta_{r,s-1}^{-1}$ are σ_t -positive or σ_t -negative for $t < s-1$. Hence the quotient $a_{p,q}^{-1} a_{r,s}$ is σ_{s-1} -positive, which implies $a_{p,q} < a_{r,s}$.

As $<$ is a linear ordering, this is enough to conclude, *i.e.*, the implications we proved above are equivalences. \square

1.4. The rotating normal form. In this section we briefly recall the construction of the rotating normal form of [12].

Definition 1.16. For $n \geq 3$ and β a braid of B_n^{+*} . The maximal braid β_1 lying in B_{n-1}^{+*} that right-divides the braid β is called the B_{n-1}^{+*} -tail of β .

Proposition 1.17. [12] Assume $n \geq 3$ and that β is a nontrivial braid in B_n^{+*} . Then there exists a unique sequence $(\beta_b, \dots, \beta_1)$ in B_{n-1}^{+*} satisfying $\beta_b \neq 1$ and

$$- \beta = \phi_n^{b-1}(\beta_b) \cdot \dots \cdot \phi_n(\beta_2) \cdot \beta_1, \quad (1.17.i)$$

$$- \text{for each } k \geq 1, \text{ the } B_{n-1}^{+*}\text{-tail of } \phi_n^{b-k}(\beta_b) \cdot \dots \cdot \phi_n(\beta_{k+1}) \text{ is trivial.} \quad (1.17.ii)$$

Definition 1.18. The unique sequence $(\beta_b, \dots, \beta_1)$ of braids introduced in Proposition 1.17 is called the ϕ_n -splitting of β . Its length, *i.e.*, the parameter b , is called the n -breadth of β .

The idea of the ϕ_n -splitting is very simple: starting with a braid β of B_n^{+*} , we extract the maximal right-divisor that lies in B_{n-1}^{+*} , *i.e.*, that leaves the n th strand unbraided, then we extract the maximal right-divisor of the remainder that leaves the first strand unbraided, and so on rotating by $2\pi/n$ at each step—see Figure 1.

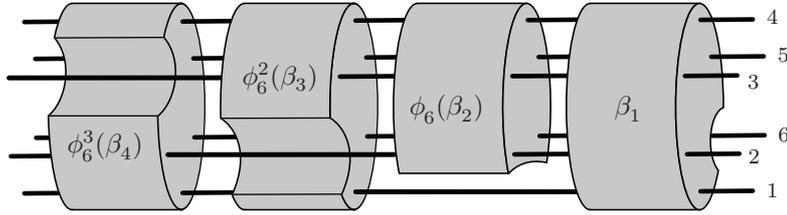


FIGURE 1. The ϕ_6 -splitting of a braid of B_6^{+*} . Starting from the right, we extract the maximal right-divisor that keeps the sixth strand unbraided, then rotate by $2\pi/6$ and extract the maximal right-divisor that keeps the first strand unbraided, etc.

As the notion of a ϕ_n -splitting is fundamental in this paper, we give examples.

Example 1.19. Let us determine the ϕ_n -splitting of the generators of B_n^{+*} , *i.e.*, of $a_{p,q}$ with $1 \leq p < q \leq n$. For $q \leq n-1$, the generator $a_{p,q}$ belongs to B_{n-1}^{+*} , then its ϕ_n -splitting is $(a_{p,q})$. Next, as $a_{p,n}$ does not lie in B_{n-1}^{+*} , the rightmost entry in its ϕ_n -splitting is trivial. As we have $\phi_n^{-1}(a_{p,n}) = a_{p-1,n-1}$ for $p \geq 2$, the ϕ_n -splitting of $a_{p,n}$ with $p \geq 2$ is $(a_{p-1,n-1}, 1)$. Finally, the braids $a_{1,n}$ and $\phi_n^{-1}(a_{1,n}) = a_{n-1,n}$ do not lie in B_{n-1}^{+*} , but $\phi_n^{-2}(a_{1,n}) = a_{n-2,n-1}$ does. So the ϕ_n -splitting of $a_{1,n}$ is $(a_{n-2,n-1}, 1, 1)$. To summarize, the ϕ_n -splitting of $a_{p,q}$ is

$$\begin{cases} (a_{p,q}) & \text{for } p < q \leq n-1, \\ (a_{p-1,n-1}, 1) & \text{for } 2 \leq p \text{ and } q = n, \\ (a_{n-2,n-1}, 1, 1) & \text{for } p = 1 \text{ and } q = n. \end{cases} \quad (1.15)$$

By Relation (1.6), the application ϕ_n maps each braid $a_{p,q}$ to another similar braid $a_{r,s}$. Using this remark, we can consider the alphabetical homomorphism, still denoted ϕ_n , that maps the letter $a_{p,q}$ to the corresponding letter $a_{r,s}$, and extends to word on the letter $a_{p,q}$. Note that, in this way, if the word w on the letter $a_{p,q}$ represents the braid β , then $\phi_n(w)$ represents $\phi_n(\beta)$.

We can now recursively define a distinguished expression for each braid of B_n^{+*} in terms of the generators $a_{p,q}$.

Definition 1.20.

- For β in B_2^{+*} , the ϕ_2 -normal form of β is defined to be the unique word $a_{1,2}^k$ that represents β .

- For $n \geq 3$ and β in B_n^{+*} , the ϕ_n -normal form of β is defined to be the word $\phi_n^{b-1}(w_b) \dots w_1$ where, for each k , the word w_k is the ϕ_{n-1} -normal form of β_k and where $(\beta_b, \dots, \beta_1)$ is the ϕ_n -splitting of β .

As the ϕ_n -splitting of a braid β lying in B_{n-1}^{+*} is the length 1 sequence (β) , the ϕ_n -normal form and ϕ_{n-1} -normal form of β coincide. Therefore, we can drop the subscript in the ϕ_n -normal form. From now on, we call *rotating normal form*, or simply *normal form*, the expression so obtained.

As each braid is represented by a unique normal word, we can unambiguously use the syntactical properties of its normal form.

We conclude this introductory section with some syntactic constraints involving ϕ_n -splittings and normal words, These results are borrowed from [12].

Definition 1.21. For β in B_n^{+*} , the *last letter* of β , denoted $\beta^\#$, is defined to be the last letter in the normal form of β .

Lemma 1.22. [12] *Assume that $(\beta_b, \dots, \beta_1)$ is a ϕ_n -splitting.*

- (i) *For $k \geq 2$, the letter $\beta_k^\#$ has the form $a_{\dots, n-1}$, unless β_k is trivial;*
- (ii) *For $k \geq 3$, the braid β_k is different from 1;*
- (iii) *For $k \geq 2$, if the normal form of β_k is $w' a_{n-2, n-1}$ with $w' \neq \varepsilon$ (the empty word), then the last letter of w' has the form $a_{\dots, n-1}$.*

2. THE ROTATING ORDERING

As explained above, we aim at proving results about the restriction of the braid ordering $<$ to the dual braid monoid B_n^{+*} . We shall do it indirectly, by first introducing an auxiliary ordering $<^*$, and eventually proving that the latter coincides with the original braid ordering.

2.1. Another ordering on B_n^{+*} . Using the ϕ_n -splitting of Definition 1.18, every braid of B_n^{+*} comes associated with a distinguished finite sequence of braids belonging to B_{n-1}^{+*} . In this way, every ordering on B_{n-1}^{+*} can be extended to an ordering on B_n^{+*} using a lexicographic extension. Iterating the process, we can start from the standard ordering on B_2^{+*} , *i.e.*, on natural numbers, and recursively define a linear ordering on B_n^{+*} .

We recall that, if (A, \prec) is an ordered set, a finite sequence s in A is called **ShortLex**-smaller than another finite sequence s' if the length of s is smaller than that of s' , or if both lengths are equal and s is lexicographically \prec -smaller than s' , *i.e.*, when both sequences are read starting from the left, the first entry in s that does not coincide with its counterpart in s' is \prec -smaller.

Definition 2.1. For $n \geq 2$, we recursively define a relation $<_n^*$ on B_n^{+*} as follows:

- For β, γ in B_2^{+*} , we declare that $\beta <_2^* \gamma$ is true for $\beta = a_{1,2}^b$ and $\gamma = a_{1,2}^c$ with $b < c$;
- For β, γ in B_n^{+*} with $n \geq 3$, we declare that $\beta <_n^* \gamma$ is true if the ϕ_n -splitting of β is smaller than the ϕ_n -splitting of γ for the **ShortLex**-extension of $<_{n-1}^*$.

Example 2.2. As was seen in Example 1.19, the n -breadth of $a_{p,q}$ with $q \leq n-1$ is 1 while the n -breadth of $a_{q,n}$ is 2 for $p \neq 1$ or 3 for $p = 1$. An easy induction on n gives $a_{p,q} <_n^* a_{r,s}$ whenever $q < s \leq n$ holds. Then, one establishes

$$1 <_n^* a_{1,2} <_n^* a_{2,3} <_n^* a_{1,3} <_n^* a_{3,4} <_n^* a_{2,4} <_n^* a_{1,4} <_n^* \dots <_n^* a_{n-1,n} <_n^* \dots <_n^* a_{1,n}.$$

We observe that, according to Lemma 1.15 and Example 2.2, the relations $<$ and $<_n^*$ agree on the generators of B_n^{+*} .

Proposition 2.3. *For $n \geq 2$, the relation $<_n^*$ is a well-ordering on B_n^{+*} . For each braid β , the immediate $<_n^*$ -successor of β is $\beta a_{1,2}$, i.e., $\beta \sigma_1$.*

Proof. The ordered monoid $(B_2^{+*}, <_2^*)$ is isomorphic to \mathbb{N} with the usual ordering, which is a well-ordering. As the ShortLex-extension of a well-ordering is itself a well-ordering—see [16]—we inductively deduce that $<_n^*$ is a well-ordering.

The result about successors immediately follows from the fact that, if the ϕ_n -splitting of β is $(\beta_p, \dots, \beta_1)$, then the ϕ_n -splitting of $\beta a_{1,2}$ is $(\beta_p, \dots, \beta_1 a_{1,2})$. \square

The connection between the ordering $<_{n-1}^*$ and the restriction of $<_n^*$ to B_{n-1}^{+*} is simple: B_{n-1}^{+*} is an initial segment of B_n^{+*} .

Proposition 2.4. *For $n \geq 3$, the monoid B_{n-1}^{+*} is the initial segment of $(B_n^{+*}, <_n^*)$ determined by $a_{n-1,n}$, i.e., we have $B_{n-1}^{+*} = \{\beta \in B_n^{+*} \mid \beta <_n^* a_{n-1,n}\}$. Moreover the braid $a_{n-1,n}$ is the smallest of n -breadth 2.*

Proof. First, by construction, every braid β of B_{n-1}^{+*} has n -breadth 1, whereas, by (1.15), the n -breadth of $a_{n-1,n}$ is 2. So, by definition, $\beta <_n^* a_{n-1,n}$ holds.

Conversely, assume that β is a braid of B_n^{+*} that satisfies $\beta <_n^* a_{n-1,n}$. As the n -breadth of $a_{n-1,n}$ is 2, the hypothesis $\beta <_n^* a_{n-1,n}$ implies that the n -breadth of β is at most 2. We shall prove, using induction on n , that β has n -breadth at most 1, which, by construction, implies that β belongs to B_{n-1}^{+*} .

Assume first $n = 3$. By definition, every ϕ_3 -splitting of length 2 has the form $(a_{1,2}^b, a_{1,2}^c)$ with $b \neq 0$. The ShortLex-least such sequence is $(a_{1,2}, 1)$, which turns out to be the ϕ_3 -splitting of $a_{2,3}$. Hence $a_{2,3}$ is the $<_3^*$ -smallest element of B_3^{+*} with 3-breadth equal to 2, and $\beta <_3^* a_{2,3}$ implies $\beta \in B_2^{+*}$.

Assume now $n > 3$. Assume for a contradiction that the n -breadth of β is 2. Let (β_2, β_1) be the ϕ_n -splitting of β . As the ϕ_n -splitting of $a_{n-1,n}$ is $(a_{n-2,n-1}, 1)$, and $\beta_1 <_{n-1}^* 1$ is impossible, the hypothesis $\beta <_n^* a_{n-1,n}$ implies $\beta_2 <_{n-1}^* a_{n-2,n-1}$. By induction hypothesis, this implies that β_2 lies in B_{n-2}^{+*} , hence $\phi_n(\beta_2)$ lies in B_{n-1}^{+*} . This contradicts Condition (1.17.ii): a sequence (β_2, β_1) with $\beta_2 <_{n-1}^* a_{n-2,n-1}$ cannot be the ϕ_n -splitting of a braid of B_n^{+*} . So the hypothesis that β has n -breadth 2 is contradictory, and β necessarily lies in B_{n-1}^{+*} . \square

Building on the compatibility result of Proposition 2.4, we hereafter drop the subscript in $<_n^*$ and simply write $<^*$. Note that $<^*$ is actually a linear order (and even a well-ordering) on B_∞^{+*} , the inductive limit of the monoids B_n^{+*} with respect to the canonical embedding of B_{n-1}^{+*} into B_n^{+*} .

2.2. Separators. By definition of $<^*$, for $b < c$, every braid in B_n^{+*} that has n -breadth b is $<^*$ -smaller than every braid that has n -breadth c . As the ordering $<^*$ is a well-ordering, there must exist, for each b , a $<^*$ -smallest braid with n -breadth b . These braids, which play the role of separators for $<^*$, are easily identified. They will play an important role in the sequel.

Proposition 2.4 says that the least upper bound of the braids with n -breadth 1 is $a_{n-1,n}$. From n -breadth 2, a periodic pattern appears.

Definition 2.5. For $n \geq 3$ and $b \geq 1$, we put $\hat{\delta}_{n,b} = \phi_n^{b+1}(a_{n-2,n-1}) \cdots \phi_n^2(a_{n-2,n-1})$.

For instance, we find $\widehat{\delta}_{6,4} = \phi_6^5(a_{4,5}) \cdot \phi_6^4(a_{4,5}) \cdot \phi_6^3(a_{4,5}) \cdot \phi_6^2(a_{4,5})$, whence $\widehat{\delta}_{6,4} = a_{3,4} a_{2,3} a_{1,2} a_{1,6}$, and, similarly, $\widehat{\delta}_{5,3} = a_{2,3} a_{1,2} a_{1,5}$.

Proposition 2.6. *For all $n \geq 3$ and $b \geq 1$,*

(i) *the ϕ_n -splitting of $\widehat{\delta}_{n,b}$ is the length $b+2$ sequence $(a_{n-2,n-1}, \dots, a_{n-2,n-1}, 1, 1)$;*

(ii) *we have $\widehat{\delta}_{n,b} = \delta_n^b \delta_{n-1}^{-b}$;*

(iii) *the braid $\widehat{\delta}_{n,b}$ is the $<^*$ -smallest braid in B_n^{+*} that has n -breadth $b+2$ —hence it is the least upper bound of all braids of n -breadth $\leq b+1$.*

Proof. (i) First, we observe that there exists no relation $a_{n-1,n} a_{n-2,n-1} = \dots$ in the presentation of the monoid B_n^{+*} . Then the word $a_{n-1,n} a_{n-2,n-1}$, which is equal to $\phi_n(a_{n-2,n-1}) a_{n-2,n-1}$, is alone in its equivalence class under the relation of B_n^{+*} . Similarly, the word $\phi_n^{b-1}(a_{n-2,n-1}) \dots a_{n-2,n-1}$, called w , is alone in its equivalence class, as no relation of the presentation may be applied to any length 2 subword of this word. As ϕ_n is an isomorphism, the same result holds for the word $\phi_n^2(w)$, which represents the braid $\widehat{\delta}_{n,b}$. As the braid $\widehat{\delta}_{n,b}$ is represented by a normal word, we deduce that $\phi_n^2(w)$ is the normal word representing $\widehat{\delta}_{n,b}$, i.e., it is its normal form.

(ii) We use an induction on b . Relation (1.9) implies $a_{1,n} = \delta_n \delta_{n-1}^{-1}$. Using (1.6), we deduce

$$\widehat{\delta}_{n,1} = \phi_n^2(a_{n-2,n-1}) = a_{1,n} = \delta_n \delta_{n-1}^{-1}.$$

Assume now $b \geq 2$. By definition, we have $\widehat{\delta}_{n,b} = \phi_n^{b+1}(a_{n-2,n-1}) \widehat{\delta}_{n,b-1}$. Then, using the induction hypothesis we have

$$\widehat{\delta}_{n,b} = \phi_n^{b+1}(a_{n-2,n-1}) \delta_n^{b-1} \delta_{n-1}^{-b+1}.$$

Pushing δ_n^{b-1} to the left using (1.7), we find:

$$\widehat{\delta}_{n,b} = \delta_n^{b-1} \phi_n^2(a_{n-2,n-1}) \delta_{n-1}^{-b+1}.$$

Relation (1.6) implies $\phi_n^2(a_{n-2,n-1}) = a_{1,n}$. So, using (1.9), we finally obtain

$$\widehat{\delta}_{n,b} = \delta_n^{b-1} \delta_n \delta_{n-1}^{-1} \delta_{n-1}^{-b+1} = \delta_n^b \delta_{n-1}^{-b}.$$

(iii) Let $(\beta_{b+2}, \dots, \beta_1)$ be the ϕ_n -splitting of a braid β that lies in B_n^{+*} and satisfies $\beta \leq^* \widehat{\delta}_{n,b}$. By definition of $<^*$, we have $\beta_{b+2} \leq^* a_{n-2,n-1}$. By Lemma 1.22(i) and (ii), the B_{n-2}^{+*} -tail of β_{b+2} is trivial, hence its $n-1$ -breadth is at least 2. Then Proposition 2.4 implies $\beta_{b+2} = a_{n-2,n-1}$. By definition of $<^*$ again, we have $\beta_{b+1} \leq^* a_{n-2,n-1}$. Using the previous argument repeatedly, we obtain $\beta_k = a_{n-2,n-1}$ for $k \geq 3$. Using the argument once more gives $\beta_2 \leq^* 1$, which implies $\beta_2 = 1$. Finally, by definition of $<^*$, we have $\beta_1 = 1$. We conclude by (i) and the uniqueness of the ϕ_n -splitting. \square

Owing to Proposition 2.6, it is coherent to extend Definition 2.5 by $\widehat{\delta}_{n,0} = a_{n-1,n}$. In this way, the result of Proposition 2.6(iii) extends to the case $b = 0$.

We now observe that the orderings $<$ and $<^*$ agree on the separators $\widehat{\delta}_{n,b}$.

Lemma 2.7. *Assume $n \geq 3$. Then $0 \leq b < c$ implies $\widehat{\delta}_{n,b} < \widehat{\delta}_{n,c}$.*

Proof. Assume $0 < b < c$. By Proposition 2.6(ii), we have

$$\widehat{\delta}_{n,b}^{-1} \cdot \widehat{\delta}_{n,c} = \delta_{n-1}^b \delta_n^{-b} \cdot \delta_n^c \delta_{n-1}^{-c} = \delta_{n-1}^b \delta_n^{c-b} \delta_{n-1}^{-c}.$$

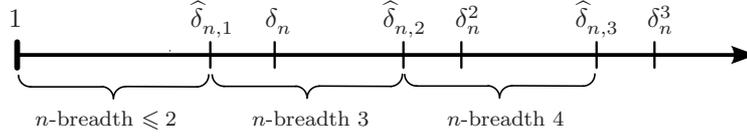


FIGURE 2. The braid $\widehat{\delta}_{n,r}$ as a separator in $(B_n^{+*}, <^*)$ —hence in $(B_n^{+*}, <)$ as well once Theorem 2.8 is proved.

The hypothesis $c - b > 0$ implies that $\widehat{\delta}_{n,b}^{-1} \widehat{\delta}_{n,c}$ is a σ_{n-1} -positive braid, since the braid δ_k is σ_{k-1} -positive. Hence we have $\widehat{\delta}_{n,b} < \widehat{\delta}_{n,c}$.

It remains to establish the result for $b = 0$. The previous case implies $\widehat{\delta}_{n,1} \leq \widehat{\delta}_{n,c}$. As the relation $<$ is transitive (Lemma 1.5), it is enough to prove $\widehat{\delta}_{n,0} < \widehat{\delta}_{n,1}$. Using Proposition 2.6(ii) and inserting $\delta_n \delta_n^{-1}$ on the left, we obtain

$$\widehat{\delta}_{n,0}^{-1} \widehat{\delta}_{n,1} = a_{n-1,n}^{-1} \delta_n \delta_n^{-1} = \delta_n \delta_n^{-1} a_{n-1,n}^{-1} \delta_n \delta_n^{-1}.$$

Relation (1.6) implies $\delta_n^{-1} a_{n-1,n}^{-1} \delta_n = \phi_n^{-1}(a_{n-1,n}^{-1}) = a_{n-2,n-1}^{-1}$. We deduce

$$\widehat{\delta}_{n,0}^{-1} \widehat{\delta}_{n,1} = \delta_n a_{n-2,n-1}^{-1} \delta_n^{-1},$$

and the latter decomposition is explicitly σ_{n-1} -positive. \square

2.3. The main result. At this point, we have two *a priori* unrelated linear orderings of the monoid B_n^{+*} , namely the standard braid ordering $<$, and the rotating ordering $<^*$ of Definition 2.1. The main technical result of this paper is:

Theorem 2.8. *For all braids β, β' in B_n^{+*} , the relation $\beta <^* \beta'$ implies $\beta < \beta'$.*

Before starting the proof of this result, we list a few consequences. First we obtain a new proof of Property **C**.

Corollary 2.9 (Property **C**). *Every non-trivial braid is σ -positive or σ -negative.*

Proof. Assume that β is a non-trivial braid of B_n . First, as B_n is a group of fractions for B_n^{+*} , there exist β', β'' in B_n^{+*} satisfying $\beta = \beta'^{-1} \beta''$. As β is assumed to be nontrivial, we have $\beta' \neq \beta''$. As $<^*$ is a strict linear ordering, one of $\beta' <^* \beta''$ or $\beta'' <^* \beta'$ holds. In the first case, Theorem 2.8 implies that $\beta'^{-1} \beta''$, i.e., β , is σ -positive. In the second case, Theorem 2.8 implies that $\beta''^{-1} \beta'$ is σ -positive, hence β is σ -negative. \square

Corollary 2.10. *The relation $<^*$ coincide with the restriction of $<$ to B_n^{+*} .*

Proof. Let β, γ belong to B_n^{+*} . By Theorem 2.8, $\beta <^* \gamma$ implies $\beta < \gamma$. Conversely, assume $\beta \not<^* \gamma$. As $<^*$ is a linear ordering, we have either $\gamma <^* \beta$, hence $\gamma < \beta$, or $\beta = \gamma$. In both cases, Property **A** implies that $\beta < \gamma$ fails. \square

Corollary 2.10 directly implies Theorem 1 stated in the introduction. Indeed, the characterization of the braid ordering given in Theorem 1 is nothing but the recursive definition of the ordering $<^*$.

Finally, we obtain a new proof of Laver's result, together with a determination of the order type.

Corollary 2.11. *The restriction of the Dehornoy ordering to the dual braid monoid B_n^{+*} is a well-ordering, and its order type is the ordinal $\omega^{\omega^{n-2}}$.*

Proof. It is standard that, if $(X, <)$ is a well-ordering of ordinal type λ , then the **ShortLex**-extension of $<$ to the set of all finite sequences of elements of X is a well-ordering of ordinal type λ^ω —see [16]. The ordinal type of $<^*$ on B_2^{+*} is ω , the order type of the standard ordering of natural numbers. So, an immediate induction shows that, for each $n \geq 2$, the ordinal type of $<^*$ on B_n^{+*} is at most $\omega^{\omega^{n-2}}$.

A priori, this is only an upper bound, because it is not true that every sequence of braids in B_{n-1}^{+*} is the ϕ_n -splitting of a braid of B_n^{+*} . However, by construction, the monoid B_n^{+*} includes the positive braid monoid B_n^+ , and it was shown in [4]—or, alternatively, in [6]—that the order type of the restriction of the braid ordering to B_n^+ is $\omega^{\omega^{n-2}}$. Hence the ordinal type of its restriction to B_n^{+*} is at least that ordinal, and, finally, we have equality. (Alternatively, we could also directly construct a type $\omega^{\omega^{n-2}}$ increasing sequence in B_n^{+*} .) \square

Remark 2.12. By construction, the ordering $<$ is invariant under left-multiplication. Another consequence of Corollary 2.10 is that the ordering $<^*$ is invariant under left-multiplication as well. Note that the latter result is *not* obvious at all from the direct definition of that relation.

3. A KEY LEMMA

So, our goal is to prove that the rotating ordering of Definition 2.1 and the standard braid ordering coincide. The result will follow from the fine properties of the rotating normal form and of the ϕ_n -splitting. The aim of this section is to establish these properties. Most of them are improvements of properties established in [12], and we shall heavily use the notions introduced in this paper.

3.1. Sigma-positive braid of type $a_{p,n}$. We shall prove Theorem 2.8 by using an induction on the number of strands n . Actually, in order to maintain an induction hypothesis, we shall prove a stronger implication: instead of merely proving that, if β is $<^*$ -smaller than γ , then the quotient braid $\beta^{-1}\gamma$ is σ -positive, we shall prove the more precise conclusion that $\beta^{-1}\gamma$ is σ -positive of type $a_{p,n}$ for some p related to the last letter in γ .

Definition 3.1. Assume $n \geq 3$.

- A braid is called $a_{p,n}$ -dangerous if it admits one decomposition of the form

$$\delta_{f(d),n-1}^{-1} \delta_{f(d-1),n-1}^{-1} \cdots \delta_{f(1),n-1}^{-1},$$

with $f(d) \geq f(d-1) \geq \dots \geq f(1) = p$.

- A braid is called σ_i -nonnegative if it is σ_i -positive or it belongs to B_i .
- For $p \leq n-2$, a braid β is called σ -positive of type $a_{p,n}$ if it can be expressed as

$$\beta^+ \cdot \delta_{p,n} \cdot \beta^-,$$

where β^+ is σ_{n-1} -nonnegative and β^- is $a_{p,n}$ -dangerous.

- A braid β is called σ -positive of type $a_{n-1,n}$ if it is 1, or equal to

$$\beta' \cdot a_{n-1,n},$$

where β' is a σ -positive braid of type $a_{1,n}$.

Note that, an $a_{p,n}$ -braid with $p \neq n-1$ is not the trivial one, *i.e.*, is different from 1, as it contains $\delta_{p,n-1}$, which is non trivial. In the other hand, the only $a_{n-1,n}$ -dangerous braid is 1.

We observe that the definition of σ -positive of type $a_{n-1,n}$ is different from the definition of σ -positive of type $a_{p,n}$ for $p < n-1$ (technical reasons make such a distinction necessary).

Saying a braid is σ -positive of type $a_{p,n}$ is motivated by the fact that $a_{p,n}$ is the simplest σ -positive braid of its type.

Lemma 3.2. *Assume that β is a σ -positive braid of type $a_{p,n}$. Then*

- (i) β is σ_{n-1} -positive,
- (ii) $\phi_{n+1}(\beta)$ is σ -positive of type $a_{p+1,n+1}$,
- (iii) if $p = 1$ then $\beta \delta_{n-1}^{-t}$ is σ -positive of type $a_{1,n}$ for all $t \geq 0$,
- (iv) if $\beta \neq a_{n-1,n}$ holds, then $\gamma\beta$ is σ -positive of type $a_{p,n}$ for every σ_{n-1} -nonnegative braid γ .

Proof. (i) An $a_{p,n}$ -dangerous braid is σ_{n-1} -nonnegative (actually it is σ_{n-2} -negative), and the braid $\delta_{p,n}$ is σ_{n-1} -positive. Therefore β is σ_{n-1} -positive.

(ii) With the notation of Definition 3.1, let $\delta_{f(d),n-1}^{-1} \dots \delta_{f(1),n-1}^{-1}$ be the decomposition of β^- , with $f(1) = p$. Then we have

$$\phi_{n+1}(\beta^-) = \delta_{f(d)+1,n}^{-1} \dots \delta_{f(1)+1,n}^{-1},$$

an $a_{p+1,n+1}$ -dangerous word. By definition, the braid β^+ can be represented by a word on the alphabet $\sigma_i^{\pm 1}$ with $i \leq n-2$. As, for $i \leq n-2$, the image of σ_i by ϕ_{n+1} is σ_{i+1} , the braid $\phi_{n+1}(\beta^+)$ is σ_n -nonnegative. So the relation $\phi_{n+1}(\beta) = \phi_{n+1}(\beta^+) \cdot \delta_{p+1,n+1} \cdot \phi_{n+1}(\beta^-)$ witnesses that $\phi_{n+1}(\beta)$ is σ -positive of type $a_{p+1,n+1}$.

Point (iii) directly follows from the fact that, if γ^- is an $a_{1,n}$ -dangerous braid, then, for each $t \geq 0$, the braid $\gamma^- \delta_{n-1}^{-t}$ is also $a_{1,n}$ -dangerous.

(iv) Assume $p \leq n-2$. Then, by definition, we have $\beta = \beta^+ \cdot \delta_{p,n} \cdot \beta^-$, where β^+ is σ_{n-1} -nonnegative and β^- is $a_{p,n}$ -dangerous. Hence, we get $\gamma\beta = \gamma\beta^+ \cdot \delta_{p,n} \cdot \beta^-$. As the product of σ_{n-1} -nonnegative braids is σ_{n-1} -nonnegative, the braid $\gamma\beta$ is σ -positive of type $a_{p,n}$.

Assume now $p = n-1$. As, by hypothesis β is different from $a_{n-1,n}$, we have $\beta = \beta' \cdot a_{n-1,n}$, where β' is σ -positive of type $a_{1,n}$. The case $p \leq n-2$ implies that the braid $\gamma\beta'$ is σ -positive of type $a_{1,n}$. Hence the braid $\gamma\beta$, which is equal to $\gamma\beta' \cdot a_{n-1,n}$, is σ -positive of type $a_{n-1,n}$. \square

Remark 3.3. For $t \geq 1$, the braid $\widehat{\delta}_{n,t}$ is σ -positive of type $a_{1,n}$. Indeed, by Proposition 2.6(ii), we have $\widehat{\delta}_{n,t} = \delta_n^{t-1} \cdot \delta_n \cdot \delta_{n-1}^{-t}$, the right-hand side being an explicit σ -positive braid of type $a_{1,n}$.

3.2. Properties of σ -positive braids of type $a_{p,n}$. We now show that the entries in a ϕ_n -splitting give raise to σ -positive braids of type $a_{p,n}$, for some p that can be effectively controlled.

Lemma 3.4. *For $n \geq 3$, every braid with last letter $a_{p,n}$ is σ -positive of type $a_{p,n}$.*

Proof. Let β be a braid of B_n^{+*} with last letter $a_{p,n}$ (Definition 1.21). Put $\beta = \beta' \cdot a_{p,n}$. Assume first $p \leq n-2$. Then, by (1.9), we have $\beta = \beta' \cdot \delta_{p,n} \cdot \delta_{p,n-1}^{-1}$, an explicit σ -positive braid of type $a_{p,n}$, since the braid β' is positive, hence σ_{n-1} -nonnegative, and $\delta_{p,n-1}^{-1}$ is $a_{p,n}$ -dangerous.

Assume now $p = n-1$. The case $\beta' = 1$ is clear. For $\beta' \neq 1$, by Lemma 1.22(iii), there is a positive braid β'' satisfying $\beta' = \beta'' \cdot a_{q,n}$ for some q . The relation $a_{1,q} a_{q,n} = a_{q,n} a_{1,n}$ implies $a_{q,n} = a_{1,q}^{-1} a_{q,n} a_{1,n}$. Using (1.9) for $a_{1,n}$ gives

$$\beta' = \beta'' a_{1,q}^{-1} a_{q,n} \cdot \delta_{1,n} \cdot \delta_{1,n-1}^{-1},$$

an explicit σ -positive braid of type $a_{1,n}$. Therefore, β is σ -positive of type $a_{1,n}$. \square

We shall see now that the normal form of every braid β of B_{n-1}^{+*} such that the B_{n-1}^{+*} -tail of $\phi_n(a_{p,n}\beta)$ is trivial contains a sequence of overlapping letters $a_{r,s}$. Words containing such sequences are what we shall call ladders.

Definition 3.5. For $n \geq 3$, we say that a normal word w is an $a_{p,n}$ -ladder lent on $a_{q-1,n-1}$, if there exists a decomposition

$$w = w_0 x_1 w_1 \dots w_{h-1} x_h w_h, \quad (3.1)$$

and a sequence $p = f(0) < f(1) < \dots < f(h) = n-1$ such that

- (i) for each $k \leq h$, the letter x_k is of the form $a_{e(k),f(k)}$ with $e(k) < f(k-1) < f(k)$,
- (ii) for each $k < h$, the word w_k contains no letter $a_{p,q}$ with $p < f(k) < q$,
- (iii) the last letter of w is $a_{q-1,n-1}$.

By convention, an $a_{n-1,n}$ -ladder lent on $a_{q-1,n-1}$ is a word on the letters $a_{p,q}$ whose last letter is $a_{q-1,n-1}$.

The concept of a ladder is easily illustrated by representing the generators $a_{p,q}$ as a vertical line from the p th line to the q th line on an n -line stave. Then, for every $k \geq 0$, the letter x_k looks like a bar of a ladder—see Figure 3.

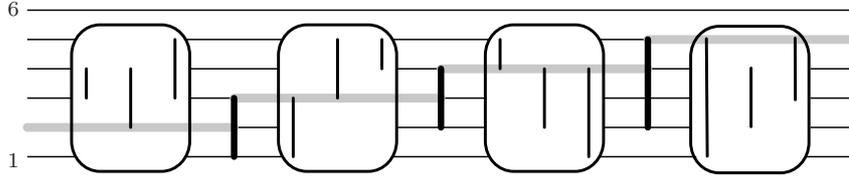


FIGURE 3. The bars of the ladder are represented by black thick vertical lines. An $a_{2,5}$ -ladder lent on $a_{3,5}$ (the last letter). The gray line starts at position 2 and goes up to position 5 using the bars of the ladder. The empty spaces between bars in the ladder are represented by a framed box. In such boxes the vertical line representing the letter $a_{p,q}$ does not cross the gray line.

Proposition 3.6. [12] Assume $n \geq 3$ and that $(\beta_b, \dots, \beta_1)$ is the ϕ_n -splitting of some braid of B_n^{+*} . Then, for each k in $\{b-1, \dots, 3\}$, the normal form of β_k is a $\phi_n(\beta_{k+1}^\#)$ -ladder lent on $\beta_k^\#$. The same results hold for $k = 2$ whenever β_2 is not 1.

A direct consequence of Proposition 5.7 of [12] is

Lemma 3.7. Assume $n \geq 3$ and let β be a braid represented by an $a_{p,n}$ -ladder lent on $a_{q-1,n-1}$ with $q \neq n-1$ and γ^- be an $a_{p,n}$ -dangerous braid. Then $\gamma^- \beta$ is a σ -positive braid of type $a_{q-1,n-1}$.

We are now ready to prove that the non-terminal entries of a ϕ_n -splitting $(\beta_b, \dots, \beta_1)$ have the expected property, namely that the braid β_k provides a protection against a $\phi_n(\beta_{k+1}^\#)$ -dangerous braid, in the sense that if γ_{k+1}^- is a $\beta_{k+1}^\#$ -dangerous braid, then the braid $\phi_n(\gamma_{k+1}^-)\beta_k$ is σ -positive of type $\beta_k^\#$.

Proposition 3.8. *Assume that $(\beta_b, \dots, \beta_1)$ is a ϕ_n -splitting. Then, for each k in $\{b-1, \dots, 3\}$ and every $\beta_{k+1}^\#$ -dangerous braid γ_{k+1}^- , the braid $\phi_n(\gamma_{k+1}^-)\beta_k$ is σ -positive of type $\beta_k^\#$. Moreover $\gamma_{k+1}^- \beta_k$ is different from $a_{n-2, n-1}$, except if β is itself $a_{n-2, n-1}$. The same result holds for $k = 2$, unless β_2 is the trivial braid1.*

Proof. Take k in $\{b-1, \dots, 3\}$. By definition of a dangerous braid, the braid $\phi_n(\gamma_{k+1}^-)$ is $\phi_n(\beta_{k+1}^\#)$ -dangerous. Assume $\beta_k^\# \neq a_{n-2, n-1}$. By Proposition 3.6, the normal form of β_k is a $\phi_n(\beta_{k+1}^\#)$ -ladder lent on $\beta_k^\#$. Then, by Lemma 3.7, the braid $\phi_n(\gamma_{k+1}^-)\beta_k$ is σ -positive of type $\beta_k^\#$.

Assume now $\beta_k^\# = a_{n-2, n-1}$ with $\beta_k \neq a_{n-2, n-1}$. By Proposition 3.6 again, the normal form of β_k is a $\phi_n(\beta_{k+1}^\#)$ -ladder lent on $a_{n-2, n-1}$. Let $w' a_{n-2, n-1}$ be the normal form of β . By definition of a ladder, as the letter $a_{n-2, n-1}$ does not satisfy the condition (i) of Definition 3.5, the word w' is an $a_{p, n}$ -ladder lent on $a_{p, n-1}$ for some p —see Lemma 1.22(iii). We denote by β'_k the braid represented by w' . Then, by Lemma 3.7, the braid $\phi_n(\gamma_{k+1}^-)\beta'_k$ is σ -positive of type $a_{p, n-1}$. Then it is the product $\beta'_k{}^+ \cdot \delta_{p, n} \cdot \beta'_k{}^-$. The relation $\delta_{1, p} \delta_{p, n-1} = \delta_{1, n-1}$ implies that the braid $\phi_n(\gamma_{k+1}^-)\beta'_k$ is equal to

$$\beta'_k{}^+ \delta_{1, p}^{-1} \cdot \delta_{1, n-1} \cdot \beta'_k{}^-,$$

where $\beta'_k{}^+ \delta_{1, p}^{-1}$ is σ_{n-2} -nonnegative and $\beta'_k{}^-$ is $a_{1, n}$ -dangerous. Then $\phi_n(\gamma_{k+1}^-)\beta'_k$ is σ -positive of type $a_{1, n-1}$. Hence $\phi_n(\gamma_{k+1}^-)\beta_k$ is σ -positive of type $a_{n-2, n-1}$.

Assume finally $\beta_k = a_{n-2, n-1}$. As the only $a_{n-2, n-1}$ -dangerous braid is trivial, the braid $\phi_n(\gamma_{k+1}^-)\beta_k$ is equal to $a_{n-2, n-1}$, a σ -positive braid of type $a_{n-2, n-1}$.

The same arguments establish the case $k = 2$ with $\beta_2 \neq 1$. \square

3.3. The Key Lemma. We arrive at our main technical result. It mainly says that, if a braid β of B_n^{+*} has n -breadth b , then the braid $\widehat{\delta}_{n, b-2}^{-1} \cdot \beta$ is either σ -positive or trivial. Actually, the result is stronger: the additional information is first that we can control the type of the quotient above, and second that a similar result holds when we replace the leftmost entry of the ϕ_n -splitting of β with another braid of B_{n-1}^{+*} that resembles it enough. This stronger result, which unfortunately makes the statement more complicated, will be needed in Section 4 for the final induction on the braid index n .

Proposition 3.9. *Assume $n \geq 3$ and that $(\beta_b, \dots, \beta_1)$ is the ϕ_n -splitting of a braid β in B_n^{+*} with $b \geq 3$. Let $a_{q, n}$ be the last letter of $\beta \beta_1^{-1}$. Whenever γ_b is a σ -positive braid of type $\beta_b^\#$, the braid*

$$\widehat{\delta}_{n, b-2}^{-1} \cdot \phi_n^{b-1}(\gamma_b) \cdot \phi_n^{b-2}(\beta_{b-1}) \cdot \dots \cdot \phi_n(\beta_2), \quad (3.2)$$

is trivial or σ -positive of type $a_{q, n}$ —the first case occurring only for $q = 1$.

Proof. Put $\beta^* = \widehat{\delta}_{n, b-2}^{-1} \cdot \phi_n^{b-1}(\gamma_b) \cdot \phi_n^{b-2}(\beta_{b-1}) \cdot \dots \cdot \phi_n(\beta_2)$ and $a_{p-1, n-1} = \beta_b^\#$ (Lemma 1.22). First, we decompose the left fragment $\widehat{\delta}_{n, b-2}^{-1} \cdot \phi_n^{b-1}(\gamma_b)$ of β^* as a product of a σ_{n-1} -nonnegative braid and a dangerous braid. By definition of a

σ -positive braid of type $\beta_b^\#$, we have

$$\gamma_b = \gamma_b^+ \delta_{p-1, n-1} \gamma_b^-,$$

where γ_b^- is an $\beta_b^\#$ -dangerous braid and where γ_b^+ is σ_{n-2} -nonnegative. Using Proposition 2.6(ii)t, we obtain

$$\widehat{\delta}_{n, b-2}^{-1} \cdot \phi_n^{b-1}(\gamma_b) = \delta_{n-1}^{b-2} \delta_n^{-b+2} \phi_n^{b-1}(\gamma_b^+ \delta_{p-1, n-1}) \phi_n^{b-1}(\gamma_b^-). \quad (3.3)$$

By (1.7), we have $\delta_n^{-b+2} \phi_n^{b-1}(\gamma_b^+ \delta_{p-1, n-1}) = \phi_n(\gamma_b^+ \delta_{p-1, n-1}) \delta_n^{-b+2}$. Using the relation $\delta_{p, n} \delta_n^{-1} = \delta_p^{-1}$, an easy consequence of (1.11), we obtain

$$\delta_n^{-b+2} \phi_n^{b-1}(\gamma_b^+ \delta_{p-1, n-1}) = \phi_n(\gamma_b^+) \delta_p^{-1} \delta_n^{-b+3}. \quad (3.4)$$

Substituting (3.4) in (3.3), we find

$$\widehat{\delta}_{n, b-2}^{-1} \cdot \phi_n^{b-1}(\gamma_b) = \delta_{n-1}^{b-2} \phi_n(\gamma_b^+) \delta_p^{-1} \delta_n^{-b+3} \phi_n^{b-1}(\gamma_b^-). \quad (3.5)$$

From there, we deduce that β^* is equal to

$$\delta_{n-1}^{b-2} \phi_n(\gamma_b^+) \delta_p^{-1} \cdot \delta_n^{-b+3} \phi_n^{b-1}(\gamma_b^-) \phi_n^{b-2}(\beta_{b-1}) \dots \phi_n(\beta_2). \quad (3.6)$$

Write $\beta^{**} = \delta_n^{-b+3} \phi_n^{b-1}(\gamma_b^-) \phi_n^{b-2}(\beta_{b-1}) \dots \phi_n(\beta_2)$. Note that the left factor of (3.6), which is $\delta_{n-1}^{b-2} \phi_n(\gamma_b^+) \delta_p^{-1}$, is σ_{n-1} -nonnegative. At this point, four cases may occur.

Case 1: $\beta_2 \notin \{1, a_{n-2, n-1}\}$. By Lemma 5.9 of [12], the braid β^{**} is equal to $\beta'' \phi_n^2(\gamma_3^-) \phi_n(\beta_2) \beta_1$, where β'' is a σ_{n-1} -nonnegative braid and where γ_3^- is a $\beta_3^\#$ -dangerous braid. Put $\beta_2' = \phi_n(\gamma_3^-) \beta_2$. By Proposition 3.8, β_2' is σ -positive of type $\beta_2^\#$ and different from $a_{n-2, n-1}$. We deduce that β^* is equal to

$$\delta_{n-1}^{b-2} \phi_n(\gamma_b') \delta_p^{-1} \beta'' \cdot \phi_n(\beta_2'). \quad (3.7)$$

The left factor of (3.7) is σ_{n-1} -nonnegative, while the right factor, namely $\phi_n(\beta_2')$, is different from $a_{n-1, n}$ and σ -positive of type $\phi_n(\beta_2^\#)$ by Lemma 3.2(ii). As, in this case, the last letter of $\beta \beta_1^{-1}$ is $\phi_n(\beta_2^\#)$, we conclude using Lemma 3.2(iv).

Case 2: $\beta_2 \in \{1, a_{n-2, n-1}\}$, $\beta_3 = \dots = \beta_{k-1} = a_{n-2, n-1}$ and $\beta_k \neq a_{n-2, n-1}$ for some $k \leq b-1$. If β_2 is trivial, then the last letter of $\beta \beta_1^{-1}$ is $a_{1, n}$; otherwise the last letter of $\beta \beta_1^{-1}$ is $\phi_n(a_{n-2, n-1})$, i.e., $a_{n-1, n}$ —a direct consequence of Lemma 1.22. As the product of a σ -positive braid of type $a_{1, n}$ with $a_{n-1, n}$ is a σ -positive braid of type $a_{n-1, n}$, it is enough to prove that the braid β^* is the product of a σ -positive braid of type $a_{1, n}$ with $\phi_n(\beta_2)$. By Lemma 5.9 of [12], the braid β^{**} is equal to

$$\beta'' \delta_n^{-k+2} \phi_n^k(\gamma_{k+1}^-) \phi_n^{k-1}(\beta_k) \phi_n^{k-2}(a_{n-2, n-1}) \dots \phi_n^2(a_{n-2, n-1}) \phi_n(\beta_2), \quad (3.8)$$

with β'' a σ_{n-1} -nonnegative braid and γ_{k+1}^- a $\beta_{k+1}^\#$ -dangerous braid. Proposition 3.8 implies that the braid $\phi_n(\gamma_{k+1}^-) \beta_k$ is σ -positive of type $\beta_k^\#$. By Corollary 3.11 of [12], the last letter of β_k is $a_{n-2, n-1}$. Then, by Lemma 3.2(ii) the braid $\phi_n^2(\gamma_{k+1}^-) \phi_n(\beta_k)$ is σ -positive of type $a_{n-1, n}$. Hence, by definition of a σ -positive braid of type $a_{n-1, n}$, we have the relation

$$\phi_n^2(\gamma_{k+1}^-) \phi_n(\beta_k) = \beta_k' \phi_n(a_{n-2, n-1}), \quad (3.9)$$

where β_k' is a σ -positive braid of type $a_{1, n}$. Substituting (3.9) in (3.8) gives that β^{**} is equal to

$$\beta'' \delta_n^{-k+2} \phi_n^{k-2}(\beta_k') \phi_n^{k-1}(a_{n-2, n-1}) \phi_n^{k-2}(a_{n-2, n-1}) \dots \phi_n^2(a_{n-2, n-1}) \phi_n(\beta_2).$$

Using $\phi_n(a_{n-2,n-1})\delta_n^{-1} = \delta_{n-1}^{-1}$ and (1.7), we obtain that the right factor of (3.6) is

$$\beta'' \beta'_k \delta_{n-1}^{-k+2} \phi_n(\beta_2).$$

As β'_k is a σ -positive braid of type $a_{1,n}$, Lemma 3.2(iii) implies that $\beta'_k \delta_{n-1}^{-k+2}$ is σ -positive of type $a_{1,n}$, and so is $\beta'' \beta'_k \delta_{n-1}^{-k+2}$ by Lemma 3.2(iv). Hence, by (3.6), the braid β^* is the product of a σ -positive braid of type $a_{1,n}$ with $\phi_n(\beta_2)$.

Case 3: $\beta_2 \in \{1, a_{n-2,n-1}\}$, $\beta_3 = \dots = \beta_{b-1} = a_{n-2,n-1}$ and $\gamma_b \neq a_{n-2,n-1}$. As in Case 2, it is enough to prove that the braid β^* is the product of a σ -positive of type $a_{1,n}$ with $\phi_n(\beta_2)$. Using Lemma 2.6(ii) and (1.7) in its definition, the braid β^* is equal to

$$\delta_{n-1}^{b-2} \cdot \phi_n(\gamma_b) \delta_n^{-1} \cdot \phi_n(a_{n-2,n-1}) \delta_n^{-1} \cdot \dots \cdot \phi_n(a_{n-2,n-1}) \delta_n^{-1} \phi_n(\beta_2).$$

By Corollary 3.11 of [12], the last letter of β_b is $a_{n-2,n-1}$, so γ_b is σ -positive of type $a_{n-2,n-1}$. Hence $\phi_n(\gamma_b)$ is σ -positive of type $a_{n-1,n}$ and is different from $a_{n-1,n}$. Then, by definition of a σ -positive braid of type $a_{n-1,n}$, there exists a σ -positive braid β'_b of type $a_{1,n}$ satisfying $\phi_n(\gamma_b) = \beta'_b a_{n-1,n}$. Using $\phi_n(a_{n-2,n-1}) \delta_n^{-1} = \delta_{n-1}^{-1}$, we deduce that the braid β^* is equal to

$$\delta_{n-1}^{b-2} \cdot \beta'_b \delta_{n-1}^{-b+2} \cdot \phi_n(\beta_2). \quad (3.10)$$

By Lemma 3.2(iii), the middle factor of (3.10), namely $\beta'_b \delta_{n-1}^{-b+2}$, is σ -positive of type $a_{1,n}$. Then β^* is the product of a σ -positive braid of type $a_{1,n}$ with $\phi_n(\beta_2)$.

Case 4: $\beta_2 \in \{1, a_{n-2,n-1}\}$, $\beta_3 = \dots = \beta_{b-1} = a_{n-2,n-1}$ and $\gamma_b = a_{n-2,n-1}$. By definition, we have

$$\beta^* = \delta_{n-1}^{b-2} \cdot \phi_n(a_{n-2,n-1}) \delta_n^{-1} \dots \phi_n(a_{n-2,n-1}) \delta_n^{-1} \phi_n(\beta_2). \quad (3.11)$$

Using $\phi_n(a_{n-2,n-1}) \delta_n^{-1} = \delta_{n-1}^{-1}$ once again, we deduce $\beta^* = \phi_n(\beta_2)$. The braid $\phi_n(\beta_2)$ is either trivial or equal to $a_{n-1,n}$, a σ -positive word of type $a_{n-1,n}$, as expected.

So the proof of the Key Lemma is complete. \square

4. PROOF OF THE MAIN RESULT

We are now ready to prove Theorem 2.8, which state that if β, γ are braids of B_n^{+*} , then

$$\beta <^* \gamma \quad \text{implies} \quad \beta < \gamma, \quad (4.1)$$

where $<^*$ refers to the ordering of Definition 2.1 and $<$ refers to the Dehornoy ordering, *i.e.*, $\beta < \gamma$ means that the quotient-braid $\beta^{-1}\gamma$ is σ -positive.

We shall split the argument into three steps. The first step consists in replacing the initial problem that involves two arbitrary braids β, γ with two problems, each of which only involves one braid. To do that, we use the separators $\widehat{\delta}_{n,t}$ of Definition 2.5, and address the problem of comparing one arbitrary braid with the special braids $\widehat{\delta}_{n,t}$. We shall prove that

$$\beta <^* \widehat{\delta}_{n,t} \quad \text{implies} \quad \beta < \widehat{\delta}_{n,t} \quad (4.2)$$

$$\widehat{\delta}_{n,t} \leq^* \beta \quad \text{implies} \quad \widehat{\delta}_{n,t} \leq \beta \quad (4.3)$$

So, essentially, we have three things to do: proving (4.2), proving (4.3), and showing how to deduce the general implication (4.1).

4.1. **Proofs of (4.2) and (4.3).** We begin with the implication (4.2). Actually, we shall prove a stronger result, needed to maintain an inductive argument in the proof of Theorem 2.8.

Proposition 4.1. *For $n \geq 3$, the implication (4.2) is true. Moreover for $t \geq 1$, the relation $\beta <^* \widehat{\delta}_{n,t}$ implies that $\beta^{-1} \widehat{\delta}_{n,t}$ is σ -positive of type $a_{1,n}$.*

Proof. Take β in B_n^{+*} and assume $\beta <^* \widehat{\delta}_{n,t}$ for some $t \geq 0$. Let $(\beta_b, \dots, \beta_1)$ be the ϕ_n -splitting of β . By Proposition 2.6(iii), we necessarily have $b \leq t+1$. If $t = 0$ holds, then the braid β lies in B_{n-1}^{+*} and the quotient $\beta^{-1} \widehat{\delta}_{n,0}$, which is $\beta^{-1} a_{n-1,n}$, is σ -positive. If $t \geq 1$ and $b \leq 1$ hold, then the braid β^{-1} is σ_{n-1} -nonnegative, and as $\widehat{\delta}_{n,t}$ is σ -positive of type $a_{1,n}$, Lemma 3.2 implies that the quotient $\beta^{-1} \widehat{\delta}_{n,t}$ is σ -positive of type $a_{1,n}$. Assume now $t \geq 1$ and $b \geq 2$. Then, by Proposition 2.6(ii), we find

$$\beta^{-1} \widehat{\delta}_{n,t} = \beta^{-1} \delta_n^t \delta_{n-1}^{-t} = \beta_1^{-1} \cdot \phi_n(\beta_2^{-1}) \dots \phi_n^{b-1}(\beta_b^{-1}) \cdot \delta_n^t \cdot \delta_{n-1}^{-t}.$$

Using Relation (1.7), we push $b-1$ factors δ_n to the left and dispatch them between the factors β_k^{-1} :

$$\begin{aligned} \beta^{-1} \widehat{\delta}_{n,t} &= \beta_1^{-1} \phi_n(\beta_2^{-1}) \dots \phi_n^{b-2}(\beta_{b-1}^{-1}) \phi_n^{b-1}(\beta_b^{-1}) \delta_n^{b-1} \delta_n^{t-b+1} \delta_{n-1}^{-t} \\ &= \beta_1^{-1} \phi_n(\beta_2^{-1}) \dots \phi_n^{b-2}(\beta_{b-1}^{-1}) \delta_n^{b-2} \delta_n \beta_b^{-1} \delta_n^{t-b+1} \delta_{n-1}^{-t} \\ &\dots \\ &= \beta_1^{-1} \delta_n \beta_2^{-1} \dots \delta_n \beta_{b-1}^{-1} \delta_n \beta_b^{-1} \delta_n^{t-b+1} \delta_{n-1}^{-t}. \end{aligned}$$

As B_{n-1}^{+*} is a Garside monoid, there exists an integer k such that $\beta_b^{-1} \delta_{n-1}^k$ belongs to B_{n-1}^{+*} . Call the latter braid β'_b . Thus, we have $\delta_n \beta_b^{-1} = \delta_n \beta'_b \delta_{n-1}^{-k}$. Relation (1.7) implies $\delta_n \beta'_b \delta_{n-1}^{-k} = \phi_n(\beta'_b) \delta_n \delta_{n-1}^{-k}$. Then the braid $\beta^{-1} \widehat{\delta}_{n,t}$ is equal to

$$\beta_1^{-1} \delta_n \beta_2^{-1} \dots \delta_n \beta_{b-1}^{-1} \phi_n(\beta'_b) \cdot \delta_n \delta_{n-1}^{-k} \delta_n^{t-b+1} \delta_{n-1}^{-t}. \quad (4.4)$$

Each braid β_k belongs to B_{n-1}^{+*} , and so its inverse β_k^{-1} does not involve the n th strand. Hence the left factor of (4.4), namely $\beta_1^{-1} \delta_n \beta_2^{-1} \dots \delta_n \beta_{b-1}^{-1} \phi_n(\beta'_b)$, is σ_{n-1} -nonnegative. If $b = t+1$ holds, the right factor of (4.4), namely $\delta_n \delta_{n-1}^{-k} \delta_n^{t-b+1} \delta_{n-1}^{-t}$, is equal to the braid $\delta_n \cdot \delta_{n-1}^{-t-k}$, which shows it is a σ -positive braid of type $a_{1,n}$. If $b \leq t$ holds, (4.4) ends with $\delta_n \delta_{n-1}^{-t}$, which is a σ -positive braid of type $a_{1,n}$, and the factor $\delta_n \delta_{n-1}^{-k} \delta_n^{t-b}$ is σ_{n-1} -nonnegative. In each case, we conclude using Lemma 3.2(iii) that $\beta^{-1} \widehat{\delta}_{n,t}$ is σ -positive of type $a_{1,n}$. \square

Using the Key Lemma of Section 3, *i.e.*, Proposition 3.9, we now establish the implication (4.3).

Proposition 4.2. *For $n \geq 3$, the implication (4.3) is true.*

Proof. Take β in B_n^{+*} and assume $\widehat{\delta}_{n,t} \leq^* \beta$. Let $(\beta_b, \dots, \beta_1)$ be the ϕ_n -splitting of β . By definition of $<^*$, the relation $\widehat{\delta}_{n,t} \leq^* \beta$ implies $t \leq b-2$. Then $\widehat{\delta}_{n,t}^{-1} \beta$ is equal to

$$\widehat{\delta}_{n,t}^{-1} \widehat{\delta}_{n,b-2} \cdot \widehat{\delta}_{n,b-2}^{-1} \beta. \quad (4.5)$$

By Lemma 2.7, the factor $\widehat{\delta}_{n,t}^{-1} \widehat{\delta}_{n,b-2}$ of (4.5) is σ -positive or trivial. By Lemma 3.4, the braid β_b is σ -positive of type $\beta_b^\#$. Then, Proposition 3.9 guarantees that the right factor of (4.5), namely $\widehat{\delta}_{n,b-2}^{-1} \beta$, is σ -positive or trivial. \square

4.2. Proof of Theorem 2.8. At this point, we know that the implications (4.2) and (4.3) are true. It is not hard to deduce that the implication (4.1), which is our goal, is true when the breadth of β is smaller than the breadth of γ , *i.e.*, in the “Short”-case of the ShortLex-ordering.

So, there remains to treat the “Lex”-case, *i.e.*, the case when β and γ have the same n -breadth, and this is what we do now. Actually, as was already mentioned, in order to maintain an induction hypothesis, we shall prove a stronger implication: instead of merely proving that the quotient braid $\beta^{-1}\gamma$ is σ -positive, we shall prove the more precise conclusion that $\beta^{-1}\gamma$ is σ -positive of type $a_{p,n}$ for some p related with the last letter of γ . That is why we shall consider the “Short”- and the “Lex”-cases simultaneously.

Proposition 4.3. *If β and γ are nontrivial braids of B_n^{+*} , the relation $\beta <^* \gamma$ implies $\beta < \gamma$. Moreover, if the B_{n-1}^{+*} -tail of γ is trivial, then the braid $\beta^{-1}\gamma$ is σ -positive of type $\gamma^\#$.*

Proof. We use induction on n . For $n = 2$, everything is obvious, as both $<$ and $<^*$ coincide with the standard ordering of natural numbers.

Assume $n \geq 3$, and $\beta <^* \gamma$ where β, γ belong to B_n^{+*} and $\beta \neq 1$ holds. Then $\gamma \neq 1$ holds as well. Let $(\beta_b, \dots, \beta_1)$ and $(\gamma_c, \dots, \gamma_1)$ be the ϕ_n -splittings of β and γ . As $\beta <^* \gamma$ holds, we have $b \leq c$. Write $\beta_c = \dots = \beta_{b+1} = 1$. Let t be the maximal integer in $\{1, \dots, c\}$ satisfying $\beta_t <^* \gamma_t$. By definition of $<^*$, such a t exists. Write $\gamma'_t = \beta_t^{-1}\gamma_t$. By induction hypothesis, the braid γ'_t is σ -positive. Moreover, if $t \geq 2$ holds, then the braid γ'_t is σ -positive of type $\gamma_t^\#$.

Assume $t = 1$. Then the braid $\beta^{-1}\gamma$ is equal to γ'_t . Hence, it is σ -positive. As the B_{n-1}^{+*} -tail of γ_1 is non-trivial, we have nothing more to prove.

Assume now $t \geq 2$. Let $a_{q,n}$ be the last letter of $\phi_n^{t-1}(\gamma_t) \cdot \dots \cdot \phi_n(\beta_2)$. The sequence $(\beta_{t-1}, \dots, \beta_1)$ is a ϕ_n -splitting of a braid of n -breadth $t-1$. Then Proposition 4.1 implies that the braid β' , that is equal to

$$\beta_1^{-1} \cdot \dots \cdot \phi_n^{t+1}(\beta_{t-1}^{-1}) \cdot \widehat{\delta}_{n,t-2},$$

is σ -positive of type $a_{1,n}$. Let γ' be the braid

$$\widehat{\delta}_{n,t-2}^{-1} \cdot \phi_n^{t-1}(\gamma'_t) \cdot \phi_n^{t-2}(\gamma_{t-1}) \cdot \dots \cdot \phi_n(\gamma_2).$$

As γ'_t is σ -positive of type $\gamma_t^\#$, Proposition 3.9 implies that the braid γ' is σ -positive of type $a_{q,n}$ or trivial (the latter occurs only for $q = 1$). Then, in any case, the braid $\beta' \gamma'$ is σ -positive of type $a_{q,n}$. As, by construction, we have $\beta^{-1}\gamma = \beta' \cdot \gamma' \cdot \gamma_1$, the braid $\beta^{-1}\gamma$ is σ -positive. Moreover, assume that the B_{n-1}^{+*} -tail of γ is trivial. Then γ_1 is trivial and the braid γ ends with $a_{q,n}$. In this case we have $\beta^{-1}\gamma = \beta' \cdot \gamma'$, a σ -positive braid of type $a_{q,n-1}$. Therefore $\beta^{-1}\gamma$ is a σ -positive braid of type $\gamma^\#$. \square

Our proof of Proposition 4.3 is therefore complete, and so is the proof of Theorem 2.8, which is strictly included in the latter.

So, we now have a complete description of the restriction of the Dehornoy ordering of braids to the Birman–Ko–Lee monoid B_n^{+*} . The characterization of Theorem 1 is inductive, connecting the ordering on B_n^{+*} to the ordering on B_{n-1}^{+*} . Actually, it is very easy to obtain a non-inductive formulation. Indeed, we can define the *iterated splitting* $T(\beta)$ of a braid β of B_n^{+*} to be the tree obtained by substituting the ϕ_{n-1} -splittings of the entries in the ϕ_n -splitting, and iterating with the ϕ_{n-2} -splittings, and so on until we reach B_2^{+*} , *i.e.*, the natural numbers. In this way, we

associate with every braid β of B_n^{+*} a tree $T(\beta)$ with branches of length $n-2$ and natural numbers labeling the leaves—see [9] for an analogous construction. Then Theorem 1 immediately implies that, for β, γ in B_n^{+*} , the relation $\beta < \gamma$ holds if and only if the tree $T(\beta)$ is **ShortLex**-smaller than the tree $T(\gamma)$.

Remark 4.4. Whether the tools developed in [12] and in the current may be adapted to the case of B_n^+ is an open question. The starting point of our approach is very similar to that of [9] and [4]. However, it seems that the machinery of ladders and dangerous braids involved in the technical results of Section 3 are really specific to the case of the dual monoids, and heavily depend on the highly redundant character of the relations connecting the Birman–Ko–Lee generators.

By contrast, a much more promising approach would be to investigate the restriction of the finite Thurston-type braid orderings of [17] to the monoids B_n^{+*} along the lines developed in [14]. In particular, it should be possible to determine the isomorphism type explicitly.

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LABORATOIRE DE MATHÉMATIQUES NICOLAS ORESME, UMR 6139 CNRS, UNIVERSITÉ DE CAEN
 BP 5186, 14032 CAEN, FRANCE
E-mail address: jean.fromentin@math.unicaen.fr