

ORTHOGONAL BUNDLES OVER CURVES IN CHARACTERISTIC TWO

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ABSTRACT. Let X be a smooth projective curve of genus $g \geq 2$ defined over a field of characteristic two. We give examples of stable orthogonal bundles with unstable underlying vector bundles and use them to give counterexamples to Behrend's conjecture on the canonical reduction of principal G -bundles for $G = \mathrm{SO}(n)$ with $n \geq 7$.

Let X be a smooth projective curve of genus $g \geq 2$ and let G be a connected reductive linear algebraic group defined over a field k of arbitrary characteristic. One associates to any principal G -bundle E_G over X a reduction E_P of E_G to a parabolic subgroup $P \subset G$, the so-called *canonical reduction* — see e.g. [Ra], [Be], [BH] or [H] for its definition. We only mention here that in the case $G = \mathrm{GL}(n)$ the canonical reduction coincides with the Harder-Narasimhan filtration of the rank- n vector bundle associated to E_G .

In [Be] (Conjecture 7.6) K. Behrend conjectured that for any principal G -bundle E_G over X the canonical reduction E_P has no infinitesimal deformations, or equivalently, that the vector space $H^0(X, E_P \times^P \mathfrak{g}/\mathfrak{p})$ is zero. Here \mathfrak{p} and \mathfrak{g} are the Lie algebras of P and G respectively. Behrend's conjecture implies that the canonical reduction E_P is defined over the same base field as E_G .

We note that this conjecture holds for the structure groups $\mathrm{GL}(n)$ and $\mathrm{Sp}(2n)$ in any characteristic, and also for $\mathrm{SO}(n)$ in any characteristic different from two — see [H] section 2. On the other hand, a counterexample to Behrend's conjecture for the exceptional group G_2 in characteristic two has been constructed recently by J. Heinloth in [H] section 5.

In this note we focus on $\mathrm{SO}(n)$ -bundles in characteristic two. As a starting point we consider the rank-2 vector bundle F_*L given by the direct image under the Frobenius map F of a line bundle L over the curve X and observe (Proposition 4.4) that the $\mathrm{SO}(3)$ -bundle $\mathcal{A} := \mathrm{End}_0(F_*L)$ is stable, but that its underlying vector bundle is unstable and, in particular, destabilized by the rank-2 vector bundle $F_*\mathcal{O}_X$. We use this observation to show that the $\mathrm{SO}(7)$ -bundle

$$F_*\mathcal{O}_X \oplus (F_*\mathcal{O}_X)^* \oplus \mathcal{A}$$

equipped with the natural quadratic form gives a counterexample to Behrend's conjecture. Replacing \mathcal{A} by $\hat{\mathcal{A}} = \mathrm{End}(F_*L)$ we obtain in the same way a counterexample for $\mathrm{SO}(8)$, and more generally for $\mathrm{SO}(n)$ with $n \geq 7$ after adding direct summands of hyperbolic planes. Note that Behrend's conjecture holds for $\mathrm{SO}(n)$ with $n \leq 6$ because of the exceptional isomorphisms with other classical groups.

The first three sections are quite elementary and recall well-known facts on quadratic forms, orthogonal groups and their Lie algebras, as well as orthogonal bundles in characteristic two. In the last section we give an example (Proposition 6.1) of an unstable $\mathrm{SO}(7)$ -bundle having its canonical reduction only defined after an inseparable extension of the base field.

This note can be considered as an appendix to the recent paper [H] by J. Heinloth, whom I would like to thank for useful discussions.

1. QUADRATIC FORMS IN CHARACTERISTIC TWO

The purpose of this section is to recall definitions and basic properties of quadratic forms in characteristic two.

Let k be an algebraically closed field of characteristic two and let V be a vector space of dimension n over k . We denote by Sym^2V and Λ^2V the symmetric and exterior square of V and by $F(V) := V \otimes_k k$ the Frobenius-twist of V . We observe that there exists a well-defined k -linear injective map

$$F(V) \hookrightarrow \text{Sym}^2V \quad v \otimes \lambda \mapsto \lambda(v \cdot v),$$

where $v \cdot v$ denotes the symmetric square of v . More precisely, we have the following exact sequence of k -vector spaces

$$0 \longrightarrow F(V) \longrightarrow \text{Sym}^2V \longrightarrow \Lambda^2V \longrightarrow 0.$$

Let σ denote the involution on $V \otimes V$ defined by $\sigma(v \otimes w) = w \otimes v$ and let $\mathbb{S}_2(V) \subset V \otimes V$ denote the σ -invariant subspace. Then one has the exact sequence

$$0 \longrightarrow \mathbb{S}_2(V) \longrightarrow V \otimes V \longrightarrow \Lambda^2V \longrightarrow 0,$$

where the last arrow denotes the canonical projection of $V \otimes V$ onto Λ^2V . By choosing a basis of V , it can be checked that the image of $\mathbb{S}_2(V)$ under the canonical projection $V \otimes V \rightarrow \text{Sym}^2V$ equals the subspace $F(V)$.

By definition a quadratic form on the vector space V is an element $Q \in \text{Sym}^2(V^*)$. Equivalently, see e.g. [Bo] section 23.5, a quadratic form can be seen as a k -valued function on V such that

$$Q(av + bw) = a^2Q(v) + b^2Q(w) + ab\beta(v, w) \quad a, b \in k; v, w \in V,$$

where β is a bilinear form on V . The bilinear form β associated to a quadratic form Q is determined by the formula

$$\beta(v, w) = Q(v + w) + Q(v) + Q(w).$$

It is clear that β is symmetric, hence β can be regarded as an element in $\mathbb{S}_2(V^*)$. The assignment $Q \mapsto \beta$ gives rise to the k -linear *polarisation map*

$$\mathcal{P} : \text{Sym}^2(V^*) \longrightarrow \mathbb{S}_2(V^*), \quad Q \mapsto \beta.$$

Over a field k of characteristic different from 2 we recall that the polarisation map \mathcal{P} is an isomorphism. In our situation one can easily work out that

$$\ker \mathcal{P} = F(V^*) \quad \text{and} \quad \text{coker } \mathcal{P} = F(V^*).$$

Quadratic forms $Q \in \ker \mathcal{P}$ correspond to squares of linear forms on V and the last equality asserts that the bilinear form β is alternating, i.e., $\beta(v, v) = 0$ for all $v \in V$.

We say that Q is *non-degenerate* if β induces an isomorphism $\tilde{\beta} : V \rightarrow V^*$ for n even, and if $\dim \ker \tilde{\beta} = 1$ and $Q|_{\ker \tilde{\beta}} \neq 0$ for n odd. We say that a linear subspace $W \subset V$ is *isotropic* for Q if $Q|_W = 0$.

2. ORTHOGONAL GROUPS AND THEIR LIE ALGEBRAS IN CHARACTERISTIC TWO

In this section we work out in detail the structure of the Lie algebras of the orthogonal groups $\text{SO}(7)$ and $\text{SO}(8)$ over a field k of characteristic two, as well as of some of their parabolic subgroups $P \subset \text{SO}(n)$. Following the principle of [H], we describe the representation $\mathfrak{so}(n)/\mathfrak{p}$ of the Levi subgroup $L \subset P$, which we will use later in section 5. The main reference is [Bo] section 23.6.

By definition the orthogonal group $\text{O}(n)$ is the subgroup of $\text{GL}(n) = \text{GL}(V)$ stabilizing a non-degenerate quadratic form $Q \in \text{Sym}^2(V^*)$. Note that in characteristic two $\text{O}(n) \subset \text{SL}(n)$. We also recall that, if n is odd, the group $\text{O}(n)$ is connected. If n is even, $\text{O}(n)$ has two connected components distinguished by the Dickson invariant ([D] page 301). In order to keep the same

notation as in characteristic $\neq 2$, we denote by $\mathrm{SO}(n)$ the connected component of $\mathrm{O}(n)$ containing the identity.

2.1. The group $\mathrm{SO}(8)$. We consider $V = k^8$ endowed with the non-degenerate quadratic form

$$Q = x_1x_7 + x_2x_8 + x_3x_5 + x_4x_6 \in \mathrm{Sym}^2(V^*).$$

We choose this non-standard quadratic form in order to obtain a simple description of the Lie algebra \mathfrak{p} of the parabolic subgroup P fixing an isotropic 2-plane, which will appear in section 5. The bilinear form β associated to Q is given by the matrix

$$\begin{pmatrix} O & O & O & I \\ O & O & I & O \\ O & I & O & O \\ I & O & O & O \end{pmatrix},$$

where I denotes the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and O the zero 2×2 matrix. A straightforward computation shows that the Lie algebra $\mathfrak{so}(8) \subset \mathrm{End}(V)$ of $\mathrm{SO}(8)$ consists of the matrices of the form

$$(1) \quad \begin{pmatrix} X_1 & X_2 & X_3 & D_1 \\ X_4 & X_5 & D_2 & {}^tX_3 \\ X_6 & D_3 & {}^tX_5 & {}^tX_2 \\ D_4 & {}^tX_6 & {}^tX_4 & {}^tX_1 \end{pmatrix},$$

where the X_i are 2×2 matrices, the tX_i denote their transpose, and the D_i denote matrices of the form $\begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}$.

We consider the maximal parabolic subgroup $P \subset \mathrm{SO}(8)$ preserving the isotropic 2-plane $W \subset V = k^8$ defined by the equations $x_i = 0$ for $i = 3, \dots, 8$. Its orthogonal space W^\perp is then defined by the equations $x_7 = x_8 = 0$. The Levi subgroup $L \subset P$ is isomorphic to $\mathrm{GL}(2) \times \mathrm{SO}(4)$. The Lie algebra \mathfrak{p} of P consists of the matrices (1) that satisfy the conditions

$$X_4 = X_6 = 0 \quad \text{and} \quad D_4 = 0.$$

The vector space $\mathfrak{so}(8)/\mathfrak{p}$ sits naturally in the exact sequence

$$(2) \quad 0 \longrightarrow \mathrm{Hom}(W, W^\perp/W) \longrightarrow \mathfrak{so}(8)/\mathfrak{p} \longrightarrow \mathcal{D} \longrightarrow 0,$$

where \mathcal{D} is the one-dimensional subspace generated by the element $e \otimes f + f \otimes e \in W^* \otimes W^* = \mathrm{Hom}(W, W^*) \cong \mathrm{Hom}(W, V/W^\perp)$, where e and f are linearly independent.

2.2. The group $\mathrm{SO}(7)$. We consider $V = k^7$ endowed with the non-degenerate quadratic form

$$Q = x_1x_6 + x_2x_7 + x_3x_4 + x_5^2 \in \mathrm{Sym}^2(V^*).$$

The bilinear form β associated to Q is given by the 7×7 matrix

$$\begin{pmatrix} O & O & 0 & I \\ O & I & 0 & O \\ 0 & 0 & 0 & 0 \\ I & O & 0 & O \end{pmatrix}.$$

The radical $\ker \tilde{\beta}$ of Q is generated by e_5 . A straightforward computation shows that the Lie algebra $\mathfrak{so}(7) \subset \mathrm{End}(V)$ of $\mathrm{SO}(7)$ consists of the matrices of the form

$$(3) \quad \begin{pmatrix} X_1 & X_2 & 0 & D_1 \\ X_3 & \Delta & 0 & {}^tX_2 \\ x_1 & x_2 & 0 & x_3 \\ D_2 & {}^tX_3 & 0 & {}^tX_1 \end{pmatrix},$$

where the x_i are 2×1 matrices and Δ is a diagonal matrix of the form $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$. We consider the maximal parabolic subgroup $P \subset \mathrm{SO}(7)$ preserving the isotropic 2-plane $W \subset V = k^7$ defined by the equations $x_i = 0$ for $i = 3, \dots, 7$. Its orthogonal space W^\perp is then defined by the equations $x_6 = x_7 = 0$. The Levi subgroup $L \subset P$ is isomorphic to $\mathrm{GL}(2) \times \mathrm{SO}(3)$. The Lie algebra \mathfrak{p} of P consists of the matrices (3) that satisfy the conditions

$$X_3 = 0, \quad x_1 = 0, \quad \text{and} \quad D_2 = 0.$$

As above we have an exact sequence

$$(4) \quad 0 \longrightarrow \mathrm{Hom}(W, W^\perp/W) \longrightarrow \mathfrak{so}(7)/\mathfrak{p} \longrightarrow \mathcal{D} \longrightarrow 0.$$

3. ORTHOGONAL BUNDLES

By definition an *orthogonal bundle* over a smooth projective curve X defined over an algebraically closed field k is a principal $\mathrm{O}(n)$ -bundle over X , which we denote by $E_{\mathrm{O}(n)}$. Equivalently a $\mathrm{O}(n)$ -bundle corresponds to a pair (\mathcal{E}, Q) , where \mathcal{E} is the underlying rank- n vector bundle and Q is a non-degenerate \mathcal{O}_X -valued quadratic form on \mathcal{E} , i.e., a non-degenerate element $Q \in H^0(X, \mathrm{Sym}^2(\mathcal{E}^*))$. By abuse of notation we will write $E_{\mathrm{O}(n)} = (\mathcal{E}, Q)$.

One observes (see e.g. [Ram] Definition 4.1) that the notion of (semi-)stability of the $\mathrm{O}(n)$ -bundle $E_{\mathrm{O}(n)}$ translates into the following condition on the pair (\mathcal{E}, Q) : we say that (\mathcal{E}, Q) is semi-stable (resp. stable) as an orthogonal bundle if and only if

$$\mu(\mathcal{F}) := \frac{\deg \mathcal{F}}{\mathrm{rk} \mathcal{F}} \leq 0 \quad (\text{resp. } < 0)$$

for all isotropic subbundles $\mathcal{F} \subset \mathcal{E}$.

A noteworthy result is the following

Proposition 3.1 ([Ram] Proposition 4.2). *Assume that the characteristic of k is different from two. The pair (\mathcal{E}, Q) is semi-stable as an orthogonal bundle if and only if the underlying vector bundle \mathcal{E} is semi-stable.*

In the next section we will show that the assumption on the characteristic can not be removed.

4. STABLE ORTHOGONAL BUNDLES WITH UNSTABLE UNDERLYING BUNDLE

We denote by $F : X \rightarrow X$ the absolute Frobenius of the curve X of genus $g \geq 2$ defined over an algebraically closed field k of characteristic two. We denote by K_X the canonical line bundle of X .

4.1. An orthogonal rank-3 bundle. Let E be a rank-2 vector bundle of arbitrary degree over X . The determinant endows the bundle $\mathrm{End}_0(E)$ of traceless endomorphisms of E with an \mathcal{O}_X -valued non-degenerate quadratic form

$$\det : \mathrm{End}_0(E) \longrightarrow \mathcal{O}_X.$$

Its associated bilinear form β is given by $\beta(v, w) = \mathrm{Tr}(v \circ w)$ for u, v local sections of $\mathrm{End}_0(E)$ and the radical of \det equals the line subbundle $\mathcal{O}_X \hookrightarrow \mathrm{End}_0(E)$ of homotheties.

Lemma 4.1. *There is an exact sequence*

$$(5) \quad 0 \longrightarrow \mathcal{O}_X \longrightarrow \mathrm{End}_0(E) \longrightarrow F^*(E) \otimes (\det E)^{-1} \longrightarrow 0,$$

where the first homomorphism is the inclusion of homotheties into $\mathrm{End}_0(E)$.

Proof. Consider $V = k^2$ and $W = \text{End}_0(V)$ equipped with the determinant. Note that for any $g \in \text{GL}(V)$ the conjugation $C_g : W \rightarrow W$ is orthogonal and leaves the identity Id invariant. Thus we obtain group homomorphisms $\text{GL}(V) \rightarrow \text{SO}(W) \rightarrow \text{GL}(W/\langle \text{Id} \rangle)$, $g \mapsto C_g \mapsto \overline{C}_g$. A straightforward computation shows that the composite map is given by $g \mapsto \frac{1}{\det g} F(g)$, where $F(g)$ is the Frobenius-twist of g . \square

Proposition 4.2. *If E is stable, then $(\text{End}_0(E), \det)$ is stable as an orthogonal bundle.*

Proof. It suffices to observe that an isotropic line subbundle M of $\text{End}_0(E)$ corresponds to a nonzero homomorphism $\phi : E \rightarrow E \otimes M^{-1}$ of rank 1. If L denotes the line bundle $\text{im} \phi \subset E \otimes M^{-1}$, we obtain the inequalities $\mu(E) < \deg L < \mu(E \otimes M^{-1})$, which implies $\deg M < 0$. \square

Let B denote the sheaf of locally exact differentials [Ray], which can be defined as the cokernel

$$(6) \quad 0 \longrightarrow \mathcal{O}_X \longrightarrow F_*\mathcal{O}_X \longrightarrow B \longrightarrow 0,$$

of the inclusion $\mathcal{O}_X \hookrightarrow F_*\mathcal{O}_X$ given by the Frobenius map. Note that in characteristic 2 the sheaf B is a theta-characteristic, i.e., $B^2 = K_X$. We denote by $e \in \text{Ext}^1(B, \mathcal{O}_X) = H^1(X, B^{-1})$ the non-zero extension class (unique up to a scalar) determined by the exact sequence (6). The next lemma immediately follows from the definition of e (see also [Ray]).

Lemma 4.3. *The extension class e generates the one-dimensional kernel of the Frobenius map, i.e.,*

$$\langle e \rangle = \ker (F : H^1(X, B^{-1}) \longrightarrow H^1(X, K_X^{-1})).$$

In particular, we have a direct sum

$$F^*(F_*\mathcal{O}_X) = \mathcal{O}_X \oplus K_X.$$

Let L be a line bundle of arbitrary degree. Then the rank-2 vector bundle F_*L has determinant equal to $B \otimes L$, is stable and is destabilized by pull-back by the Frobenius map F ([LP]). More precisely, the bundle $F^*(F_*L) \otimes B^{-1} \otimes L^{-1}$ is the unique non-split extension (see e.g. [LS])

$$(7) \quad 0 \longrightarrow B \xrightarrow{\iota} F^*(F_*L) \otimes B^{-1} \otimes L^{-1} \xrightarrow{\pi} B^{-1} \longrightarrow 0.$$

Proposition 4.4. (i) *The vector bundle $\mathcal{A} := \text{End}_0(F_*L)$ does not depend on L .*

(ii) *The vector bundle \mathcal{A} is the unique non-split extension*

$$0 \longrightarrow F_*\mathcal{O}_X \xrightarrow{\phi} \mathcal{A} \xrightarrow{\psi} B^{-1} \longrightarrow 0.$$

We denote by ϕ and ψ generators of the one-dimensional spaces $\text{Hom}(F_\mathcal{O}_X, \mathcal{A})$ and $\text{Hom}(\mathcal{A}, B^{-1})$.*

(iii) *The restriction of the quadratic form \det to the subbundle $F_*\mathcal{O}_X$ equals the evaluation morphism $F^*F_*\mathcal{O}_X \rightarrow \mathcal{O}_X$. In particular the restriction of β to $F_*\mathcal{O}_X$ is identically zero.*

(iv) *Let x and y be local sections of $F_*\mathcal{O}_X$ and \mathcal{A} . Then the two bilinear forms $\beta(\phi(x), y)$ and $\langle x, \psi(y) \rangle$ on the product $F_*\mathcal{O}_X \times \mathcal{A}$ differ by a non-zero multiplicative scalar. Here $\langle \cdot, \cdot \rangle$ denotes the standard pairing between $F_*\mathcal{O}_X$ and its dual $(F_*\mathcal{O}_X)^*$.*

Proof. First we consider the exact sequence (5) for the bundle F_*L

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \text{End}_0(F_*L) \longrightarrow F^*(F_*L) \otimes B^{-1} \otimes L^{-1} \longrightarrow 0,$$

and call its extension class f . In order to show that there exists an inclusion $F_*\mathcal{O}_X \hookrightarrow \text{End}_0(F_*L)$, it is enough to show that f pulls-back to e under the inclusion $\iota : B \hookrightarrow F^*(F_*L) \otimes B^{-1} \otimes L^{-1}$. Since by Lemma 4.3 the extension class e is characterized by the equality $F(e) = 0$ and since pull-back commutes with Frobenius, it suffices to show that the extension

$$F^*\text{End}_0(F_*L) = \text{End}_0(F^*F_*L) = \text{End}_0(F^*(F_*L) \otimes B^{-1} \otimes L^{-1})$$

contains the line subbundle $K_X = F^*B$ and that $\iota^*(f) \neq 0$. Since $F^*(F_*L) \otimes B^{-1} \otimes L^{-1}$ is the non-split extension (7), we obtain a sheaf inclusion

$$(8) \quad K_X = \text{Hom}(B^{-1}, B) \longrightarrow \text{End}_0(F^*F_*L), \quad \phi \mapsto \iota \circ \phi \circ \pi.$$

Using a degree argument, one shows that there is an exact sequence

$$(9) \quad 0 \longrightarrow \mathcal{O}_X \oplus K_X \longrightarrow \text{End}_0(F^*F_*L) \longrightarrow \text{Hom}(B, B^{-1}) = K_X^{-1} \longrightarrow 0.$$

It remains to check that $\iota^*(f) \neq 0$. Suppose on the contrary that $\iota^*(f) = 0$. Then B is a line subbundle of $\text{End}_0(F_*L)$, hence we obtain a non-zero homomorphism $F_*L \rightarrow (F_*L) \otimes B^{-1}$. But F_*L is stable and $\mu(F_*L) > \mu(F_*L \otimes B^{-1}) = \mu(F_*L) + 1 - g$, which implies $\text{Hom}(F_*L, (F_*L) \otimes B^{-1}) = 0$, a contradiction. Thus we have shown that $\text{End}_0(F_*L)$ fits into the exact sequence

$$0 \longrightarrow F_*\mathcal{O}_X \longrightarrow \text{End}_0(F_*L) \longrightarrow B^{-1} \longrightarrow 0.$$

We denote by a its extension class in $\text{Ext}^1(B^{-1}, F_*\mathcal{O}_X) = H^1(X, (F_*\mathcal{O}_X) \otimes B)$. We have $(F_*\mathcal{O}_X) \otimes B = (F_*\mathcal{O}_X)^* \otimes K_X$ and by Serre duality $H^1(X, (F_*\mathcal{O}_X) \otimes B) = H^0(X, (F_*\mathcal{O}_X)^*) = H^0(X, \mathcal{O}_X)^*$, which implies that $\dim \text{Ext}^1(B^{-1}, F_*\mathcal{O}_X) = 1$. Hence there exists (up to isomorphism) a unique non-split extension of B^{-1} by $F_*\mathcal{O}_X$. In order to show assertions (i) and (ii) it will be enough to check that $a \neq 0$. But the push-out of a under the map $F_*\mathcal{O}_X \rightarrow B$ gives the non-split extension of B^{-1} by B , which proves the claim. The proof of the equalities $\dim \text{Hom}(F_*\mathcal{O}_X, \mathcal{A}) = \dim \text{Hom}(\mathcal{A}, B^{-1}) = 1$ is standard. Note that (9) is the pull-back under F of the exact sequence in (ii).

The restriction of the bilinear form β to $F_*\mathcal{O}_X$ is identically zero, since β factorizes through the line bundle quotient B and β is alternating (see section 1). Hence the restriction of the quadratic form \det lies in $F^*(F_*\mathcal{O}_X)^* \subset \text{Sym}^2(F_*\mathcal{O}_X)^*$, i.e., corresponds to an \mathcal{O}_X -linear map $\alpha : F^*F_*\mathcal{O}_X = \mathcal{O}_X \oplus K_X \rightarrow \mathcal{O}_X$. We note that $\alpha|_{\mathcal{O}_X} = \text{id}$ and $\alpha|_{K_X} = 0$, since the summand K_X corresponds to nilpotent endomorphisms — see (8). This proves (iii).

Finally we observe that the bilinear form $(x, y) \mapsto \beta(\phi(x), y)$ factorizes through the quotient $B \times B^{-1}$, since β is identically zero on $F_*\mathcal{O}_X$ by part (iii) and since the radical of β equals \mathcal{O}_X . On the other hand it is easily seen that the bilinear form $(x, y) \mapsto \langle x, \psi(y) \rangle$ also factorizes through $B \times B^{-1}$, which proves (iv). \square

Since $F_*\mathcal{O}_X$ is stable and $\mu(F_*\mathcal{O}_X) = \frac{g-1}{2} > 0$, we deduce that the Harder-Narasimhan filtration of \mathcal{A} is

$$0 \subset F_*\mathcal{O}_X \subset \mathcal{A}.$$

However, since F_*L is stable, Proposition 4.2 implies that the orthogonal bundle (\mathcal{A}, \det) is stable.

Remark. Similarly, it can be shown that the rank-4 vector bundle $\hat{\mathcal{A}} := \text{End}(F_*L)$ does not depend on the line bundle L , that it fits into the exact sequence

$$0 \longrightarrow F_*\mathcal{O}_X \longrightarrow \text{End}(F_*L) \longrightarrow (F_*\mathcal{O}_X)^* \longrightarrow 0,$$

giving its Harder-Narasimhan filtration, and that $(\hat{\mathcal{A}}, \det)$ is a stable orthogonal bundle.

4.2. **An orthogonal rank-4 bundle.** Besides the example given in the last remark of section 3.1, we can consider the unstable decomposable rank-4 vector bundle

$$\mathcal{B} = F_*\mathcal{O}_X \oplus (F_*\mathcal{O}_X)^*$$

and endow it with the \mathcal{O}_X -valued quadratic form

$$Q(x + x^*) = q(x) + q^*(x^*) + \langle x, x^* \rangle,$$

where x, x^* are local sections of the rank-2 bundles $F_*\mathcal{O}_X$ and $(F_*\mathcal{O}_X)^*$ respectively. The bracket $\langle x, x^* \rangle$ denotes the standard pairing between $F_*\mathcal{O}_X$ and its dual $(F_*\mathcal{O}_X)^*$ and q, q^* are projections onto \mathcal{O}_X of the direct sums (see Lemma 4.3)

$$(10) \quad F^*(F_*\mathcal{O}_X) = \mathcal{O}_X \oplus K_X, \quad F^*(F_*\mathcal{O}_X)^* = \mathcal{O}_X \oplus K_X^{-1}.$$

It is clear that Q is non-degenerate and that Q restricts to q and q^* on the direct summands $F_*\mathcal{O}_X$ and $(F_*\mathcal{O}_X)^*$ respectively.

Proposition 4.5. *The orthogonal rank-4 bundle (\mathcal{B}, Q) is stable.*

Proof. Let M be an isotropic line subbundle of $F_*\mathcal{O}_X \oplus (F_*\mathcal{O}_X)^*$ and assume that $\deg M \geq 0$. By stability of $(F_*\mathcal{O}_X)^*$ the line bundle M is contained in $F_*\mathcal{O}_X$, which gives by adjunction a nonzero map $F^*M \rightarrow F^*F_*\mathcal{O}_X \rightarrow \mathcal{O}_X$, contradicting isotropy of M . Next, let S be an isotropic rank-2 subbundle of $F_*\mathcal{O}_X \oplus (F_*\mathcal{O}_X)^*$ and assume $\mu(S) \geq 0$. By the previous considerations S is stable, hence there is no non-zero map $S \rightarrow (F_*\mathcal{O}_X)^*$. Therefore S is a rank-2 subsheaf of $F_*\mathcal{O}_X$, but this contradicts isotropy of S . \square

Remark. Clearly the two bundles $\hat{\mathcal{A}}$ and \mathcal{B} are non-isomorphic.

5. COUNTEREXAMPLES TO BEHREND'S CONJECTURE

We consider the rank-7 vector bundle

$$\mathcal{E} = F_*\mathcal{O}_X \oplus \mathcal{A} \oplus (F_*\mathcal{O}_X)^*$$

equipped with the non-degenerate quadratic form Q defined by

$$Q(x + y + x^*) = \langle x, x^* \rangle + \det y,$$

where x, x^* and y are local sections of $F_*\mathcal{O}_X$, $(F_*\mathcal{O}_X)^*$ and \mathcal{A} respectively. Note that the $\mathrm{SO}(7)$ -bundle $E_{\mathrm{SO}(7)} = (\mathcal{E}, Q)$ has a reduction to the Levi subgroup

$$L = \mathrm{GL}(2) \times \mathrm{SO}(3) \subset P \subset \mathrm{SO}(7)$$

and that its L -bundle E_L given by the pair $(F_*\mathcal{O}_X, \mathcal{A})$ is stable, since $F_*\mathcal{O}_X$ and \mathcal{A} are stable $\mathrm{GL}(2)$ - and $\mathrm{SO}(3)$ -bundles respectively. Since $\mu(F_*\mathcal{O}_X) > \mu(\mathcal{A}) = 0$, we obtain that the canonical reduction of the unstable $\mathrm{SO}(7)$ -bundle (\mathcal{E}, Q) is given by the P -bundle $E_P := E_L \times^L P$. Using the exact sequence (4) we obtain the equality

$$\mathrm{Hom}(F_*\mathcal{O}_X, \mathcal{A}) = H^0(X, E_P \times^P \mathfrak{so}(7)/\mathfrak{p}).$$

Note that $H^0(X, E_P \times^P \mathcal{D}) \subset \mathrm{Hom}(F_*\mathcal{O}_X, (F_*\mathcal{O}_X)^*) = 0$ by stability of $F_*\mathcal{O}_X$. But the former space is nonzero by Proposition 4.4 (ii). Hence this provides a counterexample to Behrend's conjecture.

The following unstable $\mathrm{SO}(8)$ -bundles also provide counterexamples to Behrend's conjecture

$$F_*\mathcal{O}_X \oplus \hat{\mathcal{A}} \oplus (F_*\mathcal{O}_X)^* \quad \text{and} \quad F_*\mathcal{O}_X \oplus \mathcal{B} \oplus (F_*\mathcal{O}_X)^*,$$

with the quadratic form given by the standard hyperbolic form on the two summands $F_*\mathcal{O}_X \oplus (F_*\mathcal{O}_X)^*$ and the quadratic forms on $\hat{\mathcal{A}}$ and \mathcal{B} introduced in sections 4.1 and 4.2. Similarly we use the exact sequence (2) and Proposition 4.4 (ii) to show that the corresponding spaces $H^0(X, E_P \times^P \mathfrak{so}(8)/\mathfrak{p})$ are non-zero.

Remark. One can work out the relationship between the unstable $\mathrm{SO}(7)$ -bundle (\mathcal{E}, Q) and the unstable G_2 -bundle E_{G_2} constructed in [H] section 5: first, we recall that $G_2 \subset \mathrm{GL}(6)$. Consider the symplectic rank-6 bundle $\bar{\mathcal{E}}$ obtained from (\mathcal{E}, Q) under the purely inseparable group homomorphism $\mathrm{SO}(7) \rightarrow \mathrm{Sp}(6) \subset \mathrm{GL}(6)$. Then the bundle $\bar{\mathcal{E}}$ has a reduction to E_{G_2} .

6. NON-RATIONALITY OF THE CANONICAL REDUCTION OF AN $\mathrm{SO}(7)$ -BUNDLE

This section is largely inspired by the last remark of [H]. We consider the unstable $\mathrm{SO}(7)$ -bundle (\mathcal{E}, Q) introduced in section 5. We denote by K the field $k(t)$, by X_K the curve $X \times_k K$ and by \mathcal{E}_K the vector bundle over X_K obtained as pull-back of \mathcal{E} under the field extension K/k . We will consider the quadratic form \tilde{Q}_K on the bundle \mathcal{E}_K defined by

$$\tilde{Q}_K(x + y + x^*) = \langle x, x^* \rangle + \det y + tq(x),$$

where x, x^* and y are local sections of $F_*\mathcal{O}_X$, $(F_*\mathcal{O}_X)^*$ and \mathcal{A} respectively and q is the quadratic form on $F_*\mathcal{O}_X$ defined in (10). Note that by Proposition 4.4 (iii) one has $q(x) = \det(\phi(x))$.

Proposition 6.1. (i) *The quadratic form \tilde{Q}_K is non-degenerate.*

(ii) *The canonical reduction of the $\mathrm{SO}(7)$ -bundle $(\mathcal{E}_K, \tilde{Q}_K)$ over X_K is not defined over K , but only over the inseparable quadratic extension $K' = K[s]/(s^2 - t)$.*

Proof. We consider the vector bundle $\mathcal{E}_{K'}$ over $X_{K'}$ obtained from \mathcal{E}_K under the field extension K'/K and introduce the automorphism g_s of $\mathcal{E}_{K'}$ defined by the matrix

$$\begin{pmatrix} \mathrm{Id}_{F_*\mathcal{O}_X} & 0 & 0 \\ s\phi & \mathrm{Id}_{\mathcal{A}} & 0 \\ 0 & s\psi & \mathrm{Id}_{(F_*\mathcal{O}_X)^*} \end{pmatrix}.$$

We choose the generators $\phi \in \mathrm{Hom}(F_*\mathcal{O}_X, \mathcal{A})$ and $\psi \in \mathrm{Hom}(\mathcal{A}, B^{-1})$ such that $\beta(\phi(x), y) = \langle x, \psi(y) \rangle$ for any local sections x, y of $F_*\mathcal{O}_X$ and \mathcal{A} respectively — see Proposition (4.4) (iv). Note that g_s is an involution. Then we have

$$g_s(x, y, x^*) = (x, y + s\phi(x), x^* + s\psi(y)).$$

We also denote by Q and \tilde{Q}_K the quadratic forms on $\mathcal{E}_{K'}$ obtained from Q and \tilde{Q}_K under the field extensions K'/k and K'/K respectively. Then we have the equality

$$(11) \quad Q(g_s(x, y, x^*)) = Q(x, y, x^*) + s^2q(x) = \tilde{Q}_K(x, y, x^*),$$

i.e., the quadratic forms Q and \tilde{Q}_K on $\mathcal{E}_{K'}$ differ by the automorphism g_s . In particular \tilde{Q}_K is non-degenerate and the pair $(\mathcal{E}_K, \tilde{Q}_K)$ determines an $\mathrm{SO}(7)$ -bundle over the curve X_K .

Because of (11) the canonical reduction of the $\mathrm{SO}(7)$ -bundle $(\mathcal{E}_K, \tilde{Q}_K)$ is given by the isotropic rank-2 vector bundle $\alpha : F_*\mathcal{O}_X \hookrightarrow \mathcal{E}_{K'}$ with $\alpha(x) = (x, s\phi(x), 0)$ for any local section x of $F_*\mathcal{O}_X$. It is clear that the inclusion α is not defined over K . \square

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