

Global homotopy formulas on q -concave CR manifolds for large degrees

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It is well known that homotopy formulas are very useful in complex analysis. Such formulas were constructed by means of integral operators in the 70's by Grauert and Lieb, Henkin, Ramirez, Kerzman and Stein for the Cauchy-Riemann operator (see the historical notes in [12] for more details) and later by Airapetjan and Henkin [1], Polyakov [9], Barkatou and Laurent-Thiébaud [2] for the tangential Cauchy-Riemann operator. In most cases only local formulas were obtained. The question arises if it is possible to globalize these formulas? Gluing together local formulas, it is rather easy to get a global formula which is not yet a homotopy formula, but "almost", up to a compact perturbation. Then the main work is to eliminate this compact perturbation. A first step in that direction was done in [8] and then applied in [5] to get a global homotopy formula for the Cauchy-Riemann operator in q -concave- q^* -convex domains of a complex manifold. More recently Polyakov [10, 11] proved global homotopy formulas for the forms of small degrees for the tangential Cauchy-Riemann operator on compact q -concave CR manifolds and used them to study the embedding problem for CR manifolds. But his global operators are less regular than the local ones. Then in [7] it was obtained that, in the case of forms of small degrees, it is possible to eliminate the compact perturbation without any loss of smoothness.

In the present paper we extend the results of [7] to the case of the forms of large degree (cf. Theorem 1.1). The main tools are the same as in [7], for example the functional analytic lemma (see Lemma 1.3) and an induction lemma (see Lemma 1.5), but new difficulties appear because now the induction does not start with functions but with forms of positive degree. For that we need the Friedrichs approximation lemma for first order differential operators, well known for the L^2 -topology, in the C^k -topology. Since it seems that this approximation result does not exist in the literature, it was proved by the first author in his Diplomarbeit [3]. This proof is given at the end of this paper.

As a corollary we get a Dolbeault isomorphism type result (cf. Corollary 1.6).

In the case of the Cauchy-Riemann operator on a complex manifold, the Dolbeault isomorphism says that all the Dolbeault cohomology groups of bidegree (p, q) for currents, C^∞ -forms or C^k -forms are isomorphic to the q th cohomology group of the sheaf of germs of holomorphic p -forms. This is a consequence of the de Rham-Weil isomorphism, of the Dolbeault lemma and of the holomorphy of the $\bar{\partial}$ -closed $(p, 0)$ -currents.

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Let M be a q -concave, $q \geq 1$, CR generic submanifold of real codimension k of a complex manifold X of complex dimension n . As in the complex case, we have smoothness of $\bar{\partial}_b$ -closed $(n, 0)$ -currents and local solvability of the tangential Cauchy-Riemann equation for forms of bidegree (n, r) with $1 \leq r \leq q - 1$ and $n - k - q + 1 \leq r \leq n - k$. For the small degrees, then the Dolbeault isomorphism for the $\bar{\partial}_b$ -cohomology follows from the de Rham-Weil isomorphism. For the large degrees, i.e. bidegree (n, r) with $n - k - q + 1 \leq r \leq n - k$, $\bar{\partial}_b$ -closed currents of bidegree (n, r) need not to be smooth. Nevertheless the Dolbeault isomorphism between the $\bar{\partial}_b$ -cohomology for currents and the $\bar{\partial}_b$ -cohomology for \mathcal{C}^∞ -smooth forms is proved for $n - k - q + 2 \leq r \leq n - k$ in [6] under the additional hypothesis that the conormal bundle of M in X is trivial and in [13] without any additional hypothesis. Here we prove that, if moreover M is compact, the $\bar{\partial}_b$ -cohomology group for \mathcal{C}^l -smooth (n, r) -forms, $l \in \mathbb{N}$, and for \mathcal{C}^∞ -smooth (n, r) -forms are isomorphic in the case of the large degrees, included $r = n - k - q + 1$. In [6] the reduction to local results was based on cohomological algebra arguments, in [13] on the construction of a regularization formula for $\bar{\partial}_b$, here it uses functional analysis.

1 Global homotopy formula

In this section, X is a complex manifold and E is a holomorphic vector bundle on X . Further, $M \subseteq X$ is a generic, compact CR submanifold of class \mathcal{C}^∞ of X , k is the real codimension of M in X , and \mathcal{O} is the trivial complex line bundle on X .

If $U \subseteq M$ is an open set, then, for $0 \leq r \leq n - k$, the following notations are used:

- $\mathcal{C}_{n,r}^\infty(U, E)$ is the Fréchet space of E -valued (n, r) -forms on U which are of class \mathcal{C}^∞ , endowed with the \mathcal{C}^∞ -topology.
- $\mathcal{Z}_{n,r}^\infty(U, E)$ is the subspace of all closed forms in $\mathcal{C}_{n,r}^\infty(U, E)$, endowed with the same topology.
- $\mathcal{C}_{n,r}^{l+\alpha}(\bar{U}, E)$, $l \in \mathbb{N}$, $0 \leq \alpha < 1$, is the Banach space of l times differentiable E -valued (n, r) -forms whose derivatives up to order l admit extensions to \bar{U} which are Hölder continuous with exponent α , endowed with the $\mathcal{C}^{l+\alpha}$ -topology.
- $\mathcal{Z}_{n,r}^{l+\alpha}(\bar{U}, E)$ is the subspace of all closed forms in $\mathcal{C}_{n,r}^{l+\alpha}(\bar{U}, E)$, endowed with the same topology.
- If $r \geq 1$, then $\mathcal{B}_{n,r}^{l+\alpha-l}(M, E)$ is the space of all $f \in \mathcal{C}_{n,r}^l(M, E)$ such that $f = du$ for some $u \in \mathcal{C}_{n,r-1}^{l+\alpha}(M, E)$. Sometimes we write also

$$\mathcal{B}_{n,r}^\infty(M, E) := \mathcal{B}_{n,r}^{\infty-\infty}(M, E) := d\mathcal{C}_{n,r-1}^\infty(M, E).$$

- $(\text{Dom } d)_{n,r}^0(M, E)$ is the space of all $f \in \mathcal{C}_{n,r}^0(M, E)$ such that also df is continuous on M .

If $0 < \alpha < 1$ and q is an integer with $1 \leq q \leq n - k$, then we shall say that **condition $H(\alpha, q)$ is satisfied** if, for each point in M , there exist a neighborhood U and linear

operators

$$T_r : \mathcal{C}_{n,r}^0(M, \mathcal{O}) \rightarrow \mathcal{C}_{n,r-1}^0(U, \mathcal{O}), \quad 1 \leq r \leq q \text{ and } n - k - q + 1 \leq r \leq n - k,$$

with the following two properties:

- (i) For all $l \in \mathbb{N}$ and $1 \leq r \leq q$ or $n - k - q + 1 \leq r \leq n - k$,

$$T_r \left(\mathcal{C}_{n,r}^l(M, \mathcal{O}) \right) \subseteq \mathcal{C}_{n,r-1}^{l+\alpha}(\bar{U}, \mathcal{O})$$

and T_r is continuous as an operator between $\mathcal{C}_{n,r}^l(M, \mathcal{O})$ and $\mathcal{C}_{n,r-1}^{l+\alpha}(\bar{U}, \mathcal{O})$.

- (ii) If $f \in (\text{Dom } d)_{n,r}^0(M, \mathcal{O})$, $0 \leq r \leq q - 1$, has compact support in U , then, on U ,

$$f = \begin{cases} T_1 df & \text{if } r = 0, \\ dT_r f + T_{r+1} df & \text{if } 1 \leq r \leq q - 1 \text{ or } n - k - q + 1 \leq r \leq n - k. \end{cases} \quad (1.1)$$

If M is q -concave in the sense of Henkin [4], then it is known since 1981 [4, 1] that condition $H(\alpha, q)$ is satisfied for $0 < \alpha < 1/2$. More recently it was proved in [2] that then also condition $H(1/2, q)$ is satisfied.

Theorem 1.1. *Suppose, for some $0 < \alpha < 1$ and some integer q with $1 \leq q \leq n - k$, condition $H(\alpha, q)$ is satisfied. Then there exist finite dimensional subspaces \mathcal{H}_r of $\mathcal{Z}_{n,r}^\infty(M, E)$, $1 \leq r \leq q - 1$ and $n - k - q + 1 \leq r \leq n - k$, where $\mathcal{H}_0 = \mathcal{Z}_{n,0}^\infty(M, E)$, continuous linear operators*

$$A_r : \mathcal{C}_{n,r}^0(M, E) \rightarrow \mathcal{C}_{n,r-1}^0(M, E), \quad 1 \leq r \leq q \text{ and } n - k - q + 1 \leq r \leq n - k,$$

and continuous linear projections

$$P_r : \mathcal{C}_{n,r}^0(M, E) \rightarrow \mathcal{C}_{n,r}^0(M, E), \quad 0 \leq r \leq q - 1 \text{ and } n - k - q + 1 \leq r \leq n - k,$$

with

$$\text{Im } P_r = \mathcal{H}_r, \quad 0 \leq r \leq q - 1 \text{ and } n - k - q + 1 \leq r \leq n - k, \quad (1.2)$$

and

$$\mathcal{B}_{n,r}^{0 \rightarrow 0}(M, E) \subseteq \text{Ker } P_r, \quad 1 \leq r \leq q - 1 \text{ and } n - k - q + 1 \leq r \leq n - k, \quad (1.3)$$

such that:

- (i) For all $l \in \mathbb{N} \cup \{\infty\}$ and $1 \leq r \leq q$ or $n - k - q + 1 \leq r \leq n - k$,

$$A_r \left(\mathcal{C}_{n,r}^l(M, E) \right) \subseteq \mathcal{C}_{n,r-1}^{l+\alpha}(M, E) \quad (1.4)$$

and A_r is continuous as operator from $\mathcal{C}_{n,r}^l(M, E)$ to $\mathcal{C}_{n,r-1}^{l+\alpha}(M, E)$.

- (ii) For all $0 \leq r \leq q - 1$ or $n - k - q + 1 \leq r \leq n - k$ and $f \in (\text{Dom})_{n,r}^0(M, E)$,

$$f - P_r f = \begin{cases} A_1 df & \text{if } r = 0, \\ dA_r f + A_{r+1} df & \text{if } 1 \leq r \leq q - 1 \text{ or } n - k - q + 1 \leq r \leq n - k, \end{cases} \quad (1.5)$$

In the case of the small degrees, i.e. for $0 \leq r \leq q - 1$, Theorem 1.1 has been proven in [7], it remains to prove the case of the large degrees, i.e. $n - k - q + 1 \leq r \leq n - k$. The main ingredients are the same : first an almost homotopy formula obtained by gluing together the local formulas and a functional analytic lemma, second an inductive process.

Let us recall the almost homotopy formula, which is proven in [7] for small degrees and whose proof is exactly the same for large degrees:

Lemma 1.2. *Suppose, for some $0 < \alpha < 1$ and some integer q with $1 \leq q \leq n - k$, condition $H(\alpha, q)$ is satisfied. Then there exist linear operators*

$$T_r : \mathcal{C}_{n,r}^0(M, E) \rightarrow \mathcal{C}_{n,r-1}^0(M, E), \quad 1 \leq r \leq q \text{ and } n - k - q + 1 \leq r \leq n - k, \quad (1.6)$$

and

$$K_r : \mathcal{C}_{n,r}^0(M, E) \rightarrow \mathcal{C}_{n,r}^0(M, E), \quad 0 \leq r \leq q - 1 \text{ and } n - k - q + 1 \leq r \leq n - k, \quad (1.7)$$

with the following two properties:

(i) For all $l \in \mathbb{N}$,

$$T_r(\mathcal{C}_{n,r}^l(M, E)) \subseteq \mathcal{C}_{n,r-1}^{l+\alpha}(M, E), \quad 1 \leq r \leq q \text{ and } n - k - q + 1 \leq r \leq n - k, \quad (1.8)$$

$$K_r(\mathcal{C}_{n,r}^l(M, E)) \subseteq \mathcal{C}_{n,r}^{l+\alpha}(M, E), \quad 0 \leq r \leq q - 1 \text{ and } n - k - q + 1 \leq r \leq n - k, \quad (1.9)$$

the operators T_r , $1 \leq r \leq q$ and $n - k - q + 1 \leq r \leq n - k$, are continuous as operators acting between $\mathcal{C}_{n,r}^l(M, E)$ and $\mathcal{C}_{n,r-1}^{l+\alpha}(M, E)$, and the operators K_r , $0 \leq r \leq q - 1$ and $n - k - q + 1 \leq r \leq n - k$, are continuous as operators acting between $\mathcal{C}_{n,r}^l(M, E)$ and $\mathcal{C}_{n,r}^{l+\alpha}(M, E)$.

(ii) If $f \in (\text{Dom } d)_{n,r}^0(M, E)$, $0 \leq r \leq q - 1$ or $n - k - q + 1 \leq r \leq n - k$, then on M

$$f + K_r f = \begin{cases} T_1 df & \text{if } r = 0, \\ dT_r f + T_{r+1} df & \text{if } 1 \leq r \leq q - 1 \text{ or } n - k - q + 1 \leq r \leq n - k. \end{cases} \quad (1.10)$$

and the functional analytic lemma:

Lemma 1.3. *Let B_l , $l \in \mathbb{N}$, be a sequence of Banach spaces, and let $R : B_0 \rightarrow B_0$ be a linear operator such that, for each $l \in \mathbb{N}$:*

- $B_{l+1} \subseteq B_l$ and the imbedding $B_{l+1} \hookrightarrow B_l$ is continuous,
- $\bigcap_{\mu \in \mathbb{N}} B_\mu$ is dense in B_l ,
- $R(B_l) \subseteq B_l$ and $R|_{B_l}$ is compact as an endomorphism of B_l .

Then $I + R$ is a Fredholm endomorphism with index zero of B_0 (this is clear, because R is compact as an endomorphism of B_0), and

$$\text{Ker}(I + R) \subseteq \bigcap_{l \in \mathbb{N}} B_l. \quad (1.11)$$

from which one can deduce the next result (cf.[7], Lemma 5.1) :

Lemma 1.4. *Suppose, for some $0 < \alpha < 1$ and some integer q with $1 \leq q \leq n - k$, condition $H(\alpha, q)$ is satisfied and let K_r , $0 \leq r \leq q - 1$ or $n - k - q + 1 \leq r \leq n - k$, be the operators from lemma 1.2. Then:*

(i) *For all $0 \leq r \leq q - 1$ and $n - k - q + 1 \leq r \leq n - k$, $I + K_r$ is a Fredholm endomorphism of $\mathcal{C}_{n,r}^0(M, E)$ with index zero and*

$$\text{Ker}(I + K_r) \subseteq \mathcal{C}_{n,r}^\infty(M, E). \quad (1.12)$$

(ii) *We have*

$$Z_{n,0}^0(M, E) \subseteq \text{Ker}(I + K_0) \subseteq \mathcal{C}_{n,0}^\infty(M, E) \quad (1.13)$$

and

$$\dim Z_{n,0}^0(M, E) = \dim Z_{n,0}^\infty(M, E) < \infty. \quad (1.14)$$

(iii) *If $q \geq 2$ and $1 \leq r \leq q - 1$ or $n - k - q + 1 \leq r \leq n - k$, then*

$$(I + K_r)\left(\mathcal{Z}_{n,r}^0(M, E)\right) \subseteq \mathcal{B}_{n,r}^{\alpha \rightarrow 0}(M, E) \quad (1.15)$$

and $(I + K_r)|_{\mathcal{Z}_{n,r}^0(M, E)}$ is a Fredholm endomorphism with index zero of $\mathcal{Z}_{n,r}^0(M, E)$.

(iv) *If $q \geq 2$ and $1 \leq r \leq q - 1$ or $n - k - q + 1 \leq r \leq n - k$, then $\mathcal{B}_{n,r}^{\alpha \rightarrow 0}(M, E)$ is a closed subspace of finite codimension in $Z_{n,r}^0(M, E)$.*

(v) *If $q \geq 2$ and $1 \leq r \leq q - 1$ or $n - k - q + 1 \leq r \leq n - k$, then*

$$(I + K_r)\left(\mathcal{B}_{n,r}^{\alpha \rightarrow 0}(M, E)\right) \subseteq \mathcal{B}_{n,r}^{\alpha \rightarrow 0}(M, E) \quad (1.16)$$

and $(I + K_r)|_{\mathcal{B}_{n,r}^{\alpha \rightarrow 0}(M, E)}$ is a Fredholm endomorphism with index zero of $\mathcal{B}_{n,r}^{\alpha \rightarrow 0}(M, E)$.

We come now to the induction step, we restrict ourselves here to the case of large degrees (the case of small degrees is contained in [7]). To simplify the notations, we set $r_0 = n - k - q + 1$.

Lemma 1.5. *Suppose, for some $0 < \alpha < 1$ and some integer q with $1 \leq q \leq n - k$, condition $H(\alpha, q)$ is satisfied and let T_r , $r_0 \leq r \leq n - k$, and K_r , $r_0 \leq r \leq n - k$, be the operators from lemma 1.2. Then there exist finite dimensional continuous linear operators*

$$\begin{aligned} K_r' &: \mathcal{C}_{n,r}^0(M, E) \rightarrow \mathcal{C}_{n,r}^\infty(M, E), & r_0 \leq r \leq n - k, \\ K_r'' &: \mathcal{C}_{n,r}^0(M, E) \rightarrow \mathcal{C}_{n,r}^\infty(M, E), & r_0 \leq r \leq n - k, \\ T_r' &: \mathcal{C}_{n,r}^0(M, E) \rightarrow \mathcal{C}_{n,r-1}^\infty(M, E), & r_0 \leq r \leq n - k, \end{aligned}$$

such that with the abbreviations

$$N_r := I + K_r + K_r' + K_r'', \quad r_0 \leq r \leq n - k,$$

each N_r , $r_0 \leq r \leq n - k$, is a Fredholm endomorphism with index zero of $\mathcal{C}_{n,r}^0(M, E)$ (this is clear, because $I + K_r$ has this property), and:

(i) If $r_0 \leq r \leq n - k$ and $f \in (\text{Dom } d)_{n,r}^0(M, E)$, then

$$N_r f = d(T_r + T'_r)f + (T_{r+1} + T'_{r+1})df, \quad (1.17)$$

and hence

$$dN_r f = N_{r+1} df. \quad (1.18)$$

(ii) We have

$$\mathcal{C}_{n,r}^0(M, E) = \text{Im } N_r \oplus \text{Ker } N_r, \quad \text{if } r_0 \leq r \leq n - k, \quad (1.19)$$

$$\text{Ker } N_r \subseteq \mathcal{Z}_{n,r}^\infty(M, E), \quad \text{if } r_0 \leq r \leq n - k, \quad (1.20)$$

$$\mathcal{Z}_{n,r}^0(M, E) = \mathcal{B}_{n,r}^{\alpha \rightarrow 0}(M, E) \oplus \text{Ker } N_r \quad \text{if } r_0 \leq r \leq n - k, \quad (1.21)$$

and

$$\mathcal{B}_{n,r}^{\alpha \rightarrow 0}(M, E) = \mathcal{B}_{n,r}^{0 \rightarrow 0}(M, E), \quad \text{if } r_0 \leq r \leq n - k. \quad (1.22)$$

(iii) If $r_0 \leq r \leq n - k$, then

$$N_r(\mathcal{B}_{n,r}^{\alpha \rightarrow 0}(M, E)) = \mathcal{B}_{n,r}^{\alpha \rightarrow 0}(M, E), \quad (1.23)$$

and $N_r|_{\mathcal{B}_{n,r}^{\alpha \rightarrow 0}(M, E)}$ is an isomorphism of $\mathcal{B}_{n,r}^{\alpha \rightarrow 0}(M, E)$.

(iv) If $r_0 \leq r \leq n - k - 1$, then

$$(\text{Dom } d)_{n,r}^0(M, E) = N_r\left((\text{Dom } d)_{n,r}^0(M, E)\right) \oplus \text{Ker } K_r, \quad (1.24)$$

and hence $N_r|_{\text{Im } N_r \cap (\text{Dom } d)_{n,r}^0(M, E)}$ is an isomorphism of $\text{Im } N_r \cap (\text{Dom } d)_{n,r}^0(M, E)$.

(v) Remark: It follows from (1.19), (1.21) and (1.23) that

$$\text{Im } N_r \cap \mathcal{Z}_{n,r}^0(M, E) = \mathcal{B}_{n,r}^{\alpha \rightarrow 0}(M, E) \quad \text{if } r_0 \leq r \leq n - k. \quad (1.25)$$

Proof. We proceed by induction on r . We first construct the operators K'_{r_0} , K''_{r_0} , T'_{r_0} and T'_{r_0+1} .

We begin with the construction of K''_{r_0} and T'_{r_0} . We are looking for an operator K''_{r_0} , which satisfies

$$K''_{r_0} = dT'_{r_0},$$

where T'_{r_0} is a finite dimensional continuous linear operator from $\mathcal{C}_{n,r_0}^0(M, E)$ to $\mathcal{C}_{n,r_0-1}^\infty(M, E)$ and, if we set

$$\tilde{N}_{r_0} := I + K_{r_0} + K''_{r_0},$$

$$\text{Ker } \tilde{N}_{r_0} \cap \mathcal{B}_{n,r_0}^{0 \rightarrow 0}(M, E) = \{0\}. \quad (1.26)$$

By Lemma 1.4 the operator $(I + K_{r_0})|_{\mathcal{B}_{n,r_0}^{\alpha \rightarrow 0}(M, E)}$ is a Fredholm endomorphism with index zero of $\mathcal{B}_{n,r_0}^{\alpha \rightarrow 0}(M, E)$ and hence its kernel and cokernel are finite dimensional and of the same dimension.

Let $m = \dim \text{Ker}(I + K_{r_0})|_{\mathcal{B}_{n,r_0}^{0 \rightarrow 0}(M,E)}$. If $m = 0$ set $K_{r_0}'' = T_{r_0}' = 0$. If $m > 0$, since $\text{Ker}(I + K_{r_0}) \subseteq \mathcal{C}_{n,r_0}^\infty(M,E)$, we can choose a basis of $\text{Ker}(I + K_{r_0})|_{\mathcal{B}_{n,r_0}^{0 \rightarrow 0}(M,E)}$ made of \mathcal{C}^∞ -smooth forms $\theta_1, \dots, \theta_m$.

Moreover there exists a vector space S of dimension m such that

$$(I + K_{r_0})(\mathcal{Z}_{n,r_0}^0(M,E)) \oplus S = \mathcal{B}_{n,r_0}^{0 \rightarrow 0}(M,E).$$

Let $d\lambda_1, \dots, d\lambda_m$ be a basis of S . Then by Friedrichs lemma for \mathcal{C}^k -topology (cf. Appendix) there exists $\tilde{\lambda}_1, \dots, \tilde{\lambda}_m$ such that $\tilde{S} = \text{Vect}(d\tilde{\lambda}_1, \dots, d\tilde{\lambda}_m)$ satisfies also

$$(I + K_{r_0})(\mathcal{Z}_{n,r_0}^0(M,E)) \oplus \tilde{S} = \mathcal{B}_{n,r_0}^{0 \rightarrow 0}(M,E). \quad (1.27)$$

Taking a dual basis, we can find forms ψ_1, \dots, ψ_m of degree $n - k - r_0$ such that

$$\int_M \psi_\alpha \wedge \theta_\beta = \delta_{\alpha\beta}, \quad 1 \leq \alpha, \beta \leq m.$$

We set for $f \in \mathcal{C}_{n,r_0}^0(M,E)$

$$T_{r_0}' f = \sum_{\alpha=1}^m \left(\int_M f \wedge \psi_\alpha \right) \tilde{\lambda}_\alpha.$$

It follows from this definition that T_{r_0}' is a finite dimensional continuous linear operator from $\mathcal{C}_{n,r_0}^0(M,E)$ into $\mathcal{C}_{n,r_0-1}^\infty(M,E)$. Then we define K_{r_0}'' by $K_{r_0}'' = dT_{r_0}'$ and \tilde{N}_{r_0} by $\tilde{N}_{r_0} := I + K_{r_0} + K_{r_0}''$. The operator \tilde{N}_{r_0} is then a Fredholm operator with index 0 of $\mathcal{C}_{n,r_0}^0(M,E)$ and by Lemma 1.3

$$\text{Ker } \tilde{N}_{r_0} \subseteq \mathcal{C}_{n,r_0}^\infty(M,E).$$

It remains to prove (1.26). Let $f \in \text{Ker } \tilde{N}_{r_0} \cap \mathcal{B}_{n,r_0}^{0 \rightarrow 0}(M,E)$, then

$$(I + K_{r_0})f + K_{r_0}'' f = 0.$$

Since $(I + K_{r_0})f \in (I + K_{r_0})(\mathcal{Z}_{n,r_0}^0(M,E))$ and $K_{r_0}'' f \in \tilde{S}$, by (1.27) we get

$$f \in \text{Ker}(I + K_{r_0})|_{\mathcal{B}_{n,r_0}^{0 \rightarrow 0}(M,E)} \cap \text{Ker } K_{r_0}''$$

and hence $f = 0$ by definition of T_{r_0}' and K_{r_0}'' , which conclude the proof of (1.26).

Next we prove that

$$\mathcal{Z}_{n,r_0}^0(M,E) = \mathcal{B}_{n,r_0}^{\alpha \rightarrow 0}(M,E) \oplus \left(\text{Ker } \tilde{N}_{r_0} \cap \mathcal{Z}_{n,r_0}^0(M,E) \right). \quad (1.28)$$

Since $K_{r_0}'' = dT_{r_0}'$ and $\text{Im } T_{r_0}' \subseteq \mathcal{C}_{q-1}^\infty(M,E)$, it is clear that

$$\text{Im } K_{r_0}'' \subseteq \mathcal{B}_{n,r_0}^{\alpha \rightarrow 0}(M,E). \quad (1.29)$$

By lemma 1.4 (iv), $(I + K_{r_0})|_{\mathcal{B}_{n,r_0}^{\alpha \rightarrow 0}(M,E)}$ is a Fredholm endomorphism with index zero of $\mathcal{B}_{n,r_0}^{\alpha \rightarrow 0}(M,E)$. Since K''_{r_0} is finite dimensional and we have (1.29), this implies that $\tilde{N}_{r_0}|_{\mathcal{B}_{n,r_0}^{\alpha \rightarrow 0}(M,E)}$ has the same property. By (1.26), this means that

$$\tilde{N}_{r_0}|_{\mathcal{B}_{n,r_0}^{\alpha \rightarrow 0}(M,E)} \text{ is an isomorphism of } \mathcal{B}_{n,r_0}^{\alpha \rightarrow 0}(M,E). \quad (1.30)$$

In particular,

$$\text{Im } \tilde{N}_{r_0}|_{\mathcal{B}_{n,r_0}^{\alpha \rightarrow 0}(M,E)} = \mathcal{B}_{n,r_0}^{\alpha \rightarrow 0}(M,E). \quad (1.31)$$

Moreover, by part (iii) of lemma 1.4, $(I + K_{r_0})|_{\mathcal{Z}_{n,r_0}^0(M,E)}$ is a Fredholm endomorphism with index zero of $\mathcal{Z}_{n,r_0}^0(M,E)$, where

$$\text{Im}(I + K_{r_0})|_{\mathcal{Z}_{n,r_0}^0(M,E)} \subseteq \mathcal{B}_{n,r_0}^{\alpha \rightarrow 0}(M,E)$$

Once again since K''_{r_0} is finite dimensional and we have (1.29), this implies that also $\tilde{N}_{r_0}|_{\mathcal{Z}_{n,r_0}^0(M,E)}$ is a Fredholm endomorphism with index zero of $\mathcal{Z}_{n,r_0}^0(M,E)$, where

$$\text{Im } \tilde{N}_{r_0}|_{\mathcal{Z}_{n,r_0}^0(M,E)} \subseteq \mathcal{B}_{n,r_0}^{\alpha \rightarrow 0}(M,E).$$

Together with (1.31) this gives

$$\text{Im } \tilde{N}_{r_0}|_{\mathcal{Z}_{n,r_0}^0(M,E)} = \mathcal{B}_{n,r_0}^{\alpha \rightarrow 0}(M,E). \quad (1.32)$$

Therefore, (1.26) can be written

$$\text{Ker } \tilde{N}_{r_0} \cap \text{Im } \tilde{N}_{r_0}|_{\mathcal{Z}_{n,r_0}^0(M,E)} = \{0\}.$$

Hence

$$\text{Im } \tilde{N}_{r_0}|_{\mathcal{Z}_{n,r_0}^0(M,E)} \cap \text{Ker } \tilde{N}_{r_0}|_{\mathcal{Z}_{n,r_0}^0(M,E)} = \{0\}. \quad (1.33)$$

As the index of $\tilde{N}_{r_0}|_{\mathcal{Z}_{n,r_0}^0(M,E)}$ is zero, this yields

$$\begin{aligned} \mathcal{Z}_{n,r_0}^0(M,E) &= \text{Im } \tilde{N}_{r_0}|_{\mathcal{Z}_{n,r_0}^0(M,E)} \oplus \text{Ker } \tilde{N}_{r_0}|_{\mathcal{Z}_{n,r_0}^0(M,E)} \\ &= \text{Im } \tilde{N}_{r_0}|_{\mathcal{Z}_{n,r_0}^0(M,E)} \oplus \left(\text{Ker } \tilde{N}_{r_0} \cap \mathcal{Z}_{n,r_0}^0(M,E) \right). \end{aligned}$$

Again by (1.32), this proves (1.28).

From (1.26) and (1.28) it follows that

$$\mathcal{B}_{n,r_0}^{\alpha \rightarrow 0}(M,E) = \mathcal{B}_{n,r_0}^{0 \rightarrow 0}(M,E). \quad (1.34)$$

The construction of the operators K'_{r_0} and T'_{r_0+1} and the proof of there properties are exactly the same as for the operators K'_{q-1} and T'_q in [7]. We do not repeat it here and this ends the initialization of the induction.

Now we assume that the operators K'_r , K''_r , T'_r and T'_{r+1} of Lemma 1.5 are construct for some $r_0 \leq r \leq n - k$ and that they satisfy the properties (i) to (v) of Lemma 1.5.

We set

$$K''_{r+1} := dT'_{r+1}. \quad (1.35)$$

Since T'_{r+1} is a finite dimensional continuous linear operator from $\mathcal{C}^0_{n,r+1}(M, E)$ to $\mathcal{C}^\infty_{n,r}(M, E)$, then it is clear that also K''_{r+1} is such an operator. Set

$$\tilde{N}_{r+1} := I + K_{r+1} + K''_{r+1}.$$

Then, by lemma 1.3, \tilde{N}_{r+1} is a Fredholm endomorphism with index zero of $\mathcal{C}^0_{n,r+1}(M, E)$, and

$$\text{Ker } \tilde{N}_{r+1} \subseteq \mathcal{C}^\infty_{n,r+1}(M, E). \quad (1.36)$$

Now we first prove that

$$\text{Ker } \tilde{N}_{r+1} \cap \mathcal{B}^{0 \rightarrow 0}_{n,r+1}(M, E) = \{0\}. \quad (1.37)$$

Let $g \in \mathcal{B}^{0 \rightarrow 0}_{n,r+1}(M, E)$ with $\tilde{N}_{r+1}g = 0$ be given. Take $f \in \mathcal{C}^0_{n,r}(M, E)$ with $g = df$. Then, by definition of \tilde{N}_{r+1} and K''_{r+1} , we get

$$0 = \tilde{N}_{r+1}g = \tilde{N}_{r+1}df = (I + K_{r+1})df + K''_{r+1}df = (I + K_{r+1})df + dT'_{r+1}df.$$

By (1.10), this implies

$$0 = (dT_{r+1} + dT'_{r+1})df. \quad (1.38)$$

Since, by hypothesis of induction, the operators K'_r , K''_r , T'_r and T'_{r+1} satisfy statement (i) of lemma 1.5, we have

$$N_r f = d(T_r + T'_r)f + (T_{r+1} + T'_{r+1})df.$$

Hence $dN_r f = d(T_{r+1} + T'_{r+1})df$, which implies by (1.38) that

$$N_r f \in \mathcal{Z}^0_{n,r}(M, E). \quad (1.39)$$

By hypothesis of induction, (1.25) is valid for r , and then (1.39) implies that

$$N_r f \in \mathcal{B}^{\alpha \rightarrow 0}_{n,r}(M, E).$$

Therefore as, by hypothesis of induction, (1.23) is valid for r , we can find $\tilde{f} \in \mathcal{B}^{\alpha \rightarrow 0}_{n,r}(M, E)$ with

$$N_r f = N_r \tilde{f}.$$

Hence $f - \tilde{f} \in \text{Ker } N_r$ and since, by hypothesis of induction, (1.20) is valid for r , this implies that $f - \tilde{f} \in \mathcal{Z}^\infty_{n,r}(M, E)$. As $\tilde{f} \in \mathcal{Z}^0_{n,r}(M, E)$, this further implies that $f \in \mathcal{Z}^0_{n,r}(M, E)$. Hence $g = df = 0$. This completes the proof of (1.37).

Next in the same way as for $r = r_0$ we get

$$\mathcal{Z}^0_{n,r+1}(M, E) = \mathcal{B}^{\alpha \rightarrow 0}_{n,r+1}(M, E) \oplus \left(\text{Ker } \tilde{N}_{r+1} \cap \mathcal{Z}^0_{n,r+1}(M, E) \right), \quad (1.40)$$

and then the construction of the operators K'_r and T'_{r+1} is an exact repetition of the construction of the operators K'_{r_0} and T'_{r_0+1} . \square

End of the proof of Theorem 1.1. We set $\mathcal{H}_r = \text{Ker } N_r$ for all $0 \leq r \leq q-1$ and $n-k-q+1 \leq r \leq n-k$. Since the operators N_r are Fredholm operators and $\text{Ker } N_r \subseteq \mathcal{Z}_{n,r}^\infty(M, E)$, the spaces \mathcal{H}_r are finite subspaces of $\mathcal{Z}_{n,r}^\infty(M, E)$.

By (1.19), we have

$$\mathcal{C}_{n,r}^0(M, E) = \text{Im } N_r \oplus \mathcal{H}_r \quad (1.41)$$

and we define P_r as the linear projection in $\mathcal{C}_{n,r}^0(M, E)$ with

$$\text{Im } P_r = \mathcal{H}_r \quad \text{and} \quad \text{Ker } P_r = \text{Im } N_r, \quad 0 \leq r \leq q-1 \quad \text{or} \quad n-k-q+1 \leq r \leq n-k. \quad (1.42)$$

Since the spaces $\text{Im } N_r$ and \mathcal{H}_r are closed in the \mathcal{C}^0 -topology, these projections are continuous with respect to the \mathcal{C}^0 -topology. Since, by (1.22) and (1.23), $\mathcal{B}_{n,r}^{0 \rightarrow 0}(M, E) \subseteq \text{Im } N_r$, this implies (1.3).

Set

$$\widehat{N}_r = N_r + P_r.$$

Then, by (1.41) and (1.42), \widehat{N}_r is an isomorphism of $\mathcal{C}_{n,r}^0(M, E)$. If $0 \leq r \leq q-2$ or $n-k-q+1 \leq r \leq n-k$, then moreover

$$\widehat{N}_r \left((\text{Dom } d)_{n,r}^0(M, E) \right) = (\text{Dom } d)_{n,r}^0(M, E)$$

and therefore $\widehat{N}_r|_{(\text{Dom } d)_{n,r-1}^0(M, E)}$ is an isomorphism of $(\text{Dom } d)_{n,r}^0(M, E)$. Indeed, since $\text{Ker } K_r \subseteq \mathcal{Z}_{n,r}^\infty(M, E) \subseteq (\text{Dom } d)_{n,r}^0(M, E)$, this follows from part (iv) of lemma 1.5.

Setting

$$A_r = \begin{cases} \widehat{N}_{r-1}^{-1}(T_r + T'_r), & 1 \leq r \leq q \\ (T_r + T'_r)\widehat{N}_r^{-1}, & n-k-q+1 \leq r \leq n-k, \end{cases} \quad (1.43)$$

now we define the continuous linear operators

$$A_r : \mathcal{C}_{n,r}^0(M, E) \longrightarrow \mathcal{C}_{n,r-1}^0(M, E), \quad 1 \leq r \leq q \quad \text{or} \quad n-k-q+1 \leq r \leq n-k.$$

Proof of (i): For $0 \leq r \leq q-1$ the proof of the assertions (i) and (ii) of the theorem is contained in section 6 of [7].

Let $n-k-q+1 \leq r \leq n-k$ and $l \in \mathbb{N}$ be given. By definition, \widehat{N}_r is of the form $\widehat{N}_r = I + R$ where $R|_{\mathcal{C}_{n,r}^l(M, E)}$ is a continuous linear operator from $\mathcal{C}_{n,r}^l(M, E)$ to $\mathcal{C}_{n,r}^{l+\alpha}(M, E)$. Since $(T_r + T'_r)|_{\mathcal{C}_{n,r}^l(M, E)}$ is a continuous from $\mathcal{C}_{n,r}^l(M, E)$ to $\mathcal{C}_{n,r-1}^{l+\alpha}(M, E)$, it follows that

$$A_r = \widehat{(T_r + T'_r)} N_r^{-1}$$

is continuous from $\mathcal{C}_{n,r}^l(M, E)$ to $\mathcal{C}_{n,r-1}^{l+\alpha}(M, E)$.

Proof of (ii): Let $n-k-q+1 \leq r \leq n-k$. We first prove that

$$\widehat{N}_{r+1}^{-1} d|_{(\text{Dom } d)_{n,r}^0(M, E)} = d \widehat{N}_r^{-1}|_{(\text{Dom } d)_{n,r}^0(M, E)}. \quad (1.44)$$

Since $\widehat{N}_r|_{(\text{Dom } d)_{n,r}^0(M, E)}$ is an isomorphism of $(\text{Dom } d)_{n,r}^0(M, E)$, this is equivalent to

$$d \widehat{N}_{r-1}|_{(\text{Dom } d)_{n,r-1}^0(M, E)} = \widehat{N}_r d|_{(\text{Dom } d)_{n,r-1}^0(M, E)}. \quad (1.45)$$

Let $g \in (\text{Dom } d)_{n,r}^0(M, E)$ be given. Then, by (1.3), (1.21) and (1.18),

$$dP_{r-1}g = 0, \quad P_r dg = 0 \quad \text{and} \quad dN_{r-1}g = N_r dg.$$

Hence

$$d\widehat{N}_{r-1}g = dN_{r-1}g + dP_{r-1}g = dN_{r-1}g = N_r dg = N_r dg + P_r dg = \widehat{N}_r dg.$$

Now consider $f \in (\text{Dom } d)_{n,r}^0(M, E)$. Then, by (1.17) and definition of the operators A_r ,

$$\widehat{N}_r N_r^{-1} f = dA_r f + (T_{r+1} f + T'_{r+1}) dN_r^{-1} f \quad (1.46)$$

Since $\text{Im } N_r = \text{Ker } P_r$, we have $P_r N_r = 0$. Hence $(I - P_r)\widehat{N}_r = (I - P_r)(N_r + P_r) = N_r$ and therefore $N_r \widehat{N}_r^{-1} = I - P_r$. Therefore, (1.46) takes the form

$$f - P_r f = dA_r f + (T_{r+1} f + T'_{r+1}) dN_r^{-1} f \quad (1.47)$$

and together with (1.44) this gives (1.5). \square

As a direct consequence of Theorem 1.1, we obtain a new proof of the Dolbeault isomorphism for the $\bar{\partial}_b$ -cohomology and of the regularity theorem for the tangential Cauchy-Riemann operator in compact CR manifolds

Corollary 1.6. *Suppose, for some $0 < \alpha < 1$ and some integer q with $1 \leq q \leq n - k$, condition $H(\alpha, q)$ is satisfied. For all $1 \leq r \leq q$ or $n - k - q + 1 \leq r \leq n - k$ and $l \in \mathbb{N} \cup \{\infty\}$, the space $\mathcal{B}_{n,r}^{l+\alpha \rightarrow l}(M, E)$ is closed in $\mathcal{C}_{n,r}^l(M, E)$,*

$$\mathcal{B}_{n,r}^{l+\alpha \rightarrow l}(M, E) = \mathcal{B}_{n,r}^{0 \rightarrow l}(M, E), \quad (1.48)$$

and the natural map

$$\frac{\mathcal{Z}_{n,r}^\infty(M, E)}{d\mathcal{C}_{n,r-1}^\infty(M, E)} \rightarrow \frac{\mathcal{Z}_{n,r}^l(M, E)}{\mathcal{B}_{n,r}^{0 \rightarrow l}(M, E)} \quad (1.49)$$

is injective.

If $1 \leq r \leq q - 1$ or $n - k - q + 1 \leq r \leq n - k$, then moreover, for all $l \in \mathbb{N} \cup \{\infty\}$, there exist finite dimensional subspaces \mathcal{H}_r of $\mathcal{Z}_{n,r}^\infty(M, E)$ such that

$$\mathcal{Z}_{n,r}^l(M, E) = \mathcal{B}_{n,r}^{l+\alpha \rightarrow l}(M, E) \oplus \mathcal{H}_r, \quad (1.50)$$

and hence (1.49) is an isomorphism and the cohomology groups in (1.49) are finite dimensional.

Proof. Let $1 \leq r \leq q$ or $n - k - q + 1 \leq r \leq n - k$. It follows from (1.5) and (1.3) that

$$dA_r f = f \quad \text{for all } f \in \mathcal{B}_{n,r}^{0 \rightarrow 0}(M, E), \quad (1.51)$$

and (1.48) follows from (1.51) and (1.4).

Let $1 \leq r \leq q - 1$ or $n - k - q + 1 \leq r \leq n - k$ and $l \in \mathbb{N} \cup \{\infty\}$. From Lemma 1.2, we get that $(I + K_r)|_{\mathcal{Z}_{n,r}^l(M, E)}$ is a Fredholm operator with index zero of $\mathcal{Z}_{n,r}^l(M, E)$

with $(I + K_r)(\mathcal{Z}_{n,r}^l(M, E)) \subseteq \mathcal{B}_{n,r}^{l+\alpha-l}(M, E)$. Since $\mathcal{B}_{n,r}^{l+\alpha-l}(M, E)$ is the image of a closed linear operator this implies that $\mathcal{B}_{n,r}^{l+\alpha-l}(M, E)$ is a closed subspace for the \mathcal{C}^l -topology.

To prove that $\mathcal{B}_{n,q}^{l+\alpha-l}(M, E)$ is closed in the \mathcal{C}^l -topology, we consider a sequence $f_\nu \in \mathcal{B}_{n,q}^{l+\alpha-l}(M, E)$ which converges in the \mathcal{C}^l -topology to some $f \in \mathcal{C}_{n,q}^l(M, E)$. Since, by part (i) of Theorem 1.1, A_r is continuous as operator from $\mathcal{C}_{n,q}^l(M, E)$ to $\mathcal{C}_{n,q-1}^{l+\alpha}(M, E)$, then the sequence $A_q f_\nu$ converges in the $\mathcal{C}^{l+\alpha}$ -topology to some $g \in \mathcal{C}_{n,q-1}^{l+\alpha}(M, E)$, where, by (1.51), $dA_q f_\nu = f_\nu$ for all ν . Since the operator

$$d : \mathcal{C}_{n,q-1}^{l+\alpha}(M, E) \longrightarrow \mathcal{B}_{n,q}^{l+\alpha-l}(M, E)$$

is closed, this implies that $dg = f$, i.e. $f \in \mathcal{B}_{n,q}^{l+\alpha-l}(M, E)$.

Since, by (1.4),

$$A_r \left(\mathcal{C}_{n,r}^\infty(M, E) \cap \mathcal{B}_{n,r}^{0 \rightarrow 0}(M, E) \right) \subseteq \mathcal{C}_{n,r-1}^\infty(M, E),$$

it follows from (1.51) that

$$d\mathcal{C}_{n,r-1}^\infty(M, E) = \mathcal{C}_{n,r}^\infty(M, E) \cap \mathcal{B}_{n,r}^{0 \rightarrow l}(M, E),$$

which means that (1.49) is injective.

Now let $1 \leq r \leq q-1$ or $n-k-q+1 \leq r \leq n-k$, we define \mathcal{H}_r by $\text{Im } P_r = \mathcal{H}_r$, where P_r is the projector from Theorem 1.1. Then, by (1.21), (1.22),

$$\mathcal{Z}_{n,r}^0(M, E) = \mathcal{B}_{n,r}^{0 \rightarrow 0}(M, E) \oplus \mathcal{H}_r.$$

Since $\mathcal{H}_r \subseteq \mathcal{C}_{n,r}^\infty(M, E)$, this implies that

$$\mathcal{Z}_{n,r}^l(M, E) = \mathcal{B}_{n,r}^{0 \rightarrow l}(M, E) \oplus \mathcal{H}_r \quad \text{for all } l \in \mathbb{N} \cup \{\infty\}.$$

By (1.48) this means (1.50). □

2 Appendix

In the 40's, Friedrichs has proven a density lemma for the L^2 -topology for partial differential operators in \mathbb{R}^n . If P is such an operator he proves that the \mathcal{C}^∞ -smooth functions are dense in the domain of definition of P for the graph norm. This result has been later extend to the L^p -topology, $1 \leq p < \infty$. Here we want to generalize it to differential operators between vector bundles for the \mathcal{C}^k -topology.

Such differential operators between vector bundles appears naturally, for example the tangential Cauchy-Riemann operator on a CR generic submanifold of a complex manifold is a differential operator between two bundles of differential forms.

Let X be a paracompact differential manifold of class \mathcal{C}^∞ of real dimension n and E and F two vector bundles of class \mathcal{C}^∞ , respectively of rank p and q .

Let $\mathcal{U} = (U_i)_{i \in I}$ be a locally finite open covering of X by coordinates domains which are also trivialization domains for both E and F and $(M_{ij})_{i,j \in I}$ be the transition matrices of E on $(U_{ij} = U_i \cap U_j)_{i,j \in I}$ and $(N_{ij})_{i,j \in I}$ be the transition matrices of F on $(U_{ij} = U_i \cap U_j)_{i,j \in I}$.

For $k \in \mathbb{N} \cup \{\infty\}$ we denote by $\Gamma^k(X, E)$, respectively $\Gamma^k(X, F)$, the vector space of \mathcal{C}^k -smooth sections of E , respectively F . Some trivialization being given on \mathcal{U} , for each $f \in \Gamma^k(X, E)$, $f|_{U_i}$ is given by a p -vector of functions $f_i = (f_i^1, \dots, f_i^p)$ and on $U_i \cap U_j$ we have $f_i = M_{ij} f_j$.

A linear differential operator P of order 1 and class \mathcal{C}^k between the fiber bundles E and F is a linear map between $\Gamma^{k+1}(X, E)$ and $\Gamma^k(X, F)$ given in some trivialization of E and F by a family of linear operators P_i from $\Gamma^{k+1}(U_i, E)$ into $\Gamma^k(U_i, F)$, $i \in I$, such that, for $f \in \Gamma^{k+1}(X, E)$,

(i) $P_i(f_i) = N_{ij} P_j(M_{ij} f_j)$ on $U_i \cap U_j$,

(ii) each P_i is a system of partial differential linear equations of order 1, i.e. $P_i = (L_i^{rs})$ is given by a (q, p) -matrice of partial differential linear equations, where, if (x_1, \dots, x_n) denotes some coordinates on U_i , $D_l = \frac{\partial}{\partial x_l}$, $1 \leq l \leq n$, the partial derivative relatively to the coordinate x_l , a_l^{rs} , $1 \leq l \leq n$, $1 \leq r \leq q$ and $1 \leq s \leq p$, \mathcal{C}^{k+1} functions on \mathbb{R}^n and a_0^{rs} , $1 \leq r \leq q$ and $1 \leq s \leq p$, \mathcal{C}^k functions on \mathbb{R}^n , then

$$L_i^{rs} = \sum_{l=1}^n a_l^{rs} D_l + a_0^{rs}.$$

Let P be a linear differential operator P of order 1 and class \mathcal{C}^k between the fiber bundles E and F , we shall say that a section f of class \mathcal{C}^k of E is in the domain of definition of P , $\text{Dom } P$, if Pf , which is defined in the sense of distributions, belongs to $\Gamma^k(X, F)$.

If K is a compact subset of X and $f \in \text{Dom } P$, the graph \mathcal{C}^k -norm on K of f is defined by

$$\|f\|_{gr(K, k)} = \|f\|_{K, k} + \|Pf\|_{K, k}.$$

Theorem 2.1. Friedrichs'lemma for the \mathcal{C}^k -topology. *Let P be a linear differential operator of order 1 and class \mathcal{C}^k between two fiber bundles E and F of class \mathcal{C}^∞ over a differential manifold X of class \mathcal{C}^∞ . For each compact subset K of X , the \mathcal{C}^∞ -smooth sections of E are dense in the domain of definition of P for the graph \mathcal{C}^k -norm on K .*

Proof. First consider the case when $X = \mathbb{R}^n$, E and F are both trivial bundles of rank respectively p and q and $P = (L^{rs})$ is a system of partial differential equations. In particular for $p = q = 1$ we get

Lemma 2.2. *Let $L = \sum_{l=1}^n a_l D_l + a_0$ with $a_l \in \mathcal{C}^{k+1}(\mathbb{R}^n)$, $1 \leq l \leq n$, and $a_0 \in \mathcal{C}^k(\mathbb{R}^n)$ and $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$ be a positive smooth function with compact support in the unit ball of \mathbb{R}^n such that $\int_{\mathbb{R}^n} \varphi \, dx = 1$. Set for $\epsilon > 0$ and $x \in \mathbb{R}^n$,*

$$\varphi_\epsilon(x) = \frac{1}{\epsilon^n} \varphi\left(\frac{x}{\epsilon}\right).$$

Then for each compact subset K of \mathbb{R}^n and any $f \in \text{Dom}(L)$

$$\|L(f * \varphi_\epsilon) - L(f)\|_{K, k} \rightarrow 0,$$

when $\epsilon \rightarrow 0$.

Proof. Let $f \in \text{Dom}(L)$ and set $L(f) = \tilde{L}(f) + a_0 f$. By the classical convergence properties of the convolution, we know that $\|a_0(f * \varphi_\epsilon) - a_0 f\|_{K,k}$ tends to 0 when ϵ tends to 0. By the triangle inequality

$$\|\tilde{L}(f * \varphi_\epsilon) - \tilde{L}(f)\|_{K,k} \leq \|\tilde{L}(f * \varphi_\epsilon) - \tilde{L}(f) * \varphi_\epsilon\|_{K,k} + \|\tilde{L}(f) * \varphi_\epsilon - \tilde{L}(f)\|_{K,k}.$$

It follows again from the classical properties of the convolution that, since $\tilde{L}(f) \in \mathcal{C}^k(\mathbb{R}^n)$, $\|\tilde{L}(f) * \varphi_\epsilon - \tilde{L}(f)\|_{K,k}$ tends to 0, when ϵ tends to 0. Now by definition of \tilde{L} , we have

$$\|\tilde{L}(f * \varphi_\epsilon) - \tilde{L}(f) * \varphi_\epsilon\|_{K,k} \leq \sum_{l=1}^n \|a_l D_l(f * \varphi_\epsilon) - (a_l D_l(f)) * \varphi_\epsilon\|_{K,k}. \quad (2.1)$$

Therefore it is sufficient to prove that each term in (2.1) converges to 0 when ϵ tends to 0. Let $s = (s_1, \dots, s_n) \in \mathbb{N}^n$ be a multi-index of length less or equal to k , we set $D^s = D_1^{s_1} \dots D_n^{s_n}$. Then

$$\|f\|_{K,k} = \sum_{|s| \leq k} \|D^s\|_{K,0}.$$

From the Leibniz formula, we deduce

$$D^s(a_l D_l(f)) = \sum_{r \leq s} \binom{s}{r} (D^r a_l)(D_l(D^{s-r} f))$$

and the proof of the lemma is then reduced to the following fact :

(♣) Let $a \in \mathcal{C}^1(\mathbb{R}^n)$, $1 \leq l \leq n$ and $f \in \mathcal{C}(\mathbb{R}^n)$, then the smooth function $a D_l(f * \varphi_\epsilon) - (a D_l(f)) * \varphi_\epsilon$ converges uniformly to 0 on K , when ϵ tends to 0.

Note that it is clearly the case if the function f is of class \mathcal{C}^∞ , because then $D_l(f * \varphi_\epsilon) = D_l(f) * \varphi_\epsilon$. Let $g \in \mathcal{C}^\infty(\mathbb{R}^n)$, then by linearity

$$a D_l(f * \varphi_\epsilon) - (a D_l(f)) * \varphi_\epsilon = a D_l(g * \varphi_\epsilon) - (a D_l(g)) * \varphi_\epsilon + a D_l((f - g) * \varphi_\epsilon) - (a D_l(f - g)) * \varphi_\epsilon \quad (2.2)$$

Assume we can prove that there exists a constant C independent of $\epsilon \in]0, 1]$ such that

$$\|a D_l(f * \varphi_\epsilon) - (a D_l(f)) * \varphi_\epsilon\|_{K,0} \leq C \|f\|_{K+B(0,1),0}, \quad (2.3)$$

then from (2.2) we get

$$\|a D_l(f * \varphi_\epsilon) - (a D_l(f)) * \varphi_\epsilon\|_{K,0} \leq \|a D_l(g * \varphi_\epsilon) - (a D_l(g)) * \varphi_\epsilon\|_{K,0} + C \|f - g\|_{K+B(0,1),0}.$$

and, since continuous functions in \mathbb{R}^n can be uniformly approximate on $K + B(0, 1)$ by \mathcal{C}^∞ -smooth functions, this concludes the proof of (♣) by classical arguments.

Now let us prove (2.3). First let us recall that

$$f * \varphi_\epsilon = \frac{1}{\epsilon^n} \int_X f(x - y) \varphi\left(\frac{y}{\epsilon}\right) dy = \int_X f(x - \epsilon y) \varphi(y) dy.$$

Then

$$a D_l(f * \varphi_\epsilon) - (a D_l(f)) * \varphi_\epsilon = \int_X (a(x) - a(x - \epsilon y)) (D_l f)(x - \epsilon y) \varphi(y) dy,$$

and after an integration by parts

$$\begin{aligned} aD_l(f * \varphi_\epsilon) - (aD_l(f)) * \varphi_\epsilon &= \int_X \frac{1}{\epsilon} (a(x) - a(x - \epsilon y)) f(x - \epsilon y) \frac{\partial \varphi}{\partial y_l}(y) dy \\ &+ \int_X \frac{\partial a}{\partial y_l}(x - \epsilon y) f(x - \epsilon y) \varphi(y) dy, \end{aligned}$$

since $\frac{\partial}{\partial y_l}(f(x - \epsilon y)) = -\epsilon(D_l f)(x - \epsilon y)$.

As a is of class \mathcal{C}^1 , for $\epsilon \leq 1$, there exists a constant M_K such that for $x \in K$ and $y \in \text{supp } \varphi \subset B(0, 1)$ we have

$$|a(x) - a(x - \epsilon y)| \leq M_K \epsilon |y| \quad \text{and} \quad \left| \frac{\partial a}{\partial y_l}(x - \epsilon y) \right| \leq M_K,$$

and therefore

$$\|aD_l(f * \varphi_\epsilon) - (aD_l(f)) * \varphi_\epsilon\|_{K,0} \leq M_K \left(\int_X (|y| \left| \frac{\partial \varphi}{\partial y_l}(y) \right| + |\varphi(y)|) dy \right) \|f\|_{K+B(0,1),0}.$$

Setting $C = M_K \left(\int_X (|y| \left| \frac{\partial \varphi}{\partial y_l}(y) \right| + |\varphi(y)|) dy \right)$, we get (2.3). \square

Now if P is given by a (q, p) -matrice (L^{rs}) , using the triangle inequality, we deduce easily from Lemma 2.2 that for each compact subset K of \mathbb{R}^n and any $f \in \text{Dom}(P)$

$$\|P(f * \varphi_\epsilon) - P(f)\|_{K,k} \rightarrow 0,$$

when $\epsilon \rightarrow 0$, which proves Theorem 2.1 for a system.

Now we have to globalize the situation.

Let $(\chi_i)_{i \in I}$ be a partition of the unity subordinated to the open covering \mathcal{U} and $f \in \Gamma^k(X, E)$. After a choice of coordinates in each U_i and of a trivialization of E , we can define $\chi_i f$ as a p -vector $f_i = (f_i^1, \dots, f_i^p)$ of functions with compact support in \mathbb{R}^n . Then, for ϵ sufficiently small, $f_i * \varphi_\epsilon = (f_i^1 * \varphi_\epsilon, \dots, f_i^p * \varphi_\epsilon)$ can be identify with a section $(f_i)_\epsilon$ of E with compact support in U_i . Set $f_\epsilon = \sum_{i \in I} (f_i)_\epsilon$, then $f_\epsilon \in \Gamma^\infty(X, E)$ and for each compact subset of X $\|f - f_\epsilon\|_{K,k}$ tends to 0 when ϵ tends to 0.

Moreover on one hand $Pf = \sum_{i \in I} P(\chi_i f)$ and, after a choice of trivialization of F over U_i , $P(\chi_i f) = P_i(f_i)$ and on the other hand $Pf_\epsilon = \sum_{i \in I} P((f_i)_\epsilon)$ and, for ϵ sufficiently small, $P((f_i)_\epsilon) = P_i(f_i * \varphi_\epsilon)$. By the case of a system, which has been studied previously, $\|P_i(f_i) - P_i(f_i * \varphi_\epsilon)\|_{K,k}$ tends to 0 when ϵ tends to 0 and therefore $\|Pf - Pf_\epsilon\|_{K,k}$ tends to 0 when ϵ tends to 0, which proves the theorem. \square

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