

# Second order sliding mode and adaptive observer for synchronization of a chaotic system: a comparative study

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## Abstract:

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### 1. INTRODUCTION

### 2. PROBLEM STATEMENT

Consider the following chaotic system

$$\begin{cases} \dot{x}_1(t) = a(x_2(t) - x_1(t)) + x_2(t)x_3(t) + m_1(t) \\ \dot{x}_2(t) = b(x_1(t) + x_2(t)) - x_1(t)x_3(t) \\ \dot{x}_3(t) = -cx_3(t) - ex_4(t) + x_1(t)x_2(t) + m_1(t) \\ \dot{x}_4(t) = fx_3(t) - dx_4(t) + x_1(t)x_3(t) + m_2(t) \\ y_1(t) = x_1(t) \\ y_2(t) = x_2(t) \end{cases} \quad (1)$$

where  $x_i \in \mathbb{R}$  ( $i = 1, 2, 3, 4$ ) are the states of the system;  $m_1$  and  $m_2$  represent the information to be transmitted, which for the observation problem are considered as the unknown inputs;  $y_1$  and  $y_2$  are the outputs of the system, these are the signals to be sent by a public channel.

*Assumption. 2.1.* The signals  $m_1$  and  $m_2$  are assumed to be differentiable, bounded and with the derivative bounded.

The goal is to reconstruct the states  $x_3$  and  $x_4$  which allows to reconstruct the messages  $m_1$  and  $m_2$ . For achieving such a goal two approaches are tested, namely, an observer based on the super-twisting algorithm that basically allows to obtain information from the derivatives of the output and consequently to reconstruct the states and the messages (unknown inputs), and an adaptive observer. A comparison between these two methods will be discussed.

It should be notice that a singularity appears in the point  $x_1 = 0$ , that is, from the outputs  $x_1$  and  $x_2$  is not possible to know the states  $x_3$  and  $x_4$  and consequently neither the messages  $m_1$  and  $m_2$ .

### 3. SUPER-TWISTING OBSERVER

First in order to generate a new output, namely the variable  $x_3$ , we use the super-twisting algorithm for designing the observer for  $\hat{x}_3$  in the following way:

$$\begin{cases} \dot{x}_{a,1} = b(x_1 + x_2) + v_1 \\ v_1 = \hat{x}_{1,3} + \lambda_1 |s_1|^{1/2} \text{sign } s \\ \dot{\hat{x}}_{1,3} = \alpha_1 \text{sign } s \\ s_1 = x_2 - x_{a,1} \end{cases} \quad (2)$$

In this way, the derivative of  $s_1$  takes the form

$$\dot{s}_1 = -x_1 x_3 - v^1 \quad (3)$$

Choosing the gains  $\lambda_1 \geq \frac{(\alpha_1 + M_1)(1 + \theta)}{1 - \theta} \sqrt{\frac{2}{\alpha_1 + M_1}}$  and  $\alpha_1 > M_1 \geq \left| \frac{d}{dt}(x_1 x_3) \right|$  ( $0 < \theta < 1$ ), we get, according to (Levant, 93), (Levant, 98), and (Davila, 05), the second order sliding motion, that is,  $s(t) = 0$ ,  $\dot{s}(t) = 0$  after some finite time  $T_1$ . Notice that for  $s = 0$ ,  $v^1 = \hat{x}_3$ ; therefore, from (3), we get

$$\hat{x}_{1,3}(t) \equiv -x_1(t)x_3(t) \quad (4)$$

From (4) we can reconstruct  $x_3(t)$  provided  $x_1(t) \neq 0$ . Thus, the observer for  $x_3$  is designed in the form

$$\hat{x}_3(t) = \begin{cases} -\frac{\hat{x}_{1,3}(t)}{x_1(t)} & \text{if } |x_1(t)| \geq \varepsilon \\ \hat{x}_3(t - \tau) & \text{if } |x_1(t)| < \varepsilon \end{cases} \quad (5)$$

where  $\tau$  and  $\varepsilon$  are enough small constants<sup>1</sup>. Thus, we get the identity

$$\hat{x}_3(t) \equiv x_3(t) \text{ for } |x_1(t)| \geq \varepsilon.$$

*Reconstruction of  $x_4$*  Now, defining  $\bar{y}_3(t) = x_3(t)$  as a new output, we can rewrite the dynamic equations in (1) as a linear system with output injection and unknown inputs, that is,

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<sup>1</sup> The constant  $\tau$  is chosen enough small but bigger than the sampling time used during the realization of the observer. The constant  $\varepsilon$  should be chosen sufficiently big to avoid the singularity, but also should be notice that any estimation in this zone can not be considered as an acceptable estimation.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \underbrace{\begin{bmatrix} -a & a & 0 & 0 \\ b & b & 0 & 0 \\ 0 & 0 & -c & -e \\ 0 & 0 & f & -d \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_x + \underbrace{\begin{bmatrix} y_2 \bar{y}_3 \\ -y_1 \bar{y}_3 \\ y_1 y_2 \\ y_1 \bar{y}_3 \end{bmatrix}}_{\phi(y_1, y_2, \bar{y}_3)} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_D \underbrace{\begin{bmatrix} m_1 \\ m_2 \end{bmatrix}}_w$$

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \bar{y}_3 \end{bmatrix}}_{\bar{y}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_C x$$

(6)

Now, let  $z$  be defined by the solution of the following differential equation

$$\dot{z} = Az + \phi(y_1, y_2, \bar{y}_3)$$

Thus, defining  $e_z = x - z$  we obtain the dynamic equation for  $e_z$

$$\begin{aligned} \dot{e}_z(t) &= Ae_z(t) + Dw(t) \\ y_z &= Ce_z \end{aligned} \quad (7)$$

Hence, the system (7) is basically a linear system with unknown inputs, just the sort of systems considered in (Bejarano, 07). Then, for the reconstruction of  $x_4$  we follow in essence, but with a little modification, the algorithm proposed in (Bejarano, 07). That is, the basic idea proposed in (Bejarano, 07) is to obtain an algebraic expression of  $e_z$  in terms of the output  $y_z$  and its derivatives. Thus, we get the equalities<sup>2</sup>

$$y_z = Ce_z$$

$$\frac{d}{dt} (CD)^+ y_z = (CD)^+ CAe_z$$

thus we get<sup>3</sup>

$$e_z(t) = \begin{bmatrix} C \\ (CD)^+ CA \end{bmatrix}^+ \begin{bmatrix} y_z(t) \\ (CD)^+ y_z(t) \end{bmatrix} \quad (8)$$

Since we already know the first 3 states, we solve (8) for  $e_{z,4}$ , the fourth state of  $e_z$ . Thus,

$$e_{z,4} = \frac{1}{e} [\dot{e}_{z,1} - \dot{e}_{z,3} + a(e_{z,1} - e_{z,2}) - ce_{z,3} + e_{z,1}e_{z,2} - e_{z,2}e_{z,3}] \quad (9)$$

### 3.1 Realization of the observer using super-twisting

To reduce the fast dynamic and have a smaller gains in the super-twisting algorithm, we design the following like-linear estimator  $\tilde{x}$  whose dynamics is governed by the following differential equation

$$\dot{\tilde{x}} = \underbrace{\begin{bmatrix} -c & -e \\ f & -d \end{bmatrix}}_A \tilde{x} + \begin{bmatrix} y_1 y_2 \\ y_1 \bar{y}_3 \end{bmatrix} + L(\bar{y}_3 - \tilde{y}_3)$$

$$\tilde{y}_3 = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_C \tilde{x} \quad (10)$$

The matrix  $L$  is chosen so that the eigenvalues of the matrix  $(\bar{A} - LC)$  have negative real part. In this way we

have that the dynamic equations for  $\bar{e} = \tilde{x} - \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$  are

<sup>2</sup>  $Y^+$  is a full row rank matrix such that  $Y^+Y = 0$ .

<sup>3</sup> The matrix  $X^+$  is defined to be the pseudo-inverse of  $X$ . The matrix considered in (8) belongs to sort of matrices of full column rank. Hence, in such a case,  $X^+ = (X^T X)^{-1} X^T$ .

$$\dot{\bar{e}}(t) = (\bar{A} - LC)\bar{e}(t) + w(t)$$

where  $w$  is defined in (6). Hence we get some upper bounds

for the norm of  $\bar{e}$  and for the norm of  $\dot{\bar{e}}$  that is

$$\|\bar{e}(t)\| \leq \gamma \exp(-\lambda t) \|\bar{e}(0)\| + \mu \|w(t)\|$$

$\gamma, \lambda, \mu$  are positive constants

That is, with the estimator (10),  $\bar{e}$  is constrained to stay in a zone depending on the amplitude of  $m_1$  and  $m_2$ ; and this zone can be made smaller by moving the eigenvalues of  $(\bar{A} - LC)$  more to the left in the left half-plane.

Then, following the algebraic expression obtained in (9) we design the observer for  $x_4$  using the super-twisting algorithm as follows

$$\begin{cases} \dot{x}_{a,2} = \frac{1}{e} [a(x_2 - x_1) + c\hat{x}_3 - x_1 x_2 + x_2 \hat{x}_3] + \tilde{x}_4 + v_2 \\ v_2 = v_{2,1} + \lambda_2 |s_2| \text{sign } s_2, \dot{v}_{2,1} = \alpha \text{sign } s_2 \\ s_2 = \frac{1}{e} (x_1 - x_3) - x_{a,2} \\ \hat{x}_4(t) = \begin{cases} \tilde{x}_4 + v_{2,1} & \text{if } |x_1(t)| \geq \varepsilon \\ \hat{x}_4(t - \tau) & \text{if } |x_1(t)| < \varepsilon \end{cases} \end{cases}$$

where  $\tilde{x}_4$  is the second component of the vector  $\tilde{x}$  defined in (10). Then, the time derivative of  $s_2$  is

$$\dot{s}_2 = x_4 - \tilde{x}_4 + v_2$$

Thus, for  $\lambda_2 \geq \frac{(\alpha_2 + M_2)(1 + \theta)}{1 - \theta} \sqrt{\frac{2}{\alpha_2 + M_2}}$  and  $\alpha_2 > M_2 \geq \left| \frac{d}{dt} (x_4 - \tilde{x}_4) \right|$  ( $0 < \theta < 1$ ), after some finite time  $T_2$ , we get  $s_2 = 0$  and  $\dot{s}_2 = 0$ ; therefore,

$$\hat{x}_4 \equiv x_4 \text{ for } |x_1(t)| \geq \varepsilon.$$

### 3.2 Messages reconstruction

The reconstruction of  $m_1$  is made in the following way

$$\begin{cases} \dot{x}_{a,3} = a(x_2 - x_1) + x_2 \hat{x}_3 + v_3 \\ v_3 = v_{3,1} + \lambda_3 |s_3| \text{sign } s_3 \\ \dot{v}_{3,1} = \alpha_3 \text{sign } s_3 \\ \hat{m}_1 = \begin{cases} v_{3,1} & \text{if } |x_1(t)| \geq \varepsilon \\ \hat{m}_1(t - \tau) & \text{if } |x_1(t)| < \varepsilon \end{cases} \\ s_3 = x_1 - x_a^3 \end{cases}$$

Thus, for  $\alpha_3$  and  $\lambda_3$  satisfying  $\lambda_3 \geq \frac{(\alpha_3 + M_3)(1 + \theta)}{1 - \theta} \sqrt{\frac{2}{\alpha_3 + M_3}}$  and  $\alpha_3 > M_3 \geq |\dot{\hat{m}}_1|$  ( $0 < \theta < 1$ ), after some finite time, we get the equalities  $s_3 = 0$ ,  $\dot{s}_3 = 0$ . Thus, we get the equality

$$\hat{m}_1 \equiv m_1, \text{ for } |x_1(t)| \geq \varepsilon$$

The reconstruction of  $m_2$  is made in a similar way, that is,

$$\begin{cases} \dot{x}_{a,4} = b(x_1 + x_2) + f x_3 - d x_4 + v_4 \\ v_4 = v_{4,1} + \lambda_4 |s_4| \text{sign } s_4 \\ \dot{v}_{4,1} = \alpha_4 \text{sign } s_4 \\ \hat{m}_2 = \begin{cases} v_{4,1} & \text{if } |x_1(t)| \geq \varepsilon \\ \hat{m}_2(t - \tau) & \text{if } |x_1(t)| < \varepsilon \end{cases} \\ s_4 = x_2 + x_4 - x_{a,4} \end{cases}$$

Thus, taking into account (1) and the derivative of  $x_{a,4}$ , with the choosing of  $\alpha_4$  and  $\lambda_4$  satisfying the inequalities  $\lambda_4 \geq \frac{(\alpha_4 + M_4)(1 + \theta)}{1 - \theta} \sqrt{\frac{2}{\alpha_4 + M_4}}$  and  $\alpha_4 > M_4 \geq |\dot{\hat{m}}_2|$  ( $0 < \theta < 1$ ) (see, (Levant, 93), (Levant, 98)), we get, after some finite time,

$$\hat{m}_2 \equiv m_2 \text{ for } |x_1(t)| \geq \varepsilon$$

*Remark. 3.1.* At a glance, it seems that, during the estimation of the state  $x_3$ , it is sufficient with to use  $x_1(t) \neq 0$  instead of  $|x_1(t)| \geq \varepsilon$ , but the justification is given in the following lines. It is known that during the realization of the super-twisting algorithm  $s_1$  and  $\dot{s}_1$  are not exactly zero, hence, instead of having (4) we really have the equality  $\hat{x}_{1,3}(t) = -x_1(t)x_3(t) + \Delta(t)$ , where  $\Delta$  represents the estimation error and which does not tends to zero. Then, after dividing over  $x_1$ , it yields the equality  $\bar{x}_3 := -\frac{\hat{x}_{1,3}}{x_1} = x_3 - \frac{\Delta}{x_1}$ , which means that, when  $x_1$  is very close to zero, the error between  $\bar{x}_3$  and  $x_3$  is equal to  $O(1/x_1)$ . Therefore, in a small neighborhood of  $x_1 = 0$ , the error between  $\bar{x}_3$  and  $x_3$  is extremely big. This justify the structure of  $\hat{x}_3$ .

#### 4. ADAPTIVE OBSERVER

The model of a chaotic system (1) can be rewritten in the following interconnected compact form

$$\begin{cases} \dot{X}_1 = A_1(y_2)X_1 + g_1(y, X_1, X_2) + \phi_1(t) \\ y_1 = C_1 X_1 \end{cases} \quad (11)$$

$$\begin{cases} \dot{X}_2 = A_2(y_1)X_2 + g_2(y, X_2, X_1) + \phi_2(t) \\ y_2 = C_2 X_2 \end{cases} \quad (12)$$

where  $X_1 = (x_1, x_3, x_4, x_5)^T$  is the state of the first subsystem with  $x_5 := m_2$ ,  $X_2 = (x_2, x_3, x_6)^T$  is the state of the second subsystem with  $x_6 := m_1$ .  $y = [x_1, x_2]^T$  are the output of the whole system, and

$$A_1(y_2) = \begin{pmatrix} 0 & y_2 & 0 & 0 \\ 0 & 0 & -c & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_2(y_1) = \begin{pmatrix} 0 & -y_1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$g_1(y, X_2, X_1) = \begin{pmatrix} a(y_1 - y_2) + x_6 \\ -ex_4 + y_1 y_2 + x_6 \\ fx_3 - dx_4 + y_1 x_3 \\ 0 \end{pmatrix}$$

$$g_2(y, X_2, X_1) = \begin{pmatrix} b(y_1 + y_2) \\ -cx_3 - ex_4 + y_1 y_2 \\ 0 \end{pmatrix}$$

$$\phi_i(t) = \begin{bmatrix} 0 \\ 0 \\ \dot{m}_i \end{bmatrix}, \quad i = 1, 2$$

$$C_1 = (1 \ 0 \ 0 \ 0), \quad C_2 = (1 \ 0 \ 0).$$

*Remark. 4.1.* The choice of the variables of each subsystem has been considered in order to separate in one subsystem the message  $m_1$  and in the other one the message  $m_2$ . It is clear that other choice can be considered in order to represent these subsystems provided the necessary conditions to design an adaptive are satisfied.

Next, let us introduce an adaptive observer in order to estimate the system's state and the unknown inputs simultaneously. It is based on interconnection between several subsystems which satisfies some required properties, such that the property of inputs persistency ((Hammouri, 90), (Ghanes, 06)).

At first, let us introduce the following assumptions in order to establish the results concerning the adaptive observer design (see more details in (Ghanes, 06)).

*Assumption. 4.1.*

1. The signals  $X_1$  and  $X_2$  are assumed bounded and to be regularly persistent ((Hammouri, 90), (Ghanes, 06)) in order to guarantee the observability property of subsystems (11) and (12), respectively.
2.  $A_1(y_2)$  and  $A_2(y_1)$  are uniformly bounded.
3.  $g_1(y, X_1, X_2)$  is globally Lipschitz with respect to  $X_2$  and uniformly with respect to  $(y, X_1)$ .
4.  $g_2(y, X_2, X_1)$  is globally Lipschitz with respect  $X_1$  and uniformly with respect to  $(y, X_2)$ .
5. The unknown functions  $\dot{m}_i(t)$  ( $i = 1, 2$ ) are assumed to be bounded.

Then, an adaptive observer for interconnected subsystems (11) and (12) estimating the state and unknown parameters is given by

$$\begin{cases} \dot{Z}_1 = A_1(y_2)Z_1 + g_1(y, Z_1, Z_2) + S_1^{-1}C_1^T(y_1 - \hat{y}_1) \\ \dot{S}_1 = -\theta_1 S_1 - A_1^T(y_2)S_1 - S_1 A_1(y_2) + C_1^T C_1 \\ \hat{y}_1 = C_1 Z_1 \end{cases} \quad (13)$$

$$\begin{cases} \dot{Z}_2 = A_2(y_1)Z_2 + g_2(y, Z_2, Z_1) + S_2^{-1}C_2^T(y_2 - \hat{y}_2) \\ \dot{S}_2 = -\theta_2 S_2 - A_2^T(y_1)S_2 - S_2 A_2(y_1) + C_2^T C_2 \\ \hat{y}_2 = C_2 Z_2 \end{cases} \quad (14)$$

where  $Z_1 = (\hat{x}_1, \hat{x}_3, \hat{x}_4, \hat{x}_5)^T$ ;  $Z_2 = (\hat{x}_2, \hat{x}_3, \hat{x}_6)^T$   $S_i = S_i^T > 0$ ,  $i = 1, 2$ . Note that  $S_1^{-1}C_1^T$  and  $S_2^{-1}C_2^T$  are the gains of the observers (13) and (14), respectively.

*Remark. 4.2.* It is worth noticing that  $\|S_1\|$  and  $\|S_2\|$  are bounded for  $\theta_1$  and  $\theta_2$  large enough due to the persistency of input considered in assumption 4.1.

Now, in order to guarantee the convergence of the proposed observer, sufficient conditions are established in the following result. Denote the estimation errors:

$$\epsilon_1 = X_1 - Z_1 \quad \text{and} \quad \epsilon_2 = X_2 - Z_2$$

whose dynamics are given by

$$\begin{cases} \dot{\epsilon}_1 = [A_1(y_2) - S_1^{-1}C_1^T C_1]\epsilon_1 \\ \quad + g_1(y, X_1, X_2) - g_1(y, Z_1, Z_2) + \phi_1(t) \end{cases} \quad (15)$$

$$\begin{cases} \dot{\epsilon}_2 = [A_2(y_1) - S_2^{-1}C_2^T C_2]\epsilon_2 \\ \quad + g_2(y, X_2, X_1) - g_2(y, Z_2, Z_1) + \phi_2(t). \end{cases} \quad (16)$$

The values of  $\theta_1$  and  $\theta_2$  are chosen to satisfy the inequalities

$$\delta_1 = (\theta_1 - \Gamma\eta) > 0, \quad \delta_2 = (\theta_2 - \frac{\Gamma}{\eta}) > 0 \quad (17)$$

where  $\Gamma = \tilde{\mu}_1 + \tilde{\mu}_2$ , with  $\tilde{\mu}_i = \frac{\mu_i}{\sqrt{\lambda_{\min}(S_1)}\sqrt{\lambda_{\min}(S_2)}}$ ,

$i = 1, 2$ ; and  $\mu_1 = k_1 k_2$ ,  $\mu_2 = k_3 k_4$ ,  $\eta \in ]0, 1[$ . The parameters  $k_1, k_2, k_3, k_4$  are positive constants and  $\lambda_{\min}(S_1), \lambda_{\min}(S_2)$  are the minimal eigenvalues of  $S_1$  and  $S_2$  respectively.

*Lemma 4.1.* Consider the system (13)-(14) and that assumption 4.1 holds. Then, the system (13)-(14) is a practical exponential observer for system (11)-(12) for  $\theta_1$  and  $\theta_2$  satisfying the inequalities (17). Furthermore, the observer converges arbitrarily fast with a convergence rate fixed by a parameter  $\delta$ ,  $\delta = \min(\delta_1, \delta_2)$ .

*Sketch of proof 4.1.* Consider the following Lyapunov function candidate:

$$V_o = V_1 + V_2$$

where  $V_1 = \epsilon_1^T S_1 \epsilon_1$  and  $V_2 = \epsilon_2^T S_2 \epsilon_2$ .

From assumption 4.1, the following inequalities hold

$$\begin{aligned}
\|S_1\| &\leq k_1; \\
\|g_1(y, X_1, X_2) - g_1(y, Z_1, Z_2)\| &\leq k_2 \|\epsilon_2\|; \\
\|S_2\| &\leq k_3; \\
\|g_2(y, X_2, X_1) - g_2(y, Z_2, Z_1)\| &\leq k_4 \|\epsilon_1\|. \\
\|\phi_1\| &\leq k_5. \\
\|\phi_2\| &\leq k_6.
\end{aligned}$$

Computing the time derivative of  $V_o$ , by using the above inequalities it follows that

$$\begin{aligned}
\dot{V}_o &\leq -\theta_1 \epsilon_1^T S_1 \epsilon_1 + 2\mu_1 \|\epsilon_1\| \|\epsilon_2\| + \|\phi_1\| \\
&\quad - \theta_2 \epsilon_2^T S_2 \epsilon_2 + 2\mu_2 \|\epsilon_2\| \|\epsilon_1\| + \|\phi_2\|
\end{aligned} \quad (18)$$

Now, consider that the following inequalities are satisfied  $\lambda \min(S_i) \|\epsilon_i\|^2 \leq \|\epsilon_i\|_{S_i}^2 \leq \lambda \max(S_i) \|\epsilon_i\|^2$ ,  $i = 1, 2$ .

By writing (18) in terms of functions  $V_1$  and  $V_2$ , it follows that

$$\dot{V}_o \leq -\theta_1 V_1 - \theta_2 V_2 + 2(\tilde{\mu}_1 + \tilde{\mu}_2) \sqrt{V_1} \sqrt{V_2} + k_5 + k_6$$

where the parameters  $\tilde{\mu}_1, \tilde{\mu}_2$ , are defined just before lemma 4.1.

Next, by using the following inequality  $\sqrt{V_1} \sqrt{V_2} \leq \frac{v}{2} V_1 + \frac{1}{2v} V_2$ ,  $\forall v \in ]0, 1[$ , one get

$$\dot{V}_o \leq -(\theta_1 - \Gamma) V_1 - (\theta_2 - \frac{\Gamma}{v}) V_2 + k_5 + k_6.$$

where  $\Gamma$  is defined just before lemma 4.1.

By taking  $\delta$  and  $r$  such that  $\delta = \min(\delta_1, \delta_2)$  and  $r = k_5 + k_6$  one has

$$\dot{V}_o \leq -\delta V_o + r. \quad (19)$$

Finally, by choosing  $\theta_1$  and  $\theta_2$  such that the inequalities (17) are satisfied and sufficiently large, the inequality (19) shows that arbitrarily bounded perturbation will not result in large error estimation deviations. This ends the proof.

## 5. NUMERICAL EXAMPLE AND DISCUSSIONS

For the system (1) we use the parameters  $a = 42.5$ ,  $b = 24$ ,  $c = 13$ ,  $d = 20$ ,  $e = 50$ ,  $f = 40$ . The parameters used for the super-twisting observer are  $\alpha_1 = 7 \times 10^7$ ,  $\lambda_1 = 7 \times 10^3$ ,  $\alpha_2 = 300$ ,  $\lambda_2 = 100$ ,  $\alpha_3 = 1000$ ,  $\lambda_3 = 200$ ,  $\alpha_4 = 600$ ,  $\lambda_4 = 300$ . For the adaptive observer the parameters used are  $\theta_1 = \theta_2 = 400$ .

Figures 1 and 2 show the trajectories of the states  $x_3$  and  $x_4$  as well as the ones of  $\hat{x}_3$  and  $\hat{x}_4$  for both the super-twisting and the adaptive observers. We can see in the figures that the trajectories of the super-twisting observer converge much faster than the ones of the adaptive observer.

Figures 3 and 4 show the messages  $m_1$  and  $m_2$  together with their estimations  $\hat{m}_1$  and  $\hat{m}_2$ , respectively. In this figures we note that the singularity in the point  $x_1 = 0$  affects more the estimation of the messages made with the super-twisting than the one made with the adaptive observer. This is clear due to the explanation of Remark 3.1 and the fact that, in a ball of radius  $\varepsilon$  and center in  $x_1 = 0$ , it is not done a estimation neither of the states nor of the messages.

In order to test both observers with respect to parameters uncertainties, we introduce a variation of 1% in the nominal parameters. Figures 5 and 6 show how the parameters

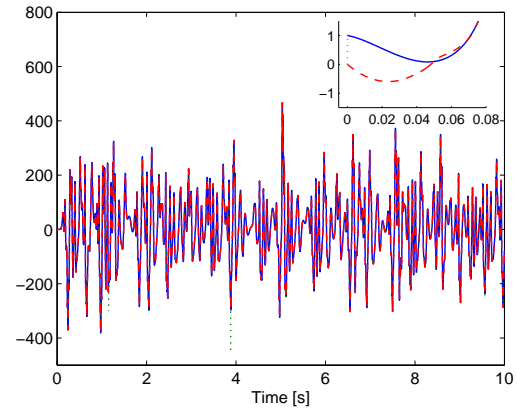


Fig. 1.  $x_3$  (solid line) and its estimation  $\hat{x}_3$  using the super-twisting (dot line) and adaptive (dash line) observers

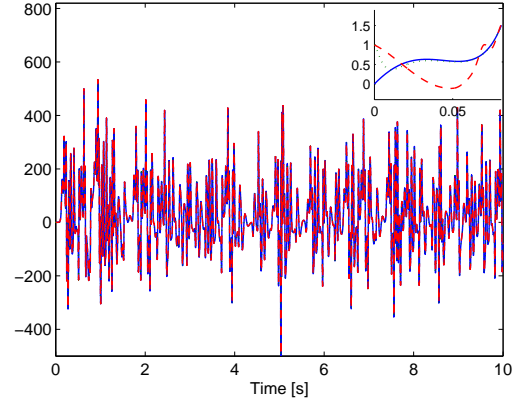


Fig. 2.  $x_4$  (solid line) and its estimation  $\hat{x}_4$  using the super-twisting (dot line) and adaptive (dash line) observers

uncertainties affects the behavior of the observers. In the case of the adaptive observer the parameter uncertainty destroy completely the estimation of the messages. Nevertheless, no matter what observer is used, in this case, the estimation of the messages can be consider unacceptable.

*Remark. 5.1.* For the super-twisting observer we do some simulations with  $\tilde{x} = 0$ , that is without using a linear estimator. In the simulations, not shown here, we obtained a bigger error in the estimation of  $x_4$  which had a big effect in the error of the estimation of  $m_2$ . It was due to the fact that using  $\tilde{x} = 0$ , the variable to be estimated with the super-twisting algorithm became much faster and for having an acceptable estimation the sampling step must be reduce considerably.

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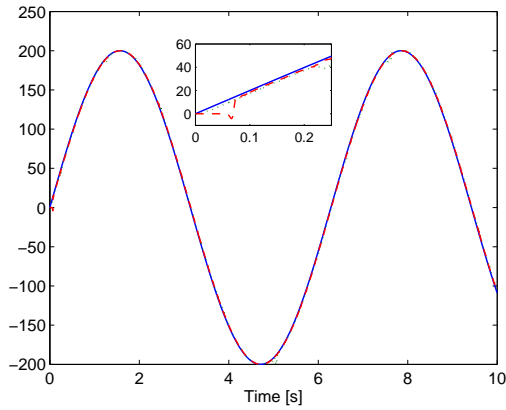


Fig. 3. Message  $m_1$  (solid line) and its estimation  $\hat{m}_1$  using the super-twisting (dot line) and adaptive (dash line) observers

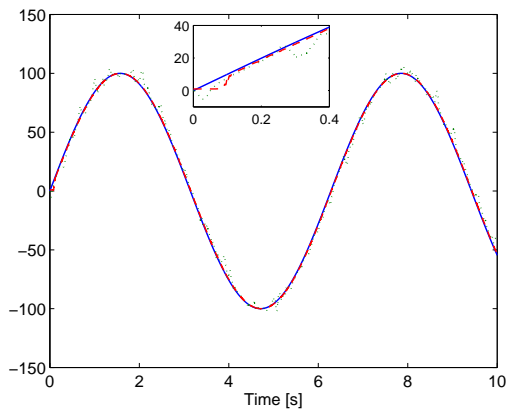


Fig. 4. Message  $m_2$  (solid line) and its estimation  $\hat{m}_2$  using the super-twisting (dot line) and adaptive (dash line) observers

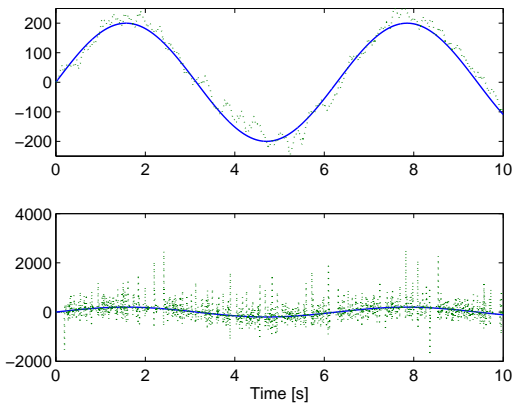


Fig. 5. Message  $m_1$  (solid line) and its estimation  $\hat{m}_1$  (dot line) for the system with 1% of uncertainty in the parameters. Above with the super-twisting observer, below with the adaptive observer

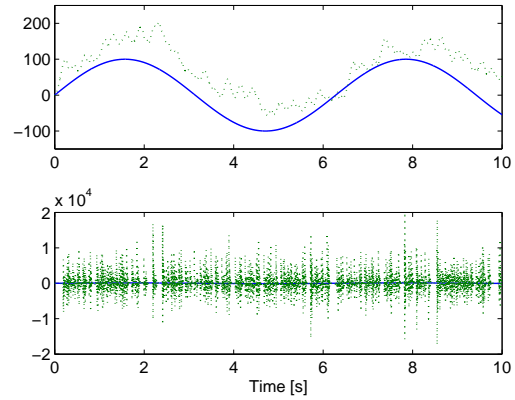


Fig. 6. Message  $m_2$  (solid line) and its estimation  $\hat{m}_2$  (dot line) for the system with 1% of uncertainty in the parameters. Above with the super-twisting observer, below with the adaptive observer

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