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An interesting family of Black-Scholes perpetuities

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Abstract We obtain the Laplace transform and integrability properties of the integral over $\mathbb{R}_+$ of the call quantity associated with geometric Brownian motion with negative drift, thus adding a new element to the list of already studied Brownian perpetuities.

Key words Bessel-McDonald functions, Sturm-Liouville equation, perpetuities.

AMS Classification 60 J 25, 60 J 65, 60 G 44.

1 Motivation, introduction

1.1 Let $(B_t, t \geq 0)$ denote 1-dimensional Brownian motion starting from 0 ; associated to B , one considers the geometric Brownian motion :

$$\left(\mathcal{E}_t := \exp \left(B_t - \frac{t}{2} \right), t \geq 0 \right)$$

which is a positive martingale converging a.s. to 0 as $t \rightarrow \infty$. In recent years, the following questions have been asked to the second author, in connection with European option pricing :

i) to express as simply as possible the quantity :

$$\int_0^\infty \lambda e^{-\lambda t} E((\mathcal{E}_t - k)^+) dt \quad (k \geq 0)$$

ii) to find the law of $\int_0^s (\mathcal{E}_t - k)^+ dt$, for fixed s

iii) to find the law of $\int_0^\infty \lambda e^{-\lambda t} (\mathcal{E}_t - k)^+ dt$

i) may be solved quite explicitly, and indeed this has led the authors, jointly with D. Madan, to write a series of papers relating European option prices and laws of last passage times of continuous positive martingales converging to 0 as $t \rightarrow \infty$ (see, e.g. [MRY1], [MRY2], [MRY3]).

Questions *ii)* and *iii)* are harder to solve explicitly, as, indeed, one may look for double Laplace transforms of either quantities, e.g. :

$$\int_0^\infty e^{-\mu t} E(e^{-\lambda \int_0^t (\mathcal{E}_s - k)^+ ds}) dt$$

In the present paper, we are studying thoroughly the law of $\int_0^\infty (\mathcal{E}_t - k)^+ dt$ which seems to be slightly less difficult than either question *ii)* or *iii)*. However, the results we obtain are not particularly simple.

1.2 These motivations having been presented, we now concentrate exclusively on the law of $\int_0^\infty (\mathcal{E}_t - k)^+ dt$. It will be convenient to consider the two-parameter process :

$$\left(\mathcal{E}_t^{(x)} := x \exp \left(B_t - \frac{t}{2} \right) = \exp \left(\log x + B_t - \frac{t}{2} \right), t \geq 0 \right) \quad (1.1)$$

$(\mathcal{E}_t^{(x)}, t \geq 0, x > 0)$ is a Markov process taking values in $\mathbb{R}_+$ which, most often, we shall denote as : $((\mathcal{E}_t, t \geq 0; P_x, x > 0)$ since $\mathcal{E}_0^{(x)} = x$.

To simplify notation, we shall write P for P_1 and $\mathcal{E}_t$ for $\mathcal{E}_t^{(1)}$.

For any $k > 0$, let us define :

$$\Pi_k^{(x)} := \int_0^\infty (\mathcal{E}_t^{(x)} - k)^+ dt \quad (1.2)$$

Again, to simplify, we shall write Π_k for $\Pi_k^{(1)}$. Since $\mathcal{E}_t^{(x)} \xrightarrow[t \rightarrow \infty]{} 0$ a.s., the integral which defines (1.2) is a.s. convergent, as $(\mathcal{E}_t^{(x)} - k)^+ = 0$ for $t \geq \mathcal{G}_k := \sup\{t; \mathcal{E}_t^{(x)} = k\}$, and $\mathcal{G}_k < \infty$ a.s.

1.3 We now explain about some reductions of the study of the laws of $\Pi_k^{(x)}$, to closely related problems.

i) From Itô's formula, we deduce :

$$\mathcal{E}_t^{(x)} = x + \int_0^t \mathcal{E}_s^{(x)} dB_s = \beta_{A_t^{(x)}} \quad (1.3)$$

where $(\beta_u, u \geq 0)$ denotes the Dubins-Schwarz Brownian motion associated with $(\mathcal{E}_t^{(x)}, t \geq 0)$ (and $\beta_0 = x$) and :

$$A_t^{(x)} = \langle \mathcal{E}^{(x)} \rangle_t = \int_0^t (\mathcal{E}_s^{(x)})^2 ds \quad (1.4)$$

Hence :

$$\Pi_k^{(x)} = \int_0^\infty (\beta_{A_s^{(x)}} - k)^+ ds = \int_0^{T_0(\beta)} \frac{(\beta_v - k)^+}{\beta_v^2} dv \quad (1.5)$$

(after the change of variable $A_s^{(x)} = v$ and where $T_0(\beta) = \inf\{u \geq 0 ; \beta_u = 0\}$)

$$= \int_k^\infty \frac{(y - k)}{y^2} L_{T_0(\beta)}^y dy \quad \left(= \int_k^\infty dz \int_z^\infty \frac{dy}{y^2} L_{T_0(\beta)}^y \right) \quad (1.6)$$

from the density of occupation formula, and where $L_{T_0}^y$ denotes the local time at time T_0 and level y of Brownian motion $(\beta_u, u \geq 0)$.

ii) When $x = k$, the first Ray-Knight Theorem allows us to write, from (1.6) :

$$\Pi_k^{(k)} = \int_k^\infty \frac{(y - k)}{y^2} \lambda_{y-k} dy \quad (1.7)$$

where $(\lambda_z, z \geq 0)$, conditionally on $\lambda_0 = l$, is a 0-dimensional squared Bessel process starting at l , and where λ_0 is an exponential variable with parameter $\frac{1}{2k}$, i.e. with expectation $2k$.

1.4 From the elementary relations :

$$(x e^{B_t - \frac{t}{2}} - k)^+ = x \left(e^{B_t - \frac{t}{2}} - \frac{k}{x} \right)^+ = k \left(\frac{x}{k} (e^{B_t - \frac{t}{2}} - 1) \right)^+ \quad (1.8)$$

valid for every $x, k > 0$, we deduce that the law of Π_k under P_x is that of $x \Pi_{\frac{k}{x}}$ under P , or that of $k \Pi_1$ under $P_{\frac{x}{k}}$. In other words, for every Borel positive function φ , we have :

$$E_x[\varphi(\Pi_k)] = E[\varphi(x \Pi_{\frac{k}{x}})] = E_{\frac{x}{k}}(\varphi(k \Pi_1)) \quad (1.9)$$

These relations allow us to reduce our study of the law of Π_k under P_x to that of Π_1 under $P_{\frac{x}{k}}$. We might as well limit ourselves to the study of Π_k under P .

1.5 It is proven in Dufresne [Duf] and Yor [Y] (see also P. Salminen and M. Yor [SY1]) that, for every $a \neq 0$ and $\nu > 0$:

$$\int_0^\infty \exp(a B_t - \nu t) dt \stackrel{(\text{law})}{=} \frac{2}{a^2 \gamma_{\frac{2\nu}{a^2}}} \quad (1.10)$$

where γ_b denotes a gamma variable with parameter b , i.e. :

$$P(\gamma_b \in dt) = \frac{1}{\Gamma(b)} e^{-t} t^{b-1} dt \quad (t \geq 0) \quad (1.11)$$

In particular, for $a = 1$ and $\nu = \frac{1}{2}$

$$\Pi_0 \stackrel{(\text{law})}{=} \frac{2}{\gamma_1} = \frac{2}{\mathfrak{e}} \quad (1.12)$$

where $\mathfrak{e}$ is a standard exponential variable.

1.6 Let $\nu \neq 0$ and :

$$\mathcal{E}_t^{(x,\nu)} := \exp \left(\nu (\log x + B_t) - \frac{\nu^2 t}{2} \right) = x^\nu \exp \left(\nu B_t - \frac{\nu^2 t}{2} \right)$$

and define, for $k > 0$:

$$\Pi_k^{(x,\nu)} := \int_0^\infty (\mathcal{E}_s^{(x,\nu)} - k)^+ ds \quad (1.13)$$

Since, by scaling, $(\mathcal{E}_t^{(x,\nu)}, t \geq 0) \stackrel{(\text{law})}{=} (x^\nu \mathcal{E}_{\nu^2 t}, t \geq 0)$, we have :

$$\begin{aligned} \Pi_k^{(x,\nu)} &\stackrel{(\text{law})}{=} \int_0^\infty (x^\nu \mathcal{E}_{\nu^2 s} - k)^+ ds = \frac{x^\nu}{\nu^2} \int_0^\infty \left(\mathcal{E}_v - \frac{k}{x^\nu} \right)^+ dv \\ &\stackrel{(\text{law})}{=} \frac{x^\nu}{\nu^2} \Pi_{\frac{k}{x^\nu}} \end{aligned} \quad (1.14)$$

Thus, the study of the law of $\Pi_k^{(x,\nu)}$ may be reduced very simply to that of $\Pi_{\frac{k}{x^\nu}}$. This is the reason why we have chosen, in this paper, to limit ourselves to $\nu = 1$.

1.7 Here are our results :

Theorem 1.1 *Let $\alpha \geq 0$. Then, for every $x > 0$, $E_x((\Pi_1)^\alpha) < \infty$ if and only if $\alpha < 1$*

Theorem 1.2 *Let $\alpha < 0$ and $x > 0$. Then :*

- i) *For every $x > 1$, $E_x((\Pi_1)^\alpha) < \infty$.*
- ii) *For every $x < 1$, $P_x(\Pi_1 = 0) = 1 - x$; hence $E_x((\Pi_1)^\alpha) = +\infty$.*
- iii) *For $x = 1$, $E_1((\Pi_1)^\alpha) < \infty$ if and only if $|\alpha| < \frac{1}{3}$.*

Theorem 1.3 *(Laplace transform of Π_1). For every $\theta \geq 0$:*

$$E_x(e^{-\frac{\theta}{2} \Pi_1}) = \begin{cases} \frac{\sqrt{x} K_\gamma(\sqrt{4\theta x})}{\frac{1}{2} K_\gamma(\sqrt{4\theta}) - \sqrt{\theta} K'_\gamma(\sqrt{4\theta})} & \text{if } x \geq 1 \\ 1 + x \frac{\frac{1}{2} K_\gamma(\sqrt{4\theta}) + \sqrt{\theta} K'_\gamma(\sqrt{4\theta})}{\frac{1}{2} K_\gamma(\sqrt{4\theta}) - \sqrt{\theta} K'_\gamma(\sqrt{4\theta})} & \text{if } 0 < x \leq 1 \end{cases} \quad (1.15)$$

where K_γ denotes the Bessel-McDonald function with index γ (see [Leb], p. 108) and where $\gamma = \sqrt{1 - 4\theta}$ if $4\theta \leq 1$ and $\gamma = i\sqrt{4\theta - 1}$ if $4\theta \geq 1$.

We shall also prove, as a consequence of Theorem 1.3 :

Theorem 1.4 *Let $(\lambda_x, x \geq 0)$ denote a squared Bessel process with dimension 0 started at l and denote its law by $Q_l^{(0)}$. Then :*

- If $4\theta \geq 1$:

$$\begin{aligned} Q_l^{(0)} \left(\exp - \frac{\theta}{2} \int_1^\infty \frac{(x-1)}{x^2} \lambda_{x-1} dx \right) &= Q_l^{(0)} \left(\exp - \frac{\theta}{2} \int_0^\infty \frac{x}{(x+1)^2} \lambda_x dx \right) \\ &= \exp \frac{l}{2} \left(\frac{1}{2} + \frac{\sqrt{\theta} K'_{i\nu}(\sqrt{4\theta})}{K_{i\nu}(\sqrt{4\theta})} \right) \end{aligned} \quad (1.16)$$

with $\nu = \sqrt{4\theta - 1}$

- If $4\theta \leq 1$:

$$Q_l^{(0)} \left(\exp - \frac{\theta}{2} \int_1^\infty \frac{x-1}{x^2} \lambda_{x-1} dx \right) = \exp \frac{l}{2} \left(\frac{1}{2} + \frac{\sqrt{\theta} K'_\nu(\sqrt{4\theta})}{K_\nu(\sqrt{4\theta})} \right)$$

with $\nu = \sqrt{1 - 4\theta}$.

In Section 3 of this work, we shall study the asymptotic behavior, as $\theta \rightarrow \infty$, of $E(e^{-\frac{\theta}{2} \Pi_1})$ and we shall obtain :

$$E(e^{-\frac{\theta}{2} \Pi_1}) \underset{\theta \rightarrow \infty}{\sim} \frac{C}{\theta^{\frac{1}{3}}} \quad (C > 0) \quad (1.17)$$

Finally, in a short Section 4, we shall indicate how Theorems 1.1, 1.2 and 1.3 extend when we replace Π_k by $\Pi_k^{(\rho)}$, with :

$$\begin{aligned} \Pi_k^{(\rho)} &:= \int_0^\infty (e^{\rho(B_t - \frac{t}{2})} - k)^+ dt \quad (\rho, k > 0) \\ &\stackrel{(\text{law})}{=} \frac{1}{\rho^2} \int_0^\infty (e^{(B_u - \frac{u}{2\rho})} - k)^+ du \quad (\text{by scaling}) \end{aligned} \quad (1.18)$$

which, from (1.18), corresponds to consider an extension of our previous perpetuities relative to $(B_t - \frac{t}{2}, t \geq 0)$ to Brownian motion with drift $-\frac{1}{2\rho}$, i.e. $(B_t - \frac{t}{2\rho}, t \geq 0)$.

1.8 For studies of other perpetuities related to Brownian motion with drift, we refer the reader to Salminen-Yor ([SY1], [SY2], [SY3]).

2 Proofs of Theorems 1.1, 1.2 and 1.3

2.1 A first proof of Theorem 1.1

2.1 i) We now prove that, for every $x > 0$, $E_x(\Pi_1) = \infty$
Indeed, from (1.6) :

$$\begin{aligned} E_1(\Pi_1) &= \int_1^\infty \frac{(y-1)}{y^2} E_1(L_{T_0}^y) dy \\ &= \int_1^\infty \frac{(y-1)}{y^2} E_1(L_{T_0}^1) dy \\ &= 2 \int_1^\infty \frac{(y-1)}{y^2} dy = +\infty \end{aligned} \quad (2.1)$$

(since, for $y \geq 1$, $E_1(L_{T_0}^y) = E_1(L_{T_0}^1) = 2$, as $L_{T_0}^1$ is an exponential variable with parameter $\frac{1}{2}$).

2.1 ii) We now prove that, for every $x > 0$, $E_x(\Pi_1) = \infty$

• For $x \geq 1$, this is clear since $\Pi_k^{(x)}$ is an increasing function of x (and a decreasing function of k). Hence :

$$E_x(\Pi_1) \geq E_1(\Pi_1) = +\infty \quad (\text{from (2.1)})$$

• For $x < 1$, from the Markov property :

$$\Pi_1^{(x)} \stackrel{(\text{law})}{=} 1_{\{T_{\log(\frac{1}{x})}^{(-\frac{1}{2})} < \infty\}} \cdot \Pi_1 \quad (2.2)$$

with $T_{\log(\frac{1}{x})}^{(-\frac{1}{2})} := \inf \left\{ t \geq 0 ; B_t - \frac{t}{2} = \log \left(\frac{1}{x} \right) \right\}$, and in (2.2), Π_1 and $T_{\log(\frac{1}{x})}^{(-\frac{1}{2})}$ are assumed independent. Hence :

$$E(\Pi_1^{(x)}) = P(T_{\log(\frac{1}{x})}^{(-\frac{1}{2})} < \infty) E(\Pi_1) = +\infty$$

from (2.1) and since $P(T_{\log(\frac{1}{x})}^{(-\frac{1}{2})} < \infty) = x$ (see Lemma 2.1 below).

2.1 iii) We prove that, for $0 \leq \alpha < 1$, $E_x((\Pi_1)^\alpha) < \infty$
We have :

$$\begin{aligned} E_x((\Pi_1)^\alpha) &= x^\alpha E((\Pi_{\frac{1}{x}})^\alpha) \quad (\text{from (1.9)}) \\ &\leq x^\alpha E((\Pi_0)^\alpha) \end{aligned}$$

(since Π_k is a decreasing function of k)

$$\leq x^\alpha \int_0^\infty \frac{2}{y^\alpha} e^{-y} dy < \infty \quad (\text{from (1.12)})$$

2.2 Second proof of Theorem 1.1

It hinges upon :

Lemma 2.1

Let $G_a^{(\nu)} := \sup\{u \geq 0 ; B_u + \nu u = a\}$ ($= 0$ if this set is empty) and
 $T_a^{(\nu)} := \inf\{u \geq 0 ; B_u + \nu u = a\}$ ($= +\infty$ if this set is empty).

i) If ν and a have the same sign :

$$\bullet \quad E(e^{\mu G_a^{(\nu)}}) < \infty \quad \text{if and only if } \mu < \frac{\nu^2}{2} \quad (2.3)$$

$$\bullet \quad \text{for every real } \alpha, \quad E((G_a^{(\nu)})^\alpha) < \infty \quad (2.4)$$

ii) If ν and a have opposite signs :

$$\bullet \quad E(e^{\mu G_a^{(\nu)}}) < \infty \quad \text{if and only if } \mu < \frac{\nu^2}{2} \quad (2.5)$$

$$\bullet \quad \text{for every real } \alpha > 0, \quad E((G_a^{(\nu)})^\alpha) < \infty \quad (2.6)$$

$$\bullet \quad P(T_a^{(\nu)} = \infty) = P(G_a^{(\nu)} = 0) = 1 - e^{2\nu a} \quad (2.7)$$

Hence, for $\alpha < 0$, $E((G_a^{(\nu)})^\alpha) = +\infty$

The proof of this Lemma is obvious. It hinges on the well-known formulae :

• If $\nu, a > 0$:

$$P(G_a^{(\nu)} \in dt) = \frac{\nu}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}(a - \nu t)^2\right) dt$$

• If $\nu, a < 0$:

$$P(G_a^{(-\nu)} \in dt) = \frac{\nu}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}(a + \nu t)^2\right) dt$$

• If ν and a have opposites signs :

$$P(G_a^{(\nu)} > 0) = P(T_a^{(\nu)} < \infty) = e^{2\nu a}$$

$$\bullet \quad G_a^{(\nu)} \stackrel{(\text{law})}{=} G_{(-a)}^{(-\nu)} \stackrel{(\text{law})}{=} \frac{1}{T_\nu^{(a)}} \quad \blacksquare$$

We now give a second proof of Theorem 1.1.

• We first show that, for $0 \leq \alpha < 1$, $E_x((\Pi_1)^\alpha) < \infty$. From the relation :

$$\begin{aligned} \Pi_1^{(x)} &= x \int_0^\infty \left(\mathcal{E}_t - \frac{1}{x}\right)^+ dt = x \int_0^{G_{-\log x}^{(-\frac{1}{2})}} \left(\mathcal{E}_t - \frac{1}{x}\right)^+ dt \\ &\leq x \left(\sup_{t \geq 0} \mathcal{E}_t\right) \cdot G_{-\log x}^{(-\frac{1}{2})} \end{aligned}$$

we deduce that, for $0 \leq \alpha < 1$, $\alpha p < 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ ($p, q > 1$) :

$$\begin{aligned} E_x((\Pi_1)^\alpha) &\leq x^\alpha E \left[\left(\sup_{t \geq 0} \mathcal{E}_t \right)^\alpha \left(G_{-\log x}^{(-\frac{1}{2})} \right)^\alpha \right] \\ &\leq x^\alpha E \left(\left(\sup_{t \geq 0} \mathcal{E}_t \right)^{\alpha p} \right)^{\frac{1}{p}} E \left(\left(G_{(-\log x)}^{(-\frac{1}{2})} \right)^{\alpha q} \right)^{\frac{1}{q}} \\ &\leq C x^\alpha E \left(\left(\sup_{t \geq 0} \mathcal{E}_t \right)^{\alpha p} \right)^{\frac{1}{p}} \quad (\text{from Lemma 2.1}) \end{aligned}$$

As $(\mathcal{E}_t, t \geq 0)$ is a positive martingale, starting from 1, and converging a.s. to 0 as $t \rightarrow \infty$, we get (see [NY] or [RY], Chap. II, Ex. 3.12, p. 73) :

$$\sup_{s \geq 0} \mathcal{E}_s \stackrel{(\text{law})}{=} \frac{1}{U} \quad (2.8)$$

with U uniform on $[0, 1]$. Thus :

$$E \left(\left(\sup_{t \geq 0} \mathcal{E}_t \right)^{\alpha p} \right) = \int_0^1 \frac{1}{u^{\alpha p}} du < \infty \quad (2.9)$$

since $\alpha p < 1$.

• We then show that $E_x(\Pi_1) = \infty$. First of all, it follows from [MRY1] that :

$$E_1((\mathcal{E}_t - k)^+) = P(G_{\log k}^{(\frac{1}{2})} \leq t) \quad (2.10)$$

Hence :

$$E_1(\Pi_1) = \int_0^\infty P(G_0^{(\frac{1}{2})} \leq t) dt = E \left(\int_{G_0^{(\frac{1}{2})}}^\infty dt \right) = +\infty$$

since $G_0^{(\frac{1}{2})} < +\infty$ a.s.

Likewise :

$$\begin{aligned} E_x(\Pi_1) &= x E_1(\Pi_{\frac{1}{x}}) \quad (\text{from (1.9)}) \\ &= x E \left(\int_{G_{\log \frac{1}{x}}^{(\frac{1}{2})}}^\infty dt \right) = \infty \end{aligned}$$

from (2.10) and since $G_{\log \frac{1}{x}}^{(\frac{1}{2})} < \infty$ a.s.

2.3 Proof of Theorem 1.2

i) We already prove point *i*) :

Let $x > 1$ and x' such that $1 < x' < x$. It is then obvious that :

$$\Pi_1^{(x)} \geq (x' - 1)T_{\log(\frac{x'}{x})}^{(-\frac{1}{2})} \quad (2.11)$$

(with $T_a^{(-\frac{1}{2})} := \inf\{t \geq 0 ; B_t - \frac{t}{2} = a\}$, since, if $t < T_{\log \frac{x'}{x}}^{(-\frac{1}{2})}$, then $e^{\log x + B_t - \frac{t}{2}} - 1 \geq x' - 1$).

Hence, with $\gamma > 0$:

$$\begin{aligned} E_x \left(\frac{1}{(\Pi_1)^\gamma} \right) &= \int_0^\infty P_x \left(\frac{1}{(\Pi_1)^\gamma} \geq t \right) dt = \int_0^\infty P_x(\Pi_1 < v) \frac{\gamma}{v^{1+\gamma}} dv \\ &\leq \int_0^\infty \frac{\gamma}{v^{1+\gamma}} dv P \left(T_{\log \frac{x'}{x}}^{(-\frac{1}{2})} \leq \frac{v}{x' - 1} \right) dv \quad (\text{from (2.11)}) \\ &= \int_0^\infty \frac{\gamma}{v^{1+\gamma}} dv \left(P(G_{\frac{1}{2}}^{(\log(\frac{x'}{x}))} \geq \frac{x' - 1}{v}) \right) dv \\ &\quad (\text{since } T_a^{(\nu)} \stackrel{(\text{law})}{=} \frac{1}{G_\nu^{(a)}} \text{ and } G_\nu^{(a)} \stackrel{(\text{law})}{=} G_{-\nu}^{(-a)}) \\ &= \int_0^\infty \frac{\gamma}{(x' - 1)^\gamma} u^{\gamma-1} du P(G_{\frac{1}{2}}^{(\log(\frac{x'}{x}))} \geq u) du \\ &= \frac{1}{(x' - 1)^\gamma} E((G_{\frac{1}{2}}^{(\log \frac{x'}{x})})^\gamma) < \infty \end{aligned} \quad (2.12)$$

(from point *i*) of Lemma 2.1).

ii) We now prove point *ii*) :

It is clear that, for $x < 1$:

$$\{\Pi_1^{(x)} = 0\} = \{T_{\log \frac{1}{x}}^{(-\frac{1}{2})} = \infty\}$$

Thus, $P\{\Pi_1^{(x)} = 0\} = P\{T_{\log \frac{1}{x}}^{(-\frac{1}{2})} = \infty\} = P\{G_{\frac{1}{2}}^{(\log x)} = 0\} = 1 - x > 0$ from Lemma 2.1 and (1.26).

iii) We now prove point *iii*) :

For this purpose, we write, for $\gamma > 0$:

$$E_1 \left(\frac{1}{(\Pi_1)^\gamma} \right) = \frac{1}{\Gamma(\gamma)} \int_0^\infty E(e^{-\theta \Pi_1}) \theta^{\gamma-1} d\theta \quad (2.13)$$

and we show, in the next Section 3, that :

$$E_1(e^{-\theta \Pi_1}) \underset{\theta \rightarrow \infty}{\sim} \frac{C}{\theta^{\frac{1}{3}}} \quad (C > 0)$$

Thus, $E \left(\frac{1}{(\Pi_1)^\gamma} \right) < \infty$ if and only if $\int_1^\infty \theta^{\gamma-1-\frac{1}{3}} d\theta < \infty$, that is, if and only if $\gamma < \frac{1}{3}$.

2.4 Proof of Theorem 1.3

a) A useful Lemma

Lemma 2.2 *Let $\theta \geq 0$ and K_γ the Bessel McDonald function with index γ , such that $\gamma = \sqrt{1-4\theta}$ if $\theta \leq \frac{1}{4}$ and $\gamma = i\sqrt{4\theta-1}$ if $\theta \geq \frac{1}{4}$.*

1) Define the function $\varphi_\theta : \mathbb{R}_+ \rightarrow \mathbb{R}$ by :

$$\varphi_\theta(y) = \sqrt{y} K_\gamma(\sqrt{4\theta y}) \quad (y \geq 0) \quad (2.14)$$

Then :

i) φ_θ is a real valued function which satisfies :

$$\varphi_\theta''(y) + \left(-\frac{\theta}{y} + \frac{\theta}{y^2}\right) \varphi_\theta(y) = 0 \quad (2.15)$$

ii) φ_θ , restricted to $[1, \infty[$ is positive, convex, bounded and decreasing.

2) We define the function $\tilde{\varphi}_\theta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by :

$$\tilde{\varphi}_\theta(y) = \begin{cases} \varphi_\theta(y) & \text{if } y \geq 1 \\ (\varphi_\theta(1) - \varphi_\theta'(1)) + y\varphi_\theta'(1) & \text{if } 0 \leq y \leq 1 \end{cases} \quad (2.16)$$

Then $\tilde{\varphi}_\theta$ is a bounded, positive, convex, decreasing function which satisfies :

$$\tilde{\varphi}_\theta''(y) + \left(-\frac{\theta}{y} + \frac{\theta}{y^2}\right) 1_{y \geq 1} \cdot \tilde{\varphi}_\theta(y) = 0 \quad (2.17)$$

b) Proof of Lemma 2.2

i) Relation (2.15) (as well as relation (2.17)) follows from a direct computation, using the equation $K_\gamma''(z) + \frac{1}{z} K_\gamma'(z) - \left(1 + \frac{\gamma^2}{z^2}\right) K_\gamma(z) = 0$ (see [Leb], p. 110) and the fact that $\gamma^2 = 1 - 4\theta$ (see Petiau [Pet], p. 306, formula (8), which needs to be corrected by replacing a by $-a$, or Kamke [Kam], p. 440).

We distinguish two cases :

Case 1 : $4\theta \leq 1$, $\gamma = \sqrt{1-4\theta}$. The function K_γ , hence also φ_θ , is positive. Furthermore φ_θ is bounded on $\mathbb{R}_+$ since :

$$\varphi_\theta(y) \underset{y \rightarrow 0}{\sim} C y^{\frac{1-\gamma}{2}}, \quad \varphi_\theta(y) \underset{y \rightarrow \infty}{\sim} C' y^{\frac{1}{4}} e^{-\sqrt{4\theta y}} \quad (\text{see [Leb], p. 123 and 136})$$

On the other hand, from (2.15), the function φ_θ is convex on the interval $[1, \infty[$. As it is convex, positive, and bounded, it is decreasing. Lemma 2.2 is thus proven in this case.

Figure 1: Graphs of φ_θ and $\tilde{\varphi}_\theta$

Case 2 : $4\theta \geq 1$, $\gamma = i\sqrt{4\theta - 1}$ and here $\varphi_\theta(y) = \sqrt{y} K_{i\nu}(\sqrt{4\theta y})$ with $\nu = \sqrt{4\theta - 1}$. From the integral representation formula (see [Leb], p. 119) :

$$K_{i\nu}(y) = \int_0^\infty e^{-y \cosh u} \cos(\nu u) du \quad (y \geq 0) \quad (2.18)$$

we deduce that $K_{i\nu}(y)$ is real valued, hence so is $\varphi_\theta(y)$.

On the other hand, φ_θ is bounded on $[0, \infty[$. Indeed, for $y \geq 0$, from (2.18) :

$$\left| \sqrt{4\theta} \varphi_\theta \left(\frac{y^2}{4\theta} \right) \right| = y |K_{i\nu}(y)| \leq y \int_0^\infty e^{-y \frac{e^u}{2}} du \leq y \int_{\frac{y}{2}}^\infty e^{-v} \frac{dv}{v}$$

Thus :

$$\begin{aligned} \left| \sqrt{4\theta} \varphi_\theta \left(\frac{y^2}{4\theta} \right) \right| &\leq e^{-\frac{y}{4}} \quad \text{for } y \text{ large enough, hence} \\ \left| \sqrt{4\theta} \varphi_\theta \left(\frac{y^2}{4\theta} \right) \right| &\xrightarrow{y \rightarrow \infty} 0 \quad \text{and} \\ \left| \sqrt{4\theta} \varphi_\theta \left(\frac{y^2}{4\theta} \right) \right| &\leq y (C + |\log(y)|) \xrightarrow{y \rightarrow 0} 0 \end{aligned}$$

On the other hand, it is clear, from (2.18), that $K_{i\nu}(y) > 0$ hence $\varphi_\theta(y) > 0$, for y large enough.

• We now show that $K_{i\nu}$ is decreasing on $[\nu, \infty[$. If not, there would exist a point $y_0 > \nu$ such that :

$$K_{i\nu}(y_0) > 0, \quad K'_{i\nu}(y_0) = 0, \quad K''_{i\nu}(y_0) \leq 0.$$

However,

$$K''_{i\nu}(y_0) + \frac{1}{y_0} K'_{i\nu}(y_0) = \left(1 - \frac{\nu^2}{y_0^2}\right) K_{i\nu}(y_0)$$

Thus $K''_{i\nu}(y_0) > 0$, which is absurd. Since $K_{i\nu}$ is decreasing on $[\nu, \infty[$, and positive near ∞ , $K_{i\nu}$ is positive on $[\nu, \infty[$. Thus, φ_θ is positive on $[1, \infty[$ (since, if $y \geq 1$, $\sqrt{4\theta y} \geq \sqrt{4\theta} \geq \sqrt{4\theta - 1} = \nu$). From the relation (2.15), we then deduce that φ_θ is convex on $[1, \infty[$. Since it is bounded, convex and positive, it is decreasing. Lemma 2.2 is proven. ■

c) **End of the proof of Theorem 1.3.**

Let $\left(M_t^\theta := \tilde{\varphi}_\theta(B_t) \exp -\frac{\theta}{2} \int_0^t \frac{\tilde{\varphi}_\theta''(B_s)}{\varphi_\theta} ds, t \geq 0\right)$. Then $(M_t^\theta, t \geq 0)$ is a local martingale. It is equal, from (2.17) to :

$$M_t^\theta = \tilde{\varphi}_\theta(B_t) \exp -\frac{\theta}{2} \int_0^t \frac{(B_s - 1)^+}{B_s^2} ds \quad (2.19)$$

and, from Lemma 2.2, for every $x \geq 0$, $(M_{t \wedge T_0}^\theta, t \geq 0)$ is bounded. Thus, from Doob's optional stopping Theorem :

$$\begin{aligned} \tilde{\varphi}_\theta(x) &= E_x(M_0^\theta) = E_x \left[\tilde{\varphi}_\theta(B_{T_0}) \exp -\frac{\theta}{2} \int_0^{T_0(B)} \frac{(B_s - 1)^+}{B_s^2} ds \right] \\ &= \tilde{\varphi}_\theta(0) E_x \left[\exp -\frac{\theta}{2} \Pi_1 \right] \quad (\text{from (1.5)}) \end{aligned}$$

Thus :

$$E_x(e^{-\frac{\theta}{2} \Pi_1}) = \frac{\tilde{\varphi}_\theta(x)}{\tilde{\varphi}_\theta(0)}$$

This is precisely Theorem 1.3, owing to formula (2.16) which yields $\tilde{\varphi}_\theta$ explicitly. ■

Remark 2.3 Theorem 1.3 allows to recover formula (1.9) : $\Pi_0 \stackrel{(\text{law})}{=} \frac{2}{\epsilon}$. Indeed, on one hand :

$$E_1(e^{-\frac{\theta}{2} \Pi_0}) = E(e^{-\frac{\theta}{2} \frac{2}{\epsilon}}) = \int_0^\infty e^{-\frac{\theta}{z} - z} dz = 2\sqrt{\theta} K_1(\sqrt{4\theta}) \quad (\text{see [Leb], p. 119})$$

and, on the other hand :

$$\begin{aligned}
E_1(e^{-\frac{\theta}{2} \Pi_0}) &= \lim_{\epsilon \downarrow 0} E_1(e^{-\frac{\theta}{2} \Pi_\epsilon}) = \lim_{\epsilon \downarrow 0} E \left(\exp -\frac{\theta}{2} \int_0^\infty (e^{B_t - \frac{t}{2}} - \epsilon)^+ dt \right) \\
&= \lim_{\epsilon \downarrow 0} E \left(e^{-\frac{\theta \epsilon}{2} \int_0^\infty (\frac{1}{\epsilon} (e^{B_t - \frac{t}{2}} - 1)^+ dt) \right) = \lim_{\epsilon \downarrow 0} E_{\frac{1}{\epsilon}}(e^{-\frac{\theta \epsilon}{2} \Pi_1}) \\
&= \lim_{\epsilon \downarrow 0} \frac{\frac{1}{\sqrt{\epsilon}} K_{\sqrt{1-4\theta\epsilon}}(\sqrt{4\theta})}{\frac{1}{2} K_{\sqrt{1-4\theta\epsilon}}(\sqrt{4\theta\epsilon}) - \frac{1}{2} \sqrt{4\theta\epsilon} K'_{\sqrt{1-4\theta\epsilon}}(\sqrt{4\theta\epsilon})} \\
&\quad \text{(from (1.5), replacing } \theta \text{ by } \theta\epsilon \text{ and } x \text{ by } \frac{1}{\epsilon}) \\
&= \lim_{\epsilon \downarrow 0} \frac{\frac{1}{\sqrt{\epsilon}} K_1(\sqrt{4\theta}) + O(1)}{\frac{1}{2} K_{\sqrt{1-4\theta\epsilon}}(\sqrt{4\theta\epsilon}) [1 - \sqrt{1-4\theta\epsilon}] + \sqrt{\theta\epsilon} K_{\sqrt{1-4\theta\epsilon}+1}(\sqrt{4\theta\epsilon})} \\
&\quad \text{(since } zK'_\nu(z) = \nu K_\nu(z) - zK_{\nu+1}(z); \text{ see [Leb], p. 110)} \\
&= \lim_{\epsilon \downarrow 0} \frac{\frac{1}{\sqrt{\epsilon}} K_1(\sqrt{4\theta}) + O(1)}{O(1) + \sqrt{\theta\epsilon} (\frac{1}{2\theta\epsilon} + O(1))} = 2\sqrt{\theta} K_1(\sqrt{4\theta})
\end{aligned}$$

■

2.5 Proof of Theorem 1.4

From Theorem 1.3, we know that, for $\theta \geq \frac{1}{4}$:

$$\begin{aligned}
E_1(e^{-\frac{\theta}{2} \Pi_1}) &= \frac{2K_{i\nu}(\sqrt{4\theta})}{K_{i\nu}(\sqrt{4\theta}) - \sqrt{4\theta} K'_{i\nu}(\sqrt{4\theta})} \\
&= \frac{1}{\frac{1}{2} - \sqrt{\theta} \frac{K'_{i\nu}(\sqrt{4\theta})}{K_{i\nu}(\sqrt{4\theta})}} = \frac{1}{2} \int_0^\infty e^{-\frac{l}{2} + \frac{l}{2} \left(\frac{1}{2} + \sqrt{\theta} \frac{K'_{i\nu}(\sqrt{4\theta})}{K_{i\nu}(\sqrt{4\theta})} \right)} dl
\end{aligned} \tag{2.20}$$

On the other hand, from (1.7) :

$$\begin{aligned}
E_1(e^{-\frac{\theta}{2} \Pi_1}) &= E_1 \left(\exp -\frac{\theta}{2} \int_1^\infty \frac{(y-1)^+}{y^2} \lambda_{y-1} dy \right) \\
&= \frac{1}{2} \int_0^\infty e^{-\frac{l}{2}} dl E \left(\exp -\frac{\theta}{2} \int_1^\infty \frac{(y-1)^+}{y^2} \lambda_{y-1} dy \middle| \lambda_0 = l \right) \\
&= \frac{1}{2} \int_0^\infty e^{-\frac{l}{2}} dl Q_l^{(0)} \left(\exp -\frac{\theta}{2} \int_0^\infty \frac{y}{(1+y)^2} \lambda_y dy \right)
\end{aligned} \tag{2.21}$$

where $Q_l^{(0)}$ denotes the expectation relative to a squared Bessel process of dimension 0, starting from l . The comparison of (2.21) and (2.20) implies Theorem 1.4 in the case

$\theta \geq \frac{1}{4}$, since the Laplace transform is one-to-one. The proof of Theorem 1.4 in the case $\theta \leq \frac{1}{4}$ is the same. $\blacksquare$

Remark 2.4 Let μ denote a positive σ -finite measure on $\mathbb{R}_+$. Let Φ denote the unique decreasing, positive solution on $\mathbb{R}_+$ of the Sturm-Liouville equation $\Phi'' = \mu\Phi$ (and such that $\Phi(0) = 1$). It is well known (see [RY], Chap. IX, p. 444) that :

$$Q_l^{(0)} \left(\exp \left(-\frac{1}{2} \int_0^\infty \lambda_y \mu(dy) \right) \right) = \exp \left(\frac{l}{2} \Phi'(0_+) \right) \quad (2.22)$$

(Observe that, as $(\lambda_y, y \geq 0)$ has compact support a.s., $\int_0^\infty \lambda_y \mu(dy) < \infty$ if μ is σ -finite on $\mathbb{R}_+$).

Theorem 1.4 may be recovered easily by applying formula (2.22) with $\mu(dy) = \frac{y}{1+y^2} dy$.

On the other hand, if $\mu(dy) = a(y)dy$, with

$$c_1 1_{[0, \gamma_1]}(y) \leq a(y) \leq c_2 1_{[0, \gamma_2]}(y) \quad (0 < c_1 \leq c_2, 0 < \gamma_1 \leq \gamma_2),$$

we deduce from [RY], Chap. XI, Corollary 1.8, that :

$$\exp -\frac{l}{2} c'_2 \sqrt{\theta} \leq E_l^{(0)} \left(e^{-\frac{\theta}{2} \int_0^\infty \lambda_y \mu(dy)} \right) \leq \exp -\frac{l}{2} c'_1 \sqrt{\theta} \quad (2.23)$$

for θ large enough, whereas, as we shall show in Section 3 :

$$Q_l^{(0)} \left(\exp -\frac{\theta}{2} \int_0^\infty \frac{y}{1+y^2} \lambda_y dy \right) \underset{\theta \rightarrow \infty}{\sim} C e^{-\frac{l}{2} \theta^{\frac{1}{3}}} \quad (2.24)$$

3 Asymptotic behavior of $E_1(e^{-\frac{\theta}{2} \Pi_1})$ as $\theta \rightarrow \infty$

We shall now end the proof of Theorem 1.2 by showing :

Theorem 3.1 *There is the equivalence result:*

$$E_1(e^{-\frac{\theta}{2} \Pi_1}) \underset{\theta \rightarrow \infty}{\sim} \frac{C}{\theta^{\frac{1}{3}}} \quad (3.1)$$

Proof of Theorem 3.1

We recall that, from Theorem 1.3 :

$$E_1(e^{-\frac{\theta}{2} \Pi_1}) = \frac{2K_{i\nu}(\sqrt{4\theta})}{K_{i\nu}(\sqrt{4\theta}) - \sqrt{4\theta} K'_{i\nu}(\sqrt{4\theta})} \quad (3.2)$$

with $\nu = \sqrt{4\theta - 1} (\theta \geq \frac{1}{4})$. We shall successively find an equivalent of the numerator and the denominator of (3.2), the difficulty arising from the fact that, in $K_{i\nu}(\sqrt{4\theta})$ (and

$K'_{i\nu}(\sqrt{4\theta})$) the argument $\sqrt{4\theta}$ and index $i\nu = i\sqrt{4\theta - 1}$ tend both to infinity as $\theta \rightarrow \infty$. To overcome this difficulty, we shall use some results about Bessel functions found in Watson ([Wat], p. 245-248), which we now recall.

i) Let $H_{i\nu}^{(1)}$ the first Hankel function (see [Leb], p. 120) ; it is related to $K_{i\nu}$ via the formula :

$$K_{i\nu}(z) = \frac{i\pi}{2} e^{-\frac{\nu\pi}{2}} H_{i\nu}^{(1)}(z e^{\frac{i\pi}{2}}) \quad (3.3)$$

We define ϵ by the formula : $i\nu = iz(1 - \epsilon)$ and assume that, as $z \rightarrow \infty$, ϵ remains bounded (with, of course, ν depending on z). Then, there is the second order asymptotic expansion :

$$H_{\nu}^{(1)}(z) = -\frac{2}{3\pi} \left\{ e^{\frac{2}{3}\pi i} \left(\sin \frac{\pi}{3} \right) \frac{\Gamma(\frac{1}{3})}{(\frac{1}{6}z)^{\frac{1}{3}}} + e^{\frac{4}{3}\pi i} (\epsilon z) \left(\sin \frac{2\pi}{3} \right) \frac{\Gamma(\frac{2}{3})}{(\frac{1}{6}z)^{\frac{2}{3}}} + o\left(\frac{1}{z^{\frac{2}{3}}}\right) \right\} \quad (3.4)$$

ii) Let us study the numerator of (3.2) :

$$\begin{aligned} N &= 2K_{i\nu}(\sqrt{4\theta}) = 2 \frac{i\pi}{2} e^{-\frac{\nu\pi}{2}} H_{i\nu}^{(1)}(e^{\frac{i\pi}{2}} \sqrt{4\theta}) \\ &= (i\pi) e^{-\frac{\nu\pi}{2}} \left(-\frac{2}{3\pi} \right) \left\{ C_1 e^{\frac{2}{3}\pi i} \frac{1}{(e^{\frac{i\pi}{2}} \sqrt{4\theta})^{\frac{1}{3}}} + o\left(\frac{1}{\theta^{\frac{1}{6}}}\right) \right\} \\ &\quad \left(\text{with } C_1 = \left(\sin \frac{\pi}{3} \right) \Gamma\left(\frac{1}{3}\right) 6^{\frac{1}{3}} \right) \end{aligned} \quad (3.5)$$

$$\underset{\theta \rightarrow \infty}{\sim} \frac{2^{\frac{2}{3}}}{3} C_1 e^{-\frac{\nu\pi}{2}} \frac{1}{\theta^{\frac{1}{6}}} \quad (3.6)$$

Here, we have used the first order expansion (3.4) with : $e^{\frac{2}{3}i\pi} \cdot e^{-\frac{i\pi}{6}} = i \left(\frac{4\pi}{6} - \frac{\pi}{6} = \frac{3\pi}{6} = \frac{\pi}{2} \right)$ and the fact that here

$$\begin{aligned} \epsilon z &= z - \nu \\ &= i\sqrt{4\theta} - i\sqrt{4\theta - 1} = i\sqrt{4\theta} \left(1 - \sqrt{1 - \frac{1}{4\theta}} \right) \xrightarrow{\theta \rightarrow \infty} 0 \end{aligned}$$

iii) We now study the denominator of (3.2)

$$\begin{aligned} D &= K_{i\nu}(\sqrt{4\theta}) - \sqrt{4\theta} K'_{i\nu}(\sqrt{4\theta}) \\ &= K_{i\nu}(\sqrt{4\theta}) + \sqrt{4\theta} K_{i\nu-1}(\sqrt{4\theta}) + i\sqrt{4\theta - 1} K_{i\nu}(\sqrt{4\theta}) \end{aligned} \quad (3.7)$$

(after using $\frac{d}{dz}(z^\mu K_\mu(z)) = -z^\mu K_{\mu-1}(z)$) ; [Leb], p. 110). Since we have already studied the asymptotic behavior of $K_{i\nu}(\sqrt{4\theta})$, it remains to study that of

$$\Delta(\theta) := \sqrt{4\theta} K_{i\nu-1}(\sqrt{4\theta}) + i\sqrt{4\theta - 1} K_{i\nu}(\sqrt{4\theta}) \quad (3.8)$$

Now, since $\sqrt{4\theta} K_{i\nu-1}(\sqrt{4\theta}) + i\sqrt{4\theta-1} K_{i\nu}(\sqrt{4\theta})$ is real (from (2.18)) and since $i\sqrt{4\theta-1} K_{i\nu}(\sqrt{4\theta})$ is purely imaginary, the development of $\Delta(\theta)$ as $\theta \rightarrow \infty$ is that of the real part of $\sqrt{4\theta} K_{i\nu-1}(\sqrt{4\theta})$. Then, we obtain :

$$\begin{aligned}\sqrt{4\theta} K_{i\nu-1}(\sqrt{4\theta}) &= \sqrt{4\theta} \frac{i\pi}{2} e^{\frac{i\pi}{2}(i\nu-1)} H_{i\nu-1}^{(1)}(e^{\frac{i\pi}{2}} \sqrt{4\theta}) \\ &= \sqrt{4\theta} \frac{\pi}{2} e^{-\frac{\nu\pi}{2}} H_{i\nu-1}^{(1)}(e^{\frac{i\pi}{2}} \sqrt{4\theta})\end{aligned}\quad (3.9)$$

from (3.3), by replacing $i\nu$ by $i\nu-1$. This time, we use the second order development (3.4) with here : $z = e^{\frac{i\pi}{2}} \sqrt{4\theta}$ and :

$$\begin{aligned}i\nu-1 = z(1-\epsilon) &= i\sqrt{4\theta}(1-\epsilon), \text{ i.e. : } \quad \epsilon z = \epsilon e^{\frac{i\pi}{2}} \sqrt{4\theta} \\ \text{or } \quad \epsilon z &= \epsilon e^{\frac{i\pi}{2}} \sqrt{4\theta} = 1 + \frac{1}{4\sqrt{\theta}} + o\left(\frac{1}{\sqrt{\theta}}\right) \quad (\nu = \sqrt{4\theta-1})\end{aligned}\quad (3.10)$$

We obtain :

$$\begin{aligned}\sqrt{4\theta} K_{i\nu-1}(\sqrt{4\theta}) &= \sqrt{4\theta} \frac{\pi}{2} e^{-\frac{\nu\pi}{2}} \left(-\frac{2}{3}\pi\right) \times \dots \\ &\dots \left\{ C_1 e^{\frac{2i\pi}{3}} \frac{1}{(e^{\frac{i\pi}{2}} \sqrt{4\theta})^{\frac{1}{3}}} + C_2 e^{\frac{4i\pi}{3}} \left(1 + \frac{i}{4\sqrt{\theta}}\right) \cdot \frac{1}{(e^{\frac{i\pi}{2}} \sqrt{4\theta})^{\frac{2}{3}}} + o\left(\frac{1}{\theta^{\frac{1}{3}}}\right) \right\}\end{aligned}\quad (3.11)$$

with $C_1 = \left(\sin \frac{\pi}{3}\right) \Gamma\left(\frac{1}{3}\right) 6^{\frac{1}{3}}$ and $C_2 = \left(\sin \frac{2\pi}{3}\right) \Gamma\left(\frac{2}{3}\right) 6^{\frac{2}{3}}$ ($C_1, C_2 > 0$)

$$= -\frac{1}{3} \sqrt{4\theta} e^{-\frac{\nu\pi}{2}} \left\{ C_1 i \frac{1}{(\sqrt{4\theta})^{\frac{1}{3}}} - C_2 \left(1 + \frac{i}{4\sqrt{\theta}}\right) \frac{1}{(\sqrt{4\theta})^{\frac{2}{3}}} + o\left(\frac{1}{\theta^{\frac{1}{3}}}\right) \right\} \quad (3.12)$$

Considering now the real part of (3.12) to obtain :

$$\begin{aligned}\Delta(\theta) &= \mathcal{R}e(\sqrt{4\theta} K_{i\nu-1}(\sqrt{4\theta}) + i\sqrt{4\theta-1} K_{i\nu}(\sqrt{4\theta})) \\ &= \frac{1}{3} \sqrt{4\theta} e^{-\frac{\nu\pi}{2}} C_2 \left(\frac{1}{(\sqrt{4\theta})^{\frac{2}{3}}} + o\left(\frac{1}{\theta^{\frac{1}{3}}}\right) \right) \\ &\underset{\theta \rightarrow \infty}{\sim} \frac{2^{\frac{1}{3}} C_2}{3} e^{-\frac{\nu\pi}{2}} \theta^{\frac{1}{6}}\end{aligned}\quad (3.13)$$

• We then gather (3.2), (3.6) and (3.13), we obtain :

$$\begin{aligned}E_1(e^{-\frac{\theta}{2} \Pi_1}) &\underset{\theta \rightarrow \infty}{\sim} \frac{2^{\frac{2}{3}} C_1 e^{-\frac{\nu\pi}{2}} \frac{1}{\theta^{\frac{1}{6}}}}{\frac{2^{\frac{2}{3}}}{3} C_1 e^{-\frac{\nu\pi}{2}} \frac{1}{\theta^{\frac{1}{6}}} + \frac{2^{\frac{1}{3}}}{3} C_2 e^{-\frac{\nu\pi}{2}} \theta^{\frac{1}{6}}} \\ &\underset{\theta \rightarrow \infty}{\sim} \frac{2^{\frac{4}{3}} C_1}{C_2} \frac{1}{\theta^{\frac{1}{3}}} = \frac{2}{3^{\frac{1}{3}}} \frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})} \frac{1}{\theta^{\frac{1}{3}}}\end{aligned}\quad (3.14)$$

from the explicit formulae for C_1 and C_2 . This is Theorem 3.1.

Remark 3.2 Using Theorem 1.3, we obtain, for $0 < x \leq 1$:

$$E_x(e^{-\frac{\theta}{2} \Pi_1}) = 1 + x \frac{\frac{1}{2} K_{i\sqrt{4\theta-1}}(\sqrt{4\theta}) + \sqrt{\theta} K'_{i\sqrt{4\theta-1}}(\sqrt{4\theta})}{\frac{1}{2} K_{i\sqrt{4\theta-1}}(\sqrt{4\theta}) - \sqrt{\theta} K'_{i\sqrt{4\theta-1}}(\sqrt{4\theta})}$$

It now follows easily from (3.6) and (3.13) that :

$$E_x(e^{-\frac{\theta}{2} \Pi_1}) \xrightarrow{\theta \rightarrow \infty} 1 - x \quad (3.15)$$

Now, on the other hand :

$$E_x(e^{-\frac{\theta}{2} \Pi_1}) \xrightarrow{\theta \rightarrow \infty} P_x(\Pi_1 = 0) \quad (3.16)$$

We recall (and recover here) that $P_x(\Pi_1 = 0) = 1 - x$. This is point *ii*) of Theorem 1.2.

4 Extending the preceding results to the variables $\Pi_k^{(\rho, x)}$ with

$$\Pi_k^{(\rho, x)} := \int_0^\infty (e^{\rho(\log x + B_t - \frac{t}{2})} - k)^+ dt \quad (4.1)$$

($\rho, x, k > 0$). In the preceding Sections, we have studied the case $\rho = 1$.

The analogue of Theorem 1.3 may be stated as :

Theorem 4.1 *The Laplace transform of $\Pi_1^{(\rho)}$ under P_x is given by :*

$$E_x(e^{-\frac{\theta}{2} \Pi_1^{(\rho)}}) \begin{cases} \frac{\sqrt{x} K_\gamma(\frac{\sqrt{4\theta}}{\rho} x^{\frac{\rho}{2}})}{\frac{1}{2} K_\gamma(\frac{\sqrt{4\theta}}{\rho}) - \sqrt{\theta} K'_\gamma(\frac{\sqrt{4\theta}}{\rho})} & \text{if } x \geq 1 \\ 1 + x \frac{\frac{1}{2} K_\gamma(\frac{\sqrt{4\theta}}{\rho}) + \sqrt{\theta} K'_\gamma(\frac{\sqrt{4\theta}}{\rho})}{\frac{1}{2} K_\gamma(\frac{\sqrt{4\theta}}{\rho}) - \sqrt{\theta} K'_\gamma(\frac{\sqrt{4\theta}}{\rho})} & \text{if } 0 < x \leq 1 \end{cases} \quad (4.2)$$

with $\gamma = \frac{\sqrt{1-4\theta}}{\rho}$ if $\theta \leq \frac{1}{4}$ and $\gamma = i \frac{\sqrt{4\theta-1}}{\rho}$ if $\theta > \frac{1}{4}$.

The proof of this Theorem 4.1 is quite similar to that of Theorem 1.3. It hinges upon the following Lemma 4.2.

Lemma 4.2

i) Let, for $\theta \geq 0$ and $\rho > 0$, the function $\varphi_\theta^{(\rho)} : \mathbb{R}_+ \rightarrow \mathbb{R}$ defined by :

$$\varphi_\theta^{(\rho)}(y) = \sqrt{y} K_\gamma \left(\frac{\sqrt{4\theta}}{\rho} y^{\frac{\rho}{2}} \right) \quad (y \geq 0) \quad (4.3)$$

Then, it satisfies :

$$\varphi_\theta^{(\rho)''}(y) + \theta \left(-y^{\rho-2} + \frac{1}{y^2} \right) \varphi_\theta^{(\rho)}(y) = 0 \quad (4.4)$$

(see [Pet], p. 306, with $\rho - 2 = m$, formula (8), after taking care of replacing a by $-a$).

ii) Let $\tilde{\varphi}_\theta^{(\rho)} : \mathbb{R}_+ \rightarrow \mathbb{R}$ defined by :

$$\tilde{\varphi}_\theta^{(\rho)}(y) = \begin{cases} \varphi_\theta^{(\rho)}(y) & \text{if } y \geq 1 \\ \varphi_\theta^{(\rho)}(1) - \varphi_\theta^{(\rho)'}(1) + y \varphi_\theta^{(\rho)'}(1) & \text{if } y \leq 1 \end{cases} \quad (4.5)$$

Then $\tilde{\varphi}_\theta^{(\rho)}$ is positive, decreasing, convex and satisfies :

$$\tilde{\varphi}_\theta^{(\rho)}(y) + \theta \left(-y^{\rho-2} + \frac{1}{y^2} \right) 1_{y \geq 1} \tilde{\varphi}_\theta^{(\rho)}(y) = 0 \quad (4.6)$$

Remark 4.1 As a check, we note that formula (4.2) allows to recover the identity (1.8) :

$$\int_0^\infty e^{\rho B_t - \frac{\rho t}{2}} dt \stackrel{(\text{law})}{=} \frac{2}{\rho^2 \gamma_{\frac{1}{\rho}}^{\frac{1}{\rho}}}.$$

Indeed, on one hand :

$$E(e^{-\frac{\theta}{2} \frac{2}{\rho^2 \gamma_{\frac{1}{\rho}}^{\frac{1}{\rho}}}}) = \frac{1}{\Gamma(\frac{1}{\rho})} \int_0^\infty e^{-\frac{\theta}{\rho^2 z} - z} z^{\frac{1}{\rho}-1} dz = \frac{2}{\Gamma(\frac{1}{\rho})} K_{\frac{1}{\rho}} \left(\frac{\sqrt{4\theta}}{\rho} \right) \left(\frac{\theta}{\rho^2} \right)^{\frac{1}{2\rho}} \quad (\text{see [Leb], p. 115})$$

On the other hand :

$$\begin{aligned} E_1(e^{-\frac{\theta}{2} \Pi_0^{(\rho)}}) &= \lim_{\epsilon \downarrow 0} E_1 \left(\exp - \frac{\theta}{2} \int_0^\infty (e^{\rho B_t - \frac{\rho t}{2}} - \epsilon)^+ dt \right) \\ &= \lim_{\epsilon \downarrow 0} E_1 \left(\exp - \frac{\theta \epsilon}{2} \int_0^\infty (e^{\rho(\frac{\log \frac{1}{\epsilon}}{\rho} + B_t) - \frac{\rho t}{2}} - 1)^+ dt \right) \\ &= \lim_{\epsilon \downarrow 0} \frac{\left(\frac{1}{\epsilon} \right)^{\frac{1}{2\rho}} K_{\frac{\sqrt{1-4\theta\epsilon}}{\rho}} \left(\frac{\sqrt{4\theta\epsilon}}{\rho} \right) \left(\frac{1}{\epsilon} \right)^{\frac{1}{\rho} \frac{\rho}{2}}}{\frac{1}{2} K_{\frac{\sqrt{1-4\theta\epsilon}}{\rho}} \left(\frac{\sqrt{4\theta\epsilon}}{\rho} \right) - \sqrt{\theta\epsilon} K'_{\frac{\sqrt{1-4\theta\epsilon}}{\rho}} \left(\frac{\sqrt{4\theta\epsilon}}{\rho} \right)} \end{aligned}$$

from (4.2), replacing θ by $\theta\varepsilon$ and x by $\left(\frac{1}{\varepsilon}\right)^{\frac{1}{\rho}}$

$$\begin{aligned}
& \stackrel{\sim}{\varepsilon \rightarrow 0} \frac{\left(\frac{1}{\varepsilon}\right)^{\frac{1}{2\rho}} K_{\frac{1}{\rho}}\left(\frac{\sqrt{4\theta}}{\rho}\right)}{\frac{1}{2} K_{\frac{\sqrt{1-4\theta\varepsilon}}{\rho}}\left(\frac{\sqrt{4\theta\varepsilon}}{\rho}\right) - \frac{\rho}{2} \left(\frac{\sqrt{4\theta\varepsilon}}{\rho}\right) K'_{\frac{\sqrt{1-4\theta\varepsilon}}{\rho}}\left(\frac{\sqrt{4\theta\varepsilon}}{\rho}\right)} \\
& \stackrel{\sim}{\varepsilon \rightarrow 0} \frac{\left(\frac{1}{\varepsilon}\right)^{\frac{1}{2\rho}} K_{\frac{1}{\rho}}\left(\frac{\sqrt{4\theta}}{\rho}\right)}{\frac{\rho}{2} \left[\frac{\sqrt{4\theta\varepsilon}}{\rho} K_{\frac{1}{\rho}+1}\left(\frac{\sqrt{4\theta\varepsilon}}{\rho}\right)\right]} \quad \left(\text{since } zK'\nu = \nu K_\nu(z) - zK_{\nu+1}(z)\right) \\
& \stackrel{\sim}{\varepsilon \rightarrow 0} \frac{\left(\frac{1}{\varepsilon}\right)^{\frac{1}{2\rho}} K_{\frac{1}{\rho}}\left(\frac{\sqrt{4\theta}}{\rho}\right)}{\frac{\rho}{2} \left(\Gamma\left(\frac{1}{\rho} + 1\right) \left(\frac{\sqrt{4\theta\varepsilon}}{\rho}\right)^{-\frac{1}{\rho}}\right)} \quad \left(\text{since } K_\nu(z) \sim \frac{1}{2} \Gamma(\nu) \left(\frac{z}{2}\right)^{-1}\right) \\
& = \frac{\left(\frac{1}{\varepsilon}\right)^{\frac{1}{2\rho}} K_{\frac{1}{\rho}}\left(\frac{\sqrt{4\theta}}{\rho}\right)}{\Gamma\left(\frac{1}{\rho}\right)} \frac{\varepsilon^{\frac{1}{2\rho}} \theta^{\frac{1}{2\rho}}}{\rho^{\frac{1}{\rho}}} \\
& \xrightarrow{\varepsilon \downarrow 0} \left(\frac{\theta}{\rho^2}\right)^{\frac{1}{2\rho}} \frac{2}{\Gamma\left(\frac{1}{\rho}\right)} K_{\frac{1}{\rho}}\left(\frac{\sqrt{4\theta}}{\rho}\right) = E\left(e^{-\frac{\theta}{2} \Pi_0^{(\rho)}}\right)
\end{aligned}$$

Remark 4.2

Taking up again the arguments of the proof of Theorem 3.1, it is not difficult to see that :

$$E_1\left(e^{-\frac{\theta}{2} \Pi_1^{(\rho)}}\right) \underset{\theta \rightarrow \infty}{\sim} \frac{C(\rho)}{\theta^{\frac{1}{3}}}$$

where $C(\rho)$ is a strictly positive constant, depending on ρ . We then deduce that, for $\alpha < 0$:

$$E_1\left((\Pi_1^{(\rho)})^\alpha\right) < \infty \text{ if and only if } |\alpha| < \frac{1}{3}$$

On the other hand, it is not difficult to see that :

- If $x > 1$, for all $\alpha < 0$, $E_1\left((\Pi_1^{(\rho)})^\alpha\right) < \infty$.
- If $x < 1$, for all $\alpha < 0$, $E_1\left((\Pi_1^{(\rho)})^\alpha\right) = +\infty$.

Concerning the positive moments of $\Pi_1^{(\rho)}$:

$$\bullet \quad \text{If } 0 < \alpha < \frac{1}{\rho}, \quad \text{then } E_x\left((\Pi_1^{(\rho)})^\alpha\right) < \infty \quad (4.7)$$

Indeed (4.7), for $x = 1$, follows from :

$$\begin{aligned}
E_1\left((\Pi_1^{(\rho)})^\alpha\right) & \leq E_1\left((\Pi_0^{(\rho)})^\alpha\right) = \frac{1}{\Gamma\left(\frac{1}{\rho}\right)} \int_0^\infty \left(\frac{2}{\rho^2 z}\right)^\alpha e^{-z} z^{\frac{1}{\rho}-1} dz \\
& < \infty \quad \text{if } \alpha < \frac{1}{\rho} \\
& \left(\text{since } \Pi_0 \stackrel{(\text{law})}{=} \frac{2}{\rho^2 \gamma_{\frac{1}{\rho}}}, \quad \text{from (1.10)}\right)
\end{aligned}$$

The fact that $E_x((\Pi_1^{(\rho)})^\alpha) < \infty$ for every $x > 0$, and every $\alpha < \frac{1}{\rho}$ may be obtained by using arguments close to those used in the proof of Theorem 1.1. We believe that, for all $\rho > 0$:

$$E_x((\Pi_1^{(\rho)})^{\frac{1}{\rho}}) = +\infty. \quad (4.8)$$

We now show (4.8), when $\rho > 1$

It then suffices, by using the arguments of the proof of Theorem 1.1, to see that (4.8) is true when $x = 1$.

$$\bullet \text{ We first show that : } 1 - E_1(e^{-\frac{\theta}{2} \Pi_1^{(\rho)}}) \underset{\theta \rightarrow 0}{\sim} C \theta^{\frac{1}{\rho}} \quad (4.9)$$

Indeed, from Theorem 4.1, we have :

$$E_1\left(\exp -\frac{\theta}{2} \Pi_1^{(\rho)}\right) = 1 + \frac{\frac{1}{2} K_\gamma\left(\frac{\sqrt{4\theta}}{\rho}\right) + \sqrt{\theta} K'_\gamma\left(\frac{\sqrt{4\theta}}{\rho}\right)}{\frac{1}{2} K_\gamma\left(\frac{\sqrt{4\theta}}{\rho}\right) - \sqrt{\theta} K'_\gamma\left(\frac{\sqrt{4\theta}}{\rho}\right)}$$

with $\gamma = \frac{\sqrt{1-4\theta}}{\rho}$ (and $\theta \leq \frac{1}{4}$). Thus :

$$\begin{aligned} 1 - E_1(e^{-\frac{\theta}{2} \Pi_1^{(\rho)}}) &= -\frac{\frac{1}{2} K_\gamma\left(\frac{\sqrt{4\theta}}{\rho}\right) + \frac{\rho}{2} \left\{ \frac{\sqrt{4\theta}}{\rho} K'_\gamma\left(\frac{\sqrt{4\theta}}{\rho}\right) \right\}}{\frac{1}{2} K_\gamma\left(\frac{\sqrt{4\theta}}{\rho}\right) - \frac{\rho}{2} \left\{ \frac{\sqrt{4\theta}}{\rho} K'_\gamma\left(\frac{\sqrt{4\theta}}{\rho}\right) \right\}} \\ &= -\frac{\frac{1}{2} K_{\frac{\sqrt{1-4\theta}}{\rho}}\left(\frac{\sqrt{4\theta}}{\rho}\right) \left(1 - \frac{\sqrt{1-4\theta}}{\rho}\right) - \frac{\sqrt{\theta}}{\rho} K_{\frac{\sqrt{1-4\theta}}{\rho}-1}\left(\frac{\sqrt{4\theta}}{\rho}\right)}{\frac{1}{2} K_{\frac{\sqrt{1-4\theta}}{\rho}}\left(\frac{\sqrt{4\theta}}{\rho}\right) \left(1 + \frac{\sqrt{1-4\theta}}{\rho}\right) + \frac{\sqrt{\theta}}{\rho} K_{\frac{\sqrt{1-4\theta}}{\rho}-1}\left(\frac{\sqrt{4\theta}}{\rho}\right)} \end{aligned}$$

(after using $z K'_\mu(z) = -\mu K_\mu(z) - z K_{\mu-1}(z)$).

Since $\rho > 1$, we replace $K_{\frac{\sqrt{1-4\theta}}{\rho}-1}$ by $K_{1-\frac{\sqrt{1-4\theta}}{\rho}}$ (since $K_\mu = K_{-\mu}$) and we deduce, from :

$$K_{1-\frac{\sqrt{1-4\theta}}{\rho}} \xrightarrow{\theta \rightarrow 0} K_{1-\frac{1}{\rho}}, \quad \text{with } 1 - \frac{1}{\rho} > 0 :$$

that :

$$\begin{aligned} 1 - E_1(e^{-\frac{\theta}{2} \Pi_1^{(\rho)}}) &\underset{\theta \rightarrow 0}{\sim} \frac{a \theta^{1-\frac{1}{2\rho}} + b \theta^{\frac{1}{2\rho}}}{a' \theta^{-\frac{1}{2\rho}} + b' \theta^{\frac{1}{2\rho}}} \quad (\text{since, for } \mu > 0 \ K_\mu(z) \underset{z \rightarrow 0}{\sim} C_\mu z^{-\mu} ; \text{ see [Leb], p. 136}) \\ &\underset{\theta \rightarrow 0}{\sim} \frac{b}{a'} \theta^{\frac{1}{\rho}} \quad \left(\text{since } \rho \geq 1 \text{ implies } 1 - \frac{1}{2\rho} \geq \frac{1}{2\rho} \right) \end{aligned}$$

• From (4.9) we deduce :

$$\begin{aligned} \int_0^\infty e^{-\frac{\theta t}{2}} P(\Pi_1^{(\rho)} \geq t) dt &= \frac{2}{\theta} (1 - E(e^{-\frac{\theta}{2} \Pi_1^{(\rho)}})) \\ &\underset{\theta \rightarrow \infty}{\sim} C' \theta^{\frac{1}{\rho}-1} \end{aligned}$$

Hence, from the Tauberian Theorem :

$$P(\Pi_1^{(\rho)} \geq t) \underset{t \rightarrow \infty}{\sim} C'' \frac{1}{t^{\frac{1}{\rho}}} \quad (4.10)$$

and

$$\begin{aligned} E((\Pi_1^{(\rho)})^{\frac{1}{\rho}}) &= \int_0^\infty P((\Pi_1^{(\rho)})^{\frac{1}{\rho}} \geq t) dt \\ &= \int_0^\infty P(\Pi_1^{(\rho)} \geq t^\rho) dt = +\infty, \quad \text{from (4.10).} \end{aligned}$$

We note that, for $\rho < 1$, the preceding argument cannot be applied since :

$$\int_1^\infty e^{-\frac{\theta t}{2}} \frac{dt}{t^{\frac{1}{\rho}}} \xrightarrow{\theta \rightarrow 0} \int_1^\infty \frac{dt}{t^{\frac{1}{\rho}}} < \infty.$$

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