

On-line simultaneous maximization of the size and the weight for degradable intervals schedules

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Abstract. We consider the problem of scheduling *on-line* a sequence of *degradable* intervals in a set of k identical machines. Our objective is to find a schedule that maximizes *simultaneously* the *Weight* (equal to the sum of processing times) and the *Size* (equal to the number) of the scheduled intervals. We propose a *bicriteria* algorithm that uses the strategies of two monocriteria algorithms (GOL [7], maximizing the *Size* and LR [4], maximizing the *Weight*) and yields two simultaneous constant competitive ratios. This work is an extension of [2] (COCOON'04), where the same model of degradable intervals was investigated in an *off-line* context and the two objectives were considered separately.

In this paper, we consider the problem of scheduling on-line degradable intervals on k identical machines. We define a *degradable* interval σ by a triplet (r, q, d) where r denotes the *release date*, q the *minimal deadline* and d the *deadline* ($r < q \leq d$). This means that σ is *scheduled* if and only if it is executed from date r to date t ($q \leq t \leq d$) on one machine. Intuitively, in this model, each interval can be shortened (with respect to the required total execution $[r, d]$). We denote by $[r, t)$ the numerical interval corresponding to the effective execution of a degradable interval σ and by $p(\sigma) = t - r$ its processing time. We define the *weight* $w(\sigma)$ of the effective execution of any interval σ by $w(\sigma) = t_\sigma - r_\sigma$. This means that the weight of an interval σ is equal to its processing time (it is known in the literature as the *proportional weight* model [8]). In our model, we consider on-line sequences of degradable intervals $\sigma_1, \dots, \sigma_n$ where the σ_i 's are *revealed* one by one in the increasing order of their release dates ($r_1 \leq r_2 \leq \dots \leq r_n$), and future intervals are not known in advance.

For any algorithm A , we denote by A^k the version of A running on k identical machines. In our model, an on-line algorithm A^k has to build at each step a *valid* schedule. A schedule is *valid* if and only if for every date t , there is at most one interval on each machine and each interval is scheduled at most once. When a new interval σ_i is revealed (at step i), the algorithm A^k can *reject* it (in this case, it is definitively lost) or *serve* it. In this second case, if the algorithm schedules σ_i on machine j , it interrupts at least the already scheduled interval intersecting σ_i on machine j . The interrupted intervals are definitively lost and no gain is obtained from them for both metrics. Thus, each step i of any on-line algorithm A^k can be decomposed into two stages: First, there is the *interrupting stage* of step i . During this stage, the algorithm interrupts a subset of the already

scheduled intervals (note that this subset can be empty). Secondly, there is the *scheduling stage* of step i . During this stage, the algorithm decides if the new interval σ_i is served or rejected, and if it is served, on which machine σ_i is served.

Notation 1 (Schedule $A^k(\sigma_1, \dots, \sigma_i)$) Let $\sigma_1, \dots, \sigma_n$ be any on-line sequence of degradable intervals and let A^k be any algorithm running on k identical machines. For every step i ($1 \leq i \leq n$), we denote by $A^k(\sigma_1, \dots, \sigma_i)$ the schedule returned by A^k at the end of step i .

We define the *size* $N(O) = |\{\sigma \in O\}|$ (i.e. the number of scheduled intervals) and the *weight* $W(O) = \sum_{\sigma \in O} w(\sigma)$ (i.e. the weight of scheduled intervals) of any schedule O . Our problem is then to find a schedule which has *size* and *weight* the largest possible. In order to evaluate the quality of a schedule for our measures (the *Size* and the *Weight*), we use the competitive ratio [5].

Definition 1 (Competitive ratio). Let $\sigma_1, \dots, \sigma_n$ be any on-line sequence of intervals. Let $A^k(\sigma_1, \dots, \sigma_i)$ be the schedule on k machines given by an algorithm A^k at step i ($1 \leq i \leq n$) and let O_i^* be the optimal (off-line) schedule on k machines of $\{\sigma_1, \dots, \sigma_i\}$ for the criterion C (here, $C = W$ or $C = N$). A^k has a competitive ratio of ρ (it is ρ -competitive) for the criterion C if and only if we have:

$$\forall i, 1 \leq i \leq n, \rho \cdots C(A^k(\sigma_1, \dots, \sigma_i)) \geq C(O_i^*)$$

An algorithm A^k is (ρ, μ) -competitive if it is *simultaneously* ρ -competitive for the *Size* and μ -competitive for the *Weight*. In this paper, we propose a $\left(\frac{k}{r}, \frac{4k}{k-r-2}\right)$ -competitive algorithm, called AB^k (with $1 \leq r < k$). For example, if we set $r = \frac{k}{2}$ (if $\frac{k}{2} \geq 3$ and k even), AB^k is $\left(2, \frac{8}{1-\frac{1}{k}}\right)$ -competitive.

Previous works. The off-line version of the bicriteria non-degradable problem has been treated in [3] where a $\left(\frac{k}{r}, \frac{k}{k-r}\right)$ -approximation algorithm ($1 \leq r < k$) has been proposed. Concerning the monocriteria non-degradable problems, they have been extensively studied for both the off-line and the on-line versions. In particular, the off-line versions are polynomial (see Faigle and Nawijn [7] for the *Size* and Carlisle and Lloyd [6] or Arkin and Silverberg [1] for the *Weight* problems). In the on-line context, the algorithm *GOL* of Faigle and Nawijn [7] is optimal for the *Size* problem. For the *Weight* problem, there is a series of works going from the paper of Woeginger [8] to the paper of Bar-Noy et al. [4], who proposed on-line algorithms with constant competitive ratios. Note that the two degradable monocriterion intervals problem has been investigated in [2].

Outline of the paper. In Section 1, we present two monocriterion algorithms (*GOL* ^{k} [7] and *LR* ^{k} [4]) in the degradable interval model. Section 2 is devoted to the description and the analysis of our on-line bicriteria algorithm AB^k using *GOL* and *LR* as subroutines.

1 Two on-line monocriteria algorithms for the *size* and the *proportional weight* metrics

In this section, we describe and analyze the competitiveness of the algorithm of Faigle and Nawijn [7] and the algorithm of Bar-Noy et al. [4]. We use them as

subroutines of our algorithm AB^k (see Section 2) in order to obtain a pair of constant competitive ratios.

The algorithm GOL^k . We describe the algorithm GOL^k of [7] in the degradable interval model by decomposing it into an interrupting stage and a scheduling stage.

Algorithm GOL^k (adaptation of [7])

When a new interval σ_i , defined by (r_i, q_i, d_i) , is revealed, choose its effective execution $\sigma_i^q = [r_i, q_i)$ (i.e. each new interval is totally degraded) and do:

Interrupting stage: If there are k served intervals intersecting the date r_i , let σ_{max} be the one with the maximum deadline.

If σ_{max} does not exist (there is a free machine), do not interrupt any interval.

Else, If $d_{max} \geq q_i$ then interrupt σ_{max} .

If $d_{max} < q_i$ then do not interrupt any interval.

Scheduling stage:

If there is a free machine, then schedule σ_i^q on it.

Else, reject σ_i .

Note that the original algorithm of [7] is described for the classical non-degradable interval model. In the following, we denote by GOL_N^k this original version of the algorithm (notice that it is the same algorithm as GOL^k , except that GOL_N^k does not degrade any interval).

Lemma 1 GOL^k is optimal for the Size in the degradable interval model.

Proof. Let $\sigma_1, \dots, \sigma_n$ be an on-line sequence of intervals such that for all i , $1 \leq i \leq n$, σ_i is defined by (r_i, q_i, d_i) . Let $\sigma_1^D, \dots, \sigma_n^D$ be the sequence such that for all i , $1 \leq i \leq n$, $\sigma_i^D = [r_i, q_i)$ (i.e. the sequence version with the intervals totally degraded). Let O_D^* be an optimal schedule of $\sigma_1, \dots, \sigma_n$ in the degradable interval model and let O_N^* be an optimal schedule of $\sigma_1^D, \dots, \sigma_n^D$ in the non-degradable interval model. By [2], we know that $N(O_D^*) = N(O_N^*)$. Furthermore, since GOL_N^k is optimal for the Size in the non-degradable interval model (See [7]), we have $N(O_N^*) = N(GOL_N^k(\sigma_1^D, \dots, \sigma_n^D))$. By definition of GOL^k we have $GOL_N^k(\sigma_1^D, \dots, \sigma_n^D) = GOL^k(\sigma_1, \dots, \sigma_n)$. If we combine all these equalities, we obtain $GOL^k(\sigma_1, \dots, \sigma_n) = GOL_N^k(\sigma_1^D, \dots, \sigma_n^D) = N(O_N^*) = N(O_D^*)$. Thus, GOL^k is optimal for the Size in the degradable interval model. \square

The algorithm LR^k . We now describe the algorithm LR^k of [4] adapted to our degradable interval model and decomposed into an interrupting stage and a scheduling stage (for all $k \geq 3$).

Algorithm LR^k (adaptation of [4])

We denote by F_t the set of scheduled intervals containing date t . When a new interval σ_i defined by (r_i, q_i, d_i) is revealed, choose the effective execution $\sigma_i^d = [r_i, d_i)$ (i.e. do not degrade any interval) and do:

Interrupting stage:

If $|F_{r_i}| < k$, then do not interrupt any interval.

If $|F_{r_i}| = k$, then

1. Sort the $k+1$ intervals of $F_{r_i} \cup \{\sigma_i^d\}$ by increasing order of release dates. If several intervals have the same release date, order them in the decreasing order of their deadlines and let L be the set of the $\lceil \frac{k}{2} \rceil$ first intervals.
 2. Sort the $k+1$ intervals of $F_{r_i} \cup \{\sigma_i^d\}$ by decreasing order of deadlines (ties are broken arbitrarily) and let R be the set of the $\lfloor \frac{k}{2} \rfloor$ first intervals.
- If $\sigma_i^d \in L \cup R$, then interrupt any interval σ_j in $F_{r_i} - L \cup R$,
Else do not interrupt any interval.

Scheduling stage:

- If $|F_{r_i}| < k$, then schedule σ_i^d on any free machine.
 If $|F_{r_i}| = k$ and $\sigma_i^d \in L \cup R$, then schedule σ_i^d on the machine where σ_j has been interrupted.
 If $|F_{r_i}| = k$ and $\sigma_i^d \notin L \cup R$, then reject σ_i .

In the following, we show that LR^k is $\left(\frac{4}{1-\frac{2}{k}}\right)$ -competitive in the degradable interval model for the *Weight* metric (note that this is very close to 4 when k is large). We first show that the weight of an optimal degradable schedule is no more than twice the weight of an optimal non-degradable schedule.

Lemma 2 *For every set of intervals $\{\sigma_1, \dots, \sigma_n\}$, let O^{*nd} be an optimal schedule of $\{\sigma_1, \dots, \sigma_n\}$ for the proportional weight metric in the non-degradable interval model (i.e. $q_i = d_i$), and let O^{*d} be an optimal schedule of $\{\sigma_1, \dots, \sigma_n\}$ for the same metric in the degradable interval model. We have:*

$$W(O^{*d}) \leq 2W(O^{*nd})$$

Proof. Let $O_1^{*d}, \dots, O_k^{*d}$ be the k sub-schedules of O^{*d} (O_i^{*d} executes the same intervals at the same dates as machine i of O^{*d}). Thus, we have:

$$W(O^{*d}) = \sum_{i=0}^k W(O_i^{*d})$$

Let $\Gamma_i = \{\sigma_j \in \{\sigma_1, \dots, \sigma_n\} : \sigma_j \in O_i^{*d}\}$. O_i^{*d} is an optimal schedule of Γ_i in the degradable interval model. Indeed, suppose, by contradiction, that there exists a valid schedule O of Γ_i such that $W(O_i^{*d}) < W(O)$. This means that the valid schedule consisting in the union of the O_j^{*d} 's, except for $j = i$, which is replaced by O , generates a weight greater than O^{*d} and is valid. This contradicts the optimality of O^{*d} .

Let us apply the 2-approximation algorithm for one machine schedules described in [2] separately on each Γ_i ($1 \leq i \leq k$). Let O_1, \dots, O_k be the obtained schedules. Thus, by Theorem 5 of [2], for each i , we have $W(O_i^{*d}) \leq 2W(O_i)$. We sum the k inequalities and we obtain $W(O^{*d}) \leq 2 \sum_{i=1}^k W(O_i)$. Moreover, since the 2-approximation algorithm of [2] does not degrade the intervals, the k machine schedule consisting in the union of the O_i 's is valid for the non-degradable interval model. This means that $\sum_{i=1}^k W(O_i) \leq W(O^{*nd})$. Combining this last inequality with $W(O^{*d}) \leq 2 \sum_{i=1}^k W(O_i)$ leads to:

$$W(O^{*d}) \leq 2W(O^{*nd}) \quad \square$$

Corollary 1 *LR^k is $\left(\frac{4}{1-\frac{2}{k}}\right)$ -competitive for the degradable interval model on $k \geq 3$ machines.*

Proof. It is known that LR^k is $(\frac{2}{1-\frac{2}{k}})$ -competitive for the non-degradable interval model (from an adaptation of the proof of [4]). Thus, by definition, we have $\frac{2}{1-\frac{2}{k}}W(LR^k(\sigma)) \geq W(O^{*nd})$. By Lemma 2, we have $2W(O^{*nd}) \geq W(O^{*d})$. Combining these two inequalities leads to $\frac{4}{1-\frac{2}{k}}W(LR^k(\sigma)) \geq W(O^{*d})$. This means that LR^k is $(\frac{4}{1-\frac{2}{k}})$ -competitive for the degradable model. \square

2 Our algorithm AB^k

Definition 2 (Cover relation). Let σ be an interval defined by the triplet (r, q, d) . Let $\sigma^1 = [r, t_1)$ and $\sigma^2 = [r, t_2)$ be two valid effective executions of σ (i.e. $q \leq t_1 \leq d$ and $q \leq t_2 \leq d$). We say that σ^1 covers σ^2 if and only if $\sigma^2 \subseteq \sigma^1$ (i.e. if and only if $t_2 \leq t_1$).

Definition 3 (Union⁺ of sets of degraded intervals). Let $\{\sigma_1, \dots, \sigma_n\}$ be a set of degradable intervals. For all i , $1 \leq i \leq n$, σ_i is defined by (r_i, q_i, d_i) . For all $\sigma_i \in \{\sigma_1, \dots, \sigma_n\}$, let $\sigma_i^1 = [r_i, t_i^1)$ and $\sigma_i^2 = [r_i, t_i^2)$ be two valid effective executions of σ_i (i.e. $q_i \leq t_i^1 \leq d_i$ and $q_i \leq t_i^2 \leq d_i$). Let $E_1 \subseteq \{\sigma_i^1 : \sigma_i \in \{\sigma_1, \dots, \sigma_n\}\}$ and $E_2 \subseteq \{\sigma_i^2 : \sigma_i \in \{\sigma_1, \dots, \sigma_n\}\}$. We define $E_1 \uplus E_2$ the union⁺ of E_1 and E_2 as follows:

- $\sigma_i^1 \in E_1 \uplus E_2$ if and only if $(\sigma_i^1 \in E_1 \text{ and } \sigma_i^2 \notin E_2)$ or $(\sigma_i^1 \in E_1 \text{ and } \sigma_i^2 \in E_2 \text{ and } \sigma_i^1 \text{ covers } \sigma_i^2)$.
- $\sigma_i^2 \in E_1 \uplus E_2$ if and only if $(\sigma_i^2 \in E_2 \text{ and } \sigma_i^1 \notin E_1)$ or $(\sigma_i^2 \in E_2 \text{ and } \sigma_i^1 \in E_1 \text{ and } \sigma_i^2 \text{ covers } \sigma_i^1)$.

Note that the union⁺ is commutative and it generalizes the usual definition of the union of two non-degradable intervals sets since in that case, $\sigma_i^1 = \sigma_i^2$. Thus, for all σ , E_1 and E_2 , if $\sigma \in E_1 \uplus E_2$, then $\sigma \in E_1 \cup E_2$.

As, by definition, the two effective executions of a same interval σ defined by (r, q, d) must start at the same release date r , the one with the smallest execution time is covered by the other.

Note that, to be completely rigorous, we should not define an interval σ_i by (r_i, q_i, d_i) , but by (r_i, q_i, d_i, i) . Indeed, let us consider the following problematic example. Let $\sigma_i^1 = [r_i, t_i^1)$ be an effective execution of $\sigma_i = (r_i, q_i, d_i)$ and $\sigma_j^1 = [r_j, t_j^1)$ be an effective execution of $\sigma_j = (r_j, q_j, d_j)$, with $i \neq j$. If we consider the particular case where $r_i = r_j$ and $t_i^1 = t_j^1$ (our model allows such a situation), then we have $\sigma_i^1 = [r_i, t_i^1) = [r_j, t_j^1) = \sigma_j^1$. Of course, in this paper, we consider that the intervals are *distinct* (i.e. $\sigma_i^1 \neq \sigma_j^1$). That is why we should define an interval σ_i by (r_i, q_i, d_i, i) and an effective execution σ_i^1 by $([r_i, t_i^1), i)$. But, in order to simplify the notations, we write $\sigma_i = (r_i, q_i, d_i)$ instead of $\sigma_i = (r_i, q_i, d_i, i)$, and $\sigma_i^1 = [r_i, t_i^1)$ instead of $\sigma_i^1 = ([r_i, t_i^1), i)$.

The algorithm AB^k . The main idea is the following. AB^k is running on k identical machines (called *real* machines because it is on these machines that

the effective schedule is built). It uses as subroutines *GOL* and *LR* (described in Section 1). For the ease of notation, we use A for *GOL* and B for *LR*. For each new submitted interval σ_i , we simulate the execution of the algorithm A^r (resp. B^{k-r}) on r (resp. $k-r$) *virtual* machines, in order to control the size (resp. the weight) of the schedule. These two simulations (for the size and for the weight) are made on machines that we call *virtual*, because they are used only in order to determine the set (potentially empty) of intervals AB^k has to interrupt and whether σ_i has to be rejected or served by AB^k (and in this last case, to decide in which degraded version AB^k has to serve the new interval). Indeed, AB^k serves σ_i on a *real* machine if and only if A^r or B^{k-r} serves σ_i (note that if both A^r and B^{k-r} serve it, AB^k chooses the effective execution of σ_i that covers the other).

In order to determine the schedule given by an algorithm after the interrupting and the scheduling stages, we introduce the following notation.

Notation 2 (Schedule returned by an algorithm on step i) For every on-line sequence $\sigma_1, \dots, \sigma_n$, for every algorithm ALG and for every step of execution i ($1 \leq i \leq n$) of ALG , let $\mathcal{O}_{i_1}(ALG)$ (resp. $\mathcal{O}_{i_2}(ALG)$) be the schedule returned ALG after the interrupting (resp. scheduling) stage of step i .

Notation 3 (Set of intervals scheduled by AB^k) For every on-line sequence $\sigma_1, \dots, \sigma_n$, for every step of execution i ($1 \leq i \leq n$) of the algorithm AB^k , let $\mathcal{R}_{i_1}(AB^k)$ (resp. $\mathcal{R}_{i_2}(AB^k)$) be the set of intervals scheduled and not interrupted after the interrupting (resp. the scheduling) stage of step i on the k machines associated to AB^k , called *real machines*.

Notation 4 (Set of intervals scheduled by A^r and B^{k-r}) For every on-line sequence $\sigma_1, \dots, \sigma_n$, for every step of execution i ($1 \leq i \leq n$) of the algorithm A^r (resp. B^{k-r}), let $\mathcal{V}_{i_1}(A^r)$ (resp. $\mathcal{V}_{i_1}(B^{k-r})$) be the set of intervals scheduled and not interrupted after the interrupting stage of step i on the r (resp. $k-r$) machines associated to A^r (resp. B^{k-r}). Let $\mathcal{V}_{i_2}(A^r)$ (resp. $\mathcal{V}_{i_2}(B^{k-r})$) be the set of intervals scheduled and not interrupted after the scheduling stage of step i on the r (resp. $k-r$) machines associated to A^r (resp. B^{k-r}). The r (resp. $k-r$) machines associated to A^r (resp. B^{k-r}) are called *virtual machines*.

We give now a formal description of the algorithm AB^k .

Input: An on-line sequence of intervals $\sigma_1, \dots, \sigma_n$ and k identical machines.

Output: After each step i ($1 \leq i \leq n$), a valid schedule $\mathcal{O}_{i_2}(AB^k)$ of $\sigma_1, \dots, \sigma_i$ on the k real machines.

Step 0: $\mathcal{V}_{0_2}(A^r) = \mathcal{V}_{0_2}(B^{k-r}) = \mathcal{R}_{0_2}(AB^k) = \emptyset$

Step i (date r_i):

1. The **interrupting stage** of AB^k :
 - (a) Execute the *interrupting stage* of A^r (resp. B^{k-r}) on the r (resp. $k-r$) virtual machines associated to A^r (resp. B^{k-r}) by submitting the new interval σ_i to A^r (resp. B^{k-r}). Note that the set of intervals scheduled and not interrupted by A^r (resp. B^{k-r}) is now $\mathcal{V}_{i_1}(A^r)$ (resp. $\mathcal{V}_{i_1}(B^{k-r})$).

- (b) Execute the *interrupting stage* of AB^k on the k real machines associated to AB^k by interrupting the subset of intervals of $\mathcal{R}_{(i-1)_2}(AB^k)$ such that:

$$\mathcal{R}_{i_1}(AB^k) = \mathcal{V}_{i_1}(A^r) \uplus \mathcal{V}_{i_1}(B^{k-r})$$
2. The **scheduling stage** of AB^k :
- (a) Execute the *scheduling stage* of A^r (resp. B^{k-r}) on the r (resp. $k-r$) virtual machines associated to A^r (resp. B^{k-r}) by serving or rejecting the new interval σ_i .
- (b) Execute the *scheduling stage* of AB^k on the k real machines associated to AB^k by switching to the appropriate case:
- i. If A^r and B^{k-r} reject σ_i , then AB^k does not schedule σ_i . Thus, we have:

$$\mathcal{R}_{i_2}(AB^k) = \mathcal{R}_{i_1}(AB^k)$$
 - ii. If A^r serves σ_i (with effective execution σ_i^A) and B^{k-r} rejects σ_i then AB^k serves σ_i^A on any free machine and we have:

$$\mathcal{R}_{i_2}(AB^k) = \mathcal{R}_{i_1}(AB^k) \cup \{\sigma_i^A\}$$
 - iii. If A^r rejects σ_i and B^{k-r} serves σ_i (with effective execution σ_i^B) then AB^k serves σ_i^B on any free machine and we have:

$$\mathcal{R}_{i_2}(AB^k) = \mathcal{R}_{i_1}(AB^k) \cup \{\sigma_i^B\}$$
 - iv. If A^r and B^{k-r} serve σ_i one with effective execution σ_i^A and the other with effective execution σ_i^B then AB^k serves the effective execution that covers the other on any free machine. If σ_i^B covers σ_i^A then we have:

$$\mathcal{R}_{i_2}(AB^k) = \mathcal{R}_{i_1}(AB^k) \cup \{\sigma_i^B\}$$

else we have:

$$\mathcal{R}_{i_2}(AB^k) = \mathcal{R}_{i_1}(AB^k) \cup \{\sigma_i^A\}$$

Intervals scheduled by the algorithm AB^k . We first present Lemma 3 which states that the algorithm AB^k schedules the same intervals as the union⁺ of the intervals scheduled by A^r and the intervals scheduled by B^{k-r} .

Lemma 3 *For each step i of execution of the algorithm AB^k , the schedule $\mathcal{O}_{i_2}(AB^k)$ is valid and $\mathcal{R}_{i_2}(AB^k) = \mathcal{V}_{i_2}(A^r) \uplus \mathcal{V}_{i_2}(B^{k-r})$.*

Proof. We prove Lemma 3 by induction on the steps of execution i of AB^k .

The basic case (step 0): By definition of AB^k , we have $\mathcal{V}_{0_2}(A^r) = \mathcal{V}_{0_2}(B^{k-r}) = \mathcal{R}_{0_2}(AB^k) = \emptyset$. Thus, $\mathcal{O}_{i_2}(AB^k)$ is valid and we have $\mathcal{R}_{i_2}(AB^k) = \mathcal{V}_{i_2}(A^r) \uplus \mathcal{V}_{i_2}(B^{k-r})$. The basic case is checked.

The main case (step i): Let us assume that $\mathcal{O}_{(i-1)_2}(AB^k)$ is valid and that $\mathcal{R}_{(i-1)_2}(AB^k) = \mathcal{V}_{(i-1)_2}(A^r) \uplus \mathcal{V}_{(i-1)_2}(B^{k-r})$ (by the assumption of the induction).

1. The interrupting stage: We first need to prove that $\mathcal{R}_{i_1}(AB^k) = \mathcal{V}_{i_1}(A^r) \uplus \mathcal{V}_{i_1}(B^{k-r})$ and that $\mathcal{O}_{i_1}(AB^k)$ is valid.
 - (a) By definition, AB^k interrupts the subset of intervals of $\mathcal{R}_{(i-1)_2}(AB^k)$ such that:

$$\mathcal{R}_{i_1}(AB^k) = \mathcal{V}_{i_1}(A^r) \uplus \mathcal{V}_{i_1}(B^{k-r}) \quad (\text{UNION})$$

We have to show that there is always a subset of intervals of $\mathcal{R}_{(i-1)_2}(AB^k)$ that can be removed such that the above equality is possible.

Since $\mathcal{V}_{i_1}(A^r) \subseteq \mathcal{V}_{(i-1)_2}(A^r)$, $\mathcal{V}_{i_1}(B^{k-r}) \subseteq \mathcal{V}_{(i-1)_2}(B^{k-r})$, and $\mathcal{R}_{(i-1)_2}(AB^k) = \mathcal{V}_{(i-1)_2}(A^r) \uplus \mathcal{V}_{(i-1)_2}(B^{k-r})$ (by the assumption of the induction), we have $\mathcal{V}_{i_1}(A^r) \uplus \mathcal{V}_{i_1}(B^{k-r}) \subseteq \mathcal{R}_{(i-1)_2}(AB^k)$.

- (b) By definition, AB^k interrupts only intervals scheduled in $\mathcal{O}_{(i-1)_2}(AB^k)$, and by assumption of induction, $\mathcal{O}_{(i-1)_2}(AB^k)$ is valid. Thus, there cannot be intervals scheduled at the same time or more than once in $\mathcal{O}_{i_1}(AB^k)$. This means that $\mathcal{O}_{i_1}(AB^k)$ is valid. (VALID)
2. The scheduling stage: We now prove that $\mathcal{R}_{i_2}(AB^k) = \mathcal{V}_{i_2}(A^r) \uplus \mathcal{V}_{i_2}(B^{k-r})$ and that $\mathcal{O}_{i_2}(AB^k)$ is valid. By definition of AB^k , several cases may happen:
- (a) If A^r and B^{k-r} reject σ_i , then AB^k does not schedule σ_i and we have:

$$\begin{aligned} \text{i. } \mathcal{R}_{i_2}(AB^k) &= \mathcal{R}_{i_1}(AB^k) = \mathcal{V}_{i_1}(A^r) \uplus \mathcal{V}_{i_1}(B^{k-r}) \\ &\quad (\text{by the definition of } AB^k \text{ and by (UNION)}) \\ &= \mathcal{V}_{i_2}(A^r) \uplus \mathcal{V}_{i_2}(B^{k-r}) \\ &\quad (\text{since } A^r \text{ and } B^{k-r} \text{ reject } \sigma_i, \text{ we have} \\ &\quad \mathcal{V}_{i_1}(A^r) = \mathcal{V}_{i_2}(A^r) \text{ and } \mathcal{V}_{i_1}(B^{k-r}) = \mathcal{V}_{i_2}(B^{k-r})) \end{aligned}$$

ii. $\mathcal{O}_{i_2}(AB^k) = \mathcal{O}_{i_1}(AB^k)$. Thus $\mathcal{O}_{i_2}(AB^k)$ is valid (because by (VALID), $\mathcal{O}_{i_1}(AB^k)$ is valid).

- (b) If A^r serves σ_i (with effective execution σ_i^A) and B^{k-r} rejects σ_i , then AB^k schedules σ_i^A on any free real machine at time r_i and we have:

$$\begin{aligned} \text{i. } \mathcal{R}_{i_2}(AB^k) &= \mathcal{R}_{i_1}(AB^k) \cup \{\sigma_i^A\} = (\mathcal{V}_{i_1}(A^r) \uplus \mathcal{V}_{i_1}(B^{k-r})) \cup \{\sigma_i^A\} \\ &\quad (\text{by the definition of } AB^k \text{ and by (UNION)}) \\ &= \mathcal{V}_{i_2}(A^r) \uplus \mathcal{V}_{i_2}(B^{k-r}) \\ &\quad (\text{union and union}^+ \text{ commute and since } A^r \text{ serves } \sigma_i \\ &\quad \text{and } B^{k-r} \text{ rejects } \sigma_i, \text{ we have } \mathcal{V}_{i_2}(A^r) = \mathcal{V}_{i_1}(A^r) \cup \{\sigma_i^A\} \\ &\quad \text{and } \mathcal{V}_{i_2}(B^{k-r}) = \mathcal{V}_{i_1}(B^{k-r})) \end{aligned}$$

ii. Since $\mathcal{O}_{i_1}(AB^k)$ is a valid schedule (by (VALID)) and $\mathcal{O}_{i_2}(AB^k)$ is built by AB^k by adding σ_i to $\mathcal{O}_{i_1}(AB^k)$ only once, the only reason for which $\mathcal{O}_{i_2}(AB^k)$ could not be valid would be because σ_i is scheduled by AB^k at time r_i whereas there is no free machine at time r_i , i.e. because there are at least $k+1$ intervals of $\mathcal{R}_{i_2}(AB^k)$ scheduled at time r_i by AB^k . Let us prove that this is impossible. Indeed, since A^r and B^{k-r} build at each time valid schedules, there are at most $r+k-r = k$ intervals of $\mathcal{V}_{i_2}(A^r) \uplus \mathcal{V}_{i_2}(B^{k-r})$ scheduled at time r_i by A^r and B^{k-r} , and thus, there are at most k intervals of $\mathcal{R}_{i_2}(AB^k)$ scheduled at time r_i by AB^k (because we just proved above that $\mathcal{R}_{i_2}(AB^k) = \mathcal{V}_{i_2}(A^r) \uplus \mathcal{V}_{i_2}(B^{k-r})$). Thus, $\mathcal{O}_{i_2}(AB^k)$ is a valid schedule.

- (c) If A^r rejects σ_i and B^{k-r} serves σ_i (with effective execution σ_i^B), then AB^k schedules σ_i^B on any free real machine at time r_i .

- i. We prove that $\mathcal{R}_{i_2}(AB^k) = \mathcal{V}_{i_2}(A^r) \uplus \mathcal{V}_{i_2}(B^{k-r})$ in the same way that we prove it in 2(b)i, except that we replace σ_i^A by σ_i^B .
 - ii. We prove that $\mathcal{O}_{i_2}(AB^k)$ is valid in the same way as in 2(b)ii.
- (d) If A^r and B^{k-r} serve σ_i one with effective execution σ_i^A and the other with effective execution σ_i^B . Let σ_i^S (resp. σ_i^L) be the shortest (resp. longest) effective execution of σ_i . Without loss of generality, we suppose that A^r schedules σ_i^S and B^{k-r} schedules σ_i^L . By definition of the algorithm, AB^k schedules σ_i^L , and we have:

$$\begin{aligned}
\text{i. } \mathcal{R}_{i_2}(AB^k) &= \mathcal{R}_{i_1}(AB^k) \cup \{\sigma_i^L\} = (\mathcal{V}_{i_1}(A^r) \uplus \mathcal{V}_{i_1}(B^{k-r})) \cup \{\sigma_i^L\} \\
&\quad (\text{by definition of } AB^k \text{ and by (UNION)}) \\
&= (\mathcal{V}_{i_1}(A^r) \uplus \mathcal{V}_{i_1}(B^{k-r})) \cup (\{\sigma_i^L\} \uplus \{\sigma_i^S\}) \\
&\quad (\text{because, by definition of union}^+, \{\sigma_i^L\} = \{\sigma_i^L\} \uplus \{\sigma_i^S\}) \\
&= \mathcal{V}_{i_1}(A^r) \uplus \mathcal{V}_{i_1}(B^{k-r}) \uplus \{\sigma_i^L\} \uplus \{\sigma_i^S\} \\
&\quad (\text{because } (\mathcal{V}_{i_1}(A^r) \uplus \mathcal{V}_{i_1}(B^{k-r})) \cap (\{\sigma_i^L\} \uplus \{\sigma_i^S\}) = \emptyset) \\
&= (\mathcal{V}_{i_1}(A^r) \cup \{\sigma_i^S\}) \uplus (\mathcal{V}_{i_1}(B^{k-r}) \cup \{\sigma_i^L\}) \\
&\quad (\text{because the union}^+ \text{ is commutative and since } \sigma_i^S \notin \mathcal{V}_{i_1}(A^r), \text{ we have } \mathcal{V}_{i_1}(A^r) \uplus \{\sigma_i^S\} = \mathcal{V}_{i_1}(A^r) \cup \{\sigma_i^S\}) \\
&= \mathcal{V}_{i_2}(A^r) \uplus \mathcal{V}_{i_2}(B^{k-r}) \\
&\quad (\text{because since } A^r \text{ and } B^{k-r} \text{ serve } \sigma_i, \text{ we have } \mathcal{V}_{i_2}(A^r) = \mathcal{V}_{i_1}(A^r) \cup \{\sigma_i^S\} \text{ and } \mathcal{V}_{i_2}(B^{k-r}) = \mathcal{V}_{i_1}(B^{k-r}) \cup \{\sigma_i^L\})
\end{aligned}$$

- ii. We prove that $\mathcal{O}_{i_2}(AB^k)$ is valid in the same way as in 2(b)ii. □

Corollary 2 *Let $A^r = GOL^r$ and $B^{k-r} = LR^{k-r}$. Let $N(\mathcal{V}_{i_2}(GOL^r)) = |\mathcal{V}_{i_2}(GOL^r)|$ and $W(\mathcal{V}_{i_2}(LR^{k-r}))$ be the sum of the weight of the intervals of $\mathcal{V}_{i_2}(LR^{k-r})$. For every input sequence $\sigma_1, \dots, \sigma_n$ and for every step i ($1 \leq i \leq n$) of the algorithm AB^k , we have:*

$$N(\mathcal{V}_{i_2}(GOL^r)) \leq N(\mathcal{R}_{i_2}(AB^k)) \quad \text{and} \quad W(\mathcal{V}_{i_2}(LR^{k-r})) \leq W(\mathcal{R}_{i_2}(AB^k))$$

Proof. By Lemma 3, for every step i of the algorithm AB^k , we have $\mathcal{R}_{i_2}(AB^k) = \mathcal{V}_{i_2}(A^r) \uplus \mathcal{V}_{i_2}(B^{k-r}) = \mathcal{V}_{i_2}(GOL^r) \uplus \mathcal{V}_{i_2}(LR^{k-r})$, thus, by definition of union⁺, Corollary 2 is checked. □

Theorem 1. *For all $k \geq 4$, for all r , $1 \leq r \leq k-3$, the algorithm AB^k applied with GOL^r and LR^{k-r} is $\left(\frac{k}{r}, \frac{4k}{k-r-2}\right)$ -competitive for the Size and Proportional weights metrics.*

Proof. Let $\sigma_1, \dots, \sigma_n$ be any on-line sequence of intervals and let O_x^{N*} (resp. O_x^{W*}) be an optimal schedule of $\{\sigma_1, \dots, \sigma_n\}$ for the size N (resp. for the proportional weight W) on $x \leq k$ machines. Let O_r^{GOL} (resp. O_{k-r}^{LR}) be the schedule returned by GOL^r (resp. LR^{k-r}) on the on-line sequence $\sigma_1, \dots, \sigma_n$ on $r \leq k-3$ (resp. $k-r \geq 3$) machines. Since, by Lemma 1, GOL^r is 1-competitive (resp. by Corollary 1, LR^{k-r} is $\left(\frac{4}{1-\frac{2}{k-r}}\right)$ -competitive), we have:

$$N(O_r^{N*}) \leq N(O_r^{GOL}) \quad (\text{resp. } W(O_{k-r}^{W*}) \leq \left(\frac{4}{1-\frac{2}{k-r}}\right) W(O_{k-r}^{LR})) \quad (1)$$

Let O'^N (resp. O'^W) be the r (resp. $k-r$) machine sub-schedule of O_k^{N*} (resp. O_k^{W*}) executing all the intervals appearing on the r (resp. $k-r$) machines of O_k^{N*} (resp. O_k^{W*}) generating the largest size (resp. weight). Since O'^N (resp. O'^W) is a r (resp. $k-r$) machine schedule, we have $N(O'^N) \leq N(O_r^{N*})$ (resp. $W(O'^W) \leq W(O_{k-r}^{W*})$), otherwise, O_r^{N*} (resp. O_{k-r}^{W*}) would not be an optimal schedule for the *size* (resp. *weight*). Combined with (1), we obtain:

$$\begin{aligned} N(O'^N) &\leq N(O_r^{N*}) \leq N(O_r^{GOL}) \\ (\text{resp. } W(O'^W) &\leq W(O_{k-r}^{W*}) \leq \left(\frac{4}{1-\frac{2}{k-r}}\right) W(O_{k-r}^{LR})) \end{aligned} \quad (2)$$

Since O'^N (resp. O'^W) is the r machine sub-schedule of O_k^{N*} (resp. O_k^{W*}) generating the largest size (resp. weight), the average size (resp. weight) per machine in O'^N (resp. O'^W) is larger than the average size (resp. weight) per machine in O_k^{N*} (resp. O_k^{W*}). Thus, we have $\frac{N(O_k^{N*})}{k} \leq \frac{N(O'^N)}{r} \Rightarrow N(O_k^{N*}) \leq \frac{k}{r} N(O'^N)$ (resp. $\frac{W(O_k^{W*})}{k} \leq \frac{W(O'^W)}{k-r} \Rightarrow W(O_k^{W*}) \leq \frac{k}{k-r} W(O'^W)$). Combined with (2), we obtain:

$$N(O_k^{N*}) \leq \frac{k}{r} N(O_r^{GOL}) \quad (\text{resp. } W(O_k^{W*}) \leq \frac{4k}{k-r-2} W(O_{k-r}^{LR})) \quad (3)$$

As $N(O_r^{GOL}) = N(\mathcal{V}_{i_2}(GOL^r))$ (resp. $W(O_{k-r}^{LR}) = W(\mathcal{V}_{i_2}(LR^{k-r}))$), by applying Corollary 2 on (3), we obtain:

$$\begin{aligned} N(O_k^{N*}) &\leq \frac{k}{r} N(\mathcal{V}_{i_2}(GOL^r)) \leq \frac{k}{r} N(\mathcal{R}_{i_2}(AB^k)) \\ (\text{resp. } W(O_k^{W*}) &\leq \frac{k}{r} W(\mathcal{V}_{i_2}(LR^{k-r})) \leq \frac{4k}{k-r-2} W(\mathcal{R}_{i_2}(AB^k))) \end{aligned}$$

This means that AB^k is $(\frac{k}{r}, \frac{4k}{k-r-2})$ -competitive. \square

Example. If $r = \frac{k}{2}$ (if $\frac{k}{2} \geq 3$ and k even), AB^k is $(2, \frac{8}{1-\frac{4}{k}})$ -competitive.

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