Free models of T-algebraic theories computed as Kan extensions

Paul-André Melliès Nicolas Tabareau *

Abstract

*One fundamental aspect of Lawvere's categorical semantics is that every algebraic theory (eg. of monoid, of Lie algebra) induces a free construction (eg. of free monoid, of free Lie algebra) computed as a Kan extension. Unfortunately, the principle fails when one shifts to linear variants of algebraic theories, like Adams and Mac Lane's PROPs, and similar PROs and PROBs. Here, we introduce the notion of T-*algebraic theory *for a pseudomonad T — a mild generalization of equational doctrine — in order to describe these various kinds of "algebraic theories". Then, we formulate two conditions (the first one combinatorial, the second one algebraic) which ensure that the free model of a T-algebraic theory exists and is computed as an Kan extension. The proof is based on Bénabou's theory of distributors, and of an axiomatization of the colimit computation in Wood's proarrow equipments.*

Keywords: Lawvere theories, PROs, PROPs, PROBs, operads, Kan extensions, distributors, enriched categories, free constructions, algebras, coalgebras.

Contents

*This work has been supported by the ANR Project INVAL "Invariants algébriques des systèmes informatiques".
Postal address: Équipe PPS, Université Paris VII, 2 place Jussieu, Case 7014, 75251 Paris Cedex 05, FRANCE. Email addresses: **mellies@pps.jussieu.fr** and **tabareau@pps.jussieu.fr**

1 Introduction

1.1 The tensor algebra

The investigation reported in this article is motivated by a basic and well-known problem in algebra, which we expose briefly here. Let k denote a commutative ring. To every k-module A is associated a particular k-algebra TA called its *tensor algebra*, defined as an infinite sum of tensorial powers:

$$TA \;=\; \bigoplus_{n\in\mathbb{N}} A^{\otimes n} \tag{1}$$

where we write $A \otimes B$ for the tensor product $A \otimes_k B$ of two k-modules A and B, and $A^{\otimes n}$ for $A \otimes \cdots \otimes A$ taken n times. Note that we take the freedom here and onwards to consider this tensor product *strictly* associative.

In the language of category theory, one says that the construction T is *functorial*. This means that it defines a functor

$$T \quad : \quad k\text{-Mod} \;\longrightarrow\; k\text{-Alg}$$

from the category k-Mod of k-modules and k-module morphisms to the category k-Alg of k-algebras and k-algebra morphisms. Besides, this functor T is left adjoint to the forgetful functor

$$U \quad : \quad k\text{-Alg} \;\longrightarrow\; k\text{-Mod} \tag{2}$$

which transports every k-algebra to its underlying k-module. The adjunction, noted

$$T \dashv U$$

indicates that every morphism $f : A \to M$ from a k-module A to a k-algebra M factors uniquely as:

$$\begin{tikzcd} & M \\ A \arrow[ur, "f"] \arrow[r, "\eta"'] & TA \arrow[u, "h"'] \end{tikzcd}$$

where h is a k-algebra morphism and η is the canonical injection induced by Equation (1). This property states precisely that TA is the *free* k-algebra generated by the k-module A.

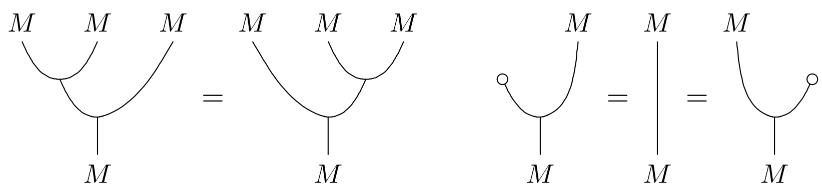
Now, recall that a k-algebra M is defined as a k-module equipped with two morphisms,

$$k \xrightarrow{e} M \xleftarrow{m} M \otimes M$$

called *unit* and *multiplication*, making the diagrams below commute:

$$\begin{tikzcd} M\otimes M\otimes M \arrow[r, "m\otimes M"] \arrow[d, "M\otimes m"'] & M\otimes M \arrow[d, "m"] \\ M\otimes M \arrow[r, "m"] & M \end{tikzcd} \qquad \begin{tikzcd} k\otimes M \arrow[r, "e\otimes M"] \arrow[dr, "\cong"'] & M\otimes M \arrow[d, "m"] & M\otimes k \arrow[l, "M\otimes e"'] \arrow[dl, "\cong"] \\ & M & \end{tikzcd} \tag{3}$$

For a more visual representation, we provide the corresponding string diagrams.



So, a k-algebra is the same thing as a *monoid object* in the category k-Mod, seen as a monoidal category equipped with the familiar tensor product $\otimes$ of k-modules. Hence, the k-algebra TA is the *free* monoid object in the category k-Mod.

1.2 A basic problem in algebra

The existence of a free k-algebra TA for every k-module A should be contrasted to the fact, which is folklore among mathematicians, that there exists (in general) no free k-bialgebra for a given k-module. This deserves further explanation. First, recall that a k-cogebra k is a k-module equipped with two morphisms

$$k \xleftarrow{u} K \xrightarrow{d} K \otimes K$$

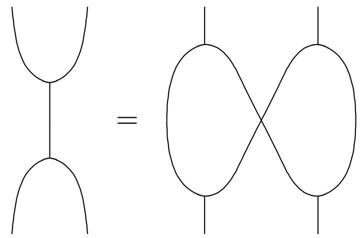
called *counit* and *comultiplication*, making the diagrams dual to (3) (reverse the direction of arrows) commute. Hence, a k-cogebra is the same thing as a *comonoid object* in the monoidal category k-Mod. This defines a category k-Cog of k-cogebras and k-cogebra morphisms.

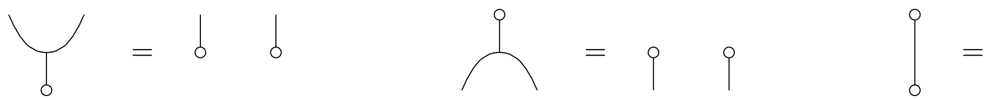
Then, a k-bialgebra H is a k-module equipped with a k-algebra and a k-cogebra structure, making the so-called *Hopf compatibility* diagrams commute:

$$\begin{tikzcd} H\otimes H \arrow[r, "m"] \arrow[d, "d\otimes d"'] & H \arrow[r, "d"] & H\otimes H \\ H\otimes H\otimes H\otimes H \arrow[rr, "H\otimes\tau\otimes H"] & & H\otimes H\otimes H\otimes H \arrow[u, "m\otimes m"'] \end{tikzcd}$$

$$\begin{tikzcd} k \arrow[r, "e"] \arrow[d, "d"'] & H \arrow[d, "d"] \\ k\otimes k \arrow[r, "e\otimes e"] & H\otimes H \end{tikzcd} \qquad \begin{tikzcd} & H \arrow[dr, "u"] & \\ k \arrow[ur, "e"] \arrow[rr, equal] & & k \end{tikzcd} \qquad \begin{tikzcd} H\otimes H \arrow[r, "u\otimes u"] \arrow[d, "m"'] & k\otimes k \arrow[d, "m"] \\ H \arrow[r, "u"] & k \end{tikzcd}$$

Here, τ denotes the symmetry $H \otimes H \to H \otimes H$ of the symmetric monoidal category k-Mod. Again, we provide a description of this compatibility through string diagrams





Note that the categories k-Alg and k-Cog inherit the monoidal structure from the category k-Mod, and that a k-bialgebra may be seen alternatively as a monoid object in the category k-Cog of comonoid objects, and as a comonoid object in the category k-Alg of monoid objects.

Now, let k-Big denote the category of k-bialgebras, and k-algebra and k-cogebra morphisms between them. It is folklore that the forgetful functor

$$U_{bij} \quad : \quad k\text{-Big} \longrightarrow k\text{-Mod} \tag{4}$$

does not have a left adjoint, for k an arbitrary commutative ring. This result states precisely that there exists no such thing as a free k-bialgebra.

In this article, we want to understand more conceptually what distinguishes the forgetful functor (2) which has a left adjoint, from the forgetful functor (4) which does not have a left adjoint. Moreover, we want to extract, in the good situations, a formula like (1) to compute the left adjoint, and hence the free construction, associated to the forgetful functor.

We carry out the investigation in Lawvere's functorial semantics, and take advantage of the extraordinary toolbox provided by categorical algebra, more specifically the concepts of algebraic theory and Kan extensions — which we explain now.

1.3 Lawvere theories

In his dissertation [8], Lawvere reformulates the familiar notion of *algebraic theory* (eg. of monoids, Lie algebras, etc.) as a category $\mathbb{L}$ with finite products, whose objects are the natural numbers: $0, 1, 2, \ldots$ and in which the categorical product of k objects $m_1, \ldots, m_k$ is provided by their arithmetic sum $m_1 + \ldots + m_k$. As will become clear in the course of the article, the underlying idea is to represent every n-ary operation of the algebraic theory as a morphism $n \to 1$ of the category $\mathbb{L}$; and to encode the equational theory on these n-ary operations inside the composition law of the category $\mathbb{L}$.

Then, an $\mathbb{L}$-model in a category $\mathcal{C}$ with finite products (noted $\times$) is defined as a finite-product preserving functor

$$A \quad : \quad \mathbb{L} \to \mathcal{C}.$$

By "finite-product preserving functor", one means that the canonical morphism

$$A[m_1 + \ldots + m_k] \ \to \ A[m_1] \times \ldots \times A[m_k] \tag{5}$$

is an isomorphism. This implies that the functor $A(-)$ transports (up to isomorphism) the natural number n to the n-th power of the object $\underline{A} = A[1]$:

$$A[n] \;\cong\; \underline{A}^{\times n}.$$

Hence, the functor A transports (up to isomorphism again) every n-ary operation $n \to 1$ of the algebraic theory to an n-ary operation on the object $\underline{A}$:

$$\underline{A}^{\times n} \;\to\; \underline{A}. \tag{6}$$

For that reason, $\underline{A}$ is called the *underlying object* of the $\mathbb{L}$-model, and the morphism (6) is called the *interpretation* of the n-ary operation $n \to 1$. The terminology is justified by the fact that the $\mathbb{L}$-model A is characterized (up to natural isomorphism) by its underlying object $\underline{A}$ and the interpretation of every n-ary operation of the algebraic theory $\mathbb{L}$.

Every algebraic theory $\mathbb{L}$ and category $\mathcal{C}$ with finite products define together a category $\mathrm{Model}(\mathbb{L}, \mathcal{C})$ whose objects are the $\mathbb{L}$-models in the category $\mathcal{C}$, and whose morphisms $A \to B$ are the natural transformations from the functor A to the functor B.

1.4 Kan extensions

An *algebraic morphism* between algebraic theories

$$j \quad : \quad \mathbb{L}_1 \longrightarrow \mathbb{L}_2$$

is defined as a finite-product preserving functor, such that $j(1) = 1$. Such an algebraic morphism induces a functor

$$U_j \quad : \quad \mathrm{Model}(\mathbb{L}_2, \mathcal{C}) \longrightarrow \mathrm{Model}(\mathbb{L}_1, \mathcal{C})$$

called the associated *forgetful functor*, which transports every $\mathbb{L}_2$-model

$$\mathbb{L}_2 \xrightarrow{B} \mathcal{C}$$

to the $\mathbb{L}_1$-model

$$\mathbb{L}_1 \xrightarrow{j} \mathbb{L}_2 \xrightarrow{B} \mathcal{C}$$

obtained by precomposing with the functor j.

By way of illustration, consider the *free* category $\mathbb{F}^{op}$ with finite products, generated by the category with one object: this defines an algebraic theory whose n-ary operations are precisely the n projection maps $\pi_1, \ldots, \pi_n : n \to 1$. The category $\mathbb{F}^{op}$ may be defined more explicitly as the opposite to the category of finite sets $[m] = \{1, \ldots, m\}$ and functions between them. This particular algebraic theory $\mathbb{F}^{op}$ is called *trivial* because a $\mathbb{F}^{op}$-model is characterized (up to natural isomorphism) by its underlying object: hence, the category $\mathrm{Model}(\mathbb{F}^{op}, \mathcal{C})$ of $\mathbb{F}^{op}$-models is equivalent to the category $\mathcal{C}$ itself.

Next, consider the algebraic theory $\mathbb{M}$ of *monoids* whose n-ary operations are the finite words (of arbitrary length) built on an alphabet $[n] = \{1, \ldots, n\}$ of n letters. By construction, the category $\mathrm{Model}(\mathbb{M}, \mathcal{C})$ is equivalent to the category of monoids and monoid morphisms in the category $\mathcal{C}$.

Now, there exists a (unique) algebraic morphism $j : \mathbb{F}^{op} \to \mathbb{M}$, which transports the i-th projection $\pi_i \in \mathbb{F}^{op}(n, 1)$ to the one-letter word $i \in \mathbb{M}(n, 1)$. It is not difficult to check that the associated forgetful functor U_j transports every monoid in the category $\mathcal{C}$ to its underlying object.

This leads to Lawvere's elegant formulation of "free constructions" in functorial semantics, using Kan extensions. Recall that the *left* Kan extension of a functor $A : \mathbb{L}_1 \to \mathcal{C}$ along a functor $f : \mathbb{L}_1 \to \mathbb{L}_2$ is defined as a functor, noted $\mathrm{Lan}_j A$, and a natural transformation:

$$\begin{tikzcd} & \mathcal{C} & \\ \mathbb{L}_1 \arrow[ur, "A"] \arrow[rr, "j"'] & \Rightarrow & \mathbb{L}_2 \arrow[ul, "\mathrm{Lan}_j A"'] \end{tikzcd}$$

inducing a bijection:

$$[\mathrm{Lan}_j A, B] \ \stackrel{\cong}{\longrightarrow} \ [A, B \circ j] \tag{7}$$

between sets of natural transformations, for every functor $B : \mathbb{L}_2 \to \mathcal{C}$.

Now, suppose that $\mathcal{C}$ is the category $\mathbf{Ens}$ of sets and functions, or to that purpose, any cartesian closed category with small colimits. Then, the left Kan extension along j of any functor $\mathbb{L}_1 \to \mathcal{C}$ exists, and a miracle happens: when j is an algebraic morphism, the left Kan extension transports a *finite-product preserving* functor A to a *finite-product preserving* functor $\mathrm{Lan}_j A$. In this way, every $\mathbb{L}_1$-model A is transported to a $\mathbb{L}_2$-model $\mathrm{Lan}_j A$, and the bijection (7) specializes to the following bijection:

$$\mathrm{Model}(\mathbb{L}_2, \mathcal{C})(\mathrm{Lan}_j A, B) \ \stackrel{\cong}{\longrightarrow} \ \mathrm{Model}(\mathbb{L}_1, \mathcal{C})(A, U_j B)$$

between sets of morphisms, for every $\mathbb{L}_2$-model B. The bijection is natural in B, which shows that the left Kan extension $\mathrm{Lan}_j A$ is the *free* $\mathbb{L}_2$-model generated by the $\mathbb{L}_1$-model A, along the algebraic morphism j.

The free construction is functorial: the left Kan extension $\mathrm{Lan}_j A$ defines a left adjoint functor to the forgetful functor:

$$\mathrm{Lan}_j \dashv U_j : \mathrm{Model}(\mathbb{L}_1, \mathcal{C}) \to \mathrm{Model}(\mathbb{L}_2, \mathcal{C}).$$

This is precisely what happens when $j : \mathbb{F}^{op} \to \mathbb{M}$ is the algebraic morphism from the trivial algebraic theory $\mathbb{F}^{op}$ to the algebraic theory $\mathbb{M}$ of monoids. In that case, the functor Lan_j transports every set A (identified here to a $\mathbb{F}^{op}$-model) to the free monoid $\mathrm{Lan}_j A$ whose underlying set $A^* = (\mathrm{Lan}_j A)[1]$ is the set of finite words on the alphabet A:

$$A^* \quad = \quad \coprod_{n \in \mathbb{N}} A^{\times n}. \tag{8}$$

This well-known equation may be obtained by direct means, or deduced from a general "coend" formula for computing left Kan extensions in the category $\mathbf{Ens}$ — a formula which appears in [10] and will be deduced in the course of the article from Bénabou's theory of "distributors".

1.5 Linear theories (PROs)

Let us compare the two formulas (1) and (8). The analogy is striking: the two formulas appear to compute *in the same way* the free monoid object TA or A^* generated by an object A in their respective categories k-Mod and $\mathbf{Ens}$. There is a fundamental difference however: the finite product $A^{\times n}$ appearing in the definition of the free monoid A^* is replaced by a tensor product $A^{\otimes n}$ in the definition of the tensor algebra TA. This motivates to adapt Lawvere's ideas, and to replace every "finite product" by a "tensor product" in all the definitions related to algebraic theories.

So, a *linear theory* $\mathbb{L}$ (also called a PRO) is defined as a monoidal category whose objects are natural numbers: 0, 1, 2, ... and in which the tensor product of k objects $m_1, \ldots, m_k$ is provided by their arithmetic sum $m_1 + \ldots + m_k$. An $\mathbb{L}$-*model* in a monoidal category $\mathcal{C}$ is then defined as a monoidal functor

$$\mathbb{L} \to \mathcal{C}.$$

By "monoidal functor" one means a functor A equipped with a family of isomorphisms:

$$A[m_1 + \ldots + m_k] \ \to \ A[m_1] \otimes \ldots \otimes A[m_k]$$

satisfying a series of expected coherence properties.

As for algebraic theories, every linear theory $\mathbb{L}$ and monoidal category $\mathcal{C}$ induce together a category $\mathrm{Model}(\mathbb{L}, \mathcal{C})$ whose objects are the $\mathbb{L}$-models in $\mathcal{C}$, and whose morphisms $\theta : A \to B$ are the *monoidal* natural transformations from A to B, seen as monoidal functors. Recall that a natural transformation θ is monoidal when the expected coherence diagram

$$\begin{tikzcd} A[m_1 + \ldots + m_k] \arrow[r] \arrow[d, "\theta_{m_1+\cdots+m_k}"'] & A[m_1] \otimes \ldots \otimes A[m_k] \arrow[d, "\theta_{m_1}\otimes\cdots\otimes\theta_{m_k}"] \\ B[m_1 + \ldots + m_k] \arrow[r] & B[m_1] \otimes \ldots \otimes B[m_k] \end{tikzcd}$$

commutes, for every natural numbers $m_1, \ldots, m_k$. It is important to notice at this point that the category $\mathrm{Model}(\mathbb{L}, \mathcal{C})$ just defined *coincides* with the category $\mathrm{Model}(\mathbb{L}, \mathcal{C})$ defined earlier for algebraic theories — when the tensor product of the two categories $\mathbb{L}$ and $\mathcal{C}$ happens to be a finite product. The reason is that (a) every finite-product preserving functor defines a monoidal functor equipped with the *canonical* family of morphisms (5) and (b) every natural transformation θ between finite-product preserving functors satisfies the coherence properties required of a *monoidal* natural transformation.

In order to carry on our running example based on monoids, we introduce here two linear theories $\mathbb{N}$ and Δ playing the role of the two algebraic theories $\mathbb{F}^{op}$ and $\mathbb{M}$ defined earlier. First, consider the *free* monoidal category $\mathbb{N}$ generated by one object: this defines the *trivial* linear theory in which the only morphisms are the identities on each object $0, 1, 2, \ldots$ Then, define the linear theory of *monoids* as the category Δ of *augmented simplices* in which a morphism $m \to n$ is a monotone function from the ordered set $[m] = \{1, \ldots, m\}$ to the ordered set $[n] = \{1, \ldots, n\}$.

Just as for the algebraic theories, the category $\mathrm{Model}(\mathbb{N}, \mathcal{C})$ is equivalent to the category $\mathcal{C}$, and the category $\mathrm{Model}(\mathbb{N}, \mathcal{C})$ is equivalent to the category of monoids and monoid morphisms in $\mathcal{C}$. The careful reader will notice for instance that the expected notion of "homomorphism" between monoids is indeed captured by the notion of *monoidal* natural transformation — and not just of natural transformation — between Δ-models.

Then, a *linear* morphism $j : \mathbb{L}_1 \to \mathbb{L}_2$ between linear theories is defined as a monoidal functor such that $j(1) = 1$. Just as in the case of algebraic theories, every linear morphism $j : \mathbb{L}_1 \to \mathbb{L}_2$ induces a forgetful functor

$$U_j \quad : \quad \mathrm{Model}(\mathbb{L}_2, \mathcal{C}) \ \to \ \mathrm{Model}(\mathbb{L}_1, \mathcal{C})$$

which transports every $\mathbb{L}_2$-model B to the $\mathbb{L}_1$-model $B \circ j$ obtained by precomposing with the functor j.

1.6 Monoidal Kan extensions

At this point, we are ready to tell *quite* the same story as for algebraic theories — at least when it comes to computing free monoids in the category $\mathcal{C} = k$-Mod of k-modules. By definition of the category $\mathbb{N}$ as a free monoidal category, there exists a *unique* linear morphism

$$j : \mathbb{N} \to \Delta. \tag{9}$$

Then, every k-module $A : \mathbb{N} \to k$-Mod (seen as a model of the trivial theory $\mathbb{N}$) is transported by left Kan extension along j to a functor defined by the general formula [10] mentioned earlier:

$$\mathrm{Lan}_j A \quad : \quad p \quad \mapsto \quad \bigoplus_{n \in \mathbb{N}} \Delta(n,p) \otimes A^{\otimes n}$$

where the k-module $\Delta(n,p) \otimes A^{\otimes n}$ means the direct sum of as many copies of the k-module $A^{\otimes n}$ as there are elements in the hom-set $\Delta(n,p)$. It appears after close inspection that this functor $\mathrm{Lan}_j A$ is *monoidal*, and thus defines a Δ-model, which coincides with the tensor algebra and free monoid TA generated by the object A in the category k-Mod. Note that monoidality of the functor $\mathrm{Lan}_j A$ amounts essentially to an isomorphism

$$\mathrm{Lan}_j A(p+q) \quad \cong \quad \mathrm{Lan}_j A(p) \otimes \mathrm{Lan}_j A(q)$$

for every pair of natural numbers p and q. The isomorphism itself derives from the bijection

$$\Delta(n, p+q) \quad \cong \quad \coprod_{k=0}^{n} \quad \Delta(k,p) \times \Delta(n-k,q)$$

between the coefficients of each part, which expresses that every monotone function $f : [n] \to [p+q]$ decomposes as a pair of monotone functions $f_1 : [k] \to [p]$ and $f_2 : [n-k] \to [q]$ for some natural number $0 \leq k \leq n$.

This success with monoids is somewhat miraculous, and should not deceive us. We are facing a serious difficulty indeed: in contrast to what happens with algebraic theories, the very fact that a $\mathbb{L}_1$-model A defines a left Kan extension along a linear morphism $j : \mathbb{L}_1 \to \mathbb{L}_2$ does not qualify the resulting functor $\mathrm{Lan}_j A$ as the free $\mathbb{L}_2$-model generated by A... There are two reasons for that defect: first, nothing ensures that the functor $\mathrm{Lan}_j A$ is monoidal, and thus, that it defines a $\mathbb{L}_2$-model; then, even when the functor $\mathrm{Lan}_j A$ happens to be monoidal, nothing ensures that this functor $\mathrm{Lan}_j A$ defines indeed what we expect: the *free* $\mathbb{L}_2$-model generated by the $\mathbb{L}_1$-model A along the linear morphism j. The reason is that the Kan extension property provides a one-to-one correspondence between the natural transformations $\mathrm{Lan}_j A \to B$ and the natural transformations $A \to U_j B$, for every $\mathbb{L}_2$-model B — whereas a correspondence between *monoidal* natural transformations is required.

At this point, it is critical to proceed conceptually, and to remember that the notion of Kan extension may be defined in any 2-category. Typically, the left Kan extension considered until now is computed in the 2-category $\mathbf{Cat}$ with

- categories as 0-cells,
- functors as 1-cells,
- natural transformations as 2-cells.

What our discussion establishes is that we need instead a *monoidal* left Kan extension, that is: a left Kan extension computed in the 2-category $\mathbf{MonCat}$ with

- monoidal categories as 0-cells,
- monoidal functors as 1-cells,
- monoidal natural transformations as 2-cells.

In this way, all the difficulties are reduced to the single question:

When is the left Kan extension of a monoidal functor A
along a monoidal functor j, a monoidal left Kan extension?

By this, we mean that the left Kan extension exists in $\mathbf{MonCat}$ and is transported by the forgetful 2-functor $\mathbf{MonCat} \to \mathbf{Cat}$ to the left Kan extension computed in $\mathbf{Cat}$. Everything works nicely when this lifting property occurs: indeed, the left Kan extension $\mathrm{Lan}_j A$ defines in that case the free $\mathbb{L}_2$-model generated by the $\mathbb{L}_1$-model A along the linear morphism j.

Interestingly, this lifting property *always* occurs with algebraic theories: this is precisely what makes these particular theories so convenient to compute free constructions. More specifically, let $\mathbf{CartCat}$ denote the 2-category with

- categories with finite products as 0-cells,
- finite-product-preserving functors as 1-cells,
- natural transformations as 2-cells.

It appears that the 2-category $\mathbf{CartCat}$ is a *full* sub-2-category of the 2-category $\mathbf{MonCat}$, in the expected sense. Moreover, the left Kan extension of a finite-product-preserving functor $\mathbb{L}_1 \to \mathbf{Ens}$ along a finite-product-preserving functor $j : \mathbb{L}_1 \to \mathbb{L}_2$ always exists in $\mathbf{CartCat}$ (and thus in $\mathbf{MonCat}$) and coincides with the left Kan extension computed in $\mathbf{Cat}$.

We will see that the lifting property holds when one computes the left Kan extension $TA = \mathrm{Lan}_j A$ of a k-module A along the linear morphism $j : \mathbb{N} \to \Delta$. This explains why the formula (1) works, and computes indeed the free monoid TA in the category of k-modules.

But the lifting property does not occur in any situation: consider for instance the linear theory of *comonoids*, defined as the category Δ^{op} obtained by reversing the maps of the category Δ. It is well-known that in a category $\mathcal{C}$ with finite products (for instance $\mathbf{Ens}$) every object defines a comonoid in a canonical and unique way; and that every morphism $A \to B$ is then a comonoid morphism. From this follows trivially that every object A in the category $\mathcal{C}$, when seen as a comonoid, defines the *free* comonoid generated by the object A. This reformulated in categorical semantics says that the commutative diagram

$$\begin{tikzcd} & \mathcal{C} & \\ \mathbb{N} \arrow[ur, "A"] \arrow[rr, "g"'] & & \Delta^{op} \arrow[ul, "A"'] \end{tikzcd}$$

defines a *monoidal* left Kan extension along the linear morphism g. When $\mathcal{C} = \mathbf{Ens}$, this says that the *monoidal* left Kan extension of a set $A : \mathbb{N} \to \mathbf{Ens}$ (seen as a model of the trivial theory $\mathbb{N}$) is the set A itself. On the other hand, the *usual* left Kan extension of $A : \mathbb{N} \to \mathbf{Ens}$ along $g : \mathbb{N} \to \Delta^{op}$ is defined by the classical formula

$$\mathrm{Lan}_g A \quad : \quad p \quad \mapsto \quad \coprod_{n \in \mathbb{N}} \Delta(p, n) \times A^{\times n}.$$

Hence, the left Kan extension of A along g computed in $\mathbf{Cat}$ does not coincide with its monoidal left Kan extension computed in $\mathbf{MonCat}$.

1.7 Symmetric theories (PROPs)

This brings us back to the discussion on free k-bialgebras which opened the article. Just like the notion of algebraic theory was replaced by the notion of linear theory, by shifting from categories with finite products to monoidal categories, the notion of linear theory may be replaced by the notion of *symmetric theory* (also called PROPs), by shifting from monoidal categories to *symmetric* monoidal categories.

The trivial symmetric theory $\mathbb{S}$ is then defined as the free symmetric monoidal category on one object, whose only morphisms are the bijections $[n] \to [n]$. And the symmetric theory of bimonoids $\mathbb{B}$ is defined as the symmetric monoidal category whose morphisms $m \to n$ are (a) the $m \times n$ matrices with integer coefficients in $\mathbb{Z}$, or equivalently, (b) the homomorphisms $\mathbb{Z}[m] \to \mathbb{Z}[n]$ between free commutative monoids — where $\mathbb{Z}[m]$ is the free commutative monoid on m elements.

At this point, we are ready to recast Loday's observation (mentioned earlier) in the language of categorical semantics. Let $\mathbf{SMonCat}$ be the 2-category with the symmetric monoidal categories as 0-cells, the symmetric monoidal functors as 1-cells, and the monoidal natural functors as 2-cells. Every k-module A may be seen as a symmetric monoidal functor $\mathbb{S} \to k$-Mod, induces a left Kan extension $\mathrm{Lan}_j(A)$ along the (unique) symmetric monoidal functor $j : \mathbb{S} \to \mathbb{B}$. Unfortunately, and this is the whole point, this left Kan extension in $\mathbf{Cat}$ is not a left Kan extension in the 2-category $\mathbf{SMonCat}$; otherwise, it would define a free construction for k-bialgebras, and a left adjoint to the forgetful functor U_j mentioned in (4): and this would contradict Loday's observation.

1.8 T-algebraic theories

In this article, we follow the ideas developed by Lawvere in his work on *equational doctrines* [9] and formulate the notion T*-algebraic theory* for a pseudomonad T on the 2-category $\mathbf{Cat}$. The notion describes in a generic way the various kinds of algebraic, linear, symmetric or even braided theories — each kind of theory described by a particular pseudomonad T. The only property required of the pseudomonad T is that it distributes over the presheaf monad, in the sense of [3, 5], so that it lifts to a pseudomonad on the bicategory of distributors.

- algebraic theories = free category with finite products,
- linear theories = free monoidal category,
- symmetric theories = free symmetric monoidal category,
- braided theories = free braided monoidal category,
- projective sketches = free category with finite limits.

Every such pseudomonad T induces a 2-category $\mathbf{Cat}^T$ with T-algebraic categories as 0-cells, T-algebraic morphisms as 1-cells, and T-algebraic natural transformations as 2-cells — where we apply the following dictionary:

Definition 1 (T-algebraic theory) *Let T be a pseudomonad on $\mathbf{Cat}$. We consider the 2-category $\mathbf{Cat}^T$ whose 0-cells are T-algebraic categories, 1-cells are T-algebraic functors and 2-cells are T-algebraic natural transformations – where the following dictionary has been used:*

- *T-algebraic category = pseudoalgebra of the pseudomonad T,*
- *T-algebraic functor = pseudomorphism of pseudoalgebras,*
- *T-algebraic natural transformation = invertible algebraic 2-cell.*

A T-algebraic theory is defines as a small *T-algebraic category – that is to say whose class of objects is a set.*

Remark that our definition of T-algebraic theory is quite generous, and does not try to characterize which algebraic theories should be accepted or rejected according to the rank of the pseudomonad T. This has to be opposed to the approach of Kelly and Power on the elaborated work on enriched Lawvere theories [6, 12].

Definition 2 (model of a T-algebraic theory) *A* model *M of a T-algebraic theory $\mathbb{L}$ ($\mathbb{L}$-model) in a T-algebraic category $\mathcal{C}$ is a T-algebraic functor*

$$A : \mathbb{L} \ \rightarrow \ \mathcal{C}.$$

We construct a category Model$(\mathbb{L}, \mathcal{C})$ of $\mathbb{L}$-models and T-algebraic natural transformations between them.

We can now reformulate the question raised by the computation of free models by the following:

When is a left Kan extension of a T-algebraic functor A
along a T-algebraic functor j, a $\underline{T\text{-algebraic}}$ Kan extension ?

1.9 Algebraic distributors at work

Besides our interest in algebraic theories, this article is motivated by our fascination for the theory of distributors — and the desire to promote this elegant theory to larger circles of algebraists. The theory of distributors was introduced by Jean Bénabou in the late 1960s, and recast in the language of enriched categories by the Sydney School, around Max Kelly and Ross Street.

There are at least two different ways to see a distributor:

- as a generalized functor,
- as a module with several objects.

Depending on the point of view, a distributor is called a *profunctor*, or a *module*. Here, we stick to the terminology introduced by Jean Bénabou, who had in mind the analogy between distributors and *distributions* in functional analysis [2].

We have already mentioned the existence of a classical formula which enables to compute the left Kan extension of a functor f along a functor j in many situations of interest. The formula appears for instance in Chapter 10 on Kan extensions of the famous monograph devoted to categories by Saunders Mac Lane [10].

It is folklore among category theorists that the theory of distributors explains this formula in a beautiful and pleasingly conceptual way. The main idea of the reconstruction is that every functor

$$f : \mathcal{A} \to \mathcal{B}$$

induces a pair of adjoint distributors $f_* \dashv f^*$ with source and target as below:

$$f_* \ : \ \mathcal{A} \nrightarrow \mathcal{B} \qquad\qquad f^* \ : \ \mathcal{B} \nrightarrow \mathcal{A}.$$

The left Kan extension of the functor $f : \mathcal{A} \to \mathcal{C}$ along the functor j is then computed by "composing" the two distributors $j^* : \mathcal{B} \nrightarrow \mathcal{A}$ and $f_* : \mathcal{A} \nrightarrow \mathcal{C}$, then by "taking the representative" of the resulting distributor – that is a functor $\mathrm{Lan}_j(f) : \mathcal{B} \to \mathcal{C}$ that satisfies

$$\mathbf{Dist}(f_* \circ j^*, g_*) \ \cong \ \mathbf{Cat}(\mathrm{Lan}_j(f), g)$$

for every functor $g : \mathcal{B} \to \mathcal{C}$. The terminology representative comes from the classical situation of an object representing a functor as described by Barr and Wells [1].

Once understood precisely how left Kan extensions are computed using distributors, there remains to understand when the resulting Kan extensions can be T-algebraic. The aim of this article is to clarify when this situation holds. We herald briefly the two recipes of our construction :

the adjunction	the T-algebraic distributor
$j_* \dashv j^*$	$f_* \circ j^* : \mathcal{A} \nrightarrow \mathcal{C}$
is T-algebraic	is "represented" by a T-algebraic functor

As we want our reasoning to be kept at an algebraic level and as we would like to explain the enriched case as well, we need to construct our argument by abstracting the properties of the bicategory $\mathbf{Dist}$ of distributors. For that reason, we introduce the formalism of *proarrow equipment* that captures the essence of Bénabou's construction.

2 Proarrow equipments

2.1 Basic definitions

The notion of proarrow equipment is an axiomatization, introduced by Richard Wood [15, 16], of the homomorphism of bicategories between $\mathbf{Cat}$ and $\mathbf{Dist}$. We used here the original notations of Wood $\mathcal{K}$ and $\mathcal{M}$ for the two bicategories of concern.

Definition 3 (proarrow equipment) *A* proarrow equipment *is a morphism of bicategories*

$$(-)_* : \mathcal{K} \ \to \ \mathcal{M}$$

that satisfies the three axioms:

1. *the object of $\mathcal{M}$ are those of $\mathcal{K}$ and $(-)_*$ is an identity on objects morphism,*

2. *$(-)_*$ is locally fully faithful, i.e.*

$$\mathcal{K}(f, g) \cong \mathcal{M}(f_*, g_*);$$

3. *for every 1-cell f of $\mathcal{K}$, f_* has a right adjoint f^*.*

Let us mention some examples of proarrow equipments:

- $\mathcal{K} = \mathcal{V}$-$\mathbf{Cat}$ et $\mathcal{M} = \mathcal{V}$-$\mathbf{Dist}$, from the enriched 2-category of enriched categories to the enriched bicategory of enriched distributors.
- $\mathcal{K} = \mathrm{Map}\,\mathbb{B}$ and $\mathcal{M} = \mathbb{B}$ for every bicategory $\mathbb{B}$, where Map $\mathbb{B}$ is the full sub-bicategory of $\mathbb{B}$ restricted to the morphisms which have a right adjoint. $(-)_*$ is then an inclusion functor,
- $\mathcal{K} = \mathbf{Ens}$ and $\mathcal{M} = \mathbf{Rel}$ where $(-)_*$ sends every function $f : A \to B$ to its graph f_*

$$af_*b \quad \text{ssi} \quad fa = b.$$

In the proarrow equipment $\mathbf{Cat} \to \mathbf{Dist}$, when a functor $f : \mathcal{C} \to \mathcal{D}$ satisfies "f^* is a retract of f_*", then we have

$$\mathcal{B}(fc, fc') \cong \mathcal{A}(c, c'),$$

that is to say f is fully faithful. The property becomes here a definition.

Definition 4 (fully faithful) *We say that a morphism $f : \mathcal{A} \to \mathcal{B}$ of $\mathcal{K}$ is* fully faithful *iff f^* is a retract of f_*.*

The notion of representability in a proarrow equipment comes from the corresponding notion for a functor. Usually, given a functor $R : \mathcal{D} \to \mathcal{C}$, we say that an object Lc represents the functor $\mathcal{C}(c, R(-))$ as soon as

$$\mathcal{C}(c, R(-)) \cong \mathcal{D}(Lc, -).$$

It is well-known that when the functor $\mathcal{C}(c, R(-))$ is representable for every object c of $\mathcal{C}$, the functor R has a left adjoint [1]. We extend this notion to the setting of proarrow equipment.

Definition 5 (representative) *A morphism $g : B \to C$ of $\mathcal{K}$ is a* representative *of a morphism $f : B \to C$ of $\mathcal{M}$ when there is a one-to-one correspondence*

$$\mathcal{M}(f, h_*) \;\cong\; \mathcal{K}(g, h)$$

for every $h : B \to C$ in $\mathcal{K}$.

The notion of representative allows to find the "closest" morphism in $\mathcal{K}$ of a morphism in $\mathcal{M}$. What we want to do is thus to compute a Kan extension in $\mathcal{K}$ by going throw $\mathcal{M}$ via the equipment, and then by going back to $\mathcal{K}$ via the representative.

2.2 Yoneda situations

We now give a recipe to compute the representative. It is well known that the computation of the representative of a presheaf on a category $\mathcal{C}$ is induced by the existence of some colimits in this category $\mathcal{C}$.

To provide an abstract point of view on this colimit computation, we introduce the concept of *Yoneda situations* in a proarrow equipment. This situation allows to describe abstractly the morphisms of $\mathbf{Cat}$ from a category $\mathcal{C}$ to a category $\overline{\mathcal{C}}$ that can be seen as restrictions of the Yoneda embedding from $\mathcal{C}$ to presheaves overt $\mathcal{C}$.

Definition 6 (Yoneda situation) *A morphism $y : \mathcal{C} \to \overline{\mathcal{C}}$ of $\mathcal{K}$ is said to be a* Yoneda situation *when*

- *y is fully faithful,*
- *y^* is pseudomonic with respect to $\mathcal{K}$, that is to say the functor*

$$y^* \circ (-)_* : \mathcal{K}_1(\mathcal{A}, \overline{\mathcal{C}}) \;\to\; \mathcal{M}_1(\mathcal{A}, \mathcal{C})$$

  is fully faithful for every object $\mathcal{A}$ of $\mathcal{K}$.

This definition of a Yoneda situation allows to give a general notion of $\overline{\mathcal{C}}$-cocompleteness on a proarrow equipment.

Definition 7 ($\overline{\mathcal{C}}$-cocompleteness) *An object $\mathcal{C}$ of $\mathcal{K}$ is said to be $\overline{\mathcal{C}}$-*cocomplete *iff there exists a Yoneda situation $y : \mathcal{C} \to \overline{\mathcal{C}}$, that has a left adjoint*

$$\textit{colim} \dashv y : \overline{\mathcal{C}} \to \mathcal{C}.$$

A famous case of $\overline{\mathcal{C}}$-cocompleteness for the proarrow equipment $\mathbf{Cat} \to \mathbf{Dist}$ is obtained when $\overline{\mathcal{C}}$ is the category of presheaves on $\mathcal{C}$ and y is the Yoneda embedding. We then say that the category $\mathcal{C}$ is said to be total. In that case, the category is not only cocomplete but also complete.

Let us take a morphism $f : \mathcal{B} \to \mathcal{C}$ in the bicategory $\mathcal{M}$. Suppose that $\mathcal{C}$ is $\overline{\mathcal{C}}$-cocomplete for the Yoneda situation $y : \mathcal{C} \to \overline{\mathcal{C}}$ and that f factories through y^* in the sense that

$$\mathcal{B} \xrightarrow{f} \mathcal{C} \;=\; \mathcal{B} \xrightarrow{\overline{f}_*} \overline{\mathcal{C}} \xrightarrow{y^*} \mathcal{C}.$$

Proposition 1 *The morphism colim $\circ \overline{f}$ is a representative of f.*

Proof : The proof follows from the following cascade of equivalences

$$\begin{array}{rcll}
\mathcal{M}(f, g_*) & = & \mathcal{M}(y^* \circ \overline{f}_*, g_*) & f = y^* \circ \overline{f}_* \\
 & \cong & \mathcal{M}(y^* \circ \overline{f}_*, y^* \circ y_* \circ g_*) & y \text{ is fully faithful} \\
 & \cong & \mathcal{K}(\overline{f}, y \circ g) & y^* \text{ is pseudomonic with respect to } \mathcal{K} \\
 & \cong & \mathcal{K}(\mathrm{colim} \circ \overline{f}, g) & \mathrm{colim} \text{ is the left adjoint of } y
\end{array}$$

■

Let us explain this construction in the paradigmatic equipment $\mathbf{Cat} \to \mathbf{Dist}$. We use here factorization system on $\mathbf{Cat}$ given by final functors and discrete fibrations [13]. Thus, every diagram $f : J \to \mathcal{C}$ can be seen as a presheaf φ given by the decomposition

$$J \xrightarrow{f} \mathcal{C} \;=\; J \xrightarrow{f_1} \mathrm{Elt}(\varphi) \xrightarrow{f_2} \mathcal{C}$$

where f_1 is a final functor and f_2 is a discrete fibration. We now follow the notion introduced by Kelly of $\mathcal{F}$-cocompleteness, and consider a cocomplete category $\mathcal{C}$ cocomplete for a given class $\mathcal{F}$ of categories, called the indices, containing the category $\mathbf{1}$. This means that there is an adjunction

$$\begin{tikzcd} \overline{\mathcal{C}} \arrow[rr, bend left, "\mathrm{colim}"] \arrow[rr, phantom, "\perp"] & & \mathcal{C} \arrow[ll, bend left, "y"] \end{tikzcd}$$

between the category $\overline{\mathcal{C}}$ of diagrams whose base is in $\mathcal{F}$ and the category $\mathcal{C}$. The functor y sends every object c of $\mathcal{C}$ to the constant diagram $y : C \to (\mathbf{1} \to \mathcal{C})$. Using the Yoneda lemma, we know that y is fully faithful and that the functor $y^* \circ (-)_*$ is equal to the identity. We directly deduce that y^* is pseudomonic with respect to $\mathbf{Cat}$. Thus, the functor y is a valid Yoneda situation.

We can now show that the notion of representative enables to compute left Kan extension in the bicategory $\mathcal{K}$.

Proposition 2 *Let $f : \mathcal{A} \to \mathcal{C}$ and $j : \mathcal{A} \to \mathcal{B}$ be two morphisms of $\mathcal{K}$. The representative of the morphism $f_* \circ j^*$ (if it exists) is the left Kan extension of f along j in $\mathcal{K}$.*

Proof: As $f_* \vdash j^*$, the morphism $f_* \circ j^*$ is the left Kan extension of f_* along j_* in $\mathcal{M}$. Note $\mathrm{Lan}_j f$ its representative. For every $g : B \to C$ in $\mathcal{K}$, we have the following cascade of bijections

$$\begin{array}{rcll} \mathcal{K}(\mathrm{Lan}_j f, g) & \cong & \mathcal{M}(f_* \circ j^*, g_*) & \quad \mathrm{Lan}_j f \text{ represents } f_* \circ j^* \\ & \cong & \mathcal{M}(f_*, g_* \circ j_*) & \quad f_* \circ j^* \text{ left Kan extension in } \mathcal{M} \\ & \cong & \mathcal{M}(f_*, (g \circ j)_*) & \quad \text{functoriality of } (-)_* \\ & \cong & \mathcal{K}(f, g \circ j) & \quad (-)_* \text{ is fully faithful} \end{array}$$

which establishes that $\mathrm{Lan}_j f$ is the left Kan extension of f along j in $\mathcal{K}$. ■

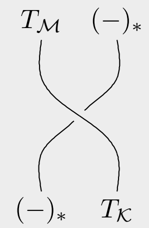
2.3 Pseudomonads in proarrow equipments

To describe the notion of law algebra in our proarrow equipment, we need to extend the notion of pseudomonads in that setting. We follow the tradition on distributive laws, and particularly the work of Nicola Gambino [5] where he defines morphisms of pseudomonads.

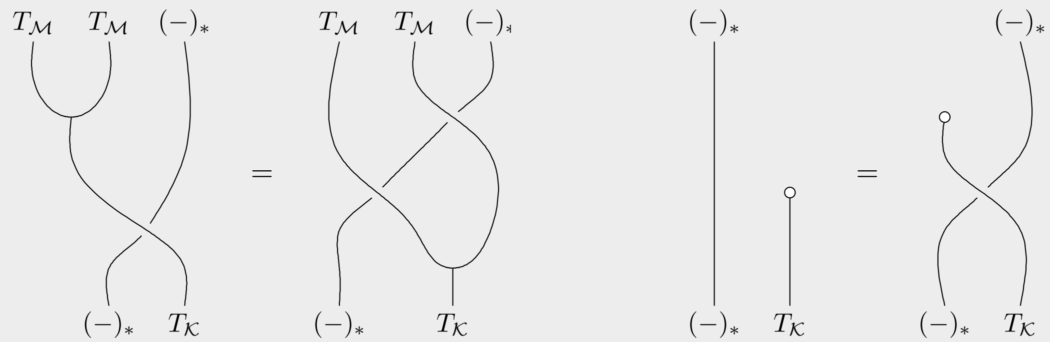
Definition 8 (pseudomonad on a proarrow equipment) *A* pseudomonad on a proarrow equipment $(-)_* : \mathcal{K} \to \mathcal{M}$ *is given by*

- *a pseudomonad $T_{\mathcal{K}}$ on the bicategory $\mathcal{K}$;*
- *a pseudomonad $T_{\mathcal{M}}$ on the bicategory $\mathcal{M}$;*

- *a pseudo natural transformation* $h : T_{\mathcal{M}} \circ (-)_* \to (-)_* \circ T_{\mathcal{K}}$ *noted*



making $((-)_*, h)$ *a morphism of pseudomonads from* $T_{\mathcal{K}}$ *to* $T_{\mathcal{M}}$ *in the sense that the following equalities hold*



A typical example of pseudomonads on a proarrow equipment is given by the pseudomonad that distribute with Yoneda. Those pseudomonads induce naturally a pseudomonad on the proarrow equipment $\mathbf{Cat} \to \mathbf{Dist}$. This situation has been studied in details by Francisco Marmolejo [11], and then by Eugenia Cheng, Martin Hyland and John Power [3]. A similar work can be done in the enriched setting. All pseudomonads that will be consider bellow, in the setting of T-algebraic theories, share the property of distributing with Yoneda.

The notion of pseudomonad on a proarrow equipment allows to state that the two pseudomonads $T_{\mathcal{K}}$ and $T_{\mathcal{M}}$ are compatible with $(-)_*$ in the following sense.

Proposition 3 *The morphism* $(-)_* : \mathcal{K} \to \mathcal{M}$ *lifts to a morphism* $(-)_*^T : \mathbf{Lax}\text{-}\mathbf{T}_{\mathcal{K}}\text{-}\mathbf{Alg} \to \mathbf{Lax}\text{-}\mathbf{T}_{\mathcal{M}}\text{-}\mathbf{Alg}$ *(that we will abusively still note* $(-)_*$*) that restricts, through the inclusion functor from pseudoalgebras to lax algebras, to a morphism* $(-)_*^T : \mathbf{Ps}\text{-}\mathbf{T}_{\mathcal{K}}\text{-}\mathbf{Alg} \to \mathbf{Ps}\text{-}\mathbf{T}_{\mathcal{M}}\text{-}\mathbf{Alg}$. *Those morphisms make the following diagram commute*

$$\begin{tikzcd}
\mathbf{Ps}\text{-}\mathbf{T}_{\mathcal{K}}\text{-}\mathbf{Alg} \arrow[r, "(-)_*^T"] \arrow[d] & \mathbf{Ps}\text{-}\mathbf{T}_{\mathcal{M}}\text{-}\mathbf{Alg} \arrow[d] \\
\mathbf{Lax}\text{-}\mathbf{T}_{\mathcal{K}}\text{-}\mathbf{Alg} \arrow[r, "(-)_*^T"] \arrow[d, "U_{\mathcal{K}}"'] & \mathbf{Lax}\text{-}\mathbf{T}_{\mathcal{M}}\text{-}\mathbf{Alg} \arrow[d, "U_{\mathcal{M}}"] \\
\mathcal{K} \arrow[r, "(-)_*"] & \mathcal{M}
\end{tikzcd}$$

where $U_{\mathcal{K}}$ and $U_{\mathcal{M}}$ are the usual forgetful morphisms.

Proof: The diagrams are easy to check and this has been done in the work of Nicola Gambino [5]. ■

3 General setting

As sketched in the introduction, we want to compute free models by Kan extension on algebraic theories, PROs, PROPS, or even projective sketches. This has lead us to the definition of a common setting for all those kinds of theories that we call T-algebraic theories for a pseudomonad T on the proarrow equipment $\mathbf{Cat} \to \mathbf{Dist}$. In the following, we use the same terminology for the straightforward generalization of T-algebraic theories to a pseudomonad on any proarrow equipment.

3.1 Our main result

We will know state the main result of this paper at the level of proarrow equipments, before we explain in more details all the needed hypotheses. From now on, we work with a pseudomonad $(T_{\mathcal{K}}, T_{\mathcal{M}}, h)$ on a proarrow equipment $(-)_* : \mathcal{K} \to \mathcal{M}$. As we have seen with Proposition 2, the left Kan extension of a $T_{\mathcal{K}}$-algebraic morphism $f : \mathcal{A} \to \mathcal{C}$ along a $T_{\mathcal{K}}$-algebraic morphism $j : \mathcal{A} \to \mathcal{B}$ in $\mathcal{K}$ is obtained by precomposing f_* with the right adjoint j^* and then by taking the representative of resulting morphism. It remains to clarify in which situation this Kan extension is also $T_{\mathcal{K}}$-algebraic. We announce briefly the two ingredients of the construction

the adjunction	the morphism $T_{\mathcal{M}}$-algebraic
$j_* \dashv j^*$	$f_* \circ j^* : \mathcal{B} \nrightarrow \mathcal{C}$
is $T_{\mathcal{K}}$-algebraic	is represented by a $T_{\mathcal{K}}$-algebraic morphism.

To capture those two situations, we introduce the following terminology.

Definition 9 (operadicity) *A morphism f on the bicategory $\mathcal{K}$ is said to be* $T_{\mathcal{K}}$-operadic *when it is $T_{\mathcal{K}}$-algebraic and when its right adjoint f^* in $\mathcal{M}$ is $T_{\mathcal{M}}$-algebraic.*

As we will see later, whereas the notion of operadicity is purely algebraic, its meaning is inherently *combinatorial*. It indicates a kind of tree decomposition property. The second property is of an *algebraic flavor*. It must be understood as a way to say, in the monoidal setting, that the morphism colim – which intuitively computes the colimits of interest – commutes with the tensor product.

Definition 10 (algebraic cocompleteness) *A $\overline{\mathcal{C}}$-cocomplete object $\mathcal{C}$ of the bicategory $\mathcal{K}$ is said to be* T-algebraically $\overline{\mathcal{C}}$-cocomplete *when $\mathcal{C}$ and $\overline{\mathcal{C}}$ are both T-pseudoalgebras and when the morphisms y, y^* and colim involved in the definition of $\overline{\mathcal{C}}$-cocompleteness are all T-algebraic morphisms.*

Thanks to those to new notions, we can now state and prove the theorem which has motivated this paper.

Theorem 1

Let $j : \mathcal{A} \to \mathcal{B}$ be a $T_{\mathcal{K}}$-operadic morphism, $\mathcal{C}$ be a T-algebraically $\overline{\mathcal{C}}$-cocomplete object of the bicategory $\mathcal{K}$ and $f : \mathcal{A} \to \mathcal{C}$ be a T-algebraic morphism. If the morphism $f_* \circ j^*$ factories as

$$\mathcal{B} \xrightarrow{f_* \circ j^*} \mathcal{C} \;=\; \mathcal{B} \xrightarrow{g_*} \overline{\mathcal{C}} \xrightarrow{y^*} \mathcal{C}$$

for every f, then the forgetful functor

$$U_f \quad : \quad \mathbf{Ps}\text{-}\mathbf{T}_{\mathcal{K}}\text{-}\mathbf{Alg}(\mathcal{B}, \mathcal{C}) \;\to\; \mathbf{Ps}\text{-}\mathbf{T}_{\mathcal{K}}\text{-}\mathbf{Alg}(\mathcal{A}, \mathcal{C})$$

has a left adjoint computed with a left Kan extension

$$\mathit{Lan}_j \quad : \quad \mathbf{Ps}\text{-}\mathbf{T}_{\mathcal{K}}\text{-}\mathbf{Alg}(\mathcal{A}, \mathcal{C}) \;\to\; \mathbf{Ps}\text{-}\mathbf{T}_{\mathcal{K}}\text{-}\mathbf{Alg}(\mathcal{B}, \mathcal{C}).$$

Proof : The morphism $f_* \circ j^*$ is the left Kan extension of f_* along j_* in $\mathcal{M}$. According to Proposition 1, the morphism colim $\circ g$ is the representative of $f_* \circ j^*$ in $\mathcal{K}$. But, according to Proposition 2, this representative is the left Kan extension $\mathrm{Lan}_j f$ of f along j in $\mathcal{K}$.

It now remains to check that $\mathrm{Lan}_j f$ is $\mathrm{T}_{\mathcal{K}}$-algebraic. As j is operadic, the morphism $f_* \circ j^*$ is $T_{\mathcal{M}}$-algebraic. But the morphism y^* is pseudomonic with respect to $\mathcal{K}$ so g is $T_{\mathcal{K}}$-algebraic. As colim is $T_{\mathcal{K}}$-algebraic, we conclude that $\mathrm{Lan}_j f$ is $T_{\mathcal{K}}$-algebraic.

The functoriality of the construction is given by th pseudmonicity of y^*. ■

For the particular case of the proarrow equipment $\mathbf{Cat} \to \mathbf{Dist}$, we deduce from Theorem 1 that the free $\mathbb{L}_2$-model on a $\mathbb{L}_1$-model A along a morphism $j : \mathbb{L}_1 \to \mathbb{L}_2$ is computed as the coend

$$\mathrm{Lan}_j A \;=\; \int^{m \in \mathbb{L}_1} \mathbb{L}_2(fm, n) \otimes A(m)$$

Since we discussed at length the conclusion of the theorem, we will directly comment the two hypotheses of operadicity and algebraic completeness.

3.2 A combinatorial hypothesis: operadicity

We have claimed that the notion of operadicity is of a combinatorial nature. Let us explain this intuition by making it more explicit in the proarrow equipment $\mathbf{Cat} \to \mathbf{Dist}$. In that situation, a T-algebraic morphism $j : \mathbb{L}_1 \to \mathbb{L}_2$ *operadic* when the canonical morphism

$$\int^{p \in T(\mathbb{L}_1)} \mathbb{L}_1(m, [p]) \otimes T(\mathbb{L}_2)(Tj(p), n) \;\longrightarrow\; \mathbb{L}_2(jm, [n]) \tag{10}$$

is an isomorphism in the category $\mathbf{Ens}$, for every object m of the category $\mathbb{L}_1$, and every object n of the category $T(\mathbb{L}_2)$.

The definition is purely algebraic, but its meaning is inherently combinatorial:

Operadicity = tree decomposition property.

To understand why, it is worth expanding the definition of operadic functor in the particular case of linear theories — keeping in mind that an object m of the linear theory $\mathbb{L}_1$ is a natural number, and an object n of the category $T(\mathbb{L}_2)$ is a finite sequence $(n_1, \cdots, n_k)$ of natural numbers. Because the underlying category is the category $\mathbf{Ens}$, the canonical morphism (10) is a function, whose domain is the set

$$\int^{p_1 \in \mathbb{L}_1} \cdots \int^{p_k \in \mathbb{L}_1} \mathbb{L}_1(m, p_1 + \cdots + p_k) \times \mathbb{L}_2(p_1, n_1) \times \cdots \times \mathbb{L}_2(p_k, n_k)$$

of pairs $(g, h_1, \cdots, h_k)$ consisting of a morphism

$$g \quad : \quad m \;\rightarrow\; p_1 + \cdots + p_k$$

in the linear theory $\mathbb{L}_1$, and a family of k morphisms

$$h_i \quad : \quad p_i \;\rightarrow\; n_i \qquad\qquad (1 \leq i \leq k)$$

in the linear theory $\mathbb{L}_2$. These pairs are considered modulo the smallest equivalence relation $\sim$ satisfying:

$$(g, h_1 \circ j(h_1'), \cdots, h_k \circ j(h_k')) \;\;\sim\;\; ((h_1' \otimes \cdots \otimes h_k') \circ g, h_1, \cdots, h_k)$$

whenever

$$h_i' \quad : \quad p_i \;\rightarrow\; q_i \qquad\qquad (1 \leq i \leq k)$$

is a family of morphisms in the linear theory $\mathbb{L}_1$. Now, the function (10) transports every such family $(g, h_1, \cdots, h_k)$ to the morphism

$$(h_1 \otimes \cdots \otimes h_k) \circ j(g) \quad : \quad m \rightarrow p_1 + \cdots + p_k \rightarrow n_1 + \cdots + n_k$$

of the linear theory $\mathbb{L}_2$. By operadic, we mean that the function (10) is a bijection, for every natural number m and sequence of natural numbers $(n_1, \cdots, n_k)$. This should be seen as a tree decomposition property, which states that every morphism

$$h \quad : \quad m \rightarrow n_1 + \cdots + n_k$$

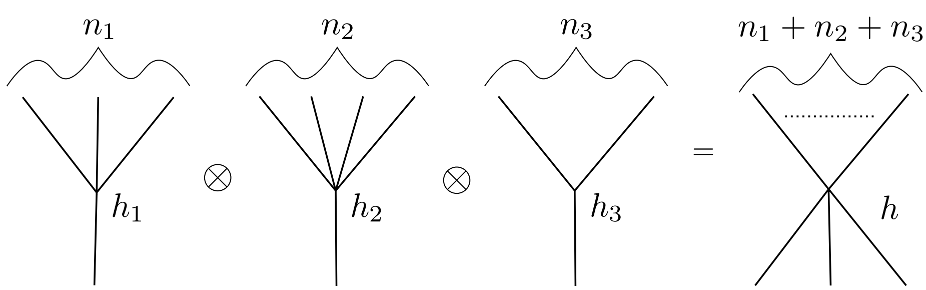
in the linear theory $\mathbb{L}_2$ decomposes uniquely as a morphism

$$j(g) \quad : \quad m \rightarrow p_1 + \cdots + p_k$$

followed by a morphism

$$h_1 \otimes \cdots \otimes h_k \quad : \quad p_1 + \cdots + p_k \rightarrow n_1 + \cdots + n_k$$

modulo the equivalence relation $\sim$ defined above. The terminology "operadic" is justified by the fact the property holds for any map of operads $j : \mathbb{L}_1 \rightarrow \mathbb{L}_2$ between operads — that is, linear theories $\mathbb{L}_1$ and $\mathbb{L}_2$ generated by an equational theory on operations $m \rightarrow 1$ with m inputs and one output. In that case, the morphism g may be taken the identity, with $m = p_1 + \cdots + p_k$, and each morphism h_i describing a particular "component" with n_i roots of the morphism h understood as "forest" of operations with $n_1 + \cdots + n_k$ roots.



In particular, the linear functor $\mathbb{N} \to \Delta$ mentioned in (9) is operadic: this explains why the formula (1) for the tensor algebra, obtained by taking left Kan extensions along j, computes indeed the free monoid object in k-Mod, and defines a left adjoint to the forgetful functor U : k-Alg $\to$ k-Mod. Another example is provided by the linear functor from the linear theory of k-algebras to the linear theory of Lie algebras, which induces the notion of enveloping k-algebra. Another less immediate example is comonoids to bimonoids in PROPs.

To conclude, we give the counter-example of the inclusion functor from the trivial linear theory $\mathbb{N}$ to the theory of bimonoids. This explains the observation mentioned in the introduction that there exists no free k-bialgebra in general.

The case of algebraic theories and projective sketches. For algebraic theories and projective sketches, it is known for a long time that the computation of a free $\mathbb{L}_2$-model on a $\mathbb{L}_1$-model is always possible. It seems then that the hypothesis of operadicity is always valid. Let us take a look at the meaning of this condition in the cartesian setting. The canonical morphism 10 becomes

$$\int^{p_1 \in \mathbb{L}_1} \cdots \int^{p_k \in \mathbb{L}_1} \mathbb{L}_1(m, p_1 \times \cdots \times p_k) \times \mathbb{L}_2(jp_1, n_1) \times \cdots \times \mathbb{L}_2(jp_k, n_k). \tag{11}$$

We can use the surjective pairing to deduce that

$$\mathbb{L}_1(m, p_1 \times \cdots \times p_k) \;\cong\; \mathbb{L}_1(m, p_1) \times \cdots \times \mathbb{L}_1(m, p_k).$$

Then, using the Yoneda lemma, we have that the object 11 is isomorphic to

$$\mathbb{L}_2(jm, n_1) \times \cdots \times \mathbb{L}_2(jm, n_k)$$

and we conclude by using again surjective pairing. But there is a more abstract reasoning that explains this property for the two pseudomonads. Those two pseudomonads are KZ-doctrines. It follows that the T-algebraic structure $a : T\mathbb{L}_1 \to \mathbb{L}_1$ of a category $\mathbb{L}_1$ is the right adjoint to the unit $\eta_{\mathbb{L}_1}$ of the pseudomonad T. This means that there is an isomorphism

$$\mathbb{L}_1(m, a(p)) \;\cong\; T\mathbb{L}_1(\eta_{\mathcal{A}}(m), p)$$

We can use this adjunction to show that the operadicity property always holds

$$\begin{array}{lll}
 & \int^{p \in T(\mathbb{L}_1)} \mathbb{L}_1(m, a(p)) \otimes T(\mathbb{L}_2)(Tj(p), n) & \\
\cong & \int^{p \in T(\mathbb{L}_1)} T\mathbb{L}_1(\eta_{\mathbb{L}_1}(m), p) \otimes T(\mathbb{L}_2)(Tj(p), n) & (a \vdash \eta_{\mathbb{L}_1})\,) \\
\cong & T\mathbb{L}_2(Tj(\eta_{\mathbb{L}_1}(m))n) & \text{(Yoneda lemma)} \\
\cong & T\mathbb{L}_2(\eta_{\mathbb{L}_2}(jm), n) & \text{(naturality of } \eta) \\
\cong & \mathbb{L}_2(jm, [n]) & (a \vdash \eta_{\mathbb{L}_1})
\end{array}$$

We deduce a well-known result, which is a corollary of Theorem 1, and which states that the computation of free models on **Ens** is always possible for algebraic theories and projective sketches.

Corollary 1 *Let j be a algebraic morphism between two algebraic theories or two projective sketches $\mathbb{L}_1$ et $\mathbb{L}_2$. Every $\mathbb{L}_1$-model A induces a free $\mathbb{L}_2$-model $Lan_j A$, we have a bijection*

$$\mathit{Model}(\mathbb{L}_2, \mathbf{Ens})(\mathit{Lan}_j A, B) \;\xrightarrow{\cong}\; \mathit{Model}(\mathbb{L}_1, \mathbf{Ens})(A, U_j B)$$

More precisely, the left Kan extension $Lan_j A$ defines a left adjoint functor to the forgetful functor:

$$\mathit{Lan}_j \dashv U_j : \mathit{Model}(\mathbb{L}_1, \mathbf{Ens}) \to \mathit{Model}(\mathbb{L}_2, \mathbf{Ens}).$$

3.3 An algebraic hypothesis: algebraic cocompleteness

A category is called cocomplete when it has all small colimits. When the category is moreover monoidal, one generally requires that "colimits commute to the tensor product". Fine... But what does this mean? We clarify the concept by reformulating it in the language of proarrow equipment, namely as a property of *algebraic cocompleteness*. Let us look at what it means in the proarrow equipment $\mathbf{Cat} \to \mathbf{Dist}$. Informally speaking,

$\overline{\mathcal{C}}$-algebraic cocompleteness = colimits computed on a diagram indexed by an element of $\overline{\mathcal{C}}$ commute with the T-algebraic structure.

Let us specialize once more and look at the case of linear theories. The category $\mathcal{C}$ is T-algebraically $\overline{\mathcal{C}}$-cocomplete when the class of indices $\mathcal{F}$ is closed under the cartesian product of categories and when the functor colim is monoidal. Given two functors $F : I \to \mathcal{C}$ and $G : J \to \mathcal{C}$ and their corresponding presheaves φ and ψ with $I, J \in \mathcal{F}$, one can form the Day's tensor product of φ and ψ as described by the coend formula

$$\varphi \otimes \psi : c \mapsto \int^{c_1 c_2} \mathcal{C}(c, c_1 \otimes c_2) \times \varphi(c_1) \times \psi(c_2).$$

The factorization system on $\mathbf{Cat}$ ensures that the diagram that corresponds to this presheaf is the functor $\otimes \circ (F \times G)$ indexed by $I \times J$ as shown by the following diagram

$$\begin{tikzcd}
I \times J \arrow[r, "\text{final}"] \arrow[dr, "F \times G"'] & \mathrm{Elt}(\varphi) \times \mathrm{Elt}(\psi) \arrow[r, "\text{final}"] \arrow[d, "\text{discrete}"', "\text{fibration}"] & \mathrm{Elt}(\varphi \otimes \psi) \arrow[d, "\text{discrete}"', "\text{fibration}"] \\
& \mathcal{C} \times \mathcal{C} \arrow[r, "\otimes"'] & \mathcal{C}
\end{tikzcd}$$

We see in that case that if the colimits of F and G exist and commute with the tensor product, then the colimit of $\otimes \circ (F \times G)$ exists and commutes with the tensor product. It is then natural to consider the category $\overline{\mathcal{C}}$ of presheaves having a colimit that commutes with the tensor product. The only remaining thing to do in concrete cases is to check that the distributor on which we compute the representative factories through this category.

4 Computing the free monoid

We will now apply our theory to the computation of the free monoid. As depicted in the introduction, this amounts to compute the free Δ-model on a $\mathbb{N}$-model in the linear theories setting. Those two theories $\mathbb{N}$ and Δ come from operads and the inclusion morphism

$$j_{\mathbb{N}} \quad : \quad \mathbb{N} \to \Delta$$

is obviously operadic. It just remains to check that the required colimits in the category $\mathcal{C}$ exist and commute with the tensor product (in what follows, we will say monoidal colimit).

4.1 A simplified verification

Let C be an object of a monoidal category $\mathcal{C}$ and $F : \mathbb{N} \to \mathcal{C}$ be the associated trivial model. Recall that Fn is equal to $c^{\otimes n}$ for all n of $\mathbb{N}$. We have to check that the distributor $F_* \circ j_{\mathbb{N}}^*$ factories through the category $\overline{\mathcal{C}}$ of presheaves having a monoidal colimit. At first glance, we must check that the presheaf

$$F_* \circ j_{\mathbb{N}}^*(-,n) = \int^{k\in\mathbb{N}} \Delta(Jk,n) \times \mathcal{C}(-,c^{\otimes k})$$

has a monoidal colimit, and this for all $n \in \Delta$. Since the distributor $F_* \circ j_{\mathbb{N}}^*$ is strong monoidal, it suffices to study the case $n = 1$ because then, the isomorphism

$$F_* \circ j_{\mathbb{N}}^*(-,n) \cong F_* \circ j_{\mathbb{N}}^*(-,1)^{\otimes n}$$

ensures that the other presheaves also have a monoidal colimit. But the object 1 is terminal in the category Δ, so the expression of $F_* \circ j_{\mathbb{N}}^*(-,1)$ can be drastically simplified into

$$F_* \circ j_{\mathbb{N}}^*(-,1) = \operatorname*{colim}_{k\in\mathbb{N}} \mathcal{C}(-,c^{\otimes k}).$$

We have already seen that a diagram corresponding to a presheaf of that kind is precisely the functor $F : \mathbb{N} \to \mathcal{C}$. We thus deduce the following proposition.

Proposition 4 *Let $\mathcal{C}$ be a monoidal category and c be an object of $\mathcal{C}$. If the colimit of diagram*

$$F : \begin{array}{ccc} \mathbb{N} & \to & \mathcal{C} \\ n & \mapsto & c^{\otimes n} \end{array}$$

exists and commutes with the tensor product, then it defines the free monoidal on C.

We have just proved, with a detour by proarrow equipments, that Equations (1) and (8) define respectively the free algebra in k-Mod and the free monoid in $\mathbf{Ens}$. But now that we have the general theory, we can deal with more complicated and less known cases.

4.2 A diagram as a colimit of diagrams

Various constructions of free monoids [4, 14, 7] shed light on the need to know that the colimit of a diagram is monoidal, by using general properties of commutation of some kinds of colimits. For example, a common hypothesis is that the considered category has all coequalizers and that they commute with the tensor product. We present here a mean to ensure that a diagram has a monoidal colimit by showing that this is the monoidal colimit of diagrams which have themselves a monoidal colimit. More formally, we have the following property.

Proposition 5 *Let $\mathcal{C}$ be a monoidal category and $\overline{\mathcal{C}}$ be the category of diagram on $\mathcal{C}$ having a monoidal colimit. Suppose that all diagrams, whose base is the category J, have a monoidal colimit. Then the category $\overline{\mathcal{C}}$ is closed for diagrams indexed on J.*

Proof : Consider a diagram $F : J \to \overline{\mathcal{C}}$ indexed on J. We have to show that the colimit of this diagram exists in $\overline{\mathcal{C}}$.

Let $\mathrm{colim} : \overline{\mathcal{C}} \to \mathcal{C}$ be a functor that associates to every diagram of $\overline{\mathcal{C}}$ its colimit in $\mathcal{C}$ and $\iota : \overline{\mathcal{C}} \to \widehat{\mathcal{C}}$ the injection from $\overline{\mathcal{C}}$ to the category of presheaves on $\mathcal{C}$. By hypothesis on J, the diagram $\mathrm{colim} \circ F$ has a monoidal colimit in $\mathcal{C}$, noted c. Since $\widehat{\mathcal{C}}$ is the free completion by colimit of $\mathcal{C}$, the diagram $\iota \circ F$ has a colimit φ in $\widehat{\mathcal{C}}$. It thus suffices then to show that the presheaf φ is represented by c. Let d be an object $\mathcal{C}$, $t_J : J \to 1$ and $y : \mathcal{C} \to \overline{\mathcal{C}}$ be the Yoneda embedding restricted to $\overline{\mathcal{C}}$. The following chain of equivalences

$$\begin{array}{rcll} \widehat{\mathcal{C}}(\varphi, \iota \circ y(d)) & \cong & [J,\widehat{\mathcal{C}}](\iota \circ F, \iota \circ y(d) \circ t_J) & \varphi \text{ is the colimit of } \iota \circ F \\ & \cong & [J,\overline{\mathcal{C}}](F, y(d) \circ t_J) & \iota \text{ is pseudomonic} \\ & \cong & [J,\mathcal{C}](\mathrm{colim} \circ F, d \circ t_J) & \mathrm{colim} \vdash y \\ & \cong & \mathcal{C}(c,d) & c \text{ is the colimit of } \mathrm{colim} \circ F \end{array}$$

shows that c is a representative of φ. We then deduce that the colimit of φ exists and is monoidal, that is to say $\varphi \in \overline{\mathcal{C}}$. ■

This proposition sheds a new light on the constructions of Eduardo Dubuc and of Bruno Vallette (refined by Steve Lack).

4.3 Th Dubuc's construction

In a 1974's paper, Eduardo Dubuc [4] gives a construction of the free monoid on a pointed object in a monoidal category $\mathcal{C}$. This construction involves a transfinite process that is not of our concern. Nevertheless, we can give here an interpretation of the ideas developed by Dubuc.

First, we have to introduce the linear theory of pointed object. This is nothing but the category Δ_{face} of augmented simplicial sets and injective maps. This is a sub-category of Δ in which we only keep non-degenerate morphisms. We note $d_i^n : n-1 \to n$ the face morphism corresponding to

$$d_i^n : \{1, \ldots, n-1\} \mapsto \{1, \ldots, i-1, i+1, \ldots, n\}.$$

Thus, on can see every pointed object $p : 1 \to A$ of $\mathcal{C}$ as a strict functor $A : \Delta_{\text{face}} \to \mathcal{C}$ which associate to n the object $A^{\otimes n}$ and

$$A(d_i^n) = \mathrm{id}_A \otimes \ldots \otimes p \otimes \ldots \mathrm{id}_A.$$

Again, this linear theory is described by an operad and the morphism

$$j_\Delta \quad : \quad \Delta_{\text{face}} \to \Delta$$

is a map of operad. As for Proposition 4, to obtain the free monoid on a pointed object $A : \Delta_{\text{face}} \to \mathcal{C}$, it is enough to check that the colimit of A exists and commutes with the tensor product. This is the case in the following general case.

Proposition 6 *Let $\mathcal{C}$ be a monoidal category such that*

- *all coequalizers exist and commute with the tensor product,*
- *all sequential colimits (in the sense of colimits of a diagram indexed on the category $\mathbb{N}$) exist and commute with the tensor product,*

Then, the free monoid on a pointed object $A : \Delta_{\mathit{face}} \to \mathcal{C}$ exists and is computed by the image in 1 of the left Kan extension of A along j_Δ:

$$\mathit{Lan}_{j_\Delta} A(1) = \operatorname*{\mathit{colim}}_{n \in \Delta_{\mathit{face}}} A(n)$$

Proof: It is enough to check that the diagram A lives in $\overline{\mathcal{C}}$. Let $\Delta_{\text{face}}(n)$ be the full sub-category of Δ_{face} of integers below n. First, we establish that the colimit of the restriction of the diagram A to $\Delta_{\text{face}}(n)$

$$A_n \quad : \quad 1 \longrightarrow A \overset{A(d_1^2)}{\underset{A(d_2^2)}{\rightrightarrows}} A^{\otimes 2} \overset{A(d_1^3)}{\underset{A(d_3^3)}{\rightrightarrows}} \cdots \overset{A(d_1^n)}{\underset{A(d_n^n)}{\rightrightarrows}} A^{\otimes n}$$

lives in $\overline{\mathcal{C}}$. Remark that this diagram has the same colimit than the diagram:

$$D_n \quad : \quad A^{\otimes n-1} \overset{A(d_1^n)}{\underset{A(d_n^n)}{\rightrightarrows}} A^{\otimes n}$$

Remark also that this diagram is computed by coequalizing the morphisms two-by-two. Indeed, because the coequalizers are monoidal, the colimit of the two arrows on the top is given by the coequalizer $e_1 : A^{\otimes 2} \to A_2$ of the two faces d_1^2 and d_2^2, tensored with $A^{\otimes n-2}$

$$A^{\otimes n-1} \underset{A(d_2^n)}{\overset{A(d_1^n)}{\rightrightarrows}} A^{\otimes n} \xrightarrow{e_1 \otimes A^{\otimes n-2}} A_2 \otimes A^{\otimes n-2}$$

Then, we compute the coequalizer of the morphism $m_1 = (e_1 \otimes A^{\otimes n-2})d_1^n = (e_1 \otimes A^{\otimes n-2}) \circ d_2^n$ with the morphism $m_3 = (e_1 \otimes A^{\otimes n-2})d_3^n$

$$A^{\otimes n-1} \underset{m_3}{\overset{m_1}{\rightrightarrows}} A_2 \otimes A^{\otimes n-2} \xrightarrow{e_2 \otimes A^{\otimes n-3}} A_3 \otimes A^{\otimes n-3} .$$

And we start again until we obtain the monoidal colimit of the diagram D_n, or equivalently of the diagram A_n. Thus, A_n lives in $\overline{\mathcal{C}}$ for all n. We conclude by noticing that A is the sequential colimit of the functor

$$\mathbb{N} \to \overline{\mathcal{C}} : n \mapsto A_n.$$

and by using Proposition 5. ■

We can refine the theorem when the category $\mathcal{C}$ is equipped with finite coproducts. It is then possible to compute the free pointed object on an object. This pointed object is simply defined by

$$A \oplus 1.$$

Proposition 7 *Let $\mathcal{C}$ be a monoidal category equipped with finite coproducts and such that*

- *all coequalizers exist and commute with the tensor product,*
- *all sequential colimits (in the sense of colimits of a diagram indexed on the category $\mathbb{N}$) exist and commute with the tensor product,*

Then, the free monoid on an object $A : \mathbb{N} \to \mathcal{C}$ exists and is computed by the image in 1 of the left Kan extension of the pointed object $A \oplus 1$ along j_Δ.

Proof : There exists an inclusion functor $j'_{\mathbb{N}} : \mathbb{N} \to \Delta_{\text{face}}$ which factories $j_{\mathbb{N}}$ through j_Δ. Since Kan extensions do compose, we can deduce that the free monoid on the free pointed object on an object A is also the free monoid on this object. According to Proposition 6, it is enough to check that $A \oplus 1$ is the free pointed object on A.

Let $f : A \to P$ be a morphism of $\mathcal{C}$ to a pointed object (P, p). We define the morphism of pointed object $f \oplus p : (A \oplus 1, i_2) \to (P, p)$. This morphism fits and is unique by universality of the coproduct. ■

Remark that when the category $\mathcal{C}$ has finite coproducts which do not commute with the tensor product, the construction of the free pointed object is not obtained by left Kan extension in $\mathbf{Cat}$. Indeed, using the left Kan extension, we get a functor $\text{Lan}_{j_{\mathbb{N}}} A$ which is not monoidal as indicated by the diagram bellow,

$$\text{Lan}_{j_{\mathbb{N}}} A(2) = 1 \oplus A \oplus A \oplus A^2 \ncong (A \oplus 1) \otimes (A \oplus 1) = \text{Lan}_{j_{\mathbb{N}}} A(1) \otimes \text{Lan}_{j_{\mathbb{N}}} A(1).$$

Thus, Proposition 7 combines a "pedestrian construction" of the monoidal left Kan extension on $j'_{\mathbb{N}} : \mathbb{N} \to \Delta_{\text{face}}$ with the computation of the left Kan extension on $j_\Delta : \Delta_{\text{face}} \to \Delta$ that is known, for abstract reasons, to be monoidal.

4.4 The Vallette/Lack's construction

Bruno Vallette notices in his paper [14] that the hypothesis that coequalizers commute with the tensor product is often too strong. He points out various situations where only reflexive co-equalizers commute with the tensor product. Let us recall briefly what is a reflexive coequalizer.

Definition 11 (reflexive coequalizer) *A pair of morphisms $f, g : A \to B$ is said to be* reflexive *when their exists a morphism $i : B \to A$ that makes the following diagram commute*

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[l, bend right, "i"'] \end{tikzcd}$$

A reflexive coequalizer *is a coequalizer computed on a reflexive pair.*

When the category of consideration is equipped with finite coproducts, it is well-known that every pair of morphisms can be replaced by a reflexive pair computing the same equalizer.

Proposition 8 *Let $\mathcal{C}$ be a category equipped with finite coproducts (noted $\oplus$) and let $f, g : A \to B$ be a pair of morphisms of $\mathcal{C}$. The coequalizer of f and g exists if and only if the reflexive coequalizer of the reflexive pair*

$$\begin{tikzcd} A \oplus B \arrow[r, shift left, "f \oplus id_B"] \arrow[r, shift right, "g \oplus id_B"'] & B \arrow[l, bend right, "i_2"'] \end{tikzcd}$$

exists (i_2 is the second injection).

We see appearing a variation on Proposition 6, where the computation of A_n using coequalizers is replaced by a computation using reflexive coequalizers. This is precisely the result stated by Bruno Vallette and then reformulated by Steve Lack.

Proposition 9 *Let $\mathcal{C}$ be a monoidal category equipped with finite coproducts and such that*

- *all reflexive coequalizers exist and commute with the tensor product,*
- *all sequential colimits (in the sense of colimits of a diagram indexed on the category $\mathbb{N}$) exist and commute with the tensor product,*

Then, the free monoid on an object $A : \mathbb{N} \to \mathcal{C}$ exists and is computed by the image in 1 of the left Kan extension of the pointed object $A \oplus 1$ along j_Δ.

Proof : We go back to the proof of Propositions 6 and 7 with the same notations. The only difference is that we can not use general coequalizers to compute the colimit of the diagram A_n and equivalently of the diagram D_n. Anyway, we will use Proposition 8 to compute the colimit of D_n using reflexive coequalizers. The first computed coequalizer can be replaced by the monoidal reflexive coequalizer as follows

$$\begin{tikzcd}
A^{n-1} \arrow[r, "i_1"] & (A \oplus A^2) \otimes A^{n-2} \arrow[r, shift left, "(A(d_1^2)\oplus \mathrm{id}_{A^2})\otimes \mathrm{id}_{A^{n-2}}"] \arrow[r, shift right, "(A(d_2^2)\oplus \mathrm{id}_{A^2})\otimes \mathrm{id}_{A^{n-2}}"'] & A^2 \otimes A^{n-2} \arrow[l, bend right, "i_2\otimes \mathrm{id}_{A^{n-2}}"'] \arrow[r, "e_1\otimes A^{n-2}"] & A_2 \otimes A^{n-2}
\end{tikzcd}$$

And so on... The rest of the proof is unchanged. ■

4.5 Computing free commutative comonoid

We are also interested in computing free commutative comonoid, that correspond to free exponential modality in linear logic. But we first have to solve a problem: since the beginning, we deal with monoids but never with comonoids. This difficulty is not deep as a comonoid in the monoidal category $\mathcal{C}$ is nothing but a monoid in the opposite category $\mathcal{C}^{\mathrm{op}}$ equipped with the induced tensor product.

One can thus recycle what has been done for the computation of the free monoid by just dualizing every diagram and computing limits instead of colimits. Let us state, for example, a direct corollary of Proposition 6.

Corollary 2 *Let $\mathcal{C}$ be a monoidal category such that*

- *the equalizers exist and commute with the tensor product,*
- *the sequential limits (in the sense of limits of diagrams indexed on the category $\mathbb{N}$) exist and commute with the tensor product.*

Then the free comonoid on the copointed object $A : {\Delta_{face}}^{op} \to \mathcal{C}$ exists and is computed by the image on 1 of the right Kan extension of A on ${j_\Delta}^{op}$:

$$Ran_{{j_\Delta}^{op}} A(1) = \lim_{n \in {\Delta_{face}}^{op}} A(n).$$

As we are interested in the commutative case, we have to move from PROs to PROPs. We have already mentioned that the category Bij of finite sets and bijections is the trivial symmetric theory. In the same way, the category FinSet of finite sets and set theoretic functions is the symmetric theory for commutative monoids.

The injection of Bij into FinSet is operadic, so we directly obtain the following proposition.

Proposition 10 *Let $\mathcal{C}$ be an algebraically cocomplete symmetric monoidal category. The free commutative comonoid $\underset{e}{!}A$ on an object A of $\mathcal{C}$ seen as a symmetric monoidal functor from Bij to $\mathcal{C}$ is computed by the following formula*

$$\underset{e}{!}A = \lim_{n \in Bij} A^n$$

For example, this construction applies to the category $\mathbf{Rel}$ of sets and relations. We get, as expected, the construction of the set of finite multisets on a set

$$\underset{e}{!}A = \coprod_{n \in \mathrm{Bij}} A^n / \sim_{\mathrm{Bij}} .$$

We find back the usual exponential of the relational model of linear logic.

5 Extension to the enriched case

6 Conclusion and future works

We have presented a framework to compute free models of various kind of theories, from algebraic, linear, symmetric theories to projective sketches. This construction, which is based on proarrow equipment, lies on two fundamental properties: operadicity and algebraic cocompleteness.

In future work, we will see how our formulation of algebraic cocompleteness enables to compute the free commutative comonoid in very poorly equipped categories such as coherence spaces or the category of Conway games.

References

[1] M. Barr, C. Wells, Toposes, triples and theories, Springer-Verlag, Berlin, Germany / Heidelberg, Germany / London, UK / etc, 1985.

[2] J. Bénabou, Les distributeurs, Séminaires de Mathématique Pure 33, Université Catholique de Louvain (1973).

[3] E. Cheng, M. Hyland, J. Power, Pseudo-distributive laws, Electronic Notes in Theoretical Computer Science 89.

[4] E. Dubuc, Free monoids, Journal of Algebra 29 (1974) 208–228.

[5] N. Gambino, On the coherence law for pseudo-distributive laws, work in progress (2006).

[6] M. Kelly, J. Power, Adjunctions whose counits are coequalizers, and presentations of finitary enriched monads, Journal of pure and applied algebra 89 (1-2) (1993) 163–179.

[7] S. Lack, Note on the construction of free monoids, Arxiv preprint arXiv:0802.1946.

[8] W. Lawvere, Functorial semantics of algebraic theories and some algebraic problems in the context of functorial semantics of algebraic theories, Ph.D. thesis, Columbia University (1963).

[9] W. Lawvere, Ordinal sums and equational doctrines, in: Lecture Notes in Mathematics, vol. 80, Springer, 1967, pp. 141–155.

[10] S. Mac Lane, Categories for the working mathematician, Springer, 1971.

[11] F. Marmolejo, Distributive laws for pseudomonads, Theory and Applications of Categories 5 (5) (1999) 91–147.

[12] J. Power, Enriched lawvere theories, Theory and Applications of Categories 6 (7) (1999) 83–93.

[13] R. Street, R. Walters, The Comprehensive Factorization Of A Functor, American Mathematical Society 79 (5).

[14] B. Vallette, Free monoid in monoidal abelian categories, arXiv : math.CT/0411543 (2004).

[15] R. Wood, Abstract proarrows i, Cahiers de Topologie et Géométrie Différentielle Catégoriques 23 (3) (1982) 279–290.

[16] R. Wood, Proarrows ii, Cahiers de Topologie et Géométrie Différentielle Catégoriques 26 (2) (1985) 135–168.