Linear continuations and duality

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Abstract

One fundamental aspect of linear logic is that its conjunction behaves in the same way as a tensor product in linear algebra. Guided by this intuition, we investigate the algebraic status of disjunction – the dual of conjunction – in the presence of linear continuations. We start from the observation that every monoidal category equipped with a tensorial negation inherits a lax monoidal structure from its opposite category. This lax structure interprets disjunction, and induces a multicategory whose underlying category coincides with the kleisli category associated to the continuation monad. We study the structure of this multicategory, and establish a structure theorem adapting to linear continuations a result by Peter Selinger on control categories and cartesian continuations.

Key words: Linear logic, monoidal categories, tensorial negation, control categories, linear continuations.

1 Introduction

Linear logic is a proof-theoretic yoga based on two basic ingredients: linearity and duality. Linearity means that every hypothesis is used exactly once in a proof. In other words, an hypothesis cannot be discarded, nor repeated in linear logic – unless the modal operator "of course" enables it explicitly on the formula. A real surprise in the 1980s, when linear logic was discovered, is that this linear policy on proofs leads to a tangible connection between proof theory and tensor algebra. The striking observation was that, once tamed by linearity, the conjunction behaves in the same way as the tensor product of linear algebra. Conceptually speaking, this enables to interpret the conjunction of linear logic as the tensor product of a monoidal category – just like the conjunction of intuitionistic logic is naturally interpreted as the cartesian product of a cartesian category.

The second main ingredient of linear logic is classical duality. Classical duality means that the negation operator $\neg$ of linear logic is involutive, in the sense that the equality

$$A \quad = \quad \neg\neg A \tag{1}$$

holds for every formula A. From this follows a duality phenomenon already familiar in classical logic: namely, that every connective of linear logic comes with a dual connective. Typically, the dual of the linear conjunction $\otimes$ is the linear disjunction $\parr$ defined by the de Morgan equality:

$$A \parr B \quad := \quad \neg (\neg A \otimes \neg B). \tag{2}$$

Moreover, it follows from the equality (1) that the disjunction is associative:

$$A_1 \parr (A_2 \parr A_3) \quad = \quad (A_1 \parr A_2) \parr A_3.$$

A logic of tensor and negation. In this article, we will outline a theory combining harmoniously linear logic and the theory of continuations in computer science. The theory itself is based on a relaxation of linear logic into a logic of tensor and negation, where the equality (1) is replaced by the existence of a canonical proof

$$A \quad \vdash \quad \neg\neg A \tag{3}$$

Given a program of type A, it is natural to think of its environment (also called evaluation context, or continuation) as a program of the dual type $\neg A$. Now, it is well-known in the theory of continuations that applying the negation another time to the type $\neg A$ does not give back the type A itself, but the slightly more liberal type $\neg\neg A$.

This phenomenon is reflected in categorical semantics by the existence of a continuation monad $\neg\neg$ associated to any tensorial negation (see definition below) in a monoidal category $\mathbf{C}$. The canonical proof (3) is then interpreted in the category $\mathbf{C}$ as the unit of the monad

$$A \quad \longrightarrow \quad \neg\neg A.$$

One central observation is that, after replacing the involutive negation by a tensorial negation, it is still possible to derive the disjunction $\parr$ from the conjunction $\otimes$ using the de Morgan equality (2). One should be careful, however, because the resulting binary connective is not associative anymore. In fact, the good way to proceed is to define *simultaneously* the family of n-ary disjunctions:

$$[\, A_1 \parr \cdots \parr A_n \,] \quad := \quad \neg (\neg A_1 \otimes \cdots \otimes \neg A_n).$$

It appears then that this family of n-ary disjunctions is associative, but in the lax sense: it defines what Leinster calls a lax monoidal product. Typically, one has canonical morphisms:

$$[A_1 \parr [A_2 \parr A_3]] \quad \longrightarrow \quad [A_1 \parr A_2 \parr A_3] \quad \longleftarrow \quad [[A_1 \parr A_2] \parr A_3]$$

which are not required to be isomorphisms. There is also a canonical morphism

$$A \quad \longrightarrow \quad [A]$$

which coincides with the unit of the continuation monad.

This leads us to replace the involutive negation of linear logic by a tensorial negation, and to study the resulting logic of tensor and negation – which we call *tensorial logic*. Our general policy, then, is to lift the theory of linear logic to this relaxed framework.

For instance, it is well-known in the theory of continuations that the monad $\neg\neg$ is strong, ie. is equipped with two morphisms

$$t_{A,B} : A \otimes \neg\neg B \to \neg\neg(A \otimes B)$$

$$t'_{A,B} : \neg\neg A \otimes B \to \neg\neg(A \otimes B)$$

natural in A and B, and compatible with the unit and multiplication of the monad, and with the symmetry of the category, in the expected sense.

In this article, we will study the structure of various kleisli categories induced by a continuation monad. Two particularly important situations have been studied comprehensively in the literature: the case when the continuation monad is commutative ; and the case when the continuation monad is not commutative, but the tensor product of the category **C** is cartesian. After reviewing in turn the two situations, we explain why one needs to consider multicategories in order to handle the general case of a linear continuation.

First situation: a commutative monad. A particularly simple but important situation is when the tensorial negation induces a commutative monad. This means that the pair of left and right strength t and t' makes the following diagram

$$\begin{array}{ccccc} & \neg\neg(\neg\neg A \otimes B) & \xrightarrow{\neg\neg(t'_{A,B})} & \neg\neg\neg\neg(A \otimes B) & \\ {}^{t_{\neg\neg A,B}}\nearrow & & & & \searrow^{\mu_{A\otimes B}} \\ \neg\neg A \otimes \neg\neg B & & & & \neg\neg(A \otimes B) \\ {}_{t'_{A,\neg\neg B}}\searrow & & & & \nearrow_{\mu_{A\otimes B}} \\ & \neg\neg(A \otimes \neg\neg B) & \xrightarrow{\neg\neg(t_{A,B})} & \neg\neg\neg\neg(A \otimes B) & \end{array}$$

commute, for all objects A and B. It is folklore that a commutative monad is the same thing as a lax monoidal monad, equipped in this case with the induced morphisms

$$m_1 : 1 \longrightarrow \neg\neg 1 \quad \text{and} \quad m_{A,B} : \neg\neg A \otimes \neg\neg B \longrightarrow \neg\neg(A \otimes B).$$

It is also folklore that the kleisli category of a monad on a monoidal category **C** is monoidal, with its monoidal structure inherited from **C**, if and only if, the monad is lax monoidal.

Masahito Hasegawa [5] has observed that a continuation monad is commutative if and only if the two morphisms

$$\neg\eta_A \quad : \quad \neg\neg\neg A \longrightarrow \neg A \qquad\qquad \eta_{\neg A} \quad : \quad \neg A \longrightarrow \neg\neg\neg A$$

are inverse of one another, for every object A. This property is equivalent to the statement that the continuation monad is *idempotent*, this meaning that every morphism

$$\mu_A \quad : \quad \neg\neg\neg\neg A \longrightarrow \neg\neg A$$

is an isomorphism, for every object A. We shall see that the kleisli category of a continuation monad is $*$-autonomous, with its monoidal structure inherited from the original category, if and only if the continuation monad is lax monoidal. Every commutative continuation monad thus defines a model of linear logic in this way. Recall that in that case, the kleisli category **K** is the category with the same objects as **C**, whose maps are described by the equality:

$$\mathbf{K}(A, B) = \mathbf{C}(A, \neg\neg B) = \mathbf{C}(\neg B, \neg A)$$

It is worth noting that many models of multiplicative linear logic are produced in this way, starting from phase spaces, coherence spaces, or more recently, finiteness spaces or KÃ¶the spaces [2]. This adapts to a proof-theoretic setting the well-known fact that the negated objects (of the form $\neg A$) of a heyting algebra define a boolean algebra.

Second situation: a cartesian tensor. The continuation monad $\neg\neg$ is not commutative in many familiar situations: typically, when $\neg A$ is defined as $A \Rightarrow \bot$ for a given object $\bot$ in a cartesian closed category. In that case, the kleisli category is not necessarily $*$-autonomous. However, an interesting phenomenon arises when the tensor product $\otimes$ is cartesian, and the category $\mathbf{C}$ has finite coproducts (noted $\oplus$). In that case, indeed, one observes that the opposite category $\mathbf{K}^{op}$ is cartesian closed. The cocartesian structure in $\mathbf{C}$ lifts to $\mathbf{K}$, and becomes a cartesian structure in $\mathbf{K}^{op}$. Hence, the cartesian product in $\mathbf{K}^{op}$ of two objects A and B is defined as $A \oplus B$. The series of natural bijections

$$\begin{aligned} \mathbf{K}^{op}(A \oplus B, C) &\cong \mathbf{C}(\neg(A \oplus B), \neg C) \\ &\cong \mathbf{C}(\neg A \otimes \neg B, \neg C) \\ &\cong \mathbf{C}(\neg A, \neg(\neg B \otimes C)) \\ &\cong \mathbf{K}^{op}(A, \neg B \otimes C) \end{aligned}$$

establishes that $\neg B \otimes C$ defines the intuitionistic implication $B \multimap C$ from the object B to the object C in the category $\mathbf{K}^{op}$. This provides a basic recipe to construct a cartesian closed category from a cartesian category $\mathbf{C}$ with negation.

At this point, one would like to clarify what particular kind of cartesian closed categories are of the form $\mathbf{K}^{op}$ for a cartesian category $\mathbf{C}$ with negation, and finite coproducts. In particular, there should remain in the category $\mathbf{K}^{op}$ something of the cartesian product $\otimes$ of the original category $\mathbf{C}$. A general result by John Power and Edmund Robinson states that to give a (bi-)strength for the monad $\neg\neg$ is to give a premonoidal structure on the kleisli category $\mathbf{K}$ such that the canonical functor $j : \mathbf{C} \rightarrow \mathbf{K}$ is a strict premonoidal functor. This means, in particular, that every object A and B of the category $\mathbf{K}$ defines two functors:

$$B \mapsto A \otimes_L B \qquad\qquad A \mapsto A \otimes_R B$$

which coincide on objects, in the sense that the object $A \otimes_L B$ is equal to the object $A \otimes_R B$ – which is generally noted $A \otimes B$. Note in particular that the diagram

$$\begin{array}{ccc} A \otimes B & \xrightarrow{f \otimes_L B} & A' \otimes B \\ {\scriptstyle A \otimes_R g}\downarrow & & \downarrow{\scriptstyle A' \otimes_R g} \\ A \otimes B' & \xrightarrow{f \otimes_L B'} & A' \otimes B' \end{array} \tag{4}$$

does not necessarily commute for $f : A \longrightarrow A'$ and $g : B \longrightarrow B'$. Intuitively, each border of the diagram captures a particular order of evaluation of the components f

and g. A map $f : A \longrightarrow A'$ is called *central* when the diagram (4) commutes for every $g : B \longrightarrow B'$. Note that a monoidal category is a premonoidal category where all the maps are central.

Hence, the cartesian product $\otimes$ of the original category $\mathbf{C}$ induces a premonoidal structure $\otimes$ on the kleisli category $\mathbf{K}$.

In his work on control categories, Peter Selinger [15] characterizes the cartesian closed categories of the form $\mathbf{K}^{op}$ for a cartesian category $\mathbf{C}$ with negation, and finite coproducts. To that purpose, he starts from the observation that the premonoidal structure ($\otimes$) just discussed on $\mathbf{K}$, may be seen alternatively as a premonoidal structure ($\parr$) in the opposite category $\mathbf{K}^{op}$. Selinger axiomatizes the properties of the premonoidal structure $\parr$ inside the cartesian closed category $\mathbf{K}^{op}$, starting from the key observation that the isomorphism

$$(\neg A \otimes B) \otimes C \cong \neg A \otimes (B \otimes C)$$

in the category $\mathbf{C}$ induces an isomorphism

$$(A \multimap B) \parr C \cong A \multimap (B \parr C)$$

in the category $\mathbf{K}^{op}$. This leads to the definition of *control category*. Selinger establishes then a structure theorem, which states that every control category $\mathbf{P}$ is of the form $\mathbf{K}^{op}$ for a given cartesian category $\mathbf{C}$ with negation, and finite coproducts. The crux of the proof is the definition of $\mathbf{C}$ as the sub-category of central maps in the control category $\mathbf{P}$. The category $\mathbf{C}$ happens to be cartesian with negation, and finite coproducts. Moreover, the associated category $\mathbf{K}^{op}$ is equivalent to the original control category $\mathbf{P}$ in a suitable 2-category of control categories.

In a later manuscript, Selinger [16] explains how to get rid of the assumption that the category $\mathbf{C}$ has finite coproducts. Recall that the category $\mathbf{K}^{op}$ has the same objects as the category $\mathbf{C}$, with

$$\mathbf{K}^{op}(A, B) = \mathbf{C}(\neg A, \neg B).$$

The idea is to replace $\mathbf{K}_1 = \mathbf{K}^{op}$ by the category $\mathbf{K}_2$ whose objects $(A_1, \ldots, A_m)$ are finite sequences of objects of the category $\mathbf{C}$, with

$$\mathbf{K}_2((A_1, \ldots, A_m), (B_1, \ldots, B_n)) = \mathbf{C}(\neg A_1 \otimes \cdots \otimes \neg A_m, \neg B_1 \otimes \cdots \otimes \neg B_n).$$

The two categories $\mathbf{K}_1 = \mathbf{K}^{op}$ and $\mathbf{K}_2$ are equivalent (as cartesian closed categories) when the original category $\mathbf{C}$ has finite coproducts. The reason is that every object $(A_1, \ldots, A_m)$ in the category $\mathbf{K}_2$ is isomorphic to the singleton object $(A_1 \oplus \cdots \oplus A_m)$. Selinger observes that, in contrast to $\mathbf{K}_1 = \mathbf{K}^{op}$, the category $\mathbf{K}_2$ is cartesian – with cartesian product given by concatenation on objects – even when the category $\mathbf{C}$ does not have finite coproducts. Moreover, the category $\mathbf{K}_2$ is cartesian *closed* because every morphism

$$\neg A_1 \otimes \cdots \otimes \neg A_m \quad \rightarrow \quad \neg B_1 \otimes \cdots \otimes \neg B_n$$

in the category $\mathbf{C}$ decomposes (by cartesianity of $\otimes$) as a sequence of n morphisms

$$\neg A_1 \otimes \cdots \otimes \neg A_m \quad \rightarrow \quad \neg B_i \tag{5}$$

from the object $(A_1, \ldots, A_m)$ to the singleton object (B_i) in the category $\mathbf{K}_2$. This enables to define the closed structure $\multimap$ in the category $\mathbf{K}_2$ in two steps. First, the closed structure is defined on singleton objects as:

$$(A_1, \ldots, A_m) \multimap B := \neg A_1 \otimes \cdots \otimes \neg A_m \otimes B \tag{6}$$

and then extended to all objects using the well-known isomorphism

$$X \multimap (B_1, \ldots, B_n) \cong (X \multimap B_1, \ldots, X \multimap B_n)$$

satisfied in any cartesian category. Selinger shows that the category $\mathbf{K}_2$ defines a control category $\mathbf{P}$, and extends in this way his structure theorem to a cartesian category $\mathbf{C}$ with negation, but without finite coproducts.

General situation. Here, we want to extend the theory of control categories to the general situation of a monoidal category with negation – without assuming that the tensor product $\otimes$ of $\mathbf{C}$ is cartesian. The situation is apparently hopeless: we have seen that the category $\mathbf{K}_1 = \mathbf{K}^{op}$ is premonoidal, but not monoidal, and that the category $\mathbf{K}_2$ is monoidal, but not closed. What category $\mathbf{K}_1$ or $\mathbf{K}_2$ should we consider?

Well, our starting point will be to consider both categories at the same time, or more precisely, the full and faithful functor

$$\iota \quad : \quad \mathbf{I} \longrightarrow \mathbf{M} \tag{7}$$

which transports the premonoidal category $\mathbf{I} = \mathbf{K}_1$ into the surrounding monoidal category $\mathbf{M} = \mathbf{K}_2$.

Every monoidal category $\mathbf{M}$ is equipped with a tensor product, defining a functor

$$\otimes \quad : \quad \mathbf{M} \times \mathbf{M} \longrightarrow \mathbf{M}.$$

Our main observation is that the category $\mathbf{I}$ induces an *exponential ideal* in the monoidal category $\mathbf{M}$. The notion of exponential ideal extends the usual notion of monoidal *closed* category, which we recall now.

Definition 1 *A monoidal closed category* $\mathbf{M}$ *is a monoidal category equipped with a functor*

$$\multimap \quad : \quad \mathbf{M}^{op} \times \mathbf{M} \longrightarrow \mathbf{M}$$

and a natural isomorphism

$$\begin{array}{ccc} \mathbf{M}^{op} \times \mathbf{M}^{op} \times \mathbf{M} & \xrightarrow{\otimes^{op} \times \mathbf{M}} & \mathbf{M}^{op} \times \mathbf{M} \\ {\scriptstyle \mathbf{M}^{op} \times \multimap} \downarrow & \cong \varphi & \downarrow {\scriptstyle \mathbf{M}(-,-)} \\ \mathbf{M}^{op} \times \mathbf{M} & \xrightarrow[\mathbf{M}(-,-)]{} & \mathbf{Set} \end{array}$$

where the functor

$$\mathbf{M}(-,-) \quad : \quad \mathbf{M}^{op} \times \mathbf{M} \longrightarrow \mathbf{Set}$$

transports every pair of objects (A, B) *to the set of morphisms* $\mathbf{M}(A, B)$.

However, the notion of monoidal closed category is sometimes too restrictive, because in many situations of interest, the arrow $A \multimap B$ is only defined for objects B of a specific subcategory $\mathbf{I}$ of the category $\mathbf{M}$. Moreover, in those situations, the object $A \multimap B$ is an element of the subcategory $\mathbf{I}$ whenever the object B is an element of the subcategory $\mathbf{I}$. This leads to the definition of *exponential ideal* $\mathbf{I}$ of a monoidal category $\mathbf{M}$ where the arrow defines a functor

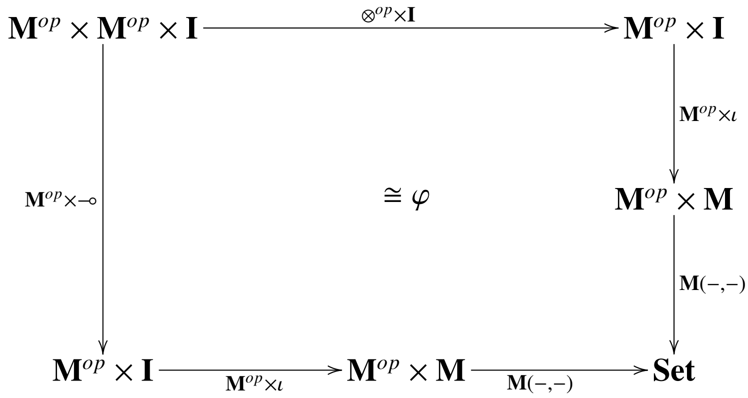
$$\multimap \quad : \quad \mathbf{M}^{op} \times \mathbf{I} \longrightarrow \mathbf{I}$$

satisfying a series of suitable properties. Here, we slightly extend the notion by replacing the subcategory $\mathbf{I}$ of the category of $\mathbf{M}$ by a functor $\iota : \mathbf{I} \longrightarrow \mathbf{M}$.

Definition 2 (exponential ideal) *An* exponential ideal *in a monoidal category* $\mathbf{M}$ *consists of a pair of functors*

$$\iota \quad : \quad \mathbf{I} \longrightarrow \mathbf{M} \qquad\qquad \multimap \quad : \quad \mathbf{M}^{op} \times \mathbf{I} \longrightarrow \mathbf{I}$$

and a natural isomorphism



Observe that a monoidal closed category $\mathbf{M}$ is the same thing as a monoidal category equipped with an exponential ideal where the functor ι is equal to the identity. In the simplest situations, the functor ι is the embedding of a subcategory $\mathbf{I}$ inside the category $\mathbf{M}$. In that case, the exponential ideal provides a natural bijection

$$\varphi \quad : \quad \mathbf{M}(A \otimes B, J) \quad \cong \quad \mathbf{M}(B, A \multimap J)$$

for every pair of objects A, B of $\mathbf{M}$, and every object J of the ideal $\mathbf{I}$. The definition also ensures that the object $A \multimap J$ is an element of $\mathbf{I}$. In fact, a simple argument based on the Yoneda lemma ensures that there exists a canonical isomorphism

$$(A \otimes B) \multimap J \quad \cong \quad A \multimap (B \multimap J).$$

Coming back to linear continuations, it appears that the full and faithful embedding $\iota : \mathbf{I} \longrightarrow \mathbf{M}$ formulated in (7) defines an exponential ideal, where

- $\mathbf{I}$ is the full subcategory consisting of the *singleton* objects in $\mathbf{M}$,
- the linear implication $\multimap$ is defined as:

$$(A_1, \ldots, A_m) \multimap B \quad \stackrel{\text{def}}{=} \quad \neg A_1 \otimes \cdots \otimes \neg A_m \otimes B.$$

Observe that the definition of linear implication coincides with (6) and thus generalizes the situation induced by a *cartesian* category with negation.

The task of the article is to define a notion of *linear control category* axiomatizing the various properties satisfied by the three connectives $\otimes$, $\multimap$ and $⅋$; and then, to establish a structure theorem for linear control categories generalizing the result by Selinger on control categories.

We are guided by the observation that the functor ι induces a multicategory $\mathcal{M}$

whose objects are the objects of the category **I** and whose morphisms from a n-tuple of objects $(A_1, \ldots, A_n)$ to an object B are defined as the elements of:

$$\begin{array}{rcl} \mathcal{M}(A_1, \ldots, A_n; B) & \stackrel{\text{def}}{=} & \mathbf{M}((A_1, \cdots, A_n), (B)) \\ & = & \mathbf{C}(\neg A_1 \otimes \cdots \otimes \neg A_n, \neg B). \end{array}$$

At this point, one feels in familiar grounds again. Remember that the control category $\mathbf{K}^{op}$ is the opposite of the kleisli category **K** induced by the continuation monad on the category **C**. We are in a similar situation here, except that the continuation monad has been replaced by the lax disjunction:

$$[\, A_1 \parr \cdots \parr A_n \,] \quad := \quad \neg \; (\; \neg A_1 \;\; \otimes \;\; \cdots \;\; \otimes \;\; \neg A_n \;\;).$$

discussed at the very beginning of the article. Categorically speaking, this lax disjunction defines a lax monoidal structure on the category **C**. Just like every monad induces a kleisli category, every such lax monoidal structure induces a kleisli multicategory. And just like the control category $\mathbf{K}^{op}$ is the opposite of the kleisli category **K**, the multicategory $\mathcal{M}$ is the opposite of the kleisli multicategory induced by the lax disjunction:

$$\begin{array}{rcl} \mathcal{M}(A_1, \ldots, A_n; B) & = & \mathbf{C}(B, \neg(\neg A_1 \otimes \cdots \otimes \neg A_n)) \\ & = & \mathbf{C}(B, [A_1 \parr \cdots \parr A_n]). \end{array}$$

Hence, our analysis of linear continuations requires to generalize the traditional notion of continuation monad, and *at the same time* the traditional notion of kleisli category.

We also advocate this extension from monads to lax monoidal structures because it is inherently justified by a 2-categorical point of view. On the one hand, a monad in **C** is the same thing as a lax algebra structure on **C**, for the identity 2-monad Id on **Cat**. On the other hand, a lax monoidal structure on **C** is the same thing as a lax algebra structure, for the "free monoidal category" 2-monad T on **Cat**. There exists a unique 2-monad morphism $Id \longrightarrow T$ because the 2-monad Id is initial in the category of 2-monads on **Cat**. This enables to transport every lax monoidal structure (that is, a lax algebra for T) to a monad (that is, a lax algebra for Id) by "change of basis" along the map $Id \longrightarrow T$. In our case, the lax disjunction is transported to the continuation monad, understood now as the unary case of the lax disjunction:

$$[A] \quad = \quad \neg\neg A.$$

We benefit today of the rich formal theory of monads developed by Ross Street and his collaborators [17,7]. One motivation for our work is to encourage the development of an equivalent formal theory of lax monoidal structures, and more generally, of lax monoidal algebras for various 2-monads.

Related works. Yves Lafont, Bernhard Reus and Thomas Streicher [8] observe that the fragment of intuitionistic logic with negation instead of implication is sufficient to interpret the λ-calculus, using a continuation passing style translation.

Later on, Martin Hofmann and Thomas Streicher [6] define a semantics of the call-by-name $\lambda\mu$-calculus in terms of categorical continuation models induced by response categories. They establish moreover that these continuation models are complete among all the models of the call-by-name $\lambda\mu$-calculus.

The first attempt to characterize algebraically those categories of continuation is made by Hayo Thielecke [18] in his study of $\otimes\neg$-categories. To that purpose, Thielecke introduces the important idea that these categories should be premonoidal, instead of monoidal. He also observes that negation defines a self-adjoint functor, whose associated monad is the continuation monad.

Peter Selinger introduces the notion of control category [15] and establishes a fundamental structure theorem, stating that every such category is the continuation category of a particular response category. Our work is largely inspired from this result, and the techniques Selinger introduced to that purpose.

A different class of models for the call-by-name $\lambda\mu$-calculus, based on fibrations, was defined by Luke Ong for the original $\lambda\mu$-calculus [12] and later extended to the disjunctive $\lambda\mu$-calculus by David Pym and Eike Ritter [14]. These models clarify the fibered nature of the $\lambda\mu$-calculus with respect to control contexts. These fibered categories offer a rich algebraic structure, in which the $\lambda\mu$-calculus is recovered as an internal language. There is a back-and-forth translation between these fibered models of the $\lambda\mu$-calculus and control categories, based on the idea that an object A in a fiber Δ should be identified with the object $A \parr \Delta$ in the control category. John Power and Edmund Robinson observe that this recipe is an instance of a general construction, defining a fibration from any premonoidal structure, with fibers provided by the comonoid objects of the category [13].

In his PhD thesis, Olivier Laurent [9] suggests a notion of linear control category, standing half way between the notions of control category and linearly distributive category [1]. Laurent establishes then that every such category provides a model of MALLP, a polarized variant of multiplicative and additive linear logic. In contrast to control categories, the definition of linear control category does not attempt to capture the precise properties of the continuation models induced by a monoidal response category. This task of this article is precisely to answer this open question.

Plan of the article. In section 2, we recall the definition of tensorial logic and its connection to polarized linear logic and ludics. In section 3, we introduce multicategories and a notion of footprints that enables to see a multicategory as a functor from a category to a monoidal category. In section 4, we define the notion of linear control categories and categories of linear continuation, and prove a structure theorem between them.

2 Tensorial logic

In this section, we introduce a logic of tensor and negation – called *tensorial logic*. We proceed in reverse order to the usual order in proof theory: instead of defining the logic first, we start by explaining its categorical semantics. The reason is that the algebraic setting is basic and concise: it simply consists of a *tensorial negation* on a symmetric monoidal category. Once the categorical situation settled, we formulate the sequent calculus of the resulting logic – this leading us to tensorial logic.

From that point of view, tensorial logic may be seen as a very natural relaxation of linear logic, where the assumption that negation is involutive

$$A \quad \cong \quad \neg\neg A \tag{8}$$

is lifted. The idea of relaxing linear logic into tensorial logic is originally motivated by our categorical analysis of game semantics [11]. In the game-theoretic account of logic, every negation of the formula is interpreted as a turn in the game – where Proponent gives the hand to Opponent, or conversely. The assumption (8) that negation is involutive implies that every turn between Proponent and Opponent becomes invisible in linear logic. Hence, the idea behind tensorial logic is simply to relax this assumption.

Interestingly, the resulting logic of tensor and negation provides just another account of polarities in logic. The idea that logic is regulated by polarities is far from new: it was discovered by Jean-Yves Girard in his seminal work on LC [4]. Our presentation simply clarifies the fact that polarities and continuations describe the same duality phenomenon in their own logical and computational language, respectively. In particular, we take the opportunity to clarify here in what sense the sequent calculus underlying ludics is related to the notion of lax disjunction discussed in the introduction.

Tensorial negation. A tensorial negation in a symmetric monoidal category $(\mathbf{C}, \otimes, 1)$ is defined as a functor

$$\neg \quad : \quad \mathbf{C} \longrightarrow \mathbf{C}^{op} \tag{9}$$

equipped with a family of bijections

$$\psi_{A,B,C} \quad : \quad \mathbf{C}(A \otimes B, \neg\, C) \quad \cong \quad \mathbf{C}(A, \neg\,(B \otimes C))$$

natural in A, B and C and such that the following diagram

$$\begin{array}{ccc}
\mathbf{C}(A \otimes (B \otimes C), \neg\, D) & \xrightarrow{\mathbf{C}(\alpha_{A,B,C}, \neg C)} & \mathbf{C}((A \otimes B) \otimes C, \neg\, D) \\
 & & \downarrow{\scriptstyle \psi_{A \otimes B, C, D}} \\
\downarrow{\scriptstyle \psi_{A, B \otimes C, D}} & & \mathbf{C}(A \otimes B, \neg\,(C \otimes D)) \\
 & & \downarrow{\scriptstyle \psi_{A,B,C \otimes D}} \\
\mathbf{C}(A, \neg\,((B \otimes C) \otimes D)) & \xrightarrow[\mathbf{C}(A, \neg \alpha^{-1}_{B,C,D})]{} & \mathbf{C}(A, \neg\,(B \otimes (C \otimes D)))
\end{array}$$

commutes for all objects A, B, C and D in $\mathbf{C}$, where α denotes the associativity of the tensor product in $\mathbf{C}$. A symmetric monoidal category equipped with a tensorial negation is called a *dialogue category*.

Given a negation, it is customary to define the formula *false* as the object $\bot \stackrel{\text{def}}{=} \neg\, 1$ obtained by "negating" the unit object 1 of the monoidal category. Note that the bijection $\varphi_{A,B,1}$ provides then the category $\mathbf{C}$ with a one-to-one correspondence

$$\varphi_{A,B,1} \, : \, \mathbf{C}(A \otimes B, \bot) \cong \mathbf{C}(A, \neg B)$$

for all objects A and B. For that reason, the definition of a negation $\neg$ is often replaced by thestatement that "the object $\bot$ is exponentiable" in the symmetric monoidal category $\mathbf{C}$, with negation $\neg A$ noted $\bot^A$. In the theory of continuation, such an object is called an object of *responses*. It appears that the existence of an exponential object in a monoidal category $\mathbf{C}$ is equivalent to the presence of a tensorial negation in that category.

The starting point of this work is to think of a tensorial negation $\neg$ as a pair of functors:

$$L \quad : \quad \mathbf{C} \longrightarrow \mathbf{C}^{op} \qquad\qquad R \quad : \quad \mathbf{C}^{op} \longrightarrow \mathbf{C}$$

defined by the functor L given in (9) and its opposite functor R. This formulation reveals that the functor L is left adjoint to the functor R,

$$L \ : \ \mathbf{C} \ \rightleftarrows \ \mathbf{C}^{op} \ : \ R \tag{10}$$

this reflecting the existence of a bijection

$$\mathbf{C}(A, \neg B) \quad \cong \quad \mathbf{C}(B, \neg A)$$

natural in A and B, described by the adjunction (10) as a bijection

$$\mathbf{C}(A, RB) \quad \cong \quad \mathbf{C}^{op}(LA, B).$$

This adjunction $L \dashv R$ induces a monad $\neg\neg = R \circ L$ on the category $\mathbf{C}$, called the *continuation monad* associated to the tensorial negation. This monad has been thoroughly studied in computer science, because it captures the essential features of continuations in programming languages. The monad has also a purely logical status. Typically, its unit provides a natural transformation

$$\eta_A \quad : \quad A \quad \longrightarrow \quad \neg\neg A$$

which reflects the fact that every formula A implies its double negation $\neg\neg A$.

Tensorial logic. The algebraic situation induced by a tensorial negation is reflected in proof theory by a logic of tensor and negation – called *tensorial logic*. We formulate below the sequent calculus of the logic. The formulas of tensorial logic are defined by the following grammar:

$$A \, , \, B \quad ::= \quad \mathbf{true} \mid A \otimes B \mid \neg A.$$

where **true** will very soon play the role of the tensor unit. The rules of the sequent calculus of the logic is given in Figure 1.

Polarities. The relationship between polarities and continuations is nicely captured by reformulating the bilateral sequent calculus above as a monolateral sequent calculus. To that purpose, one needs indeed to introduce polarities on formula. The formulas A that were on the right in the bilateral presentation remain there, and are

$$\frac{}{A \vdash A}\text{ Axiom} \qquad \frac{\Gamma \vdash A \qquad A\,,\,\Delta \vdash B}{\Gamma\,,\,\Delta \vdash B}\text{ Cut}$$

$$\frac{\Gamma\,,\,A\,,\,B \vdash C}{\Gamma\,,\,A \otimes B \vdash C}\text{ Left }\otimes \qquad \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma\,,\,\Delta \vdash A \otimes B}\text{ Right }\otimes$$

$$\frac{\Gamma \vdash A}{\Gamma\,,\,\mathbf{true} \vdash A}\text{ Left }\mathbf{true} \qquad \frac{}{\vdash \mathbf{true}}\text{ Right }\mathbf{true}$$

$$\frac{\Gamma \vdash A}{\Gamma\,,\,\neg A \vdash}\text{ Left }\neg \qquad \frac{\Gamma\,,\,A \vdash}{\Gamma \vdash \neg A}\text{ Right }\neg$$

$$\frac{A_1\,,\,\ldots\,,\,A_n \vdash}{A_{\sigma(1)}\,,\,\ldots\,,\,A_{\sigma(n)} \vdash}\text{ Permutation (for any permutation }\sigma)$$

Fig. 1. The sequent calculus for multiplicative tensorial logic: bilateral presentation

called *positive*. On the other hand, the formulas on the left of the sequent jump on the right of the sequent, and are now called *negative*.

To distinguish between positive and negative formulas, we have to clone each construct $1, \otimes$ into itself: $1, \otimes$ and its dual: $\bot, \parr$. The negation $\neg$ itself is cloned in two operations $\downarrow$ and $\uparrow$, each of them with a specific effect:

- $\downarrow$ transports the positive formulas into the negative formulas,
- $\uparrow$ transports the negative formulas into the positive formulas.

We use the letters P and Q for the positive formulas, the letters M and N for the negative formulas, and the letters Γ, Δ for the contexts of negative formulas. Formulas are constructed by the following grammar:

$$\begin{array}{ll} \text{Positives} & 1 \mid {\downarrow}N \mid P \otimes Q \\ \text{Negatives} & \bot \mid {\uparrow}P \mid M \parr N \end{array}$$

Every positive formula P has a dual negative formula $P^{\perp}$, obtained by dualizing every logical construct appearing in the formula P.

There are two kinds of sequents in the bilateral presentation of tensorial logic: (1) the sequents $\Gamma \vdash A$ with a conclusion, and (2) the sequents $\Gamma \vdash$ without a conclusion. Consequently, there are two kinds of sequents in the monolateral presentation: (1) the sequents $\vdash \Gamma, P$ containing exactly one positive formula P, and (2) the sequents

$$\frac{}{\vdash P^\perp, P}\ \text{Axiom} \qquad\qquad \frac{\vdash \Gamma, P \qquad \vdash P^\perp, \Delta, Q}{\vdash \Gamma, \Delta, Q}\ \text{Cut}$$

$$\frac{\vdash \Gamma, P \qquad \vdash \Delta, Q}{\vdash \Gamma, \Delta, P \otimes Q}\ \text{Tensor} \qquad\qquad \frac{\vdash \Gamma_1, L, M, \Gamma_2, P}{\vdash \Gamma_1, L \parr M, \Gamma_2, P}\ \text{Par}$$

$$\frac{}{\vdash 1}\ \text{One} \qquad\qquad \frac{\vdash \Gamma}{\vdash \Gamma, \bot}\ \text{Bottom}$$

$$\frac{\vdash \Gamma, N}{\vdash \Gamma, \downarrow N}\ \text{Linear strengthening} \qquad\qquad \frac{\vdash \Gamma, P}{\vdash \Gamma, \uparrow P}\ \text{Linear dereliction}$$

$$\frac{\vdash\ N_1\ ,\ \ldots\ ,\ N_n}{\vdash\ A_{\sigma(1)}\ ,\ \ldots\ ,\ A_{\sigma(n)}}\ \text{Permutation (for any permutation } \sigma\text{)}$$

Fig. 2. The sequent calculus for multiplicative tensorial logic: monolateral presentation

$\vdash \Gamma$ where Γ contains only negative formulas.

Once understood the general recipe, the monolateral sequent calculus follows immediately from the bilateral sequent calculus as depicted in Figure 2.

This monolateral formulation clarifies in what sense tensorial logic, a logic of linear continuations, is at the same time a polarized logic. In fact, tensorial logic coincides with the multiplicative fragment of MALLP, a linear variant of polarized linear logic introduced by Olivier Laurent in his PhD thesis [9]. The interested reader will find in [11] how this multiplicative fragment of tensorial logic is extended to additives and exponentials.

3 Multicategories and fingerprints

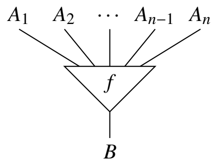
As depicted in the introduction, adapting control categories to a linear setting leads naturally to the study of a multicategory $\mathcal{M}$ induced by a functor

$$\iota\ :\ \mathbf{I} \longrightarrow \mathbf{M}.$$

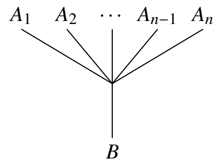
We recall here the basics of the theory of multicategories. We also introduce a notion of fingerprint $\mathbf{I} \longrightarrow \mathbf{M}$ which captures the fact that a multicategory $\mathcal{M}$ can be formulated as a functor from a category $\mathbf{I}$ – describing the objects of the multicategory – to a monoidal category $\mathbf{M}$ – describing the morphisms of the multicategory. We refer the reader to the Tom Leinster's book *Higher Operads, Higher categories*

[10] for a comprehensive work on multicategories.

Multicategories A multicategory $\mathcal{M}$ consists of (1) a class $\mathcal{M}_0$ whose elements are called the objects of $\mathcal{M}$, (2) for each $n \in \mathbb{N}$ and $A_1, \ldots, A_n, B \in \mathcal{M}_0$, a class of morphisms $\mathcal{M}(A_1, \ldots, A_n; B)$ whose elements f are depicted graphically as



or simply (when f is implicit)



(3) for each $n, k_1, \ldots, k_n \in \mathbb{N}$ and $A_i, A_i^j, B \in \mathcal{M}_0$, a function

$$\mathcal{M}(A_1, \ldots, A_n; B) \times \mathcal{M}(A_1^1, \ldots, A_1^{k_1}; A_1) \times \cdots \times \mathcal{M}(A_n^1, \ldots, A_1^{k_n}; A_n)$$
$$\longrightarrow \mathcal{M}(A_1^1, \ldots, A_1^{k_1}, \ldots, A_n^1, \ldots, A_1^{k_n}; B)$$

called composition and written

$$(g, f_1, \ldots, f_n) \mapsto g \circ (f_1, \ldots, f_n)$$

and (4) for each $A \in \mathcal{M}_0$, an element $id_A \in \mathcal{M}(A; A)$ called the identity on A, satisfying to the usual identity and associativity axioms.

A multicategory $\mathcal{M}$ is called *symmetric* when it is equipped with a bijection

$$\mathcal{M}(A_1, \ldots, A_n; A) \quad \tilde{\rightarrow} \quad \mathcal{M}(A_{\sigma(1)}, \ldots, A_{\sigma(n)}; A)$$

for any objects $A_1, \ldots, A_n, A \in \mathcal{M}_0$ and permutation $\sigma \in S_n$. One requires that this bijection is the identity when σ is the identity, and is compatible with the composition of permutations, and the composition of morphisms in the multicategory.

Maps of multicategories. Let $\mathcal{M}$ and $\mathcal{N}$ be multicategories. A *map (or homomorphism)* of multicategories $F : \mathcal{M} \to \mathcal{N}$ consists of a function $F_0 : \mathcal{M}_0 \to \mathcal{N}_0$ (usually just written F) together with a function

$$\mathcal{M}(A_1, \ldots, A_n; B) \quad \longrightarrow \quad \mathcal{N}(F(A_1), \ldots, F(A_n); F(B))$$

for each $A_1, \ldots, A_n, B \in \mathcal{M}_0$, such that identities and composition are preserved. The map of multicategories F is called *symmetric* when the two underlying multicategories are symmetric, and F preserves the symmetry.

This defines a category **Multicat** of multicategories and homomorphisms.

Every monoidal category **M** gives rise to a multicategory $\mathscr{U}(\mathbf{M})$ where commas are understood as tensors

$$A_1, \ldots, A_n \to B \quad \stackrel{\text{def}}{=} \quad A_1 \otimes \cdots \otimes A_n \to B.$$

When the monoidal category is not strict, there is an ambiguity in what we mean by the object $A_1 \otimes \cdots \otimes A_n$. A nice way to get rid of this ambiguity is to work with an unbiased formulation of our monoidal category (see definition 3.1.1, page 67 of [10]). For simplicity, we will ignore this issue in the rest of the paper. The construction $\mathscr{U}$ defines a functor

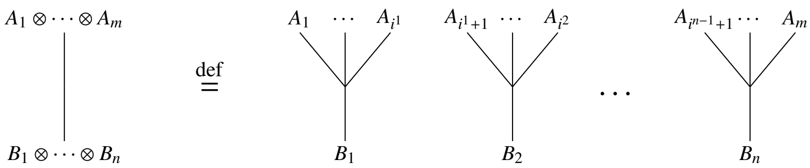
$$\mathscr{U} \quad : \quad \mathbf{MonCat} \longrightarrow \mathbf{Multicat}$$

from the category **MonCat** of monoidal categories and strict monoidal functors, to the category **Multicat**.

Conversely, every multicategory $\mathcal{M}$ gives rise to a monoidal category $\mathscr{F}(\mathcal{M})$ whose objects are finite sequences $(A_1, \ldots, A_n)$ of objects of $\mathcal{M}$ concatenated by the tensor product

$$A_1 \otimes \cdots \otimes A_n \quad \stackrel{\text{def}}{=} \quad (A_1, \ldots, A_n)$$

and whose morphisms are sequences of morphisms in $\mathcal{M}$ put side by side, as depicted informally below:



It is folklore that this construction defines a functor

$$\mathscr{F} \quad : \quad \mathbf{Multicat} \longrightarrow \mathbf{MonCat}$$

left adjoint to the functor $\mathscr{U}$. We explain now how this adjunction enables to see every multicategory $\mathcal{M}$ as a functor from a category of objects to a monoidal category of morphisms – what we call the *fingerprint* of the multicategory $\mathcal{M}$.

Fingerprint. Suppose given three categories **A**, **B** and **C** and two adjunctions

$$L_1 : \mathbf{A} \rightleftarrows \mathbf{B} : R_1 \qquad\qquad L_2 : \mathbf{B} \rightleftarrows \mathbf{C} : R_2$$

The adjunctions $L_1 \dashv R_1$ and $L_2 \dashv R_2$ induce a comonad $L_1 \circ R_1$ and a monad $R_2 \circ L_2$ on the category **B**. Every object B of the category **B** is thus equipped with a pair of morphism

$$L_1 \circ R_1(B) \xrightarrow{\varepsilon_B} B \xrightarrow{\eta_B} R_2 \circ L_2(B) \tag{11}$$

provided by the counit ε of the comonad and the unit η of the monad, both of them instantiated at the object B. Now, imagine that the composite morphism

$$L_1 \circ R_1(B) \longrightarrow R_2 \circ L_2(B) \tag{12}$$

characterizes the object B, up to isomorphism. In that case, we declare that the composite morphism is the *fingerprint* of the object B.

Typically, this situation arises when the category **B** is equipped with a factorization system $(\mathscr{E}, \mathscr{M})$ in the sense of Freyd and Kelly [3] and moreover, the counit morphism ε_B is element of $\mathscr{E}$ and the unit morphism η_B is element of $\mathscr{M}$. In that case, the pair of morphisms (11) is recovered, up to isomorphism, by factoring the morphism (12) as a composite of a morphism in $\mathscr{E}$ and a morphism in $\mathscr{M}$.

Remark that, by adjunction, the fingerprint (12) may be seen equivalently as a morphism

$$R_1(B) \longrightarrow R_1 \circ R_2 \circ L_2(B)$$

in the category **A**, or as a morphism

$$L_2 \circ L_1 \circ R_1(B) \longrightarrow L_2(B)$$

in the category **C**.

Fingerprints for multicategories. An illustration of this situation is when **A** is the category **Cat** of categories and functors, **B** is the category **MultiCat** of multicategories and homomorphisms, and **C** is the category **MonCat** of monoidal categories and strict monoidal functors; and the factorization system in **MultiCat** is provided by the set of bijective on objects homomorphisms for $\mathscr{E}$ and the set of full and faithful homomorphisms for $\mathscr{M}$.

The functor

$$L_1 \quad : \quad \mathbf{Cat} \longrightarrow \mathbf{Multicat}$$

transports every category A to the multicategory $L_1(A)$ with the same objects, and the same morphisms. In particular, the multicategory $L_1(A)$ is *filiform* in the sense that it does not contain any morphism with n inputs for $n \neq 1$. The functor

$$R_1 \quad : \quad \mathbf{Multicat} \longrightarrow \mathbf{Cat}$$

is obtained by transporting every multicategory B to the category $R_1(B)$ with the same objects, obtained by restricting B to its morphisms with exactly one input.

The adjunction $L_2 \dashv R_2$ is the adjunction $\mathscr{F} \dashv \mathscr{U}$ described above.

This demonstrates that every multicategory $\mathcal{M}$ may be seen alternatively as a particular functor

$$\mathcal{M} : \mathbf{I} \to \mathbf{M}.$$

We find the multicategory back by taking as objects the objects of $\mathbf{I}$ and as morphisms of type $A_1, \ldots, A_n \to A$ the morphisms of type

$$\mathcal{M}A_1 \otimes \cdots \otimes \mathcal{M}A_n \to \mathcal{M}A$$

in $\mathbf{M}$. Remark that we can assume – and we will do so in the rest of the paper – without lost of generality that the functor $\mathcal{M}$ is the identity on objects.

4 Linear control categories

Premonoidal categories It is a well-known fact that the category **Cat** of categories has exactly two symmetric monoidal closed structures.The traditional one – where the tensor product is cartesian and the exponential is the category of functors and natural transformations – gives rise to a cartesian 2-category, also noted **Cat**. The notion of monoidal category follows, in the sense that a monoidal category is the same thing as a pseudo-monoid in **Cat**.

The other monoidal closed category is generally noted $\mathbf{Cat}'$. Informally speaking, its tensor product $\mathbf{A} \otimes \mathbf{B}$ has the same objects as $\mathbf{A} \times \mathbf{B}$, and its morphisms are finite alternating sequences of non-identity morphisms of $\mathbf{A}$ and $\mathbf{B}$. The tensor product $\mathbf{A} \otimes \mathbf{B}$ may be defined conceptually the following pushout in **Cat** [13]

$$\begin{array}{ccc} |\mathbf{A}| \times |\mathbf{B}| & \longrightarrow & \mathbf{A} \times |\mathbf{B}| \\ \downarrow & & \downarrow \\ |\mathbf{A}| \times \mathbf{B} & \longrightarrow & \mathbf{A} \otimes \mathbf{B} \end{array}$$

where $|\mathbf{A}|$ denotes the discrete category with same objects as the category $\mathbf{A}$. Its exponential $[\mathbf{A}, \mathbf{B}]$ is the category of functors and transformations from $\mathbf{A}$ to $\mathbf{B}$. Intuitively, the objects of $\mathbf{C} \otimes \mathbf{D}$ are $|\mathbf{C}| \times |\mathbf{D}|$ and its morphisms are finite alternating sequences of non-identity arrows of $\mathbf{C}$ and $\mathbf{D}$. The tensor is then obtained by concatenation and cancellation according to the composition in $\mathbf{C}$ and $\mathbf{D}$.

For instance, a sesqui-category is a category enriched over the symmetric monoidal category $\mathbf{Cat}'$, just like a 2-category is a category enriched over the symmetric monoidal category $\mathbf{Cat}$. Consequently, a sesqui-category is essentially a 2-category for which the *interchange* formula is not valid, that is horizontal and vertical compositions of 2-cells do not commute.As the category $\mathbf{Cat}'$ is symmetric closed, it may be also seen as a sesqui-category with categories as objects, functors as morphisms, and transformations (without naturality) as 2-cells.

John Power and Edmund Robinson [13] observe that a monoid in $\mathbf{Cat}'$ is the same thing as a strict premonoidal category. Usually, one obtains the non-strict version of the structure by considering pseudomonoids instead of monoids. However, this pseudo-machinery does not work directly here because the structural 2-cells of a premonoidal category are required to be natural and central. Hence, considering pseudomonoid is not enough as the structural 2-cells have no reason to be central. We propose here to introduce the definition of a natural 2-cell in a sesqui-category, in order to provide a 2-dimensional definition of (non necessarily strict) premonoidal categories.

However, observe that a transformation is natural if and only if it satisfies the interchange law with every precomposed cell. From now on, we call *natural 2-cell* in a sesqui-category, any cell commuting to every precomposed 2-cell. Then, we define a premonoidal category as a pseudomonoid in the sesqui-category $\mathbf{Cat}'$ such that every 2-cell generated by the structure is natural.

This provides a purely conceptual definition of premonoidal categories. However, it is often convenient to work with the more concrete definition given below.

Let $⅋$ be a functor from $\mathbf{C} \otimes \mathbf{C}$ to $\mathbf{C}$. A morphism $f : A \to A'$ is called *central* if for any morphism $g : B \to B'$, the two composite $(f ⅋ B') \circ (A ⅋ g)$ and $(A' ⅋ g) \circ (f ⅋ B)$ agree, and the two composite $(B' ⅋ f) \circ (g ⅋ A)$ and $(g ⅋ A') \circ (B ⅋ f)$ agree. In this case, we shall use the notation $f ⅋ g$ and $g ⅋ f$ respectively.

A *premonoidal category* is a category $\mathbf{C}$, together with a functor $⅋ : \mathbf{C} \otimes \mathbf{C} \to \mathbf{C}$ together with an object $\bot$ and central natural isomorphisms $\alpha_{A,B,C} : (A ⅋ B) ⅋ C \to$

$A \parr (B \parr C)$, $l_A : A \to A \parr \bot$ and $r_A : A \to \bot \parr A$ subject to the same coherence conditions as for monoidal categories: the Mac Lane's pentagon expressing coherence of α, and the triangle expressing coherence of l and r with respect to α.

A *symmetric premonoidal category* has in addition a family of central natural isomorphisms $\sigma_{a,b} : a \parr b \to b \parr a$, satisfying $\sigma \circ \sigma = id$ and the evident compatibility conditions with respect to α, l and r.

The central morphism of a premonoidal category $\mathbf{C}$ form a monoidal category $\mathcal{Z}(\mathbf{C})$. In particular, a premonoidal category is monoidal if and only if $\mathcal{Z}(\mathbf{C}) = \mathbf{C}$.

Linear control categories. Shifting from a cartesian category $\mathbf{C}$ to a monoidal category in the theory of control categories, introduces a new issue: the fact that $\mathbf{K}_2$ is monoidal, but not closed. This is handled by considering not only the monoidal category $\mathbf{K}_2$ but the whole functor

$$\iota \quad : \quad \mathbf{K}_1 \longrightarrow \mathbf{K}_2$$

as explained in the introduction and in Section 3. Actually, we will see later that the equivalence between linear control categories and linear continuation categories is more precisely stated at the level of the underlying multicategory $\mathcal{M}$ defined by the functor ι. We temporarily postpone this issue to give a simple and intuitive definition of a linear control category.

Definition 1 (linear control category) *A* linear control category *is a symmetric monoidal category* $(\mathbf{M}, \otimes, 1)$ *together with a functor*

$$\mathcal{P} : \mathbf{I} \longrightarrow \mathbf{M}$$

that embeds a symmetric premonoidal category $(\mathbf{I}, \parr, \bot)$ *into the symmetric monoidal category* $\mathbf{M}$ *and defines an exponential ideal*

$$\multimap \quad : \quad \mathbf{M}^{op} \times \mathbf{I} \to \mathbf{I}.$$

Besides, these structures are related by a transformation (called the exponential strength)

$$s_{A,B,C} : (A \multimap B) \parr C \xrightarrow{\cong} A \multimap (B \parr C)$$

natural in A *in* $\mathbf{M}$ *and in* B, C *in* $\mathbf{I}$*, satisfying the following coherence conditions in* *the category* **I***:*

$$\begin{array}{ccccc}
((A \multimap B) \parr C) \parr D & \xrightarrow{s \parr D} & (A \multimap (B \parr C)) \parr D & \xrightarrow{s} & A \multimap ((B \parr C) \parr D) \\
\Big\downarrow{\scriptstyle \alpha} & & & & \Big\downarrow{\scriptstyle A \multimap \alpha} \\
(A \multimap B) \parr (C \parr D) & & \xrightarrow{\qquad\qquad s \qquad\qquad} & & A \multimap (B \parr (C \parr D))
\end{array}$$

$$\begin{array}{ccc}
& A \multimap B & \\
\swarrow{\scriptstyle l} & & \searrow{\scriptstyle A \multimap l} \\
(A \multimap B) \parr \bot & \xrightarrow{\qquad s \qquad} & A \multimap (B \parr \bot)
\end{array}$$

Finally, using the symmetry σ of the premonoidal category **I***, we define*

$$s'_{A,B,C} = \sigma_{C,A \multimap B}; s_{A,B,C}; A \multimap \sigma_{B,C} : C \parr (A \multimap B) \stackrel{\sim}{\to} A \multimap (C \parr B)$$

and require that the following coherence condition holds in the category **I**

$$\begin{array}{ccccc}
(A \multimap B) \parr (C \multimap D) & & \xrightarrow{\qquad\qquad s' \qquad\qquad} & & C \multimap ((A \multimap B) \parr D) \\
\Big\downarrow{\scriptstyle s} & & & & \Big\downarrow{\scriptstyle C \multimap s} \\
A \multimap (B \parr C \multimap D) & \xrightarrow{A \multimap s'} & A \multimap (C \multimap (B \parr D)) & \xrightarrow{smcc} & C \multimap (A \multimap (B \parr D))
\end{array}$$

As explained in Section 3, the functor $\mathcal{P}$ can be seen as the fingerprint of a multicategory that we will still denote by $\mathcal{P}$. The starting point of our analysis is to work exclusively on the multicategory $\mathcal{P}$ but using the structure provided by the underlying monoidal and premonoidal categories.

Example : categories of linear continuation. Before studying more deeply the structure of a linear control category, we check that the axioms above are satisfied by a category of linear continuation associated to a dialogue category **C**.

Given such a symmetric monoidal category **C**, we have already seen in the introduction that the opposite of the kleisli category $\mathbf{K}_1 = \mathbf{K}^{op}$ is symmetric premonoidal, and that the category $\mathbf{K}_2$ is symmetric monoidal – with monoidal structure inherited from **C**. The category $\mathbf{K}_2$, together with the functor $\iota : \mathbf{K}_1 \longrightarrow \mathbf{K}_2$ is then called the *category of linear continuation*.

To show that ι is a linear control category, it remains to define the linear implication $\multimap$ of the exponential ideal, and its strength. As discussed in the introduction, they are simply given by

$$A \multimap B \;=\; \neg A \otimes B$$
$$(A \multimap B) \parr C \;=\; (\neg A \otimes B) \otimes C \;\cong\; \neg A \otimes (B \otimes C) \;=\; A \multimap (B \parr C)$$

Once these data properly defined, it is routine to check that they define a linear control category.

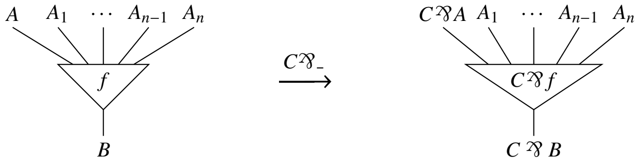
The key bijection of the center. Our goal is to show that the monoidal category $\mathcal{Z}(\mathbf{I})$ is equipped with a negation. To do that, we will establish a bijection between arrows in $\mathcal{Z}(\mathbf{I})$ and arrows in $\mathcal{P}$. A first step toward this bijection is to examine what remains of the premonoidality of $\mathbf{I}$ and of the exponential ideal $\multimap$ in the multicategory $\mathcal{P}$. The exponential ideal induces a family of bijections natural in $A_1, \ldots A_n$, and B,

$$\varphi_{A,A_1,\ldots,A_n,B} : \frac{A, A_1, \ldots, A_n \to B}{A_1, \ldots, A_n \to A \multimap B}$$

One can think of these bijections as part of a definition of a closure on the multicategory $\mathcal{P}$. Nevertheless, we do not believe that it is worth making this intuition more precise – even if we will use this terminology later in the text – as it will introduce a large bench of technicality.

Note that in the theory of control categories, the morphism s defining the exponential strength is induced by a distributivity law between the monoidal product and the premonoidal product. Here, on the other hand, there is no such linear distributivity law. Hence, we introduce the morphism s as an additional structure satisfying the suitable coherence properties. However, there is a more conceptual perspective on the presence of the exponential strength.

Indeed, the exponential strength enables to extends the premonoidal structure to the whole multicategory in the following sense. In any linear control category $\mathcal{P}$, we have a map



dinatural in central C, natural in $A_1, \ldots, A_n$ and natural in central A, B. This map is obtained by first applying the closure n-times to get an arrow $A \to A_1 \multimap (\ldots \multimap$

$(A_n \multimap B))$ in **I**. Then we apply $C \parr \_$ in **I** and use the exponential strength to get an arrow $C \parr A \to A_1 \multimap (\ldots \multimap (A_n \multimap C \parr B))$. Finally, we transport the A_i's back through the closure. Remark that this construction is compatible with the closure in the following sense

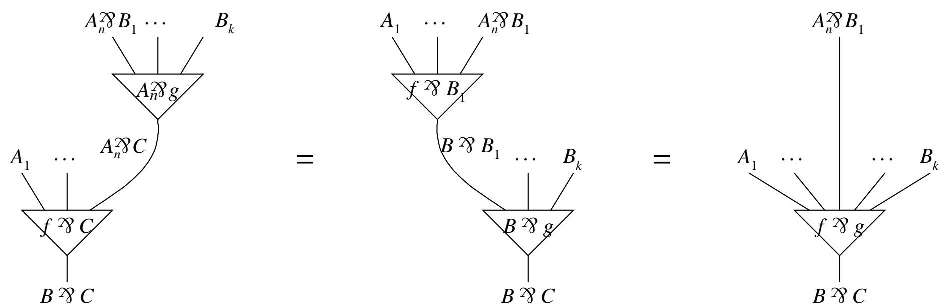
$$\varphi(f) \parr C; s = \varphi(C \parr f)$$

Again, one can think of this structure as part of the definition of what should be a premonoidal multicategory, but we let this prickly question to a later treatment.

Whatever the formulation of premonoidal multicategories, the map $C \parr \_$ (and its symmetrical counterpart $\_ \parr C$) enables to extend the notion of central morphism of **I** to arrows of $\mathcal{P}$ in the following way.

The only difference is that one now needs to specify in which argument the map in $\mathcal{P}$ is central as the multicategory is symmetric and the $\parr$ can be applied anywhere. This extended notion of centrality is preserved by the closure in the following sense.

Definition 3 *A morphism $f : A_1, \ldots, A_n \to B$ of a linear control category $\mathcal{P}$ is said to be* central *in A_n when for every morphism $g : B_1, \ldots, B_k \to C$, the two composites $(f \parr C) \circ (A_n \parr g)$ et $(B \parr g) \circ (f \parr B_1)$ are equal. In that case, we can use the notation $f \parr g$ that is not ambiguous. Graphically, this equality becomes*



It is not necessary to define left and right central morphisms as all considered tensor products are symmetric. It is important to notice that this extended notion of centrality is preserved by the closure coming from the exponential ideal, as indicated by the following lemma.

Lemma 1 *In a symmetric control multicategory, a morphism $f : A, A_1, \ldots, A_n \to B$ is central in A_n iff $\varphi(f) : A_1, \ldots, A_n \to A \multimap B$ is central in A_n.*

Proof: The commutativity of the diagram

$$\begin{array}{ccc} A, A_1, \ldots, A_n \parr C & \xrightarrow{f \parr C} & B \parr C \\ {\scriptstyle A,A_1,\ldots,A_n \parr g}\downarrow & & \downarrow{\scriptstyle B \parr g} \\ A, A_1, \ldots, A_n \parr D & \xrightarrow[f \parr D]{} & B \parr D \end{array}$$

is equivalent to the commutativity of

$$\begin{array}{ccc} A_1, \ldots, A_n \parr C & \xrightarrow{\varphi(f \parr C)} & A \multimap (B \parr C) \\ {\scriptstyle A_1,\ldots,A_n \parr g}\downarrow & & \downarrow{\scriptstyle A \multimap (B \parr g)} \\ A_1, \ldots, A_n \parr D & \xrightarrow[\varphi(f \parr D)]{} & A \multimap (B \parr D) \end{array}$$

by naturality of φ, which is equivalent to the commutativity of

$$\begin{array}{ccc} A_1, \ldots, A_n \parr C & \xrightarrow{\varphi(f) \parr C} & (A \multimap B) \parr C \\ {\scriptstyle A_1,\ldots,A_n \parr g}\downarrow & & \downarrow{\scriptstyle (A \multimap B) \parr g} \\ A_1, \ldots, A_n \parr D & \xrightarrow[\varphi(f) \parr D]{} & (A \multimap B) \parr D \end{array}$$

by compatibility of $\parr$ with the closure ■ In the same way, we can prove that when we compose a morphism f central in A with morphisms on inputs of f that differ from A, the resulting morphism is still central in A. More formally, the following lemma holds.

Lemma 2 *In a linear control category $\mathcal{P}$, when a morphism $f : A_1, \ldots, A_n, A \to B$ is central in A then any composed morphism $(f_1, \ldots, f_n, id_a) \circ f$ is central in A, where $f_i : C_{j_1}, \ldots, C_{j_i} \to A_i$.*

Before stating the key bijection between central maps and ordinary arrow of $\mathcal{P}$, we study the status of the excluded middle induced by the exponential strength. Indeed, all axioms of a linear control category are intuitionistically valid except for the presence of an inverse for the exponential strength. This inverse induces an nullary morphism *tnd* (for *tertium non datur*) representing the excluded middle in the multicategory $\mathcal{P}$

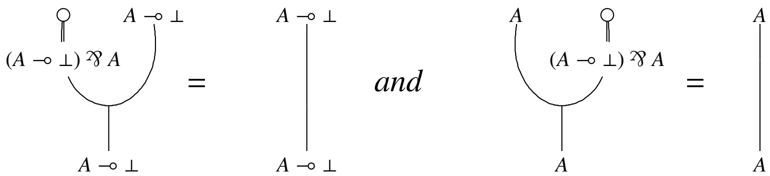
$$\xrightarrow{tnd} (A \multimap \bot) \parr A \quad \stackrel{\text{def}}{=} \quad \xrightarrow{\varphi_{A,A}(id_A)} A \multimap A \xrightarrow{s^{-1}_{A,\bot,A}} (A \multimap \bot) \parr A$$

defined by applying the closure on the identity morphism and then the exponential strength. Moreover, this arrow is dinatural in A, but it is not central in general.

Lemma 3 *The following equalities hold*

$$[(A \multimap \bot) \parr \varphi(id_{A\multimap\bot})] \circ (tnd, id_{A\multimap\bot}) = id_{A\multimap\bot}$$
$$[\varphi(id_{A\multimap\bot}) \parr A] \circ (id_A, tnd) = id_A$$

Graphically, we get



At this stage, we are ready to exhibit a one-to-one relationship between some particular central morphisms of $\mathbf{I}$ and the nullary morphisms of $\mathcal{P}$. Following Selinger's terminology, we call it the *key bijection of the center*.

Proposition 2 (key bijection of the center) *For every A and B in* $\mathbf{I}$*, there exists a bijection*

$$\mathcal{Z}(\mathbf{I})(A \multimap \bot; B) \cong \mathcal{P}(; B \parr A)$$

natural in A and natural in central B.

Proof: We define below the two components

$$\theta_{A,B} \quad : \quad \mathcal{P}(; B \parr A) \quad \longrightarrow \quad \mathcal{Z}(\mathbf{I})(A \multimap \bot; B)$$
$$\rho_{A,B} \quad : \quad \mathcal{Z}(\mathbf{I})(A \multimap \bot; B) \quad \longrightarrow \quad \mathcal{P}(; B \parr A)$$

of the bijection.

$\theta_{A,B}$ **:** Let f be in $\mathcal{P}(; B \parr A)$. By lemma 1 and properties of $\parr$, the morphism h_B defined as

$$h_B \quad \stackrel{\text{def}}{=} \quad B \parr \varphi(id_{A\multimap\bot}) \quad \in \quad \mathcal{P}(B \parr A, A \multimap \bot; B)$$

is central in its second argument $A \multimap \bot$. By lemma 2, the composed morphism

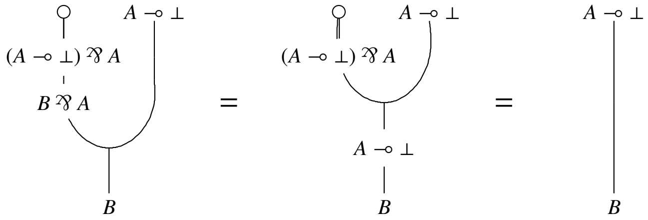
$$\theta_{A,B}(f) \quad \stackrel{\text{def}}{=} \quad h_B \circ (f, A \multimap \bot)$$

is still central.

$\rho_{A,B}$ **:** Let g be in $\mathcal{Z}(\mathbf{I})(A \multimap \bot; B)$. We define $\rho_{A,B}$ by

$$\rho_{A,B}(g) \quad \stackrel{\text{def}}{=} \quad (g \parr A) \circ tnd$$

To see that $\theta \circ \rho(f) = f$ holds for all f, consider the following equality chain.



The equality on the left holds by naturality of $B \parr \_$ in central B. The equality on the right holds by lemma 3 (i).

To see that $\rho \circ \theta(g) = g$ holds, it suffices to use lemma 3 (ii) together with the equality $B \parr id_A = id_{B \parr A}$. ∎

Structure theorem. Suppose that one starts with a dialogue category **C** and its associated category of linear continuation

$$\iota \quad : \quad \mathbf{K}_1 \longrightarrow \mathbf{K}_2,$$

and that one considers it as a linear control category as explained above. The functor ι is the fingerprint of a multicategory denoted $\mathcal{M}$. It appears that it is not possible to reconstruct the category **C** from the multicategory $\mathcal{M}$, but another dialogue category **D** of "central maps" living inside $\mathcal{M}$. From this dialogue category **D** follows another category of linear continuation

$$\iota' \quad : \quad \mathbf{K}'_1 \longrightarrow \mathbf{K}'_2$$

whose associated multicategory coincides with $\mathcal{M}$. The point is that, although the functors ι and ι' are different in general, they induce the same multicategory. So, the structure theorem is stated with respect to the underlying multicategory $\mathcal{M}$, not with respect to the category of linear continuation $\iota : \mathbf{K}_1 \longrightarrow \mathbf{K}_2$ itself. This justifies to work with multicategories, since the canonical object is *not* the category of linear continuation, but its underlying multicategory. The structure theorem may be also understood as "normalizing" the linear continuation $\iota : \mathbf{K}_1 \longrightarrow \mathbf{K}_2$ into the linear continuation $\iota' : \mathbf{K}'_1 \longrightarrow \mathbf{K}'_2$ canonically generated by the multicategory $\mathcal{M}$.

In order to state our structure theorem, we need a notion of strong functors and equivalence of linear control categories. Those notions will be defined at the level of the underlying multicategories. For example, a functor between two linear control categories will be described by a map on the underlying multicategories preserving the structure. The following definitions are the expected one, the only subtlety lies in the fact that all structural isomorphisms are required to be central.

A *strong functor of linear control categories* from $\mathcal{P}$ to $\mathcal{P}'$ is a (symmetric) map of multicategories $F : \mathcal{P} \to \mathcal{P}'$ that respects the structure up to central isomorphisms, together with central natural isomorphisms

$$FA \multimap FB \cong F(A \multimap B)$$

$$FA \parr FB \cong F(A \parr B)$$

$$\bot \cong F(\bot)$$

for any A and B in $\mathcal{P}$, preserving the structural morphisms in all the evident ways. For example, the functor must preserved the exponential strength in the sense that the following diagram commutes.

$$\begin{array}{ccccc} F((A \multimap B) \parr C) & \xrightarrow{\cong} & F(A \multimap B) \parr FC & \xrightarrow{\cong} & (FA \multimap FB) \parr FC \\ \downarrow{\scriptstyle Fs_{A,B,C}} & & & & \downarrow{\scriptstyle s_{FA,FB,FC}} \\ F(A \multimap (B \parr C)) & \xrightarrow{\cong} & FA \multimap F(B \parr C) & \xrightarrow{\cong} & FA \multimap (FB \parr FC) \end{array}$$

A *natural transformation* θ between two morphisms F and G of linear control categories is a natural transformation between the underlying morphisms of multicategories, such that θ_a is central for every A et such that the structural morphisms are preserved in all the evident ways. For example, the following diagram

$$\begin{array}{ccc} F(A \parr B) & \xrightarrow{\cong} & FA \parr FB \\ \downarrow{\scriptstyle \theta_{A \parr B}} & & \downarrow{\scriptstyle \theta_A \parr \theta_B} \\ G(A \parr B) & \xrightarrow{\cong} & GA \parr GB \end{array}$$

commutes for every A and B of $\mathcal{P}$.

Those two notions defines a 2-category of linear control categories from which we extract a notion of equivalence between two linear control categories. Namely, an *equivalence of linear control categories* $\mathcal{P}$ and $\mathcal{P}'$ is given by a pair of strong functors of linear control categories, $F : \mathcal{P} \to \mathcal{P}'$ and $G : \mathcal{P}' \to \mathcal{P}$, together with two natural isomorphisms $G \circ F \cong id_{\mathcal{P}}$ and $F \circ G \cong id_{\mathcal{P}'}$.

We are now ready to show that every linear control category is equivalent, as a linear control category, to the multicategory associated to a category of linear continuation. We already now that every category of linear continuation induces a linear control category. For the converse direction, we will show that the center of a linear control category defined a dialogue category whose category of linear continuation is equivalent to the linear control category we have started with.

Let **D** be the opposite category of $\mathcal{Z}(\mathbf{I})$. Like $\mathcal{Z}(\mathbf{I})$, it is a symmetric monoidal category (with tensor product noted $\otimes$). More interestingly, the key bijection of the centre indicates that **D** is a dialogue category.

Proposition 3 *The functor* $\neg : \mathbf{D} \to \mathbf{D}^{op}$ *defined on objects and arrows of* **D** *by*

$$\neg A \overset{\text{def}}{=} A \multimap \bot \quad \textit{and} \quad \neg f \overset{\text{def}}{=} f \multimap \bot$$

defines a tensorial negation on the symmetric monoidal category **D**.

Proof: First of all, we prove that $\neg$ sends central morphisms to central morphisms. This is established by the commutativity of the diagram

$$\begin{array}{ccc}
(B \multimap C) \parr D & \xrightarrow{(f \multimap C) \parr D} & (A \multimap C) \parr D \\
{\scriptstyle (B \multimap C) \parr g} \downarrow & & \downarrow {\scriptstyle (A \multimap C) \parr g} \\
(B \multimap C) \parr E & \xrightarrow[(f \multimap C) \parr E]{} & (A \multimap C) \parr E
\end{array}$$

which follows from the commutativity of

$$\begin{array}{ccc}
B \multimap (C \parr D) & \xrightarrow{f \multimap (C \parr D)} & A \multimap (C \parr D) \\
{\scriptstyle B \multimap (C \parr g)} \downarrow & & \downarrow {\scriptstyle A \multimap (C \parr g)} \\
B \multimap (C \parr E) & \xrightarrow[f \multimap (C \parr E)]{} & A \multimap (C \parr E)
\end{array}$$

by naturality of the strength and functoriality of $\multimap$. Then, there remains to construct the bijection characterizing a negation. We will use the correspondence between exponential object ad tensorial negation, and just show that $\bot$ is exponentiable. The following sequence of bijections

$$\begin{aligned}
\mathbf{D}(A \otimes B, \bot) &\cong \mathcal{Z}(\mathbf{I})(\bot, A \parr B) \\
&\cong \mathcal{P}(; A \parr B) \\
&\cong \mathcal{Z}(\mathbf{I})(B \multimap \bot; A) \\
&\cong \mathbf{D}(A, \neg B)
\end{aligned}$$

natural in A and B shows that $\bot$ is exponentiable. ■

Now that we know that **D** is a dialogue category, it remains to show that the associated category of linear continuation.

Theorem 4 (Structure theorem) *Any linear control category $\mathcal{P}$ is equivalent to the multicategory associated to a category of linear continuation coming a dialogue category.*

Proof: Let $\mathcal{N}$ be the multicategory defined by the category of linear continuation associated to the dialogue category

$$\neg : \mathbf{D} \to \mathbf{D}^{op}.$$

The map of multicategories F from $\mathcal{P}$ to $\mathcal{N}$ is the identity on objects. The only difficulty is to construct a bijection between maps of $\mathcal{P}$ and maps of $\mathcal{N}$ by

$$\begin{aligned}
\mathcal{P}(A_1, \ldots, A_n; B) &\cong \mathcal{P}(; A_n \multimap (\ldots \multimap (A_1 \multimap B))) \\
&\cong \mathcal{P}(; (A_n \multimap \bot) \parr \ldots \parr (A_1 \multimap \bot) \parr B) \\
&\cong \mathcal{Z}(\mathbf{I})(\neg B; \neg A_n \parr \ldots \parr \neg A_1) \\
&\cong \mathbf{D}(\neg A_1 \otimes \ldots \otimes \neg A_n, \neg B) \\
&\cong \mathcal{N}(A_1, \ldots, A_n; B)
\end{aligned}$$

natural in $A_1, \ldots, A_n$. So F defines an equivalence of multicategories. It is then routine to check that it preserves the structure of a linear control category. ■

5 Discussion.

Connections to $\lambda\mu$-calculus.

Connections to ludics. Every tensorial negation in a symmetric monoidal category **C** induces a lax monoidal structure, defined as a family of n-ary connectives on the objects of **C**:

$$[\, A_1 \parr \cdots \parr A_n \,] \quad := \quad \neg(\neg A_1 \otimes \cdots \otimes \neg A_n)$$

alternatively defined as

$$[\, A_1 \parr \cdots \parr A_n \,] \quad := \quad R(LA_1 \otimes \cdots \otimes LA_n).$$

As mentioned in the introduction, the multicategory $\mathcal{M}$ has the same objects as the category C, and its n-ary morphisms are defined as

$$\mathcal{M}(A_1, \ldots, A_n; B) \quad \stackrel{\text{def}}{=} \quad \mathbf{C}(B, [\, A_1 \parr \cdots \parr A_n \,]).$$

We indicate how this multicategory reflects the logical engine of ludics. The sequent calculus of ludics is based on two different kinds of sequents:

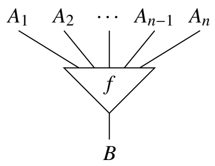
- the sequents with a fork

  $$B \vdash A_1, \ldots, A_n$$

  are interpreted as

  $$B \longrightarrow [\, A_1 \parr \cdots \parr A_n \,]$$

  in the category $\mathbf{C}$, and thus as a n-ary morphism



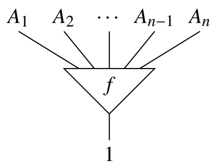
  in the multicategory $\mathcal{M}$,
- the sequents without a fork

  $$\vdash A_1, \ldots, A_n$$

  interpreted as

  $$1 \longrightarrow [\, A_1 \parr \cdots \parr A_n \,]$$

  in the category $\mathbf{C}$, and thus as a n-ary morphism



  in the multicategory $\mathcal{M}$.

6 Conclusion.

This work is part of a wider research program, aiming at an elegant synthesis between linear logic, and the theory of continuations. We believe that this synthesis is possible, and that it will be extremely fruitful. Our ambition is to extend every aspect of the beautiful and rich theory of linear logic, to the relaxed setting of linear continuations. Typically, the fundamental idea that every continuation monad is the unary component of a lax disjunction $[A_1 \parr \cdots \parr A_n]$ is the product of a crossbreeding of the two theories. One lesson of this article is that a successful hybridization will certainly require to enrich the mathematical toolbox of each field – integrating the recent developments of the theory of 2-dimensional algebra. In this way, we hope to see slowly emerge a monoidal theory of computational effects, combining linear logic, game semantics, and higher dimensional algebra.

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