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On the stabilization of permanently excited linear systems*

Yacine Chitour[†] Mario Sigalotti[‡]

Abstract

We consider control systems of the type $\dot{x} = Ax + \alpha(t)bu$, where $u \in \mathbf{R}$, (A, b) is a controllable pair and α is an unknown time-varying signal with values in $[0, 1]$ satisfying a permanent excitation condition i.e., $\int_t^{t+T} \alpha \geq \mu$ for every $t \geq 0$, with $0 < \mu \leq T$ independent on t . We prove that such a system is stabilizable with a linear feedback depending only on the pair (T, μ) if the real part of the eigenvalues of A is non-positive. The stabilizability does not hold in general for matrices A whose eigenvalues have positive real part. Moreover, the question of whether the system can be stabilized or not with an arbitrarily large rate of convergence gives rise to a bifurcation phenomenon in dependence of the parameter μ/T .

1 Introduction

The present paper is a continuation of [9], where the study of general n -dimensional linear systems subject to scalar persistently excited PE-signals was initiated. Such a linear time-dependent system is given by

$$\dot{x} = Ax + \alpha(t)Bu, \quad (1)$$

where the function α is a *scalar* PE-signal i.e., α takes values in $[0, 1]$ and there exist two positive constants μ, T such that, for every $t \geq 0$,

$$\int_t^{t+T} \alpha(s)ds \geq \mu. \quad (2)$$

Given two positive real numbers $\mu \leq T$, we use $\mathcal{G}(T, \mu)$ to denote the class of all PE signals verifying (2).

In (1), the PE-signal α can be seen as an input perturbation modeling the fact that the instants where the control u acts on the system are not exactly known. If α only takes the values 0 and 1, then (1) actually switches between the uncontrolled system $\dot{x} = Ax$ and the controlled one $\dot{x} = Ax + Bu$. In that context, the PE condition (2) is designed to guarantee some action on the system. (For a more detailed discussion on the interpretation of permanently excited systems and on the related literature, see [9].)

Our main concern will be the global asymptotic stabilization of system (1) with a constant linear feedback $u = -Kx$ where the gain matrix K is required to be the same *for all* signals in the

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considered class $\mathcal{G}(T, \mu)$ i.e., K depends only on A, b, T, μ and not on a specific element of $\mathcal{G}(T, \mu)$. We refer to such a gain matrix K as a (T, μ) -stabilizer. It is clear that (A, B) must be stabilizable for hoping that a (T, μ) -stabilizer exists and we will suppose that throughout the paper. Moreover, by standard linear algebra considerations, the stabilizability analysis can be reduced to the case where (A, B) is controllable.

The questions studied in this paper find their origin in a problem stemming from identification and adaptive control (cf. [3]). Such a problem deals with the linear system $\dot{x} = -P(t)u$, where the matrix $P(\cdot)$ is symmetric non-negative definite and plays the role of α . If $P \equiv I$, then $u^* = x$ trivially stabilizes the system exponentially. But what if $P(t)$ is only semi-positive definite for all t ? Under which conditions on P does $u^* = x$ still stabilize the system? The answer for this particular case, can be found in the seminal paper [12] which asserts that, if $x \in \mathbf{R}^n$ and $P \geq 0$ is bounded and has bounded derivative, it is *necessary and sufficient*, for the global exponential stability of $\dot{x} = -P(t)x$, that P is also *persistently exciting* (PE) i.e., that there exist $\mu, T > 0$ such that

$$\int_t^{t+T} \xi^\top P(\tau) \xi \geq \mu, \quad (3)$$

for all unitary vectors $\xi \in \mathbf{R}^n$ and all $t \geq 0$. Therefore, as regards the stabilization of (1), the notion of *persistent excitation* seems to be a reasonable additional assumption on the signals α .

Let us recall the main results of [9]. We first addressed the issue of controllability of (1), uniformly with respect to $\alpha \in \mathcal{G}(T, \mu)$. We proved that, if the pair (A, B) is controllable, then (1) is (completely) controllable in time t if and only if $t > T - \mu$. We next focused on the existence of (T, μ) -stabilizers. We first treated the case where A is neutrally stable and we showed that in this case the gain $K = B^T$ is a (T, μ) -stabilizer for system (1) (see also [3]). Note that in the neutrally stable case K does not even depend on the specific class $\mathcal{G}(T, \mu)$. We next turned to the case where A is not stable. In such a situation, even in the one-dimensional case, a stabilizer K cannot be chosen independently of (T, μ) . In [9], we considered the first nontrivial unstable case, namely the double integrator $\dot{x} = J_2 x + \alpha b_0 u$, with J_2 , the 2×2 Jordan block and $b_0 = (0, 1)^T$. We showed that, for every pair (T, μ) , there exists a (T, μ) stabilizer for $\dot{x} = J_2 x + \alpha b_0 u$, $\alpha \in \mathcal{G}(T, \mu)$.

In this paper, we provide two sets of results. The first one concerns the stabilization of (1) when the control is scalar-valued. Given two arbitrary constants $0 < \mu \leq T$, we prove the existence of a (T, μ) -stabilizer when the eigenvalues of A have non-positive real part. The second set of results concerns the possibility of obtaining an arbitrary rate of convergence once the stabilization is achieved. We essentially focus on the two-dimensional case and we point out an interesting phenomenon: there exists $\rho_* \in (0, 1)$ so that, for every controllable two-dimensional pair (A, b) and $0 < \mu \leq T$ such that $\mu/T \leq \rho_*$, the maximal rate of convergence of $\dot{x} = Ax + \alpha bu$, $\alpha \in \mathcal{G}(T, \mu)$, is finite. Here maximality is evaluated with respect to all possible (T, μ) -stabilizers. As a consequence, we prove the existence of matrices A (e.g. $J_2 + \lambda \text{Id}_2$ with λ large enough) such that for every $0 < \mu \leq T$ with $\mu/T \leq \rho_*$, the PE system $\dot{x} = Ax + \alpha bu$, $\alpha \in \mathcal{G}(T, \mu)$, does not admit (T, μ) -stabilizers. The latter result is rather surprising when one compares it with the following two facts: let $\rho \in (0, 1]$; (i) a sequence (α_n) , with $\alpha_n \in \mathcal{G}(T_n, \rho T_n)$ and $\lim_{n \rightarrow +\infty} T_n = 0$, has at least one weak- \star limit point α_* taking values in $[\rho, 1]$ (see Lemma 2.5) and (ii) the two-dimensional switching system $\dot{x} = J_2 x + \alpha_* b_0 u$ can be stabilized, uniformly with respect to $\alpha_* \in L^\infty(\mathbf{R}_{\geq 0}, [\rho, 1])$, with an arbitrary rate of convergence. The weak- \star convergence considered in (i) is the natural one in this context since it renders the input-output mapping continuous.

Let us briefly comment on the technics used in this paper. First of all, it is clear that the notion of *common Lyapunov function*, rather powerful in the realm of switched systems, cannot be of (direct) help here since, at the differential level, one can evolve with an unstable dynamics $\dot{x} = Ax$, when $\alpha = 0$ takes the value zero. More refined tools as multiple and non-monotone Lyapunov functions (see, e.g., [1, 2, 7, 10, 13, 15]) do not seem well-adapted to permanently excited systems, at least for

what concerns the proof of their stability. It seems to us that one must rather perform a trajectory analysis, on a time interval of length at least equal to T , in order to achieve any information which is uniform with respect to $\alpha \in \mathcal{G}(T, \mu)$. This viewpoint is more similar to the geometric approach to switched system behind the results in [4, 5, 6]. As a second consideration, notice that point (i) described above, which is systematically used in the paper, presents formal similarities with the technic of *averaging* but is rather different from it, since T is not the period of a PE signal. Moreover, for a given permanently excited system, T is fixed and thus it does not tend to zero.

The paper is organized as follows. In Section 2, we introduce the notations of the paper, the basic definitions and some useful technical lemmas. We gather in Section 3 the stabilization results for matrices with spectrum having non-positive real part. Finally, the analysis of the maximal rates of convergence and divergence is the object of Section 4. Since many of our results give rise to further challenging questions, we propose in the course of the paper several conjectures and open problems.

2 Notations and definitions

Let \mathbf{N} denote the set of positive integers. Given n and m belonging to \mathbf{N} , we use $0_{n \times m}$ to denote the $n \times m$ matrix made of zeroes, $M_n(\mathbf{R})$ the set of real-valued $n \times n$ matrices, and Id_n the $n \times n$ identity matrix. We also write 0_n for $0_{n \times 1}$, $\sigma(A)$ for the spectrum of a matrix $A \in M_n(\mathbf{R})$, and $\Re(\lambda)$ (respectively, $\Im(\lambda)$) for the real (respectively, imaginary) part of a complex number λ .

Definition 2.1 (PE signal and (T, μ) -signal) *Let $\mu \leq T$ be positive constants. A (T, μ) -signal is a measurable function $\alpha : \mathbf{R}_{\geq 0} \rightarrow [0, 1]$ satisfying*

$$\int_t^{t+T} \alpha(s) ds \geq \mu, \quad \forall t \in \mathbf{R}_{\geq 0}. \quad (4)$$

We use $\mathcal{G}(T, \mu)$ to denote the set of all (T, μ) -signals. A PE signal is a measurable function $\alpha : \mathbf{R}_{\geq 0} \rightarrow [0, 1]$ such that there exist T, μ positive real numbers for which α is a (T, μ) -signal.

Definition 2.2 (PE system) *Given two positive constants $\mu \leq T$ and a controllable pair $(A, b) \in M_n(\mathbf{R}) \times \mathbf{R}^n$, we define the PE system associated to T, μ, A , and b as the family of linear control systems given by*

$$\dot{x} = Ax + \alpha b, \quad \alpha \in \mathcal{G}(T, \mu). \quad (5)$$

Given a PE system (5), we address the following problem. We want to stabilize (5) *uniformly* with respect to every (T, μ) -signal α , i.e., we want to find a vector $K \in \mathbf{R}^n$ which makes the origin of

$$\dot{x} = (A - \alpha(t)bK^T)x \quad (6)$$

globally asymptotically stable, with K depending only on A, b, T and μ .

More precisely, referring to $x(\cdot; t_0, x_0, K, \alpha)$ as the solution of (6) with initial condition $x(t_0; t_0, x_0, K, \alpha) = x_0$, we introduce the following definition.

Definition 2.3 ((T, μ) -stabilizer) *Let $\mu \leq T$ be positive constants. The gain K is said to be a (T, μ) -stabilizer for (5) if (6) is globally asymptotically stable, uniformly with every (T, μ) -signal α . Since (6) is linear in x , this is equivalent to say that (6) is exponentially stable, uniformly with respect to $\alpha \in \mathcal{G}(T, \mu)$, i.e., there exist $C, \gamma > 0$ such that every solution $x(\cdot; t_0, x_0, K, \alpha)$ of (6) satisfies*

$$\|x(t; t_0, x_0, K, \alpha)\| \leq C e^{-(t-t_0)\gamma} \|x_0\|, \quad \forall t \geq t_0.$$

The next two lemmas collect some properties of PE signals.

Lemma 2.4 1. If $\alpha(\cdot)$ is a (T, μ) -signal, then, for every $t_0 \geq 0$, $\alpha(t_0 + \cdot)$ is a (T, μ) -signal as well.

2. If $0 < \rho' < \rho$ and $T > 0$ then $\mathcal{G}(T, \rho T) \subset \mathcal{G}(T, \rho' T)$.

3. For $\eta \in (0, \mu)$, $\mathcal{G}(T, \mu) \subset \mathcal{G}(T + \eta, \mu) \cap \mathcal{G}(T - \eta, \mu - \eta)$.

4. If $T \geq \tau > 0$ and $\rho > 0$, then $\mathcal{G}(\tau, \rho\tau) \subset \mathcal{G}(T, (\rho/2)T)$.

5. For every $0 < \rho' < \rho$ there exists $M > 0$ such that for every $T \geq M\tau > 0$ one has $\mathcal{G}(\tau, \rho\tau) \subset \mathcal{G}(T, \rho' T)$.

Proof. We only provide an argument for points 4 and 5. Fix $t \geq 0$, $T \geq \tau$, $\rho > 0$ and $\alpha \in \mathcal{G}(\tau, \rho\tau)$. Let l be the integer part of T/τ . Since $l \geq \max(1, T/\tau - 1)$, then $\int_t^{t+T} \alpha \geq l\rho\tau \geq \max(\tau, T - \tau)\rho \geq T\rho/2$. For $\rho' \in (0, \rho)$ and T/τ large enough, then $\max(\tau, T - \tau) \geq (\rho'/\rho)T$ and so $\int_t^{t+T} \alpha \geq \rho' T$. ■

Let

$$b_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Recall that an element f of $L^\infty(\mathbf{R}_{\geq 0}, [0, 1])$ is the weak- \star limit of a sequence $(f_k)_{k \in \mathbf{N}}$ of elements of $L^\infty(\mathbf{R}_{\geq 0}, [0, 1])$ if, for every $g \in L^1(\mathbf{R}_{\geq 0}, \mathbf{R})$,

$$\int_0^\infty f(s)g(s)ds = \lim_{k \rightarrow \infty} \int_0^\infty f_k(s)g(s)ds. \quad (7)$$

It is well known that $L^\infty(\mathbf{R}_{\geq 0}, [0, 1])$ endowed with the weak- \star topology is compact (see, for instance, [8]). Hence, $\mathcal{G}(T, \mu)$ is weak- \star compact for every $0 < \mu \leq T$. Unless specified, limit points of sequences of PE signals are to be understood as limits of subsequences with respect to the weak- \star topology of $L^\infty(\mathbf{R}_{\geq 0}, [0, 1])$.

Lemma 2.5 Let $(\alpha^{(n)})_{n \in \mathbf{N}}$ and $(\nu_n)_{n \in \mathbf{N}}$ be, respectively, a sequence of (T, μ) -signals and an increasing sequence of positive real numbers such that $\lim_{n \rightarrow \infty} \nu_n = \infty$.

1. Define α_n as the $(T/\nu_n, \mu/\nu_n)$ -signal given by $\alpha_n(t) = \alpha^{(n)}(\nu_n t)$ for $t \geq 0$. If α_\star is a limit point of the sequence $(\alpha_n)_{n \in \mathbf{N}}$, then α_\star takes values in $[\mu/T, 1]$ almost everywhere.

2. Let $j_0 \in \{0, 1\}$ and $h \in \mathbf{N}$. Let ω_j , $j = j_0, \dots, h$ be real numbers with $\omega_j = 0$ if and only if $j = 0$ and $\{\pm\omega_j\} \neq \{\pm\omega_l\}$ for $j \neq l$. For every $t \geq 0$, let

$$v(t) = \begin{pmatrix} 1 \\ e^{\omega_1 A_0 t} b_0 \\ \vdots \\ e^{\omega_h A_0 t} b_0 \end{pmatrix} \text{ if } j_0 = 0 \quad \text{or} \quad v(t) = \begin{pmatrix} e^{\omega_1 A_0 t} b_0 \\ \vdots \\ e^{\omega_h A_0 t} b_0 \end{pmatrix} \text{ if } j_0 = 1.$$

For every signal α and every $t \geq 0$, define

$$\alpha^C(t) = \alpha(t)v(t)v(t)^T. \quad (8)$$

Then α^C is a time-dependent non-negative symmetric $(2h + 1 - j_0) \times (2h + 1 - j_0)$ matrix with $\alpha^C \leq \text{Id}_{2h+1-j_0}$ and there exists $\xi > 0$ only depending on T, μ and $\omega_{j_0}, \dots, \omega_h$ such that, for every $t \geq 0$,

$$\int_t^{t+T} \alpha^C(\tau) d\tau \geq \xi \text{Id}_{2h+1-j_0}. \quad (9)$$

Therefore, α^C may be considered as a matrix-valued PE signal. Moreover, define $\alpha_n^C(t) = (\alpha^{(n)})^C(\nu_n t)$ for every $t \geq 0$ and every $n \in \mathbf{N}$. If α_\star^C is a limit point of the sequence $(\alpha_n^C)_{n \in \mathbf{N}}$ for the weak- \star topology of $L^\infty(\mathbf{R}_{\geq 0}, M_{2h+1-j_0}(\mathbf{R}))$, then $\alpha_\star^C \geq (\xi/T)\text{Id}_{2h+1-j_0}$ almost everywhere.

Proof. Let us first prove point 1. Let α_\star be the weak- \star limit of some sequence $(\alpha_{n_k})_{k \geq 1}$. For every interval $J \subset \mathbf{R}_{\geq 0}$ of finite length $|J| > 0$, apply (7) by taking as g the characteristic function of J . Since each α_{n_k} is a $(T/\nu_{n_k}, \mu/\nu_{n_k})$ -signal, it follows that

$$\frac{1}{|J|} \int_J \alpha_\star(s) ds = \lim_{k \rightarrow \infty} \frac{1}{|J|} \int_J \alpha_{n_k}(s) ds \geq \liminf_{k \rightarrow \infty} \frac{\mu}{|J|\nu_{n_k}} \mathcal{I} \left(\frac{|J|\nu_{n_k}}{T} \right) = \frac{\mu}{T},$$

where $\mathcal{I}(\cdot)$ denotes the integer part. Since α_\star is measurable and bounded (actually, L^1_{loc} would be enough), almost every $t > 0$ is a Lebesgue point for α_\star , *i.e.*, the limit

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} \alpha_\star(s) ds$$

exists and is equal to $\alpha_\star(t)$ (see, for instance, [14]). We conclude that, as claimed, $\alpha_\star(t) \geq \mu/T$ almost everywhere.

For the first part of point 2 fix $t \geq 0$ and notice that the map

$$\alpha \mapsto \int_t^{t+T} \alpha^C(\tau) d\tau$$

is continuous with respect to the weak- \star topology and takes values in the set of non-negative symmetric matrices. If we prove that all such matrices are positive definite then, by weak- \star compactness of $\mathcal{G}(T, \mu)$, we deduce the existence of $\xi > 0$ independent of α such that (9) holds true. (The independence of ξ with respect to t follows from the shift-invariance of $\mathcal{G}(T, \mu)$ pointed out in Lemma 2.4, point 1.) Assume by contradiction that there exist $\alpha \in \mathcal{G}(T, \mu)$ and $x_0 \in \mathbf{R}^{2h+1-j_0} \setminus \{0_{2h+1-j_0}\}$ such that $\int_t^{t+T} x_0^T \alpha^C(\tau) x_0 d\tau = 0$. Then, for almost every $s \in [t, t+T]$, we would have $\alpha(s) x_0^T v(s) = 0$. Since $\alpha(s) \neq 0$ for s in a set of positive measure, we deduce that the real-analytic function $x_0^T v(\cdot)$ takes the value zero on a set of positive measure, *i.e.* it is identically equal to zero. Let $A_0^C = \text{diag}(1, \omega_1 A_0, \dots, \omega_h A_0)$ if $j_0 = 0$ or $A_0^C = \text{diag}(\omega_1 A_0, \dots, \omega_h A_0)$ if $j_0 = 1$. Then $x_0^T (A_0^C)^j v(0) = 0$ for every non-negative integer j . The contradiction is reached, since $(A_0^C, v(0))$ is a controllable pair and $x_0 \neq 0_{2h+1-j_0}$.

As for the second part of point 2, we follow the same argument used to prove point 1, noticing that, for every $t \geq 0$,

$$\int_t^{t+\frac{T}{\nu_n}} \alpha_n^C(\tau) d\tau \geq \frac{\xi}{\nu_n} \text{Id}_{2h+1-j_0}. \quad \blacksquare$$

3 Spectra with non-positive real part

We consider here below the problem of whether a controllable pair (A, b) gives rise to a PE system that can be (T, μ) -stabilized for every choice of $0 < \mu \leq T$. We will see in the Section 4 that this cannot in general be done if the real part of the eigenvalues of A is too large. The scope of this section is to study the case in which each eigenvalue of A has non-positive real part.

Theorem 3.1 *Let $(A, b) \in M_n(\mathbf{R}) \times \mathbf{R}^n$ be a controllable pair and assume that the eigenvalues of A have non-positive real part. Then, for every $0 < \mu \leq T$ there exists a (T, μ) -stabilizer for (1).*

Since the proof of Theorem 3.1 is technical, we prefer to clarify its main ideas by first showing the theorem in the special case of the n -integrator.

Let $J_n \in M_n(\mathbf{R})$ be defined as

$$J_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \ddots & \vdots \\ 0 & \cdots & & & 0 & 1 \\ 0 & \cdots & \cdots & & & 0 \end{pmatrix}.$$

Theorem 3.2 *Let $A = J_n$ and $b = (0, \dots, 0, 1)^T \in \mathbf{R}^n$. Then, for every $0 < \mu \leq T$ there exists a (T, μ) -stabilizer for (1).*

Proof. In the special case of the n -integrator system (6) becomes

$$\begin{cases} \dot{x}_j &= x_{j+1}, \quad \text{for } j = 1, \dots, n-1, \\ \dot{x}_n &= -\alpha(t)(k_1x_1 + \cdots + k_nx_n), \end{cases} \quad (10)$$

where $K = (k_1, \dots, k_n)^T$.

For every $\nu > 0$, define $D_{n,\nu} = \text{diag}(\nu^{n-1}, \dots, \nu, 1)$. As done in [9] in the case $n = 2$, one easily checks that, in accordance with

$$\nu D_{n,\nu}^{-1} J_n D_{n,\nu} = J_n, \quad D_{n,\nu} b = b, \quad (11)$$

the time-space transformation

$$x_\nu(t) = D_{n,\nu}^{-1} x(\nu t), \quad \forall t \geq \frac{t_0}{\nu}, \quad (12)$$

of the trajectory $x(\cdot) = x(\cdot; t_0, x_0, K, \alpha)$ satisfies

$$\frac{d}{dt} x_\nu(t) = J_n x_\nu(t) - \alpha(\nu t) \nu b K^T D_{n,\nu} x_\nu(t),$$

that is,

$$x_\nu(\cdot) = x(\cdot; t_0/\nu, D_{n,\nu}^{-1} x_0, \nu D_{n,\nu} K, \alpha(\nu \cdot)).$$

As a consequence, (10) admits a (T, μ) -stabilizer if and only if it admits a $(T/\nu, \mu/\nu)$ -stabilizer. (More precisely, K is a (T, μ) -stabilizer if and only if $\nu D_{n,\nu} K$ is a $(T/\nu, \mu/\nu)$ -stabilizer.)

We prove Theorem 3.2 by contradiction, fixing a suitable gain K and assuming that it is not a $(T/\nu, \mu/\nu)$ -stabilizer, whatever the value of ν . Therefore, for every $\nu > 0$ there exists $\alpha_\nu \in \mathcal{G}(T/\nu, \mu/\nu)$ that destabilizes, in a suitable sense, (10). Considering all the possible limit points of $(\alpha_\nu)_{\nu>0}$ as $\nu \rightarrow \infty$ (see point 1 of Lemma 2.5), we introduce the *limit switched system*

$$\begin{cases} \dot{x}_j &= x_{j+1}, \quad \text{for } j = 1, \dots, n-1, \\ \dot{x}_n &= -\alpha_\star(t)(k_1x_1 + \cdots + k_nx_n), \quad \alpha_\star(t) \in [\mu/T, 1]. \end{cases} \quad (13)$$

The gain K will be selected by asking it to stabilize (13). In order to check the existence of such a stabilizer, assume that $k_1 \neq 0$ and define $X_1 = k_1x_1 + \cdots + k_nx_n$, $X_2 = k_1x_2 + \cdots + k_{n-1}x_n$, \dots , $X_n = k_1x_n$. Although such a change of variables depends on K , the uniformly exponential global asymptotic stabilizability of (13) is clearly equivalent to that of

$$\dot{X}_j = X_{j+1} - \alpha_\star \bar{k}_j X_1, \quad j = 1, \dots, n, \quad \alpha_\star(t) \in [\mu/T, 1], \quad (14)$$

where $\bar{k}_j = k_{n+1-j}$ and, by convention, $X_{n+1} = 0_n$. The stabilizability of (14) results from the existence of $\bar{K} \in \mathbf{R}^n$, a scalar $\gamma > 0$ and a symmetric positive definite $n \times n$ matrix S such that

$$(J_n - \bar{\alpha}\bar{K}(1, 0, \dots, 0))^T S + S(J_n - \bar{\alpha}\bar{K}(1, 0, \dots, 0)) \leq -\gamma \text{Id}_n, \quad (15)$$

for every (constant) $\bar{\alpha} \in [\mu/T, 1]$. The existence of such \bar{K} , γ and S is proved in Gauthier and Kupka [11, Lemma 4.0] (where it is attributed to W.P. Dayawansa).

This proves the stabilizability of (13). Fix a stabilizing feedback K and a positive definite matrix S' such that $V(x) = x^T S' x$ defines a quadratic Lyapunov function that decreases uniformly on every trajectory of (13). In particular, there exists a time τ such that every trajectory of (13) starting in $B_2^V = \{x \in \mathbf{R}^n \mid V(x) \leq 2\}$ at time 0 lies in $B_1^V = \{x \in \mathbf{R}^n \mid V(x) \leq 1\}$ for every time larger than τ . Assume now by contradiction that, for every $l \in \mathbf{N}$, K does not uniformly contract every trajectory corresponding to a $(T/l, \mu/l)$ -signal starting from B_2^V into B_1^V in time 2τ . In particular, there exist a sequence of initial conditions $(x_{0,l})_{l \in \mathbf{N}} \subset B_2^V$, a sequence of times $(t_l)_{l \in \mathbf{N}} \subset [2\tau, 4\tau]$ and a sequence of signals $(\alpha_l)_{l \in \mathbf{N}}$ such that $\alpha_l \in \mathcal{G}(T/l, \mu/l)$ and

$$x(t_l; 0, x_{0,l}, K, \alpha_l) \notin B_1^V \quad \text{for every } l \in \mathbf{N}. \quad (16)$$

By compactness of $B_2^V \times [2\tau, 4\tau]$ and by weak- \star compactness of $L^\infty(\mathbf{R}_{\geq 0}, [0, 1])$, we can assume that, up to extracting a subsequence, $x_{0,l} \rightarrow x_{0,\star} \in B_2^V$, $t_l \rightarrow t_\star \in [2\tau, 4\tau]$ and α_l converges weakly- \star to $\alpha_\star \in L^\infty(\mathbf{R}_{\geq 0}, [0, 1])$ as l goes to infinity. Then $x(t_l; 0, x_{0,l}, K, \alpha_l)$ converges, as l goes to infinity, to $x(t_\star; 0, x_\star, K, \alpha_\star)$ (see [9, Appendix] for details). Since $\alpha_\star \geq \mu/T$ almost everywhere (point 1 of Lemma 2.5), then α_\star can be taken as an admissible signal in (13).

By homogeneity of the linear system (13) and because $t_\star \geq 2\tau$, we have that

$$V(x(t_\star; 0, x_\star, K, \alpha_\star)) \leq 1/2.$$

Therefore, for l large enough $x(t_l; 0, x_{0,l}, K, \alpha_l) \in B_1^V$ contradicting (16). \blacksquare

Let us now follow the same scheme of proof as above to tackle Theorem 3.1. The main technical difficulties come from the fact that A may have several Jordan blocks of different sizes.

Proof of Theorem 3.1. Fix a controllable pair $(A, b) \in M_n(\mathbf{R}) \times \mathbf{R}^n$. Thanks to simple linear algebra considerations (see [9, Lemma 9] for details), we can assume, with no loss of generality, that all eigenvalues of A lie on the imaginary axis. Denote the distinct eigenvalues of A by $\pm i\omega_j$, $j \in \{j_0, j_0 + 1, \dots, h\}$, where $j_0 = 1$ if $0 \notin \sigma(A)$ and $j_0 = 0$ with $\omega_0 = 0$ otherwise. For every $j \in \{0, \dots, h\}$, let r_j be the multiplicity of $i\omega_j$, with the convention that $r_0 = 0$ if $0 \notin \sigma(A)$.

Assume that A is decomposed in Jordan blocks. Since (A, b) is controllable, then A has a unique (complex) Jordan block associated with each $\{i\omega_j, -i\omega_j\}$, $j_0 \leq j \leq h$. (Otherwise, the rank of the matrix $(A - i\omega_j \text{Id}_n \mid b)$ would be strictly smaller than n , contradicting the Hautus test for controllability.) Therefore, for every $j = 1, \dots, h$, the Jordan block associated to $i\omega_j$ is $\omega_j A^{(j)} + J_{r_j}^C$, where $A^{(j)} = \text{diag}(A_0, \dots, A_0) \in M_{2r_j}(\mathbf{R})$ and $J_{r_j}^C \in M_{2r_j}(\mathbf{R})$ is defined as

$$J_{r_j}^C = \begin{pmatrix} 0_{2 \times 2} & \text{Id}_2 & 0_{2 \times 2} & \cdots & \cdots & 0_{2 \times 2} \\ 0_{2 \times 2} & 0_{2 \times 2} & \text{Id}_2 & 0_{2 \times 2} & \cdots & 0_{2 \times 2} \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0_{2 \times 2} & & \cdots & 0_{2 \times 2} & 0_{2 \times 2} & \text{Id}_2 \\ 0_{2 \times 2} & & \cdots & \cdots & 0_{2 \times 2} & 0_{2 \times 2} \end{pmatrix},$$

that is, in terms of the Kronecker product, $J_{r_j}^C = J_{r_j} \otimes \text{Id}_2$.

All controllable linear control systems associated with a pair (A, b) that have in common the eigenvalues of A , counted according to their multiplicity, are state-equivalent, since they can be transformed by a linear transformation of coordinates into the same system under companion form. We exploit such an equivalence to deduce that, up to a linear transformation of coordinates, (1) can be written as

$$\begin{cases} \dot{x}_0 &= J_{r_0}x_0 + \alpha b^0 u, \\ \dot{x}_j &= (\omega_j A^{(j)} + J_{r_j}^C)x_j + \alpha b^j u, \quad \text{for } j = 1, \dots, h, \end{cases} \quad (17)$$

where b^0 and b^j are respectively the vectors of \mathbf{R}^{r_0} and \mathbf{R}^{2r_j} with all coordinates equal to zero except the last one that is equal to one. Here $x_0 \in \mathbf{R}^{r_0}$ and $x_j \in \mathbf{R}^{2r_j}$ for $j = 1, \dots, h$

Write the feedback law as $u = -K^T x = -K_0^T x_0 - \sum_{l=1}^h K_l^T x_l$ with $K_0 \in \mathbf{R}^{r_0}$ and $K_j \in \mathbf{R}^{2r_j}$ for every $1 \leq j \leq h$.

For every $\nu > 0$ consider the following change of time-space variables: applying to x_0 the same transformation as in (12), let

$$y_0(t) = D_{r_0, \nu}^{-1} x_0(\nu t).$$

Define, moreover,

$$y_j(t) = (D_{r_j, \nu}^C)^{-1} e^{-\nu t A^{(j)}} x_j(\nu t), \quad \text{for } 1 \leq j \leq h,$$

where

$$D_{r_j, \nu}^C = D_{r_j, \nu} \otimes \text{Id}_2 \in M_{2r_j}(\mathbf{R}).$$

In accordance with

$$\nu (D_{r_j, \nu}^C)^{-1} J_{r_j}^C D_{r_j, \nu}^C = J_{r_j}^C, \quad D_{r_j, \nu}^C b^j = b^j,$$

we end up with the following linear time-varying system

$$\begin{cases} \dot{y}_0 &= J_{r_0} y_0 - \alpha_\nu(t) b^0 (K_{0, \nu}^T y_0 + \sum_{l=1}^h K_{l, \nu}^T e^{\nu t \omega_l A^{(l)}} y_l), \\ \dot{y}_j &= J_{r_j}^C y_j - \alpha_\nu(t) b^{j, \nu}(t) (K_{0, \nu}^T y_0 + \sum_{l=1}^h K_{l, \nu}^T e^{\nu t \omega_l A^{(l)}} y_l), \quad \text{for } j = 1, \dots, h, \end{cases} \quad (18)$$

where $K_{0, \nu} = \nu D_{r_0, \nu} K_0$, $K_{j, \nu} = \nu D_{r_j, \nu}^C K_j$ and $b^{j, \nu}(t) = e^{-\nu t \omega_j A^{(j)}} b^j$ for $j = 1, \dots, h$.

Given $\nu > 0$, (6) admits a (T, μ) -stabilizer if and only if (18) admits a $(T/\nu, \mu/\nu)$ -stabilizer. As done in the proof of Theorem 3.2, we prove the (T, μ) -stabilizability of (6) reasoning by contradiction. This is done by fixing a suitable gain K and assuming that $K_\nu = K$ is not a $(T/\nu, \mu/\nu)$ -stabilizer of (18), whatever the value of ν . We then consider the possible weak- \star limit points as ν tends to infinity of the time-dependent coefficients in (18) (with an arbitrary choice of $\alpha_\nu \in \mathcal{G}(T/\nu, \mu/\nu)$ for every $\nu > 0$). These limit points can be characterized using point 2 of Lemma 2.5. We associate with (18) a set of limit points as ν tends to infinity, which are linear time-dependent systems of the type $\dot{y} = A_\star(K, t)y$. The time-dependent matrix $A_\star(K, \cdot)$ happens to take values in a compact set \mathcal{A}_K of matrices only depending on T, μ, A and K . The final contradiction is reached by proving the existence of K only depending on T, μ and A so that all the matrices in \mathcal{A}_K are Hurwitz and admit a common quadratic Lyapunov function.

In order to identify the possible weak- \star limit points of system (18), let us compute the weak- \star limit points of $\alpha_\nu(t) b^{j, \nu}(t) K_l^T e^{t \nu \omega_l A^{(l)}}$, $j_0 \leq j, l \leq h$, as ν tends to infinity.

For each $l = 1, \dots, h$, we chose the feedback K_l^T of the form $(0, k_1^l, \dots, 0, k_{r_l}^l)$, that is,

$$K_l^T = \mathcal{K}_l \otimes (0, 1), \quad \mathcal{K}_l = (k_1^l, \dots, k_{r_l}^l).$$

For uniformity of notations, we also write $\mathcal{K}_0 = K_0^T$.

Recall that $(\alpha_\nu)_{\nu > 0}$ is any family of signals satisfying $\alpha_\nu \in \mathcal{G}(T/\nu, \mu/\nu)$ for every $\nu > 0$.

Consider a sequence $(\nu_n)_{n \in \mathbf{N}}$ going to infinity as $n \rightarrow \infty$ such that the matrix-valued curve $\alpha_{\nu_n}^C(\cdot)$ (defined as in (8)) has a weak- \star limit as $n \rightarrow \infty$ in $L^\infty(\mathbf{R}_{\geq 0}, M_{2h+1-j_0}(\mathbf{R}))$. Denote the weak- \star limit by C_\star . Using point 2 of Lemma 2.5, one deduces that $C_\star(t)$ is symmetric and

$$C_\star(t) \geq \xi \text{Id}_{2h+1-j_0},$$

for almost every $t \geq 0$, for some positive scalar ξ only depending on T, μ and $\sigma(A)$.

Define the 2×2 time-dependent matrices C_{jl} , $1 \leq j, l \leq h$, the 1×2 time-dependent matrices C_{0j} , $1 \leq j \leq h$, and the scalar time-dependent signal C_{00} by the relation

$$C_\star = (C_{jl})_{j_0 \leq j, l \leq h}.$$

Then all coefficients of system (18) are weakly- \star convergent along the sequence $(\nu_n)_{n \in \mathbf{N}}$ (with $K_{\nu_n} = K$ for every n). The limit system obtained taking all such limits is equal to

$$\begin{cases} \dot{y}_0 &= J_{r_0} y_0 - b^0 (C_{00} \mathcal{K}_0 y_0 + \sum_{l=1}^h C_{0l} (\mathcal{K}_l \otimes \text{Id}_2) y_l), \\ \dot{y}_j &= J_{r_j}^C y_j - (b^j \otimes \text{Id}_2) \left(C_{0j}^T \mathcal{K}_0 y_0 + \sum_{l=1}^h C_{jl} (\mathcal{K}_l \otimes \text{Id}_2) y_l \right), \quad \text{for } j = 1, \dots, h. \end{cases} \quad (19)$$

In the sequel, we only treat the case where 0 is not an eigenvalue of A . The general case presents no extra mathematical difficulties and can be treated similarly. Then system (19) takes the form

$$\dot{y}_j = J_{r_j}^C y_j - (b^j \otimes \text{Id}_2) \sum_{l=1}^h C_{jl} (\mathcal{K}_l \otimes \text{Id}_2) y_l, \quad \text{for } j = 1, \dots, h. \quad (20)$$

We also assume that the multiplicities r_1, \dots, r_h of the eigenvalues of A form a non-increasing sequence.

Let us impose a further restriction on the structure of the feedback K . Assume that there exist $\bar{k}_1, \dots, \bar{k}_{r_1} \in \mathbf{R}$, each of them different from zero, such that

$$k_\xi^l = \bar{k}_{r_l+1-\xi}, \quad \text{for } 1 \leq l \leq h \text{ and } 1 \leq \xi \leq r_l.$$

We find it useful to provide an equivalent representation of system (20) in a higher dimensional vector space, introducing some redundant variables. In order to do so, for $1 \leq l \leq r_1$, associate to $y = (y_1, \dots, y_h)$ the $2h$ -vector

$$Y_l = \begin{pmatrix} (\mathcal{K}_1 \otimes \text{Id}_2) (J_{r_1}^C)^{l-1} y_1 \\ \vdots \\ (\mathcal{K}_h \otimes \text{Id}_2) (J_{r_h}^C)^{l-1} y_h \end{pmatrix}.$$

Notice that the last $2h - 2m_l$ coordinates of Y_l are equal to zero, where m_l denotes the number of Jordan blocks of size at least l , that is,

$$m_l = \#\{j \mid 1 \leq j \leq h, r_j \geq l\}.$$

For $1 \leq l \leq r_1$, let p_l be the orthogonal projection of \mathbf{R}^{2h} onto $\mathbf{R}^{2m_l} \times \{0_{2h-2m_l}\}$, i.e.,

$$p_l = \text{diag}(\text{Id}_{2m_l}, 0_{(2h-2m_l) \times (2h-2m_l)}).$$

By construction we have $p_1 = \text{Id}_{2r_1}$ and $p_l Y_j = Y_j$ for $1 \leq l \leq j \leq r_1$.

Notice that the map $y = (y_1, \dots, y_h) \mapsto (Y_1, \dots, Y_{r_1})$ is a bijection between \mathbf{R}^n and the subspace of \mathbf{R}^{2hr_1} defined as follows

$$E_{m_1, \dots, m_{r_1}}^h = \{(Y_1, \dots, Y_{r_1}) \mid Y_l \in \mathbf{R}^{2h} \text{ and } p_l Y_l = Y_l \text{ for } l = 1, \dots, r_1\}.$$

Indeed, the matrix corresponding to the transformation is upper triangular, with the \bar{k}_l 's as elements of the diagonal, if one considers the following choice of coordinates on $E_{m_1, \dots, m_{r_1}}^h$: take the first two coordinates of the first copy of \mathbf{R}^{2h} , then the first two of its second copy and so on until the last copy; then start again with the third and fourth coordinates of the first copy of \mathbf{R}^{2h} and repeat the procedure until its r_2^{th} copy; and so on.

If y evolves along the dynamics of system (20), then $Y = (Y_1, \dots, Y_{r_1})$ is a trajectory in $E_{m_1, \dots, m_{r_1}}^h$ satisfying the system of equations

$$\dot{Y}_l = Y_{l+1} - \bar{k}_l p_l C_\star Y_1, \quad \text{for } l = 1, \dots, r_1, \quad (21)$$

where, by convention, $Y_{r_1+1} = 0_{2h}$.

We are left to prove that there exist $\bar{k}_1, \dots, \bar{k}_{r_1} \neq 0$ such that system (21), restricted to $E_{m_1, \dots, m_{r_1}}^h$, is exponentially stable uniformly with respect to C_\star . Notice that such a system is well defined, since $E_{m_1, \dots, m_{r_1}}^h$ is invariant for the dynamics of (21).

The following proposition, calqued on [11, Lemma 4.0], states the stabilizability of all systems of the type (21) defined on $E_{m_1, \dots, m_{r_1}}^h$ for an arbitrary choice of $h, r_1 \in \mathbf{N}$ and $h \geq m_1 \geq m_2 \geq \dots \geq m_{r_1} \geq 0$, uniformly with respect to any measurably varying time-dependent symmetric matrix C_\star satisfying $\xi \text{Id}_{2h} \leq C_\star(t) \leq \text{Id}_{2h}$ almost everywhere.

Proposition 3.3 *For every $h, r_1 \in \mathbf{N}$, for every non-increasing sequence of non-negative numbers m_1, \dots, m_{r_1} such that $m_1 \leq h$ and for every $\xi > 0$, there exist $\lambda, \bar{k}_1, \dots, \bar{k}_{r_1} > 0$ and a symmetric positive definite $2hr_1 \times 2hr_1$ matrix S such that, for every $C_\star \in L^\infty(\mathbf{R}_{\geq 0}, M_{2h}(\mathbf{R}))$, if $C_\star(t)$ is symmetric and satisfies $\xi \text{Id}_{2h} \leq C_\star(t) \leq \text{Id}_{2h}$ almost everywhere, then any solution $Y : \mathbf{R}_{\geq 0} \rightarrow E_{m_1, \dots, m_{r_1}}^h$ of (21) satisfies for almost every $t \geq 0$ the inequality*

$$\frac{d}{dt} (Y(t)^T S Y(t)) \leq -\lambda \|Y(t)\|^2.$$

Proof. The proof is similar to that of [11, Lemma 4.0] and goes by induction on r_1 .

We start the argument for $r_1 = 1$, with $h \in \mathbf{N}$, $0 \leq m_1 \leq h$ and $\xi > 0$ arbitrary. In that case the system reduces to

$$\dot{Y}_1 = -\bar{k}_1 p_1 C_\star Y_1,$$

with $Y_1 \in E_{m_1}^h = \mathbf{R}^{2m_1} \times \{0_{2h-2m_1}\}$. The conclusion follows by taking $\bar{k}_1 = 1$ and $S = \text{Id}_{2h}$.

Let r_1 be a positive integer. Assume that the proposition holds true for every positive integer $j \leq r_1$ and for every $h \in \mathbf{N}$, $0 \leq m_1 \leq \dots \leq m_{r_1} \leq h$ and $\xi > 0$. Consider system (21) where l runs between 1 and $r_1 + 1$.

Set $Y = (Y_2^T, \dots, Y_{r_1+1}^T)^T$. Note that if $(Y_1^T, \dots, Y_{r_1+1}^T)^T \in E_{m_1, \dots, m_{r_1+1}}^h$, then $Y \in E_{m_2, \dots, m_{r_1+1}}^h$. The dynamics of (Y_1, Y) are given by

$$\begin{cases} \dot{Y}_1 = -\bar{k}_1 C_\star Y_1 + \Pi_1 Y, \\ \dot{Y} = -\bar{K} C_\star Y_1 + \mathcal{J} Y, \end{cases}$$

where

$$\begin{aligned} \Pi_1 &= (\text{Id}_{2h}, 0_{2h \times 2h(r_1-1)}), \\ \bar{K} &= \begin{pmatrix} \bar{k}_2 p_2 \\ \vdots \\ \bar{k}_{r_1+1} p_{r_1+1} \end{pmatrix}, \\ \mathcal{J} &= J_{r_1} \otimes \text{Id}_{2h}. \end{aligned}$$

Define the linear change of variables (Z_1, Z) given by

$$Z_1 = Y_1, \quad Z = Y + \Omega Y_1,$$

where

$$\Omega = \begin{pmatrix} \eta_2 p_2 \\ \vdots \\ \eta_{r_1+1} p_{r_1+1} \end{pmatrix}$$

and the η_l 's are scalar constants to be chosen later. Note that Z belongs to $E_{m_2, \dots, m_{r_1+1}}^h$ if Y does. The dynamics of (Z_1, Z) are given by

$$\begin{cases} \dot{Z}_1 = (-\bar{k}_1 C_\star + \Pi_1 \Omega) Z_1 + \Pi_1 Z, \\ \dot{Z} = -((\bar{K} + \bar{k}_1 \Omega) C_\star + (\mathcal{J} + \Omega \Pi_1) \Omega) Z_1 + (\mathcal{J} + \Omega \Pi_1) Z. \end{cases} \quad (22)$$

Let us apply the induction hypothesis to the system

$$\dot{Z} = -(\mathcal{J} + \Omega \Pi_1) Z, \quad (23)$$

which is well defined on $E_{m_2, \dots, m_{r_1+1}}^h$ and has the same structure as system (21). (Here $\mathbf{C}_\star \equiv \text{Id}_{2h}$ and therefore one can take as ξ any positive constant smaller than one.) We deduce the existence of $\lambda > 0$, $\eta_l < 0$, $2 \leq l \leq r_1 + 1$ and a symmetric positive definite matrix S such that $\dot{V}(t) \leq -\lambda \|Z(t)\|^2$ where $V(t) = Z(t)^T S Z(t)$ and $Z(t)$ is any trajectory of (23) in $E_{m_2, \dots, m_{r_1+1}}^h$. Therefore,

$$- [(\mathcal{J} + \Omega \Pi_1)^T S + S(\mathcal{J} + \Omega \Pi_1)] \Big|_{E_{m_2, \dots, m_{r_1+1}}^h} \leq -\lambda \text{Id}_{E_{m_2, \dots, m_{r_1+1}}^h}.$$

Since Ω is fixed, for every $\bar{k}_1 > 0$ there exists a unique $\bar{K}(\bar{k}_1)$ such that $\bar{K}(\bar{k}_1) + \bar{k}_1 \Omega = 0_{2r_1 h \times 2h}$. Assume that $\bar{K} = \bar{K}(\bar{k}_1)$ and notice that the corresponding $\bar{k}_2, \dots, \bar{k}_{r_1+1}$ are positive. Our freedom of choice is now restricted to \bar{k}_1 alone.

Choose $S' = (1/2) \text{diag}(\text{Id}_{2h}, S)$ and define the corresponding Lyapunov function $W(Z_1, Z) = \|Z_1\|^2/2 + Z^T S Z$. If (Z_1, Z) is a trajectory of (22), then

$$\begin{aligned} \frac{d}{dt} W(Z_1, Z) &= -Z_1^T ((\bar{k}_1 C_\star - \Pi_1 \Omega) Z_1 - \Pi_1 Z) - Z^T S ((\mathcal{J} + \Omega \Pi_1) \Omega Z_1 + (\mathcal{J} + \Omega \Pi_1) Z) \\ &\leq Z_1^T (-\bar{k}_1 C_\star + \Pi_1 \Omega) Z_1 - \lambda \|Z\|^2 + (\|\Pi_1\| + \|S(\mathcal{J} + \Omega \Pi_1) \Omega\|) \|Z_1\| \|Z\| \\ &\leq (-\bar{k}_1 \xi + \delta_1) \|Z_1\|^2 - \lambda \|Z\|^2 + \delta_2 \|Z_1\| \|Z\|, \end{aligned}$$

where the constants $\delta_1, \delta_2 > 0$ do not depend on \bar{k}_1 . Since

$$\|Z_1\| \|Z\| \leq \varepsilon^2 \|Z_1\|^2 + \frac{\|Z\|^2}{\varepsilon^2}$$

for every $\varepsilon > 0$, then

$$\frac{d}{dt} W(Z_1, Z) \leq \left(-\bar{k}_1 \xi + \delta_1 + \frac{\delta_2}{\varepsilon^2} \right) \|Z_1\|^2 + (-\lambda + \varepsilon^2 \delta_2) \|Z\|^2.$$

Choosing ε^2 small enough in order to have $-\lambda + \varepsilon^2 \delta_2 \leq -\lambda/2$ and \bar{k}_1 large enough, we have

$$\frac{d}{dt} W(Z_1, Z) \leq -\frac{\lambda}{2} (\|Z_1\|^2 + \|Z\|^2).$$

The proof is concluded, since (Z_1, Z) and (Y_1, Y) are equivalent systems of coordinates on the space $E_{m_1, \dots, m_{r_1+1}}^h$. \blacksquare

The last step of the contradiction argument for the proof of Theorem 3.1 is now completed. \blacksquare

4 Maximal rates of exponential convergence and divergence

Let $(A, b) \in M_n(\mathbf{R}) \times \mathbf{R}^n$ be a controllable pair, $T \geq \mu > 0$ and $K \in \mathbf{R}^n$. For $\alpha \in \mathcal{G}(T, \mu)$ let $\lambda^+(\alpha, K)$ and $\lambda^-(\alpha, K)$ be, respectively, the maximal and minimal Lyapunov exponents associated with $\dot{x} = (A - \alpha b K^T)x$, i.e.,

$$\lambda^+(\alpha, K) = \sup_{\|x_0\|=1} \limsup_{t \rightarrow +\infty} \frac{\log(\|x(t; 0, x_0, K, \alpha)\|)}{t}, \quad \lambda^-(\alpha, K) = \inf_{\|x_0\|=1} \liminf_{t \rightarrow +\infty} \frac{\log(\|x(t; 0, x_0, K, \alpha)\|)}{t}.$$

The *rate of convergence* (respectively, the *rate of divergence*) associated with the family of systems $\dot{x} = (A - \alpha b K^T)x$, $\alpha \in \mathcal{G}(T, \mu)$, is defined as

$$\text{rc}(A, b, T, \mu, K) = - \sup_{\alpha \in \mathcal{G}(T, \mu)} \lambda^+(\alpha, K) \quad (\text{respectively, } \text{rd}(A, b, T, \mu, K) = \inf_{\alpha \in \mathcal{G}(T, \mu)} \lambda^-(\alpha, K)). \quad (24)$$

Notice that

$$\text{rc}(A, b, T, \mu, K) \leq \min_{\bar{\alpha} \in [\mu/T, 1]} \min\{-\Re(\sigma(A - \bar{\alpha} b K^T))\}, \quad (25)$$

and

$$\text{rd}(A, b, T, \mu, K) \leq \min_{\bar{\alpha} \in [\mu/T, 1]} \min\{\Re(\sigma(A - \bar{\alpha} b K^T))\}.$$

Moreover, since a linear change of coordinates $x' = Px$ does not affect Lyapunov exponents, then

$$\text{rc}(A, b, T, \mu, K) = \text{rc}(PAP^{-1}, Pb, T, \mu, (P^{-1})^T K), \quad (26)$$

and

$$\text{rd}(A, b, T, \mu, K) = \text{rd}(PAP^{-1}, Pb, T, \mu, (P^{-1})^T K). \quad (27)$$

Define the maximal rate of convergence associated with the PE system $\dot{x} = Ax + \alpha bu$, $\alpha \in \mathcal{G}(T, \mu)$, as

$$\text{RC}(A, T, \mu) = \sup_{K \in \mathbf{R}^n} \text{rc}(A, b, T, \mu, K), \quad (28)$$

and similarly, the maximal rate of divergence as

$$\text{RD}(A, T, \mu) = \sup_{K \in \mathbf{R}^n} \text{rd}(A, b, T, \mu, K). \quad (29)$$

Notice that the definition makes sense, since neither $\text{RC}(A, T, \mu)$ nor $\text{RD}(A, T, \mu)$ depend on b , as it follows from (26) and (27).

Remark 4.1 Let us collect some properties of RC and RD that follow directly from their definition. First of all, one has

$$\text{RC}(A + \lambda \text{Id}_n, T, \mu) = \text{RC}(A, T, \mu) - \lambda, \quad \text{RD}(A + \lambda \text{Id}_n, T, \mu) = \text{RD}(A, T, \mu) + \lambda. \quad (30)$$

Then, a simple time-rescaling shows that

$$\text{RC}(A, T, \rho T) = \text{RC}(A/T, 1, \rho), \quad \text{RD}(A, T, \rho T) = \text{RD}(A/T, 1, \rho). \quad (31)$$

Notice moreover that, thanks to (11), both $\text{RC}(J_n, T, \rho T)$ and $\text{RD}(J_n, T, \rho T)$ only depend on ρ and thus are equal to $\text{RC}(J_n, 1, \rho)$ and $\text{RD}(J_n, 1, \rho)$, respectively. Finally, because of point 2 in Lemma 2.4, RC and RD are monotone with respect to their third argument.

Remark 4.2 Given a controllable pair (A, b) and a class $\mathcal{G}(T, \mu)$ of PE signals, whether or not RC and RD are both infinite can be understood as whether or not a pole-shifting type property holds true for the PE control system $\dot{x} = Ax + \alpha bu$, $\alpha \in \mathcal{G}(T, \mu)$.

The study of the pole-shifting type property for two-dimensional PE systems actually reduces to that of their maximal rates of convergence as a consequence of the following property.

Proposition 4.3 Consider the two-dimensional PE systems $\dot{x} = Ax + \alpha bu$, $\alpha \in \mathcal{G}(T, \mu)$, with (A, b) controllable. Then $\text{RC}(A, T, \mu) = +\infty$ if and only if $\text{RD}(A, T, \mu) = +\infty$.

Proof. According to (26), (27) and (30), it is enough to prove the result for (A, b) in companion form and with $\text{Tr}(A) = 0$. Let then

$$A = \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix} \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (32)$$

with $a = -\det A$.

Assume that $\text{RC}(A, T, \mu) = +\infty$. By definition, for every $C > 0$ there exists $K \in \mathbf{R}^2$ such that $\text{rc}(A, b, T, \mu, k) > C$. Therefore, by definition of rc ,

$$\limsup_{t \rightarrow +\infty} \frac{\log(\|x(t; 0, x_0, K, \alpha)\|)}{t} < -C, \quad \forall \alpha \in \mathcal{G}(T, \mu), \forall \|x_0\| = 1. \quad (33)$$

Moreover, due to (25), for C large enough we can assume that k_1, k_2 and k_1/k_2 are large positive numbers.

Let $K_- = (k_1, -k_2)$. We claim that if C is large enough then $\text{RD}(A, b, T, \mu, K_-) \geq C$. Assume by contradiction that there exists $\bar{\alpha} \in \mathcal{G}(T, \mu)$ such that $\lambda^-(\bar{\alpha}, K_-) < C$. Then there exists $\bar{x} \in \mathbf{R}^2$ of norm one and an increasing sequence $(t_n)_{n \in \mathbf{N}}$ of positive times going to infinity such that

$$\frac{\log(\|x(t_n; 0, \bar{x}, K_-, \bar{\alpha})\|)}{t_n} < C, \quad \forall n \in \mathbf{N}.$$

Notice that for every $t \in [0, t_n]$,

$$x(t; 0, \bar{x}, K_-, \bar{\alpha}(\cdot)) = \text{diag}(1, -1)x(t_n - t; 0, x_n, K, \bar{\alpha}(t_n - \cdot)),$$

where $x_n = \text{diag}(1, -1)x(t_n; 0, \bar{x}, K_-, \bar{\alpha})$.

Therefore, by homogeneity,

$$\frac{\log\left(\left\|x\left(t_n; 0, \frac{x_n}{\|x_n\|}, K, \bar{\alpha}(t_n - \cdot)\right)\right\|\right)}{t_n} = -\frac{\log(\|x_n\|)}{t_n} = -\frac{\log(\|x(t_n; 0, \bar{x}, K_-, \bar{\alpha})\|)}{t_n} > -C. \quad (34)$$

This would contradict (33) if, for some positive integer n , $x_n/\|x_n\| = \bar{x}$ and the signal obtained by repeating $\bar{\alpha}|_{[0, t_n]}$ by periodicity over $\mathbf{R}_{\geq 0}$ belonged to $\mathcal{G}(T, \mu)$. Indeed, in such a case,

$$\frac{\log(\|x(kt_n; 0, \bar{x}, K, \tilde{\alpha}(\cdot))\|)}{kt_n} > -C \quad (35)$$

for every $k \geq 1$, where $\tilde{\alpha} \in \mathcal{G}(T, \mu)$ denotes the signal obtained by repeating $\bar{\alpha}|_{[0, t_n]}(t_n - \cdot)$ by periodicity over $\mathbf{R}_{\geq 0}$.

In order to recover the periodic case, we are going to extend $\bar{\alpha}$ backwards in time over an interval $[-2\mu - \tau_n, 0)$ as follows. First set $A_1^- = A - bK_-^T$. We take $\bar{\alpha} = 1$ on the intervals $[-\mu, 0)$ and $[-2\mu - \tau_n, -\mu - \tau - n)$ and we extend $\bar{\alpha}$ on $[-\mu - \tau_n, -\mu)$ in such a way that the

trajectory corresponding to $\bar{\alpha}|_{[-\mu-\tau_n, -\mu]}$ and to the gain K_- connects the half-line $\mathbf{R}_{\geq 0}x_n^+$ to \bar{x}^- , where $x_n^+ = \exp(\mu A_1^-)\text{diag}(1, -1)x_n$ and $\bar{x}^- = \exp(-\mu A_1^-)\bar{x}$. We show below that this can be done fulfilling the PE condition and with τ_n upper bounded by a constant independent of n . Hence, the signal obtained extending $\bar{\alpha}|_{[-2\mu-\tau_n, t_n]}$ by periodicity belongs to $\mathcal{G}(T, \mu)$ and we have

$$\begin{aligned} x(t_n + 2\mu + \tau_n; 0, x_n, K, \bar{\alpha}(t_n + 2\mu + \tau_n - \cdot)) &\in \mathbf{R}_{\geq 0}x_n \\ \log \left(\left\| x \left(t_n + 2\mu + \tau_n; 0, \frac{x_n}{\|x_n\|}, K, \bar{\alpha}(t_n + 2\mu + \tau_n - \cdot) \right) \right\| \right) &= \log(\|\tilde{x}\|) - \log(\|x(t_n; 0, \bar{x}, K_-, \bar{\alpha})\|), \end{aligned}$$

where $\tilde{x} = x(\tau_n + 2\mu; 0, \text{diag}(1, -1)\bar{x}, K, \bar{\alpha}|_{[-2\mu-\tau_n, 0]}(-\cdot))$. Note that $\log(\|\tilde{x}\|)$ can be lower bounded independently of n , because of the uniform boundedness of τ_n . Therefore,

$$\frac{\log(\|x(t_n + 2\mu + \tau_n; 0, x_n, K, \bar{\alpha}(t_n + 2\mu + \tau_n - \cdot))\|)}{t_n + 2\mu + \tau_n} > \frac{\log(\|\tilde{x}\|)}{t_n + 2\mu + \tau_n} - \frac{Ct_n}{t_n + 2\mu + \tau_n}$$

is larger than $-C$ for n large enough and we can conclude as in (35).

We are left to prove that the control system on the unit circle whose admissible velocities are the projections of the linear vector fields $x \mapsto (A - \xi b K_-^T)x$, $\xi \in [0, 1]$, is completely controllable in finite time by controls $\xi = \xi(t)$ satisfying the PE condition. Notice that the equilibria of the projection of a linear vector field $x \mapsto A'x$ on the unit circle are given by the eigenvalues of A' . All other trajectories are heteroclinic connections between the equilibria, unless the eigenvalues of A' are non-real, in which case the phase portrait is given by a single periodic trajectory.

Denote by θ a point on the unit circle, identified with $\mathbf{R}/2\pi\mathbf{Z}$. Then, the above mentioned control system on the unit circle can be written

$$\dot{\theta} = a \cos^2(\theta) - \sin^2(\theta) + \xi \cos(\theta) (k_2 \sin(\theta) - k_1 \cos(\theta)), \quad \xi \in [0, 1]. \quad (36)$$

Note that the required controllability of (36) is proved if we exhibit a trajectory $\bar{\theta}$ of (36) corresponding to a PE control $\bar{\xi}$, starting at some $\theta_0 \in \mathbf{R}/2\pi\mathbf{Z}$, making a complete turn and going back in finite time to θ_0 .

We next build an input function $\bar{\xi}$ giving rise to such a trajectory $\bar{\theta}$. The PE condition will be verified by checking that the control $\bar{\xi} = 0$ is applied for a total time that is smaller than $T - \mu$. Define the angle $\theta_K \in (0, \pi/2)$ by

$$\tan(\theta_K) = 2 \frac{k_2}{k_1}.$$

Notice that the eigenvectors of A_1^- are proportional to the vectors $(2, k_2 \pm \sqrt{k_2^2 - 4(k_1 - a)})$. Therefore, assuming that k_1 is larger than a , the angle between any real eigenvector of A_1^- and the vertical axis is smaller than θ_K .

Take $\theta_0 = \pi/2$ and apply $\bar{\xi} = 0$ until $\bar{\theta}$ reaches $\pi/2 - \theta_K$. Since k_2/k_1 is small and θ_K is of the same order as k_2/k_1 , then we can assume that $a \cos^2(\theta) - \sin^2(\theta) < -1/2$ for $\theta \in [\pi/2 - \theta_K, \pi/2]$. Therefore, the time needed to go from $\pi/2$ to $\pi/2 - \theta_K$ can be assumed to be smaller than $(T - \mu)/2$. When the trajectory $\bar{\theta}$ reaches $\pi/2 - \theta_K$, switch to $\bar{\xi} = 1$ and apply it until $\bar{\theta}$ reaches (in finite time) $-\pi/2$. This is possible since either the eigenvectors of A_1^- are non-real or they are contained in the cone

$$\{(r \cos \theta, r \sin \theta) \mid r > 0, \theta \in (\pi/2 - \theta_K + m\pi, \pi/2 + m\pi), m \in \mathbf{Z}\}.$$

In both cases the dynamics of (36) with $\xi = 1$ describe a non-singular clockwise rotation on the arc of the unit circle corresponding to $[\pi/2, \pi/2 - \theta_K]$. The trajectory is completed, by homogeneity, taking $\bar{\xi} = 0$ until $\bar{\theta}$ reaches $-\pi/2 - \theta_K$ and finally $\bar{\xi} = 1$ until $\bar{\theta}$ reaches $-3\pi/2 = \pi/2 \pmod{2\pi}$. As required, the sum of the lengths of the intervals on which $\bar{\xi} = 0$ does not exceed $T - \mu$.

This concludes the proof that $\text{RC}(A, T, \mu) = +\infty$ implies $\text{RD}(A, T, \mu) = +\infty$. The converse can be proven by a perfectly analogous argument. \blacksquare

Open problem 1 Does Proposition 4.3 still hold true in dimension bigger than two? Notice that our argument essentially relies on the controllability of (36) in finite time.

4.1 Arbitrary rates of convergence and divergence for ρ large enough

This section aims at proving that for ρ large enough a permanently excited system can be either stabilized with an arbitrarily large rate of exponential convergence or destabilized with an arbitrarily large rate of exponential divergence. This will be done by adapting the classical high-gain technique.

Proposition 4.4 Let n be a positive integer. There exists $\rho^* \in (0, 1)$ (only depending on n) such that for every controllable pair $(A, b) \in M_n(\mathbf{R}) \times \mathbf{R}^n$, every $T > 0$ and every $\rho \in (\rho^*, 1]$ one has $\text{RC}(A, T, \rho T) = \text{RD}(A, T, \rho T) = +\infty$.

Proof. Let $T > 0$ and $(A, b) \in M_n(\mathbf{R}) \times \mathbf{R}^n$ be a controllable pair in companion form. According to (30), it is enough to establish the result with the extra hypothesis that $\text{Tr}(A) = 0$. We therefore assume in the sequel that $b = (0, \dots, 0, 1)^T$, $A = J_n + bK_A^T$ and $K_A^T b = 0$.

We first prove the stabilization result. Fix $K \in \mathbf{R}^n$ such that $J_n - bK^T$ is Hurwitz. Let P be the unique positive definite $n \times n$ matrix that solves the Lyapunov equation

$$(J_n - bK^T)^T P + P(J_n - bK^T) = -\text{Id}_n.$$

Define $V(x) = x^T P x$. Then, for every $\alpha \in L^\infty(\mathbf{R}, [0, 1])$ and every solution of $\dot{x} = (J_n - \alpha bK^T)x$, one has

$$\frac{d}{dt} V(x(t)) \leq -C_1 V(x(t)) + C_2(1 - \alpha(t))V(x(t)),$$

with C_1, C_2 two positive constants only depending on K . Choose $\rho \in (0, 1)$ and assume that α is a $(T, T\rho)$ -signal. Then, for every $t \geq 0$,

$$V(x(t+T)) \leq V(x(t)) \exp(-T(C_1 - C_2(1 - \rho))).$$

Therefore, if $\rho > 1 - (C_1/2C_2)$ then $\text{RC}(J_n, T, T\rho) \geq C_1/2 > 0$. For every $\gamma > 0$, set $K_\gamma = \gamma D_\gamma K$ (where, as in the previous section, $D_\gamma = \text{diag}(\gamma^{n-1}, \dots, \gamma, 1)$). Recall that J_n and D_γ satisfy (11). Take a solution of $\dot{x} = (A - \alpha bK_\gamma^T)x$ with $\alpha \in \mathcal{G}(T, \rho T)$. Set $z(\cdot) = D_\gamma x(\cdot)$ and notice that for every $\gamma > 1$

$$\frac{d}{dt} V(z(t)) \leq \gamma(-C_1 + C_2(1 - \alpha(t)) + C_A/\gamma^2)V(z(t)),$$

where C_A only depends on K_A and P . Then clearly $\text{RC}(A, T, T\rho) \geq \gamma C_1/3$ for $\rho > 1 - (C_1/2C_2)$ and γ large enough. Thus, $\text{RC}(A, T, T\rho) = +\infty$ and one can choose $\rho^* \geq 1 - (C_1/2C_2)$.

The destabilization result can be obtained by a similar argument based on the Lyapunov equation

$$(J_n - bL^T)^T Q + Q(J_n - bL^T) = \text{Id}_n,$$

verified for some $L \in \mathbf{R}^n$ and some symmetric positive definite matrix Q . ■

Open problem 2 It would be interesting to investigate the dependance of ρ^* on the dimension n as $n \rightarrow \infty$.

4.2 Finite maximal rate of convergence for ρ small enough

In this section we restrict our attention to the case $n = 2$.

Proposition 4.5 *There exists $\rho_* \in (0, 1)$ such that for every controllable pair $(A, b) \in M_2(\mathbf{R}) \times \mathbf{R}^2$, every $T > 0$ and every $\rho \in (0, \rho_*)$ one has $\text{RC}(A, T, \rho T) < +\infty$.*

Proof. Thanks to Remark 4.1, it suffices to show that there exists $\rho_* \in (0, 1)$ such that, for every controllable pair $(A, b) \in M_2(\mathbf{R}) \times \mathbf{R}^2$ with $\text{Tr}(A) = 0$, one has $\text{RC}(A, 1, \rho_*) < +\infty$.

As in (32), take (A, b) in companion form, ie,

$$A = J_2 + aH, \quad b = (0, 1)^T,$$

with $a = -\det A$ and $H = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

For $\theta \in [-\pi, \pi)$ set $e_\theta = (\sin \theta, \cos \theta)^T$ and define $y_0 = (-1, 0)^T$. Every gain can be written as

$$K_{\theta, \gamma} = \gamma D_\gamma e_\theta,$$

with $\gamma \geq 0$ and $\theta \in [-\pi, \pi)$.

Moreover, if $A - bK^T$ is Hurwitz with $K = \gamma D_\gamma e_\theta$ then the sum and the product of its two eigenvalues are, respectively, $\gamma \cos \theta > 0$ and $\gamma^2 \sin \theta - a > 0$. In particular, $\theta \in (-\pi/2, \pi/2)$ and $\gamma^2 \sin \theta > a$. If $\theta \in (-\pi/2, 0]$ with $A - bK^T$ Hurwitz, then $|a - \sin \theta \gamma^2| \leq |a| = -a$ and therefore the convergence rate of $A - bK^T$ is upper bounded by a constant only depending on a .

Let $\Omega_0 = (0, \pi/2) \times (0, \infty)$. We show in the following the existence of $\rho > 0$ and $\Omega \subset \Omega_0$ such that

$$\text{if } (\theta, \gamma) \in \Omega_0 \text{ and } K_{\theta, \gamma} \text{ is a } (1, \rho)\text{-stabilizer of } \dot{x} = Ax + \alpha bu, \text{ then } (\theta, \gamma) \in \Omega, \quad (37)$$

and

$$\sup_{(\theta, \gamma) \in \Omega} \min\{-\Re(\sigma(A - bK_{\theta, \gamma}^T))\} < +\infty, \quad (38)$$

and the conclusion then follows from (25).

The strategy of the proof consists of determining $\rho > 0$ and a function $\gamma(\theta)$, defined on $(0, \pi/2)$, such that, for every gain $K_{\theta, \gamma}$ with $\gamma > \gamma(\theta)$, there exists a $(1, \rho)$ -signal α and a trajectory of $\dot{x} = Ax - \alpha b K_{\theta, \gamma}^T$ escaping to infinity. The set Ω is then defined as

$$\Omega = \{(\theta, \gamma) \mid 0 < \theta < \pi/2, 0 < \gamma < \gamma(\theta)\},$$

and one finally checks condition (38).

Fix $\theta \in (0, \pi/2)$. In order to find γ destabilizing

$$\dot{x} = Ax - \alpha b K_{\theta, \gamma} x, \quad \alpha \in \mathcal{G}(1, \rho),$$

(i.e., such that there exist $\alpha \in \mathcal{G}(1, \rho)$ and an initial condition x_0 whose corresponding trajectory escapes to infinity), we apply the transformation $y_\gamma(\cdot) = D_\gamma x(\cdot/\gamma)$: the problem is now to determine γ large enough so that there exist $\alpha \in \mathcal{G}(\gamma, \rho\gamma)$ and a trajectory of

$$\dot{y} = \left(J_2 + \frac{a}{\gamma^2} H \right) y - \alpha b e_\theta y, \quad (39)$$

escaping to infinity.

Due to the homogeneity of the system, the latter fact reduces to determine τ large enough and $\alpha \in \mathcal{G}(\tau, 2\rho\tau)$ such that the solution $y(\cdot; 0, y_0, e_\theta, \alpha)$ of (39) satisfies $y(\tau; 0, y_0, \alpha) = -\xi y_0$ with $\xi > 1$.

Indeed, for every $\gamma > \tau$ the extension of $\alpha|_{[0,\tau]}$ by periodicity is a $(\gamma, \rho\gamma)$ -signal (see point 4 in Lemma 2.4) and the sequence $\|y(m\tau; 0, y_0, \alpha)\| = \xi^m$ goes to infinity as m goes to infinity.

Set

$$M_\theta = J_2 - be_\theta^T, \quad N_{a,\theta,\gamma} = J_2 + \frac{a}{\gamma^2}H - be_\theta^T.$$

Consider $h > 0$ small to be fixed later. We distinguish two cases depending on whether $\theta \in (0, h)$ or not.

The case $\theta \in [h, \pi/2)$.

We construct a PE signal α as follows: starting at y_0 take $\alpha = 1$ until the trajectory $y(\cdot; 0, y_0, e_\theta, \alpha)$ of (39) reaches, at time T_1 , the switching line $\sin(\theta)x + \cos(\theta)y = 0$. In order to ensure that the switching line is reached in finite time and, moreover, that T_1 is lower and upper bounded by two positive constants only depending on h (and not on $\theta \in [h, \pi/2)$), it suffices to choose $\gamma > \Gamma_1(a, h) > 0$ with $\Gamma_1(a, h)$ only depending on a and h . (Indeed, the bounds hold for all matrices in a neighborhood of $\{M_\theta \mid \theta \in [h, \pi/2)\}$ and it suffices to ensure that $N_{a,\theta,\gamma}$ belongs to such neighborhood.)

From $y(T_1; 0, y_0, e_\theta, \alpha)$ set $\alpha = 0$ until the first coordinate of $y(\cdot; 0, y_0, e_\theta, \alpha)$ takes, at time $T_1 + T_2$, the value 1. Finally, take $\alpha = 1$ until the second coordinate of $y(\cdot; 0, y_0, e_\theta, \alpha)$ reaches, at time $T_1 + T_2 + T_3$, the value 0. (See Figure 1.)

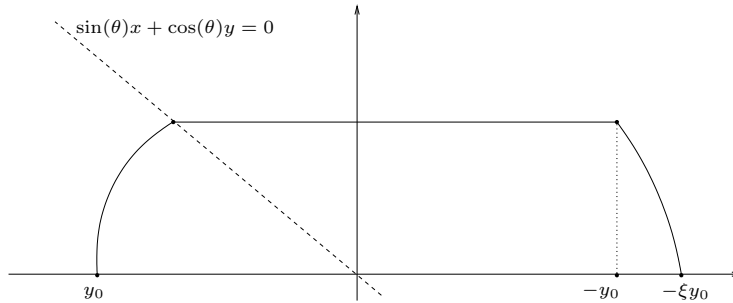


Figure 1: The trajectory $y(\cdot; 0, y_0, e_\theta, \alpha)$ when $\theta \in [h, \pi/2)$

Analogously to what happens for T_1 , the values T_2 and T_3 admit lower and upper positive bounds only depending on h .

Define $\tau = T_1 + T_2 + T_3$ and notice that it admits an upper bound $\mathcal{T}_1(h)$ only depending on h . Finally $\frac{T_1+T_3}{T_1+T_2+T_3}$ admits a lower bound ρ_1 only depending on h . The construction of the required $(\tau, \rho_1\tau)$ -signal is achieved and we set

$$\gamma(\theta) \equiv \max(\Gamma_1(a, h), \mathcal{T}_1(h)). \quad (40)$$

The case $\theta \in (0, h)$.

Notice that the condition for $N_{a,\theta,\gamma}$ to be Hurwitz is that $\gamma^2 > |a|/\sin\theta$. Choose $\gamma > \Gamma_2(a, \theta) = M\sqrt{|a|/\sin\theta}$ with M large (to be fixed later independently of all parameters). In particular, for M large enough and $h_0 > 0$ small enough (independent of all parameters), for every $\theta \in (0, h_0)$ and every $\gamma > \Gamma_2(a, \theta)$ the matrix $N_{a,\theta,\gamma}$ has two real eigenvalues, denoted by $\mu_+(a, \theta, \gamma) > \mu_-(a, \theta, \gamma)$ and

$$-2 < \mu_-(a, \theta, \gamma) < -1/2, \quad -2\sin\theta < \mu_+(a, \theta, \gamma) < -\sin\theta/2. \quad (41)$$

From now on we assume $h \in (0, h_0)$.

Similarly to what has been done above, we construct a PE signal α as follows: starting at y_0 take $\alpha = 1$ in (39) for a time $T_1 = \bar{\rho}M/|\mu_+(a, \theta, \gamma)|$ with $\bar{\rho} \in (0, 1)$ to be fixed later. Set $y_1 = y(T_1; 0, y_0, e_\theta, \alpha)$.

From y_1 set $\alpha = 0$ for a time $T_2 = M/|\mu_+(a, \theta, \gamma)|$ and denote by y_2 the point $y(T_1+T_2; 0, y_0, e_\theta, \alpha)$. Finally, take $\alpha = 1$ until the second coordinate of $y(\cdot; 0, y_0, e_\theta, \alpha)$ assumes, at time $T_1 + T_2 + T_3$, the value 0. (See Figure 2.)

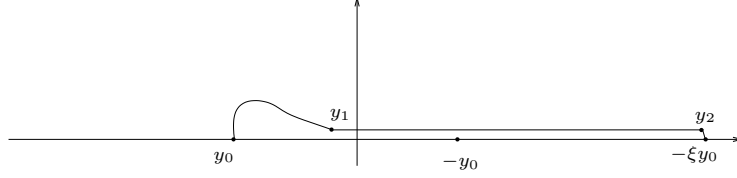


Figure 2: The trajectory $y(\cdot; 0, y_0, e_\theta, \alpha)$ when $\theta \in (0, h)$

We next show that there exist $\bar{\rho}$ and M independent of θ and a such that T_3 is well defined and $y(T_1 + T_2 + T_3; 0, y_0, e_\theta, \alpha) = -\xi y_0$ with $\xi > 1$.

An easy computation yields

$$\begin{aligned} y_1 &= \frac{1}{\mu_-(a, \theta, \gamma) - \mu_+(a, \theta, \gamma)} \begin{pmatrix} e^{\mu_-(a, \theta, \gamma)T_1} \mu_+(a, \theta, \gamma) - e^{\mu_+(a, \theta, \gamma)T_1} \mu_-(a, \theta, \gamma) \\ \mu_-(a, \theta, \gamma) \mu_+(a, \theta, \gamma) (e^{\mu_-(a, \theta, \gamma)T_1} - e^{\mu_+(a, \theta, \gamma)T_1}) \end{pmatrix} \\ &= e^{-\bar{\rho}M} \begin{pmatrix} -1 \\ \mu_+(a, \theta, \gamma) \end{pmatrix} + O(\theta^2), \end{aligned}$$

with $\|O(\theta^2)\| \leq C\theta^2$ and C only depending on M and $\bar{\rho}$. (Similarly, in the sequel the symbol $O(\theta)$ stands for a function of θ majorized by $C\theta$ with C only depending on M and $\bar{\rho}$.)

In addition, one also gets that the first coordinate of y_2 is equal to

$$\begin{cases} e^{-M\bar{\rho}}(M - 1) + O(\theta) & \text{if } a = 0, \\ e^{-M\bar{\rho}} \left(M \frac{\mu_+(a, \theta, \gamma)}{\sin \theta} \sinh \left(\frac{\sin \theta}{\mu_+(a, \theta, \gamma)} \right) - \cosh \left(\frac{\sin \theta}{\mu_+(a, \theta, \gamma)} \right) \right) + O(\theta) & \text{if } a > 0, \\ e^{-M\bar{\rho}} \left(M \frac{\mu_+(a, \theta, \gamma)}{\sin \theta} \sin \left(\frac{\sin \theta}{\mu_+(a, \theta, \gamma)} \right) - \cos \left(\frac{\sin \theta}{\mu_+(a, \theta, \gamma)} \right) \right) + O(\theta) & \text{if } a < 0. \end{cases}$$

Using (41) one deduces that the first coordinate of y_2 is larger than

$$\begin{cases} e^{-M\bar{\rho}}(M/2 \sinh(1/2) - \cosh(2)) + O(\theta) & \text{if } a > 0, \\ e^{-M\bar{\rho}}(M/2 \sin(1/2) - \cos(2)) + O(\theta) & \text{if } a < 0. \end{cases}$$

Then in all three cases the first coordinate of y_2 becomes larger than

$$e^{-M\bar{\rho}}(MC_0 - C_1 + O(\theta)),$$

and one also gets that the second coordinate of y_2 can always be lower bounded by

$$\sin \theta e^{-M\bar{\rho}}(C_1 - C_0/M + O(\theta)),$$

with $C_0 > 0$ and $C_1 > 0$ independent of all the parameters.

Fix M large and $\bar{\rho} \in (0, 1)$ such that

$$e^{-M\bar{\rho}}(MC_0 - C_1) \geq 2, \quad e^{-M\bar{\rho}}(C_1 - C_0/M) \geq C_1/2.$$

Finally, by eventually reducing h in order to make each $O(\theta)$ uniformly small, one can ensure that the first coordinate of y_2 remains larger than one and that its second coordinate is positive.

Similar computations to the ones provided above show that it is possible to further ensure that $T_3 \leq 2T_1$.

Define $\tau = T_1 + T_2 + T_3$. Then $M/(2 \sin \theta) < \tau < 8M/\sin \theta = \mathcal{T}_2(\theta)$. Choose now

$$\gamma(\theta) = M(8 + \sqrt{a})/\sin \theta \geq \max(\mathcal{T}_2(\theta), \Gamma_2(a, \theta)). \quad (42)$$

By construction, $\alpha \in \mathcal{G}(\tau, \bar{\rho}\tau)$. To conclude the proof it is clearly enough to check condition (38) on

$$\Omega_* = \{(\theta, \gamma) \mid 0 < \theta < h, 0 < \gamma < \gamma(\theta)\}.$$

For $(\theta, \gamma) \in \Omega_*$ define

$$A_{\theta, \gamma}^{\text{stab}} = A - bK_{\gamma, \theta}^T = \begin{pmatrix} 0 & 1 \\ a - \gamma^2 \sin \theta & -\gamma \cos \theta \end{pmatrix}.$$

Then

$$0 < \det(A_{\theta, \gamma}^{\text{stab}}) \leq C_0 |\text{Tr}(A_{\theta, \gamma}^{\text{stab}})| + |a|,$$

with $C_0 = 2M(8 + \sqrt{|a|})$. Then (38) easily follows. \blacksquare

The following corollary is a direct consequence of Remark 4.1 and Proposition 4.5.

Corollary 4.6 *Take ρ_* as in the statement of Proposition 4.5. For every controllable pair $(A, b) \in M_2(\mathbf{R}) \times \mathbf{R}^2$, every $T > 0$ and every $\rho < \rho_*$, if $\lambda > 0$ is large enough, then $(A + \lambda \text{Id}_2, b)$ is not $(T, \rho T)$ -stabilizable. Moreover, if $0 < \rho < \rho_*$ and $\lambda > \text{RC}(J_2, 1, \rho)$, then $(J_2 + \lambda \text{Id}_2, b_0)$ is not $(T, \rho T)$ -stabilizable for every $T > 0$.*

The above corollary establishes the existence of non-stabilizable PE systems if the ratio $\rho = \mu/T > 0$ is small enough and regardless of T . This is especially intriguing when one recalls, on the one hand, that any weak- \star limit point α_* of a sequence (α_n) , with $\alpha_n \in \mathcal{G}(T_n, \rho T_n)$ and $\lim_{n \rightarrow +\infty} T_n = 0$, takes values in $[\rho, 1]$ (see point 1 of Lemma 2.5) and, on the other hand, that the switching system $\dot{x} = J_2 x + \alpha_* b_0 u$, $u \in [\rho, 1]$, can be stabilized with an arbitrary rate of convergence by taking the feedback law $u_\gamma = -\gamma D_\gamma K x$, where $\gamma > 0$ is arbitrarily large and K is provided by [11, Lemma 4.0].

Remark 4.7 One possible interpretation of Proposition 4.5 goes as follows. Consider the destabilizing signals built in the argument of the proposition back in the original time-scale, i.e., as $(1, \rho)$ -signals. These signals take only the values 0, 1 over time intervals of length proportional to $1/\gamma$. Therefore, the fundamental solution associated to $\dot{x} = (A - \alpha b_0 K_{\gamma, \theta})x$ is a power of the product $A_1 A_2 A_3$, where $A_1 = \exp(T_1(A - b_0 K_{\gamma, \theta})/\gamma)$, $A_2 = \exp(T_2 A/\gamma)$ and $A_3 = \exp(T_3(A - b_0 K_{\gamma, \theta})/\gamma)$. In some sense, the stabilizing effect of $A - b_0 K_{\gamma, \theta}$ is compensated by the overshoot phenomenon occurring when the exponential of $A - b_0 K_{\gamma, \theta}$ is taken only over small intervals of time. Then Proposition 4.5 says that, if γ is large enough, the procedure of systematically introducing the overshoot eventually destabilizes $\dot{x} = (A - \alpha b_0 K_{\gamma, \theta})x$.

Open problem 3 *We conjecture that Proposition 4.5 should hold true in dimension $n > 2$. Note however that the proof given in the 2D case cannot be easily extended to the case in which $n > 2$. Indeed, our strategy is based on a complete parameterization of the candidate feedbacks for stabilization and on the explicit construction of a destabilizing signal α for every value of the parameter θ , which varies in a subset of the one-dimensional sphere. In the general case, the parameter would belong to an $(n - 1)$ -dimensional manifold and an explicit construction, if possible, would be more intricate.*

4.3 Further discussion on the maximal rate of convergence

Let $(A, b) \in M(n, \mathbf{R}) \times \mathbf{R}^n$ be a controllable pair. Define

$$\rho(A, T) = \inf\{\rho \in (0, 1] \mid \text{RC}(A, T, T\rho) = +\infty\}.$$

Notice that $\rho(A, T) = \rho(A/T, 1)$ and does not depend on $\text{Tr}(A)$ (see Remark 4.1).

Proposition 4.4 implies that $\rho(A, T) \leq \rho^*$ for some $\rho^* \in (0, 1)$ only depending on n . In the case $n = 2$, moreover Proposition 4.5 establishes a uniform lower bound $\rho(A, T) \geq \rho_* > 0$.

The following lemma collects some further properties of the function $T \mapsto \rho(A, T)$, defined on $(0, +\infty)$.

Lemma 4.8 *Let $(A, b) \in M_n(\mathbf{R}) \times \mathbf{R}^n$ be a controllable pair. Then (i) $T \mapsto \rho(A, T)$ is locally Lipschitz; (ii) there exist $\lim_{T \rightarrow +\infty} \rho(A, T) = \sup_{T > 0} \rho(A, T)$ and $\lim_{T \rightarrow 0} \rho(A, T) = \inf_{T > 0} \rho(A, T)$.*

Proof. In order to prove (i), notice that point 3 in Lemma 2.4 implies that if $\text{RC}(A, T, \rho T) < +\infty$ then for every $\eta \in (0, \rho T)$,

$$\text{RC}\left(A, T + \eta, \frac{\rho T}{T + \eta}(T + \eta)\right), \text{RC}\left(A, T - \eta, \frac{\rho T - \eta}{T - \eta}(T - \eta)\right) < +\infty.$$

Therefore, for $\eta \in (0, \rho(A, T)T)$,

$$\rho(A, T + \eta) \geq \frac{\rho(A, T)T}{T + \eta}, \quad \rho(A, T - \eta) \geq \frac{\rho(A, T)T - \eta}{T - \eta}.$$

The second inequality can be rewritten as

$$\rho(A, T) \geq \frac{\rho(A, T + \eta)(T + \eta) - \eta}{T}$$

and holds for every $\eta \in (0, \rho(A, T + \eta)(T + \eta))$. We conclude that, for η small enough,

$$|\rho(A, T + \eta) - \rho(A, T)| \leq \frac{\eta}{T}.$$

As for point (ii), it suffices to deduce from point 5 in Lemma 2.4 that for $0 < \rho' < \rho < 1$ there exists $M > 0$ such that whenever $\text{RC}(A, T, \rho T) = +\infty$ one has $\text{RC}(A, \gamma, \rho'\gamma) = +\infty$ for every $\gamma > 0$ such that $\gamma/T > M$. ■

The case $A = J_n$ can be tackled thanks to the equality (11) and the time-space transformation (12): it easily follows that $\rho(J_n, T)$ is constant with respect to T (and positive in the case $n = 2$).

Open problem 4 *It is a challenging question to determine whether the function $T \mapsto \rho(A, T)$ is constant for a general matrix A . If this is true, one may wonder whether the constant value depends on A . Otherwise, a natural question would be to understand the dependence of $\lim_{T \rightarrow +\infty} \rho(A, T) \leq \rho^* < 1$ on the matrix A .*

4.4 Remarks on the pole shifting property for classes of PE control systems

Following Remark 4.2, one can interpret the results of the previous sections as follows. In dimension two, the PE control system $\dot{x} = Ax + \alpha bu$, $\alpha \in \mathcal{G}(T, \mu)$, does not have the pole-shifting property for μ/T small. It makes therefore sense to investigate additional conditions to impose on the PE signals (periodicity, positive dwell-time, uniform bounds on the derivative of the PE signal, etc) so that the pole-shifting property holds true for these restricted classes of PE signals, regardless of the

ratio μ/T . Proposition (4.3) guarantees that the issue reduces, at least in dimension two, to checking that the rate of convergence can be made arbitrary large. First of all, the subclass of periodic PE signals must be excluded, since the destabilizing inputs constructed in Proposition 4.5 are periodic. It is also clear that, for the subclass of $\mathcal{G}(T, \mu)$ given by all signals with a positive dwell time $t_d > 0$, one gets arbitrary rate of convergence with a linear constant feedback, for every choice of T, μ, t_d . We conclude with yet another conjecture.

Open problem 5 *Given $T, M > 0$ and $\rho \in (0, 1]$, let $\mathcal{D}(T, \rho, M)$ be the subset of $\mathcal{G}(T, \rho T)$ whose signals are globally Lipschitz over $[0, +\infty)$ with Lipschitz constant bounded by M . Then, given a controllable pair (A, b) , we conjecture that it is possible to stabilize by a linear feedback the system $\dot{x} = Ax + \alpha bu$, $\alpha \in \mathcal{D}(T, \rho, M)$, with an arbitrarily large rate of convergence, i.e., we conjecture that for every $C > 0$ there exists a gain K such that for every $\alpha \in \mathcal{D}(T, \rho, M)$ the maximal Lyapunov exponent of $\dot{x} = (A - \alpha b K^T)x$ is smaller than $-C$.*

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