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Limit Theorems and Coexistence Probabilities for the Curie-Weiss Potts Model with an external field*

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Abstract

The Curie-Weiss Potts model is a mean field version of the well-known Potts model. In this model, the critical line $\beta = \beta_c(h)$ is explicitly known and corresponds to a first order transition when $q > 2$. In the present paper we describe the fluctuations of the density vector in the whole domain $\beta \geqslant 0$ and $h \geqslant 0$, including the conditional fluctuations on the critical line and the non-Gaussian fluctuations at the extremity of the critical line. The probabilities of each of the two thermodynamically stable states on the critical line are also computed. Similar results are inferred for the Random-Cluster model on the complete graph.

1 Introduction

The Curie-Weiss Potts model is a model of statistical mechanics which, being a mean-field model, can be studied by means of analytic tools. First it was shown in [1] that at $h = 0$, the model undergoes a phase transition at the critical inverse temperature

$$\beta_c = \begin{cases} q & \text{if } q \leqslant 2 \\ 2\frac{q-1}{q-2}\log(q-1) & \text{if } q > 2. \end{cases}$$

When $q > 2$ this transition is first order. The case of non-zero external field was considered in [2] and it appeared that the first-order transition remains on a critical line. Recently this critical line was computed explicitly [3].

On the critical line, two or more states can coexist. One of the issue we address in the present work is the computation of the probabilities of these stable states. We also obtain a description of the limit distribution of the empirical vector of the spin variables that extend previous results on the Curie-Weiss Ising model [4] (see also [5, 6]), and previous results on the Curie-Weiss Potts model with no external field [7].

The Curie-Weiss Potts model is connected as well to the random-cluster model. In that model, the first order phase transition for $q > 2$ was described in [8] and it appeared that at criticality, two possible structures of the random graph are possible. The probability for each structure was latter computed in [9]. A consequence of our results we present a simple way of computing these probabilities when $q > 2$ is integer.

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2 The Curie-Weiss Potts model

The Curie-Weiss Potts model is a spin model on the complete graph. The probability of observing the configuration $\sigma \in \{1, \dots, q\}^n$ at inverse temperature β, in an exterior field $H = h/\beta$ equals

$$\mu_{\beta,h,n}(\sigma) = \frac{1}{Z_{\beta,h,n}} \exp\left(\frac{\beta}{n} \sum_{1 \leqslant i < j \leqslant n} \delta_{\sigma_i,\sigma_j} + h \sum_{i=1}^{n} \delta_{\sigma_i,1} \right)$$

where δ is the Kronecker symbol and $Z_{\beta,h,n}$ the partition function

$$Z_{\beta,h,n} = \sum_{\sigma \in \{1,\cdots,q\}^n} \exp\left(\frac{\beta}{n} \sum_{1 \leqslant i < j \leqslant n} \delta_{\sigma_i,\sigma_j} + h \sum_{i=1}^{n} \delta_{\sigma_i,1} \right).$$

Our interest is in the limit distribution of the empirical vector

$$\boldsymbol{N} = (N_1, \dots, N_q) = \left(\sum_{i=1}^{n} \delta_{\sigma_i,1}, \dots, \sum_{i=1}^{n} \delta_{\sigma_i,q} \right) \tag{2.1}$$

that represents the number of spins of each color for a given configuration σ. The normalized vector $\boldsymbol{N}/n$ belongs to the set of probability vectors

$$\Omega = \{\boldsymbol{x} \in \mathbb{R}^q : x_1 + \cdots + x_q = 1 \text{ and } x_i \geqslant 0, \forall i\}. \tag{2.2}$$

The large deviation principle for $\boldsymbol{N}/n$ is an immediate application of Stirling's formula (see for instance Lemma 4.1). If we consider $f_{\beta,h}$ the microcanonical free energy of the model:

$$f_{\beta,h}(\boldsymbol{x}) = \sum_{i=1}^{q} x_i \log x_i - \frac{\beta}{2} \sum_{i=1}^{q} x_i^2 - h x_1, \quad \forall \boldsymbol{x} \in \Omega \tag{2.3}$$

with the convention that $0 \log 0 = 0$, then we have the following classical large deviation result (see for instance [10], and also [8, 11] for LDP concerning the closely related random cluster model).

Theorem 2.1 *Assume that $\beta_n \to \beta$ and $h_n \to h$. Then, the vector $\boldsymbol{N}/n \in \Omega$ distributed according to the measure $\mu_{\beta_n,h_n,n}$ follows a large deviation principle with speed n and good rate function $f_{\beta,h} - \min_\Omega f_{\beta,h}$.*

This large deviation principle leads to a law of large number: when $f_{\beta,h}$ has a unique global minimizer, $\boldsymbol{N}/n$ converges towards that minimizer. The structure of the minimizers of $f_{\beta,h}$ was determined in the papers [1, 12, 2]. Here we give some further details:

Proposition 2.2 *Let $\beta, h \geqslant 0$ and let $\boldsymbol{x}$ be a global minimizer of $f_{\beta,h}$ in Ω.*

i. The vector $\boldsymbol{x}$ has the coordinate $\min(x_i)$ repeated $q-1$ times at least.

ii. If $h > 0$, then $x_1 > x_i$, for all $i \in \{2, \dots, q\}$.

iii. The inequality $\min(x_i) > 0$ holds.

iv. For any $q \geqslant 3$, or $q = 2$ and $(\beta, h) \neq (\beta_c, 0)$, one has $\min(x_i) < 1/\beta$.

Because of the simple structure of the global minimizers of the free energy, the problem of finding them reduces to a one-dimensional optimization problem. The usual parametrization consists in taking $x_1 = (1 + (q-1)s)/q$, $x_2 = \cdots = x_q = (1-s)/q$ where $s \in [0, 1]$ is a parameter called the magnetization. Another equivalent parametrization permitted in [3] the explicit computation of the critical line

$$h_T = \left\{ (\beta, h) : 0 \leqslant h < h_0 \text{ and } h = \log(q-1) - \beta \frac{q-2}{2(q-1)} \right\} \tag{2.4}$$

with extremities $(\beta_c, 0)$ and (β_0, h_0), where

$$\beta_0 = 4\frac{q-1}{q} \quad \text{and} \quad h_0 = \log(q-1) - 2\frac{q-2}{q}$$

were already determined in [2]. The key observation in [3] was that the free energy $f_{\beta,h}(\boldsymbol{x}_z)$ at

$$\boldsymbol{x}_z = \left(\frac{1+z}{2}, \frac{1-z}{2(q-1)}, \cdots, \frac{1-z}{2(q-1)}\right), \qquad z \in [\pm 1] \tag{2.5}$$

is easily split into its even and odd parts:

$$\begin{aligned} f_{\beta,h}(\boldsymbol{x}_z) &= \frac{1+z}{2}\log\frac{1+z}{2} + \frac{1-z}{2}\log\frac{1-z}{2} - \frac{1}{2}\log(q-1) - \frac{\beta(1+z^2)}{8}\left[1+\frac{1}{q-1}\right] - \frac{1}{2}h \\ &\quad + \frac{z}{2}\left[\log(q-1) - \beta\frac{q-2}{2(q-1)} - h\right] \end{aligned}$$

showing that, on the critical line h_T, the free energy $f_{\beta,h}(\boldsymbol{x}_z)$ is an *even* function of z. It is strictly convex for $\beta < \beta_0$ but not for $\beta \geqslant \beta_0$. Indeed, the second derivative of $z \mapsto f_{\beta,h}(\boldsymbol{x}_z)$ is

$$\frac{\mathrm{d}^2 f_{\beta,h}(\boldsymbol{x}_z)}{\mathrm{d}z^2} = \frac{1}{1-z^2} - \frac{\beta q}{4(q-1)}, \tag{2.6}$$

thus, for $\beta \geqslant \beta_0$, the function is strictly convex on $[-1, -z_i)$ and on $(z_i, 1]$, concave on $(-z_i, z_i)$ where

$$z_i = \sqrt{1 - \beta_0/\beta}. \tag{2.7}$$

Depending on the parameters (β, h) the free energy presents one or several global minimizers. The following is a summary of the works [1] (for $h = 0$) and [3] (for $h > 0$):

Theorem 2.3 *Let $\beta, h \geqslant 0$.*

i. If $h > 0$ and $(\beta, h) \notin h_T$, the free energy $f_{\beta,h}$ has a unique global minimizer in Ω. This minimizer is analytic in β and h outside of $h_T \cup \{(\beta_0, h_0)\}$.

ii. If $h > 0$ and $(\beta, h) \in h_T$, the free energy $f_{\beta,h}$ has two global minimizers in Ω. More precisely, for any $z \in (0, (q-2)/q)$, the two global minimizers of f_{β_z,h_z} at

$$\beta_z = 2\frac{q-1}{q}\frac{1}{z}\log\frac{1+z}{1-z} \quad \text{and} \quad h_z = \log(q-1) - \frac{q-2}{2(q-1)}\beta_z$$

are the points $\boldsymbol{x}_{\pm z}$. Furthermore, $\boldsymbol{x}_z$ (resp. $\boldsymbol{x}_{-z}$) is the limit of the unique global minimizer of $f_{\beta,h}$ as $(\beta, h) \to (\beta_z, h_z)$ above (resp. below) the line h_T.

iii. If $h = 0$ and $\beta < \beta_c$, the unique global minimizer of $f_{\beta,h}$ is $(1/q, \ldots, 1/q) = \boldsymbol{x}_{-(q-2)/q}$.

iv. If $h = 0$ and $\beta > \beta_c$, there are q global minimizers of $f_{\beta,h}$, which all equal $\boldsymbol{x}_z$ up to a permutation of the coordinates, for some appropriate $z \in ((q-2)/q, 1)$.

v. If $h = 0$ and $\beta = \beta_c$, there are $q+1$ global minimizers of $f_{\beta,h}$: the symmetric one $(1/q, \ldots, 1/q) = \boldsymbol{x}_{-(q-2)/q}$ together with the permutations of

$$\left(\frac{q-1}{q}, \frac{1}{q(q-1)}, \cdots, \frac{1}{q(q-1)}\right) = \boldsymbol{x}_{(q-2)/q}.$$

3 Statement of the results

In this paper we address essentially two questions. According to Theorem 2.1 the distribution of $\boldsymbol{N}/n$ is concentrated, as $n \to +\infty$, on the set of global minimizers of the free energy. First, we study the fluctuations of the empirical vector $\boldsymbol{N}$ around its typical value. Second, when several global minimizers exists we explicit the weight of each of them.

These questions were answered in several very interesting papers for particular cases of the model. The case of the Curie-Weiss Ising model ($q = 2$) was reported in [4] (see [5, 6] for the proofs), while the Curie-Weiss Potts model was treated at zero external field in [7].

Our approach is similar to that of the former references, with the technical difference that our computations are based on Stirling's formula while the former works are based on the fact that the law of $\boldsymbol{N}/n + \boldsymbol{W}/\sqrt{n}$, where $\boldsymbol{W}$ is a Gaussian vector in $\mathbb{R}^q$ with distribution $\mathcal{N}(0, \beta^{-1} I_q)$, can be explicitly computed (see for instance Lemma 3.2 in [7]).

We also permit that the parameters β and h fluctuate with n, and take in the sequel $(\beta_n, h_n) \to (\beta, h)$. This will be useful for applying our results to related model such as the random cluster model on the complete graph.

Our first result concerns the fluctuations of the empirical vector $\boldsymbol{N}$ outside of the critical line. The fluctuations belong to the hyperplane

$$\mathcal{H} \quad = \quad \left\{ \boldsymbol{w} \in \mathbb{R}^d : \sum_{i=1}^{d} w_i = 0 \right\}. \tag{3.1}$$

Not surprisingly, these fluctuations are Gaussian. This generalizes Theorem 2.4 in [7] to the case of positive external fields. The way that (β_n, h_n) converges to (β, h) is able to shift the center of the distribution.

Theorem 3.1 *Assume that* $(\beta_n, h_n) \to (\beta, h)$ *for some* $\beta, h \geqslant 0$ *with* $(\beta, h) \neq (\beta_0, h_0)$. *Assume that there is a unique global minimizer* $\boldsymbol{x} = (x_1, x_q, \ldots, x_q)$ *of the free energy* $f_{\beta,h}$. *For every* n, *let* $\boldsymbol{d}_n$ *the smallest* $\boldsymbol{d} \in \mathcal{H}$ *such that* $\boldsymbol{x} + \boldsymbol{d} \in \Omega$ *is a local minimizer of* f_{β_n,h_n}. *Let* $\boldsymbol{W}$ *be the random variable in* $\mathcal{H}$ *such that*

$$\boldsymbol{N} = n\boldsymbol{x} + n\boldsymbol{d}_n + n^{1/2}\boldsymbol{W} \tag{3.2}$$

where the distribution of $\boldsymbol{N}$ *is given according to the measure* $\mu_{\beta_n,h_n,n}$. *Then,* $\boldsymbol{W}$ *converges in law towards the centered Gaussian vector with covariance matrix*

$$\left(\frac{1}{x_1 x_q} - q\beta\right)^{-1} \begin{pmatrix} q-1 & -1 & \cdots & -1 \\ -1 & 1+(q-2)\frac{\frac{1}{x_1}-\beta}{\frac{1}{x_q}-\beta} & & -\frac{\frac{1}{x_1}-\beta}{\frac{1}{x_q}-\beta} \\ \vdots & & \ddots & \\ -1 & -\frac{\frac{1}{x_1}-\beta}{\frac{1}{x_q}-\beta} & & 1+(q-2)\frac{\frac{1}{x_1}-\beta}{\frac{1}{x_q}-\beta} \end{pmatrix} \tag{3.3}$$

which has rank $q-1$.

Remark 3.2 *The vector* $\boldsymbol{d}_n$ *is* $O\left(|\beta_n - \beta| + |h_n - h|\right)$ *when the quadratic term in the Taylor expansion of* $f_{\beta,h}$ *is definite, that is for any* $(\beta, h) \neq (\beta_0, h_0)$ *– see Lemmas 4.5, 4.6 and 4.7 below. Hence, for* $\beta_n - \beta = o(n^{-1/2})$ *and* $h_n - h = o(n^{-1/2})$ *the vector* $n\boldsymbol{d}_n$ *is negligible with respect to* $n^{1/2}\boldsymbol{W}$ *and could be removed from the definition of* $\boldsymbol{W}$ *at (3.2). It is remarkable also that on the line* $\beta < \beta_c$, $h = 0$ *we have* $\boldsymbol{x} = (1/q, \ldots, 1/q)$, *hence for* $h_n = 0$ *and* $\beta_n \to \beta$, *the vector* $\boldsymbol{d}_n$ *is exactly zero.*

Remark 3.3 *In the range of validity of Theorem 2.4 in [7], that is* $\beta_n = \beta < \beta_c$ *and* $h_n = h = 0$, *we have* $x_1 = \ldots = x_q = 1/q$ *thus the covariance matrix simplifies to*

$$\frac{1}{q^2 - q\beta} \begin{pmatrix} q-1 & & -1 \\ & \ddots & \\ -1 & & q-1 \end{pmatrix}.$$

We have checked the correspondence with the covariance matrix that appears in [7].

The matrix (3.3) gives a special emphasis on the first coordinate since it corresponds to the case $x_2 = \ldots = x_q$. Before stating the next theorem we give a more symmetric definition for the covariance matrix: we let

$$K(\boldsymbol{x}) \quad = \quad \left(\frac{1}{\min(x_i)\max(x_i)} - q\beta\right)^{-1} \begin{pmatrix} 1+(q-2)\alpha(x_1,x_1) & & -\alpha(x_1,x_q) \\ & \ddots & \\ -\alpha(x_q,x_1) & & 1+(q-2)\alpha(x_q,x_q) \end{pmatrix} \tag{3.4}$$

where

$$\alpha(x,y) = \frac{\max(x_i)^{-1} - \beta}{\max(x,y)^{-1} - \beta}.$$

When the free energy has several global minimizers, that is when $(\beta, h) \in h_T$ or $\beta \geqslant \beta_c$ and $h = 0$, the empirical vector $\boldsymbol{N}/n$ is close to either one or the other of the minimizers of the free energy $f_{\beta,h}$. We first determine the conditional fluctuations (this extends Theorem 2.5 of [7]):

Theorem 3.4 *Assume that $(\beta_n, h_n) \to (\beta, h)$ with $\beta, h \geqslant 0$. Assume that the free energy $f_{\beta,h}$ has multiple global minimizers $\boldsymbol{x}, \boldsymbol{x}', \dots$ and let $\varepsilon > 0$ smaller than the distance between any two global minimizers of $f_{\beta,h}$. Let $\boldsymbol{d}_n$ the smallest $\boldsymbol{d} \in \mathcal{H}$ such that $\boldsymbol{x} + \boldsymbol{d} \in \Omega$ is a local minimizer of f_{β_n,h_n}. Then, under the conditional measure*

$$\mu_{\beta_n,h_n,n}\left(\cdot \,\middle|\, \frac{\boldsymbol{N}}{n} \in B(\boldsymbol{x}, \varepsilon)\right),$$

the variable $\boldsymbol{W}$ defined by $\boldsymbol{N} = n\boldsymbol{x} + n\boldsymbol{d}_n + n^{1/2}\boldsymbol{W}$ converges in law to the centered Gaussian vector with covariance matrix $K(\boldsymbol{x})$.

Additionally we compute the limit probabilities that $\boldsymbol{N}/n$ be close to a given global minimizer of the free energy, generalizing Theorem 2.3 of [7] with an explicit formula.

Theorem 3.5 *Assume that there are $\beta, h \geqslant 0$ and $\lambda, \nu \in \mathbb{R}$ such that*

$$(\beta_n, h_n) = (\beta, h) + n^{-1}(\lambda, \nu) + o(n^{-1}),$$

and assume that the free energy $f_{\beta,h}$ has multiple global minimizers $\boldsymbol{x}, \boldsymbol{x}', \dots$ If $\varepsilon > 0$ is smaller than the distance between any two global minimizers of $f_{\beta,h}$, then

$$\lim_{n\to\infty} \mu_{\beta_n,h_n,n}\left(\frac{\boldsymbol{N}}{n} \in B(\boldsymbol{x}, \varepsilon)\right) = \frac{\tau(\boldsymbol{x})}{\tau(\boldsymbol{x}) + \tau(\boldsymbol{x}') + \cdots} \tag{3.5}$$

where

$$\tau(\boldsymbol{x}) = \left(1 - \beta \min_{i=1}^{q}(x_i)\right)^{\frac{2-q}{2}} \exp\left(\frac{\lambda}{2}\sum_{i=1}^{q} x_i^2 + \nu x_1\right). \tag{3.6}$$

Remark 3.6 *On the critical line h_T one can parametrize the formula (3.5) according to the second point in Theorem 2.3 : when $(\beta, h) = (\beta_z, h_z)$ with $z \in (0, (q-2)/q)$ the two global minimizers are $\boldsymbol{x}_{\pm z}$. In particular, when $z \to 0$ (i.e. (β, h) on h_T close to (β_0, h_0)), the probability of each corresponding state converges to $1/2$.*

We also describe the fluctuations at the extremity (β_0, h_0) of the critical line. This extends for instance Theorem 2 in [4] that applies to the case of the Curie-Weiss Ising model at criticality, namely $q = 2$ and $(\beta_0, h_0) = (\beta_c, 0)$. We recall that $\mathcal{H}$ defined at (3.1) is the hyperplane parallel to Ω. Given a vector $\boldsymbol{u} \in \mathbb{R}^q$, we denote by $\boldsymbol{u}^\perp$ the vector space made of all vectors orthogonal to $\boldsymbol{u}$ in the Euclidean space $\mathbb{R}^q$.

Theorem 3.7 *Assume that $(\beta_n, h_n) \to (\beta_0, h_0)$ with $\beta_n - \beta = o(n^{-3/4})$ and $h_n - h = o(n^{-3/4})$ and let $\boldsymbol{x} = (1/2, 1/2(q-1), \dots, 1/2(q-1))$ be the unique minimizer of f_{β_0,h_0}. Let $\boldsymbol{u} = (1-q, 1, \dots, 1)$. If the random variables $T \in \mathbb{R}$ and $\boldsymbol{V} \in \mathcal{H} \cap \boldsymbol{u}^\perp$ are defined by*

$$\boldsymbol{N} = n\boldsymbol{x} + n^{3/4}T\boldsymbol{u} + n^{1/2}\boldsymbol{V}, \tag{3.7}$$

then $(T, \boldsymbol{V})$ converges in law. The limit has the following properties:

i. T and $\boldsymbol{V}$ are asymptotically independent

ii. T converges in law to the probability measure on $\mathbb{R}$ proportional to

$$\exp\left(-\frac{4(q-1)^4}{3}t^4\right)\mathrm{d}t$$

iii. $\boldsymbol{V}$ converges in law towards the centered Gaussian vector with covariance matrix

$$\frac{q}{2(q-1)^2(q-2)}\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & q-2 & & -1 \\ \vdots & & \ddots & \\ 0 & -1 & & q-2 \end{pmatrix}$$

of rank $q-2$.

We conclude the summary of our results with two claims on the random-cluster model $G(n,p,q)$ on the complete graph K_n with n vertices. In that model, a configuration $\omega \in \{0,1\}^{E(K_n)}$ has a probability proportional to

$$\prod_{e\in E(K_n)} p^{\omega_e}(1-p)^{1-\omega_e}q^{C(\omega)}$$

where $C(\omega)$ stands for the number of connected components of the sub-graph with edge set $\{e \in E(K_n) : \omega_e = 1\}$. This model is closely related to the Potts model after the well known Fortuin-Kasteleyn representation (see for instance [13]). We take a spin configuration $\sigma \in \{1,\ldots,q\}^{V(K_n)}$ under the measure $\mu_{\beta_n,h_n,n}$, then let $\omega_e = 1$ with probability $p_n = 1-\exp(-\beta_n/n)$ only if $\sigma_i = \sigma_j$, where i,j are the extremities of the edge e (and else $\omega_e = 0$). The resulting configuration ω follows the distribution of the random cluster model $G(n,p_n,q)$.

First we have a Corollary of Theorem 3.5: we compute the probability that there exists a giant component in $G(n,p_n,q)$, that is a connected component for ω of size $\Theta(n)$, when p_n is close to the critical value β_c/n. This completes part (b) of Theorem 2.3 in [8], with a simpler proof than that of Theorem 19 of [9].

Corollary 3.8 *Let $q > 2$ integer and consider p_n such that*

$$p_n = \frac{\beta_c}{n} + \frac{\gamma}{n^2} + o\left(\frac{1}{n^2}\right).$$

Then, with a probability that converges to

$$\frac{1}{1+\frac{1}{q}\left(\frac{1-\beta_c/q}{1-\beta_c/(q(q-1))}\right)^{\frac{2-q}{2}}\exp\left(-\left(\frac{\beta_c^2}{4}+\frac{\gamma}{2}\right)\frac{(q-2)^2}{q(q-1)}\right)}$$

the graph $G(n,p_n,q)$ contains a giant component.

The description of the Gaussian fluctuations also enable fine computations of the partition function of the random-cluster model

$$Z^{\mathrm{RC}}_{p,q,n} = \sum_{\omega\in\{0,1\}^{E(K_n)}} \prod_{e\in E(K_n)} p^{\omega_e}(1-p)^{1-\omega_e}q^{C(\omega)}. \tag{3.8}$$

For instance,

Proposition 3.9 *The partition function of the random cluster model for integer $q \geqslant 2$ and*

$$p_n = \frac{\beta}{n} + \frac{\gamma}{n^2} + o\left(\frac{1}{n^2}\right)$$

with $0 \leqslant \beta < \beta_c$ and $\gamma \in \mathbb{R}$ satisfies

$$Z^{\mathrm{RC}}_{p_n,q,n} = (1+o_n(1))\left(1-\frac{\beta}{q}\right)^{-\frac{q-1}{2}} q^n \exp\left(-\frac{2n\beta+2\gamma+\beta^2}{4}\left(\frac{q-1}{q}\right)\right). \tag{3.9}$$

Remark 3.10 *Although our Theorem 3.5 agrees with Theorem 2.3 of [7] when $h = 0$, Corollary 3.8 and Proposition 3.9 do not give exactly the same conclusions as, respectively, Theorem 19 and Theorem 9 (i) in [9]. The latter Theorem states an equivalent to the partition function restricted to the set of configurations made of trees and unicyclic components, which, for $\beta < \beta_c$, is equivalent to the whole partition function. The ratio of the equivalent in Theorem 9 (i) in [9] over (3.9) is*

$$\exp\left(-\frac{3}{4}+\frac{\beta}{2}+\frac{\beta^2}{4q}\right)$$

(the formulas do coincide at the exponential order). We could not check the proofs in [9], yet we were surprised to find that Theorem 9 (i) would not permit to recover $Z^{\mathrm{RC}}_{p_n,1,n} = 1$ for $q = 1$.

4 Proofs

This section is organized as follows. First we describe the asymptotics of the distribution using Stirling's formula. We also prove Proposition 2.2. Then we address successively the limit distribution at $(\beta, h) \neq (\beta_0, h_0)$, at the extremity (β_0, h_0) of the critical line, and finally we give the proofs of the related results for the random-cluster model.

4.1 Asymptotic density & limit of the uniform measure

In this Section we give an equivalent to the density of the Potts model, prove Proposition 2.2 and describe the limit of the uniform measure on the set of possible realizations of $\boldsymbol{N}/n$.

For any $\varepsilon \geqslant 0$, we let

$$\Omega_\varepsilon = \{\boldsymbol{x} \in \Omega : \min x_i \geqslant \varepsilon\} \quad \text{and} \quad \Omega_\varepsilon^n = \{\boldsymbol{x} \in \Omega_\varepsilon : n\boldsymbol{x} \in \mathbb{N}^q\},$$

where Ω is the set of probability vectors, see (2.2). We also write $\Omega_{0+} = \bigcup_{\varepsilon>0} \Omega_\varepsilon$ and $\Omega^n = \Omega_0^n$. In our first Lemma we give an equivalent to the density of the Potts model with respect to the counting measure on Ω^n. We use nothing else than Stirling's formula

$$n! \quad = \quad (1 + o_n(1))\sqrt{2\pi n}\left(\frac{n}{e}\right)^n .$$

For any $\boldsymbol{x} \in \Omega_{0+}$ and $\beta \geqslant 0$, we let

$$A_\beta(\boldsymbol{x}) = (2\pi)^{-\frac{q-1}{2}} \prod_{i=1}^q x_i^{-1/2} \exp\left(-\frac{\beta}{2}\right). \tag{4.1}$$

We also recall that the free energy $f_{\beta,h}$ was defined at (2.3). We have:

Lemma 4.1 *For any β, h, any $n \geqslant 1$ and $\boldsymbol{x} \in \Omega_{0+}^n$, define $r_{\beta,h,n}(\boldsymbol{x})$ by*

$$Z_{\beta,h,n}\mu_{\beta,h,n}(\boldsymbol{N} = n\boldsymbol{x}) = (1 + r_{\beta,h,n}(\boldsymbol{x}))n^{-\frac{q-1}{2}} A_\beta(\boldsymbol{x}) \exp(-nf_{\beta,h}(\boldsymbol{x})).$$

Then, for any $\varepsilon > 0$, $\sup_{x\in\Omega_\varepsilon^n} \sup_{\beta,h} |r_{\beta,h,n}(\boldsymbol{x})|$ goes to 0 as $n \to \infty$.

Proof Given $\boldsymbol{x} \in \Omega_\varepsilon^n$ we write $n\boldsymbol{x} = (n_1, \ldots, n_q) = \boldsymbol{n}$. It is a vector with positive integer coordinates. There are exactly $n!/\prod_{i=1}^q n_i$ ways of choosing the spin configuration that satisfy the constraint $(N_1, \ldots, N_q) = (n_1, \ldots, n_q)$, hence

$$\begin{aligned} Z_{\beta,h,n}\mu_{\beta,h,n}(\boldsymbol{N} = \boldsymbol{n}) &= \frac{n!}{n_1! \cdots n_q!} \exp\left(\frac{\beta}{n}\sum_{i=1}^q \frac{n_i(n_i-1)}{2} + hn_1\right) \\ &= \frac{n!}{n_1! \cdots n_q!} \exp\left(-\frac{\beta}{2} + n\left[\frac{\beta}{2}\sum_{i=1}^q x_i^2 + hx_1\right]\right). \end{aligned}$$

Thus

$$\begin{aligned} 1 + r_{\beta,h,n}(\boldsymbol{x}) &= \frac{Z_{\beta,h,n}\mu_{\beta,h,n}(\boldsymbol{N} = n\boldsymbol{x})}{n^{-\frac{q-1}{2}} A_\beta(\boldsymbol{x}) \exp(-nf_{\beta,h}(\boldsymbol{x}))} \\ &= \frac{n!}{n_1! \cdots n_q!} \frac{\prod_{i=1}^q \sqrt{2\pi n x_i}}{\sqrt{2\pi n}} \exp\left(n \sum_{i=1}^q x_i \log x_i\right) \end{aligned}$$

which does not depend on β nor on h. Applying Stirling's formula yields the conclusion as all the n_i go to infinity uniformly over $\boldsymbol{x} \in \Omega_\varepsilon^n$. □

Remark 4.2 *Theorem 2.1 is a consequence of Lemma 4.1 as $f_{\beta_n,h_n}(\boldsymbol{x}) \underset{n\to\infty}{\longrightarrow} f_{\beta,h}(\boldsymbol{x})$ when $(\beta_n, h_n) \to (\beta, h)$, uniformly over $\boldsymbol{x} \in \Omega$ (for the uniformity, see Lemma 4.6 below).*

Now we give the proof of Proposition 2.2. For completeness we repeat some arguments from [12, 2].

Proof (Proposition 2.2). Call $g_\beta(z) = \log(z) - \beta z$. As $\boldsymbol{x}$ is a minimizer of the free energy in $\mathcal{H}$ one has

$$\begin{aligned} 0 &= \frac{\partial f_{\beta,h}}{\partial x_i}(\boldsymbol{x}) - \frac{\partial f_{\beta,h}}{\partial x_j}(\boldsymbol{x}) \\ &= g_\beta(x_i) - g_\beta(x_j) - h\left[\mathbf{1}_{\{i=1\}} - \mathbf{1}_{\{j=1\}}\right], \qquad \forall i,j \in \{1,\dots q\} \qquad (4.2) \\ \text{and } 0 &\leqslant \frac{\partial^2 f_{\beta,h}}{\partial x_i^2}(\boldsymbol{x}) + \frac{\partial^2 f_{\beta,h}}{\partial x_j^2}(\boldsymbol{x}) \\ &= g'_\beta(x_i) + g'_\beta(x_j). \qquad (4.3) \end{aligned}$$

First we assume $h = 0$. As g is concave, (4.2) implies that the set $\{x_i : i = 1,\dots,q\}$ contains at most two values. Equation (4.3) implies that at most one of the x_i has $g'_\beta(x_j) < 0$. As

$$g'_\beta(x) = \frac{1}{x} - \beta$$

is positive on $(0, 1/\beta)$ and negative on $(1/\beta, 1)$, the first point of Proposition 2.2 follows together with the inequality

$$\min(x_i) \leqslant \frac{1}{\beta}. \qquad (4.4)$$

Assume now that $h > 0$ and that (4.4) does not hold for some $i \in \{2,\dots,q\}$. If $x_i > x_1$ with $i \in \{2,\dots,q\}$, the vector $\tilde{\boldsymbol{x}}$ with x_1 and x_i permuted has $f_{\beta,h}(\tilde{\boldsymbol{x}}) < f_{\beta,h}(\boldsymbol{x})$, a contradiction, therefore $x_1 \geqslant x_i$. The equality $x_1 = x_2$ is impossible in view of (4.2), yielding the second point of Proposition 2.2. Now we conclude the proof of the first point of Proposition 2.2 when $h > 0$: the inequality $x_i \geqslant 1/\beta$, that implies $g'_\beta(x_i) \leqslant 0$ and $g'_\beta(x_1) < 0$ since $x_1 > x_i$, would contradict (4.3). Hence all the x_i belong to $(0, 1/\beta)$ where there is at most one reciprocal image of $g_\beta(x_1) - h$ by g_β, hence $x_2 = \ldots = x_q < 1/\beta$.

Now we address the third point. If $\min(x_i) = 0$, one can find $i, j \in \{1,\dots,q\}$ such that $x_i = 0$ and $x_j > 0$ as $\boldsymbol{x} \in \Omega$. Hence $\boldsymbol{x}^t = \boldsymbol{x} + t(\boldsymbol{e}_i - \boldsymbol{e}_j)$ belongs to Ω_{0+} for small enough $t > 0$. Yet,

$$\frac{\mathrm{d}}{\mathrm{d}t} f_{\beta,h}(\boldsymbol{x}^t) = g_\beta(x_i + t) - g_\beta(x_j - t) + h\left[\mathbf{1}_{\{i=1\}} - \mathbf{1}_{\{j=1\}}\right]$$

goes to $-\infty$ as $t \to 0^+$, a contradiction.

Remains the strict inequality in (4.4). We let $\boldsymbol{x}^\beta = (1 - (q-1)/\beta, 1/\beta, \dots, 1/\beta)$ (which is in Ω for $\beta \geqslant q - 1$, and satisfies the case of equality in (4.4) for $\beta \geqslant q$) and derive conditions for $\boldsymbol{x}_\beta$ being a minimizer of the free energy. Equation (4.2) for $i = 1, j = 2$ gives

$$h = \log(\beta x_1) - \beta x_1 + 1$$

which is negative unless x_1 also equals $1/\beta$. Yet, $x_1 = 1/\beta$ implies $\beta = q$ and $h = 0$. But $\boldsymbol{x} = (1/q, \dots, 1/q)$ is a minimizer of the free energy $f_{\beta,0}$ only for $\beta \leqslant \beta_c$. As $q > 2 \Rightarrow \beta_c < q$, the only case of equality is $q = \beta = 2$ and $h = 0$. □

In a second Lemma we compare the counting measure on Ω^n with the Lebesgue measure. This will help in the proofs of Theorems 3.1, 3.4, 3.5 and 3.7. We denote by $\mathcal{L}$ the Lebesgue measure on hyperplanes.

Lemma 4.3 *Let $\Pi : \mathbb{R}^q \to \mathbb{R}^q$ be a affine and one-to-one transformation. Let $P = [0,1]^d \cap \mathcal{H}$. Then, for any $f : \mathbb{R}^q \to \mathbb{R}$ bounded,*

$$\sum_{\boldsymbol{X} \in \Omega^n} f(\boldsymbol{X}) \leqslant \frac{1}{\mathcal{L}(\Pi(n^{-1}P))} \int_{\Pi(\Omega + n^{-1}P)} \varphi \mathrm{d}\mathcal{L}$$

where

$$\varphi(\boldsymbol{z}) = \sup_{\Pi^{-1}(\boldsymbol{z}) - n^{-1}P} f, \qquad \forall \boldsymbol{z} \in \mathbb{R}^q.$$

Remark 4.4 *Applying this to $-f$ one obtains a useful lower bound.*

Proof For any $\boldsymbol{z} \in \Pi\left(\boldsymbol{x} + n^{-1}P\right)$, one has $\Pi^{-1}(\boldsymbol{z}) \in \boldsymbol{x} + n^{-1}P$ hence $\boldsymbol{x} \in \Pi^{-1}(\boldsymbol{z}) - n^{-1}P$. Thus $\varphi(\boldsymbol{z}) \geqslant f(\boldsymbol{x})$, and

$$f(\boldsymbol{x}) \quad \leqslant \quad \frac{1}{\mathcal{L}(\Pi(n^{-1}P))} \int_{\Pi(\boldsymbol{x}+n^{-1}P)} \varphi d\mathcal{L}.$$

The claim follows when we sum over $\boldsymbol{x} \in \Omega^n$, as $\Omega^n + n^{-1}P = \Omega + n^{-1}P$. □

4.2 Gaussian fluctuations

The limit theorems will be proved as consequences of a Taylor expansion of the free energy. First we consider a second order expansion of $f_{\beta,h}$, that will be enough to describe the Gaussian fluctuations at $(\beta, h) \neq (\beta_0, h_0)$.

This section is organized as follows. First we give a series of Lemmas that permit to establish Proposition 4.8 below. Then we give the proofs of Theorems 3.1, 3.4 and 3.5.

4.2.1 Taylor expansion of the free energy

The Taylor-Lagrange formula applied to the $\mathcal{C}^\infty$ function $t \in [0,1] \mapsto f_{\beta,h}(\boldsymbol{x} + t\boldsymbol{w})$ yields:

Lemma 4.5 *Let $\boldsymbol{x} \in \Omega$ be a global minimizer of $f_{\beta,h}$ and $\boldsymbol{w} \in \mathcal{H}$ such that $\boldsymbol{x} + \boldsymbol{w} \in \Omega_{0^+}$. Then, there exists $\alpha \in (0,1)$ such that*

$$f_{\beta,h}(\boldsymbol{x} + \boldsymbol{w}) \quad = \quad f_{\beta,h}(\boldsymbol{x}) + \frac{1}{2} \sum_{i=1}^{q} \left(\frac{1}{x_i + \alpha w_i} - \beta \right) w_i^2. \tag{4.5}$$

On the other hand, the influence of β_n and h_n is immediate to characterize:

Lemma 4.6 *For any β, β_n, h, h_n and any $\boldsymbol{x} \in \Omega$, the following equality holds:*

$$f_{\beta_n,h_n}(\boldsymbol{x}) \quad = \quad f_{\beta,h}(\boldsymbol{x}) - \frac{\beta_n - \beta}{2} \sum_{i=1}^{q} x_i^2 - (h_n - h)x_1. \tag{4.6}$$

4.2.2 The quadratic form

Given $\boldsymbol{x} \in \Omega_{0^+}$ and $\beta \geqslant 0$ we consider the quadratic form $Q_{\boldsymbol{x},\beta} : \mathcal{H} \mapsto \mathbb{R}$ defined by

$$Q_{\boldsymbol{x},\beta}(\boldsymbol{w}) \quad = \quad \sum_{i=1}^{q} \left(\frac{1}{x_i} - \beta \right) w_i^2. \tag{4.7}$$

This is the quadratic form that appears in Lemma 4.5. When it is positive definite it determines the fluctuations. We have:

Lemma 4.7 *Let $\beta, h \geqslant 0$ and $\boldsymbol{x}$ be some global minimizer of $f_{\beta,h}$.*

i. The quadratic form $Q_{\boldsymbol{x},\beta}$ is positive definite on $\mathcal{H}$ if and only $(\beta, h) \neq (\beta_0, h_0)$.

ii. When $(\beta, h) = (\beta_0, h_0)$ and $\boldsymbol{x} = \boldsymbol{x}_0$, the kernel of $Q_{\boldsymbol{x},\beta}$ is $\mathrm{Vect}(\boldsymbol{u})$ where

$$\boldsymbol{u} = (1 - q, 1, \ldots, 1).$$

Proof First we assume that $(\beta, h) \neq (\beta_0, h_0)$ and prove that $Q_{\boldsymbol{x},\beta}$ is positive definite. According to Proposition 2.2 the vector $\boldsymbol{x}$ has one coordinate repeated at least $q-1$ times. Let $j \in \{1, \ldots, q\}$ be the smallest index such that $x_j = \max(x_i)$, and $J = \{1, \ldots, q\} \setminus \{j\}$. For any $\boldsymbol{w} \in \mathcal{H}$ one has $w_j = -\sum_{i \in J} w_i$, hence

$$Q_{\boldsymbol{x},\beta}(\boldsymbol{w}) \quad = \quad \left(\frac{1}{\max(x_i)} - \beta \right) \left(\sum_{i \in J} w_i \right)^2 + \left(\frac{1}{\min(x_i)} - \beta \right) \sum_{i \in J} w_i^2.$$

Now we let

$$\alpha_j(\boldsymbol{w}) \quad = \quad \frac{1}{q-1}\frac{\left(\sum_{i\in I} w_i\right)^2}{\sum_{i\in J} w_i^2},$$

which belongs to the interval $[0,1]$ according to Cauchy-Schwarz inequality, and obtain

$$Q_{\boldsymbol{x}}(\boldsymbol{w}) \quad = \quad \left[\left(\frac{1}{\max(x_i)} - \beta\right)(q-1)\alpha_j(\boldsymbol{w}) + \left(\frac{1}{\min(x_i)} - \beta\right)\right]\sum_{i\in J} w_i^2. \tag{4.8}$$

Hence the quadratic form $Q_{\boldsymbol{x}}$ is positive definite on $\mathcal{H}$ if and only if the factor in (4.8) is strictly positive at both $\alpha = 0$ and $\alpha = 1$, that is to say if

$$\frac{1}{\min(x_i)} - \beta \quad > \quad 0 \tag{4.9}$$

$$\text{and} \quad \frac{q-1}{\max(x_i)} + \frac{1}{\min(x_i)} - q\beta \quad > \quad 0. \tag{4.10}$$

Condition (4.9) is true as $(\beta, h) \neq (\beta_0, h_0)$, cf. Proposition 2.2. Condition (4.10) is equivalent to

$$1 - q\beta\min(x_i)\max(x_i) \quad > \quad 0$$

as $(q-1)\min(x_i) + \max(x_i) = 1$. The reader will remark that $\min(x_i)\max(x_i)$ is constant over all the minimizers of $f_{\beta,h}$ described in Theorem 2.3. Hence we might take the minimizer of the form $\boldsymbol{x} = \boldsymbol{x}_z$ as in (2.5), that is $x_1 = (1+z)/2$ and $x_2 = \cdots = x_q = (1-z)/(q-1)$, which reveals that condition (4.10) is equivalent to $z \mapsto f_{\beta,h}(\boldsymbol{x}_z)$ having a positive second derivative at its minima, as

$$\frac{\mathrm{d}^2 f_{\beta,h}(\boldsymbol{x}_z)}{\mathrm{d}z^2} = \frac{1}{1-z^2} - \frac{\beta q}{4(q-1)},$$

which is the case again as $(\beta, h) \neq (\beta_0, h_0)$ (see the discussion after (2.6)).

Assume now that $(\beta, h) = (\beta_0, h_0)$. If $q = 2$, the quadratic form $Q_{\boldsymbol{x},\beta}$ is identically zero on $\mathcal{H} = \mathrm{Vect}(\boldsymbol{u})$. If $q \geqslant 3$, we have $h = h_0 > 0$ hence $j = 1$. The quadratic form vanishes at $\boldsymbol{w} \in \mathcal{H}$ if and only if $\alpha_1(\boldsymbol{w}) = 1$. In view of the definition of α, this is the case of equality in the Cauchy-Schwarz inequality: $\alpha_1(\boldsymbol{w}) = 1 \Leftrightarrow w_2 = \cdots = w_q \Leftrightarrow \boldsymbol{w} \in \mathrm{Vect}(\boldsymbol{u})$. □

4.2.3 Centering of the fluctuations

As in Theorems 3.1 and 3.4 we let $\boldsymbol{d}_n$ the smallest $\boldsymbol{d} \in \mathcal{H}$ such that $\boldsymbol{x} + \boldsymbol{d} \in \Omega$ is a global minimizer of f_{β_n,h_n}. We have:

Proposition 4.8 *Assume that $(\beta_n, h_n) \to (\beta, h)$ and let $\boldsymbol{x} \in \Omega$ be a global minimizer of $f_{\beta,h}$.*

i. For any $R > 0$,

$$nf_{\beta_n,h_n}\left(\boldsymbol{x} + \boldsymbol{d}_n + n^{-1/2}\boldsymbol{w}\right) \quad = \quad nf_{\beta_n,h_n}(\boldsymbol{x} + \boldsymbol{d}_n) + \frac{1}{2}\boldsymbol{Q}_{\boldsymbol{x},\beta}(\boldsymbol{w}) + o_n(1) \tag{4.11}$$

uniformly over $\boldsymbol{w} \in \mathcal{H} \cap B(0,R)$.

ii. If $(\beta, h) \neq (\beta_0, h_0)$, for small enough $\varepsilon > 0$ there is $\lambda > 0$ such that, for n large enough and any $\boldsymbol{w} \in \mathcal{H}$ with $\|\boldsymbol{w}\| \leqslant \varepsilon n^{1/2}$,

$$nf_{\beta_n,h_n}\left(\boldsymbol{x} + \boldsymbol{d}_n + n^{-1/2}\boldsymbol{w}\right) \quad \geqslant \quad nf_{\beta_n,h_n}(\boldsymbol{x} + \boldsymbol{d}_n) + \lambda\|\boldsymbol{w}\|^2. \tag{4.12}$$

Proof We begin with an application of Lemma 4.5 at the global minimum point $\boldsymbol{x} + \boldsymbol{d}_n$:

$$nf_{\beta_n,h_n}\left(\boldsymbol{x} + \boldsymbol{d}_n + n^{-1/2}\boldsymbol{w}\right) \quad = \quad nf_{\beta_n,h_n}(\boldsymbol{x} + \boldsymbol{d}_n) + \frac{1}{2}\boldsymbol{Q}_{\boldsymbol{x}+\boldsymbol{d}_n+\alpha n^{-1/2}\boldsymbol{w},\beta}(\boldsymbol{w})$$

for some $\alpha \in (0,1)$ depending on n and $\boldsymbol{w}$. For (4.11) we only have to notice that $\boldsymbol{d}_n + \alpha n^{-1/2}\boldsymbol{w} = o(1)$. For (4.12) we remark that as $\boldsymbol{d}_n \to 0$ (cf. Remark 3.2), for all n large enough and $\|\boldsymbol{w}\| \leqslant \varepsilon n^{1/2}$,

$$\left\|\boldsymbol{d}_n + \alpha n^{-1/2}\boldsymbol{w}\right\| \quad \leqslant \quad 2\varepsilon.$$

As this can be made arbitrary small, for small enough ε the quadratic form $\boldsymbol{Q}_{\boldsymbol{x}+\boldsymbol{d}_n+\alpha n^{-1/2}\boldsymbol{w},\beta}$ dominates $\boldsymbol{Q}_{x,\beta}/2$, which is definite positive after Lemma 4.7. □

4.2.4 Some linear algebra

The next Lemma will be useful at the time of computing inverses or determinants. Denote by I_n the $n \times n$ unitary matrix and by A_n the $n \times n$ matrix with all entries equal to 1.

Lemma 4.9 *Let $M = aA_n + bI_n$.*

i. The determinant of M is

$$\det(M) = b^{n-1}(b + na).$$

ii. When M is invertible, it has

$$M^{-1} = \begin{cases} (a+b)^{-1} & \text{if } n = 1 \\ \frac{1}{b}\left(I_n - \frac{a}{na+b}A_n\right) & \text{if } n \geqslant 2. \end{cases}$$

Proof We prove the first point as follows: let $P(\lambda) = \det(\lambda I_n - A_n)$ be the characteristic polynomial for the matrix A_n. The matrix A_n has rank 1 and eigenvalues $0, \ldots, 0, n$, which are the roots of the unitary polynomial P, thus $P(\lambda) = \lambda^{n-1}(\lambda - n)$. The second point follows from an immediate computation. □

4.2.5 Proof of Theorems 3.1, 3.4 and 3.5

As a consequence of Proposition 4.8 we give the proof of Theorems 3.1, 3.4 and 3.5.
Proof (Theorem 3.1). First we condition $\boldsymbol{W} = n^{-1/2}(\boldsymbol{N} - n\boldsymbol{x} - n\boldsymbol{d}_n)$ on $\|\boldsymbol{W}\| < R$ for some positive R. For $g : \mathbb{R}^q \mapsto \mathbb{R}$ continuous bounded, Lemma 4.1 and (4.11) in Proposition 4.8 yield

$$\begin{aligned} Z_{\beta_n,h_n,n}\mu_{\beta_n,h_n,n}\left(g(\boldsymbol{W})\mathbf{1}_{\{\|\boldsymbol{W}\| \leqslant R\}}\right) &= (1 + o_n(1))n^{-\frac{q-1}{2}}A_\beta(\boldsymbol{x})e^{-nf_{\beta_n,h_n}(\boldsymbol{x}+\boldsymbol{d}_n)} \\ &\quad \times \sum_{\boldsymbol{N}/n \in \Omega^n} g(\boldsymbol{W})\mathbf{1}_{\{\|\boldsymbol{W}\| \leqslant R\}}e^{-\frac{1}{2}\boldsymbol{Q}_{\boldsymbol{x},\beta}(\boldsymbol{W})}. \end{aligned}$$

The transformation $\boldsymbol{X} = \boldsymbol{N}/n \mapsto \Pi(\boldsymbol{X}) = \boldsymbol{W}$ is affine. The image of Ω by Π is greater than $\mathcal{H} \cap B(0, R)$ for large enough n as $\boldsymbol{x} \in \Omega_{0+}$, and on the other hand $\Pi(\boldsymbol{X} + n^{-1}P) = \Pi(\boldsymbol{X}) + n^{-1/2}P$, that is to say the dimensions of the image of the lattice element P go to zero. Hence Lemma 4.3 gives

$$\begin{aligned} Z_{\beta_n,h_n,n}\mu_{\beta_n,h_n,n}\left(g(\boldsymbol{W})\mathbf{1}_{\{\|\boldsymbol{W}\| \leqslant R\}}\right) &= (1 + o_n(1))A_\beta(\boldsymbol{x})e^{-nf_{\beta_n,h_n}(\boldsymbol{x}+\boldsymbol{d}_n)} \times \\ &\quad \frac{1}{\mathcal{L}(P)}\int_{\mathcal{H} \cap B(0,R)} g(\boldsymbol{w})e^{-\frac{1}{2}\boldsymbol{Q}_{\boldsymbol{x},\beta}(\boldsymbol{w})}\mathrm{d}\mathcal{L}(\boldsymbol{w}). \end{aligned} \tag{4.13}$$

In other words the law of $\boldsymbol{W}$ conditioned on $\|\boldsymbol{W}\| \leqslant R$ converges to the distribution on $\mathcal{H} \cap B(0, R)$ with density proportional to

$$\boldsymbol{w} \mapsto e^{-\frac{1}{2}Q_{\boldsymbol{x},\beta}(\boldsymbol{w})} \tag{4.14}$$

with respect to the Lebesgue measure on $\mathcal{H} \cap B(0, R)$.

Now we show that the variable $\boldsymbol{W}$ is tight. Thanks to Theorem 2.1 we know already that for any $\varepsilon > 0$,

$$\limsup_n \mu_{\beta_n,h_n,n}(\|\boldsymbol{N} - n\boldsymbol{x}\| \geqslant \varepsilon n) = 0.$$

Thus it is enough to show that, for small enough $\varepsilon > 0$,

$$\lim_{\kappa \to \infty} \limsup_n \mu_{\beta_n,h_n,n}\left(\|\boldsymbol{W}\| \geqslant \kappa \middle| \|\boldsymbol{W}\| \leqslant \varepsilon n^{1/2}\right) = 0. \tag{4.15}$$

According to Lemma 4.1 and Proposition 4.8, for small enough $\varepsilon > 0$ and large enough κ there is $\lambda > 0$ such that

$$\begin{aligned} \mu_{\beta_n,h_n,n}\left(\|\boldsymbol{W}\| \geqslant \kappa \middle| \|\boldsymbol{W}\| \leqslant \varepsilon n^{1/2}\right) &\leqslant \frac{\mu_{\beta_n,h_n,n}\left(\kappa \leqslant \|\boldsymbol{W}\| \leqslant \varepsilon n^{1/2}\right)}{\mu_{\beta_n,h_n,n}(\|\boldsymbol{W}\| \leqslant \kappa)} \\ &\leqslant (1 + o_n(1))\frac{\sum_{\boldsymbol{N}/n \in \Omega^n : \kappa \leqslant \|\boldsymbol{W}\| \leqslant \varepsilon n^{1/2}} e^{-\lambda\|\boldsymbol{W}\|^2}}{\sum_{\boldsymbol{N}/n \in \Omega^n : \|\boldsymbol{W}\| \leqslant \kappa} e^{-\frac{1}{2}\boldsymbol{Q}_{\boldsymbol{x},\beta}(\boldsymbol{W})}} \\ &\leqslant (1 + o_n(1))\frac{\int_{\mathcal{H} \setminus B(0,\kappa)} e^{-\lambda\|\boldsymbol{w}\|^2}\mathrm{d}\mathcal{L}(\boldsymbol{w})}{\int_{\mathcal{H} \cap B(0,\kappa)} e^{-\frac{1}{2}\boldsymbol{Q}_{x,\beta}(\boldsymbol{w})}\mathrm{d}\mathcal{L}(\boldsymbol{w})}. \end{aligned}$$

after Lemma 4.3. Since $\boldsymbol{Q}_{\boldsymbol{x},\beta}$ is positive definite the ratio goes to 0 as $\kappa \to \infty$, giving (4.15).

Let us show that this limit distribution is as well the distribution of the centered Gaussian vector with covariance matrix (3.3). We take $\boldsymbol{V}$ a random vector in $\mathcal{H}$ with the density (4.14) with respect to the Lebesgue measure on $\mathcal{H}$. The law of $\boldsymbol{V}$ is also proportional to

$$e^{-\frac{1}{2}Q_{\boldsymbol{x},\beta}(\boldsymbol{v})}\mathrm{d}v_2 \cdots \mathrm{d}v_q.$$

This density can be expressed only in terms of the truncated vector $\tilde{\boldsymbol{V}} = (V_2, \ldots, V_q)$. Indeed, if we take

$$H = \left(\frac{1}{x_1} - \beta\right) A_{q-1} + \left(\frac{1}{x_q} - \beta\right) I_{q-1} \tag{4.16}$$

we have $Q_{\boldsymbol{x},\beta}(\boldsymbol{v}) = {}^t\tilde{\boldsymbol{v}} H \tilde{\boldsymbol{v}}$ and thus the covariance matrix of $\tilde{\boldsymbol{V}}$ is

$$H^{-1} = \left(\frac{1}{x_1} - \beta\right)^{-1} \times \left(I_{q-1} - \frac{\frac{1}{x_1} - \beta}{\frac{1}{x_1 x_q} - q\beta} A_{q-1}\right)$$

according to Lemma 4.9. Using the relation $V_1 = -\sum_{i=2}^q V_i$ we compute the remaining covariance coefficients, leading to the completed matrix (3.3). The rank of the matrix is not less that that of H, that is $q-1$, and it is also strictly less than q because of the linear constraint $\boldsymbol{V} \in \mathcal{H}$. □

Proof (Theorem 3.4). The former proof can be repeated almost verbatim. One has to take care however that $\boldsymbol{x}$ needs not be of the particular form $x_2 = \cdots = x_q$ (although it still has a coordinate repeated $q-1$ times), and that the variable which is tight is $\boldsymbol{W}$ *conditioned* on $\boldsymbol{N}/n \in B(\boldsymbol{x}, \varepsilon)$. □

Proof (Theorem 3.5). The tightness of $\boldsymbol{W}$ conditioned on $\boldsymbol{N}/n \in B(\boldsymbol{x}, \varepsilon)$ and the convergence of the law of $\boldsymbol{W}$ on bounded sets (cf. the proof of Theorem 3.1) imply that for any $\varepsilon > 0$ smaller than the distance between any two minimizers of $f_{\beta,h}$,

$$\frac{\mu_{\beta_n,h_n,n}\left(\frac{\boldsymbol{N}}{n} \in B(\boldsymbol{x}, \varepsilon)\right)}{\mu_{\beta_n,h_n,n}\left(\frac{\boldsymbol{N}}{n} \in B(\boldsymbol{x}', \varepsilon)\right)} = (1 + o_n(1)) \frac{A_\beta(\boldsymbol{x}) e^{-n f_{\beta_n,h_n}(\boldsymbol{x} + \boldsymbol{d}_n(\boldsymbol{x}))} \int_{\mathcal{H}} e^{-\frac{1}{2}\boldsymbol{Q}_{\boldsymbol{x},\beta}(\boldsymbol{w})} \mathrm{d}\mathcal{L}(\boldsymbol{w})}{A_\beta(\boldsymbol{x}') e^{-n f_{\beta_n,h_n}(\boldsymbol{x}' + \boldsymbol{d}_n(\boldsymbol{x}'))} \int_{\mathcal{H}} e^{-\frac{1}{2}\boldsymbol{Q}_{\boldsymbol{x}',\beta}(\boldsymbol{w})} \mathrm{d}\mathcal{L}(\boldsymbol{w})}.$$

Hence we call

$$C_{\beta_n,h_n,n}(\boldsymbol{x}) = A_\beta(\boldsymbol{x}) e^{-n f_{\beta_n,h_n}(\boldsymbol{x} + \boldsymbol{d}_n(\boldsymbol{x}))} \int_{\mathcal{H}} e^{-\frac{1}{2}\boldsymbol{Q}_{\boldsymbol{x},\beta}(\boldsymbol{w})} \mathrm{d}\mathcal{L}(\boldsymbol{w})$$

and give an equivalent to $C_{\beta_n,h_n,n}(\boldsymbol{x})$. First we compute the integral up to a constant factor. We pick $j \in \{1, \ldots, q\}$ such that $x_j = \max(x_i)$ and let $J = \{1, \ldots, q\} \setminus \{j\}$. The Lebesgue measure on $\mathcal{H}$ is proportional to the measure induced on $\boldsymbol{w}$ by $\prod_{i \in J} \mathrm{d}w_i$, given $w_j = -\sum_{i \in J} w_i$. As in (4.16) we let

$$H_{\boldsymbol{x},\beta} = \left(\frac{1}{\max(x_i)} - \beta\right) A_{q-1} + \left(\frac{1}{\min(x_i)} - \beta\right) I_{q-1}$$

and $\tilde{\boldsymbol{w}} = (w_i)_{i \in J}$, thus ${}^t\tilde{\boldsymbol{w}} H_{\boldsymbol{x},\beta} \tilde{\boldsymbol{w}} = Q_{\boldsymbol{x},\beta}(\boldsymbol{w})$ and therefore

$$\begin{aligned}
\int_{\mathcal{H}} e^{-\frac{1}{2}\boldsymbol{Q}_{\boldsymbol{x},\beta}(\boldsymbol{w})} \prod_{i \in J} \mathrm{d}w_i &= \sqrt{2\pi}^{q-1} \sqrt{\det H_{\boldsymbol{x},\beta}^{-1}} \\
&= \sqrt{2\pi}^{q-1} \left[\left(\frac{1}{\min(x_i)} - \beta\right)^{q-2} \left(\frac{1}{\min(x_i)} + \frac{q-1}{\max(x_i)} - q\beta\right)\right]^{-1/2}
\end{aligned}$$

according to Lemma 4.9. If we multiply with the prefactor $A_\beta(\boldsymbol{x})$ we obtain

$$A_\beta(\boldsymbol{x}) \int_{\mathcal{H}} e^{-\frac{1}{2}\boldsymbol{Q}_{\boldsymbol{x},\beta}(\boldsymbol{w})} \prod_{i \in J} \mathrm{d}w_i = e^{-\frac{\beta}{2}} (1 - \beta \min(x_i))^{\frac{2-q}{2}} (1 - q\beta \max(x_i) \min(x_i))^{-1/2} \tag{4.17}$$

as $\max(x_i) + (q-1)\min(x_i) = 1$. Then we use Lemma 4.6:

$$\begin{aligned}
n f_{\beta_n,h_n}(\boldsymbol{x} + \boldsymbol{d}_n(\boldsymbol{x})) &= n f_{\beta,h}(\boldsymbol{x} + \boldsymbol{d}_n(\boldsymbol{x})) - n \frac{\beta_n - \beta}{2} \sum_{i=1}^q (x_i + d_{n,i}(\boldsymbol{x}))^2 - n(h_n - h)(x_1 + d_1(\boldsymbol{x})) \\
&= n f_{\beta,h}(\boldsymbol{x}) - \frac{\lambda}{2} \sum_{i=1}^q x_i^2 - \nu x_1 + o_n(1)
\end{aligned}$$

as $\boldsymbol{d}_n(\boldsymbol{x}) = O(1/n)$ and $\boldsymbol{x}$ is a global minimizer of $f_{\beta,h}$. Thus we have shown that

$$\begin{aligned} C_{\beta_n,h_n,n}(\boldsymbol{x}) &= (1+o_n(1))\frac{\mathrm{d}\mathcal{L}}{\prod_{i\in J}\mathrm{d}w_i}\exp\left(-nf_{\beta,h}(\boldsymbol{x})+\frac{\lambda}{2}\sum_{i=1}^q x_i^2+\nu x_1-\frac{\beta}{2}\right)\\ &\quad\times(1-\beta\min(x_i))^{\frac{2-q}{2}}\left(1-q\beta\max(x_i)\min(x_i)\right)^{-1/2}\end{aligned}$$

where the factor $\mathrm{d}\mathcal{L}/\prod_{i\in J}\mathrm{d}w_i$ does not depend on J. The claim follows from the remark that the product $\min(x_i)\max(x_i)$ is constant over all the global minimizers $\boldsymbol{x},\boldsymbol{x}',\dots$ of the free energy $f_{\beta,h}$ at any $(\beta,h)\in h_T$, cf. Theorem 2.3. □

4.3 Limit theorems at criticality

The proof of Theorem 3.7 relies again on a Taylor expansion of the free energy:

Lemma 4.10 *Let $(\beta,h)=(\beta_0,h_0)$ and $\boldsymbol{x}=\boldsymbol{x}_0=(1/2,1/2(q-1),\dots,1/2(q-1))$ be the unique minimizer of $f_{\beta,h}$. Let $\boldsymbol{u}=(1-q,1,\dots,1)$. For all $t\in\mathbb{R}$ and $\boldsymbol{v}\in\mathcal{H}\cap\boldsymbol{u}^\perp$ such that $\boldsymbol{x}+t\boldsymbol{u}+\boldsymbol{v}\in\Omega_{0+}$, there are $\alpha,\alpha'\in(0,1)$ such that*

$$f_{\beta,h}(\boldsymbol{x}+t\boldsymbol{u}+\boldsymbol{v}) = f_{\beta,h}(\boldsymbol{x})+\frac{1}{2}Q_{\boldsymbol{x}+t\boldsymbol{u}+\alpha\boldsymbol{v},\beta}(\boldsymbol{v})+\frac{t^4}{12}\sum_{i=1}^q\frac{u_i^4}{(x_i+\alpha' tu_i)^3}.$$

Furthermore,

$$\begin{aligned}\frac{1}{2}Q_{\boldsymbol{x},\beta}(\boldsymbol{v}) &= \frac{(q-1)(q-2)}{q}\|\boldsymbol{v}\|^2\\ \textit{and}\quad \frac{1}{12}\sum_{i=1}^q\frac{u_i^4}{x_i^3} &= \frac{4}{3}(q-1)^3\end{aligned}$$

Proof A second-order Taylor expansion in $\boldsymbol{v}$ yields

$$f_{\beta,h}(\boldsymbol{x}+t\boldsymbol{u}+\boldsymbol{v}) = f_{\beta,h}(\boldsymbol{x}+t\boldsymbol{u})+\nabla f_{\beta,h}(\boldsymbol{x}+t\boldsymbol{u})\cdot\boldsymbol{v}+\frac{1}{2}Q_{\boldsymbol{x}+t\boldsymbol{u}+\alpha\boldsymbol{v},\beta}(\boldsymbol{v})$$

for some $\alpha\in(0,1)$. The last $q-1$ coordinates of the gradient $\nabla f_{\beta,h}(\boldsymbol{x}+t\boldsymbol{u})$ are equal, hence it is orthogonal to $\boldsymbol{v}$. Then a fourth order expansion in t gives

$$f_{\beta,h}(\boldsymbol{x}+t\boldsymbol{u}) = f_{\beta,h}(\boldsymbol{x})+\frac{t^4}{12}\sum_{i=1}^q\frac{u_i^4}{(x_i+\alpha' tu_i)^3} \qquad (4.18)$$

for some $\alpha'\in(0,1)$. Indeed, the first order term is zero as $\boldsymbol{x}$ is the global minimizer of $f_{\beta,h}$. The second order term is $Q_{\boldsymbol{x},\beta}(t\boldsymbol{u})/2=0$ in view of Lemma 4.7. Hence the third order term is 0, yielding (4.18).

Let us prove the last two formulas. The assumption $\boldsymbol{v}\in\mathcal{H}\cap\boldsymbol{u}^\perp$ implies $v_1=0$, hence

$$\begin{aligned}\frac{1}{2}Q_{\boldsymbol{x},\beta}(\boldsymbol{v}) &= \frac{1}{2}\sum_{i=2}^q\left(\frac{1}{x_i}-\beta\right)v_i^2\\ &= \frac{1}{2}\sum_{i=2}^q\left(2(q-1)-4\frac{q-1}{q}\right)v_i^2\\ &= \frac{(q-1)(q-2)}{q}\sum_{i=1}^q v_i^2.\end{aligned}$$

On the other hand:

$$\sum_{i=1}^q\frac{u_i^4}{x_i^3} = 8(q-1)^4+(q-1)8(q-1)^3 = 16(q-1)^4.$$

□

Using Lemma 4.10 we establish the analog of Proposition 4.8:

Proposition 4.11 *Assume that $(\beta_n, h_n) \to (\beta_0, h_0)$ with $\beta_n - \beta_0 = o(n^{-3/4})$ and $h_n - h_0 = o(n^{-3/4})$, and let $\boldsymbol{x}_0 \in \Omega$ be the unique global minimizer of f_{β_0,h_0}.*

i. For any $R > 0$,

$$nf_{\beta_n,h_n}\left(\boldsymbol{x} + n^{-1/4}t\boldsymbol{u} + n^{-1/2}\boldsymbol{v}\right) = nf_{\beta,h}(\boldsymbol{x}) - n\frac{\beta_n - \beta_0}{2}\|\boldsymbol{x}\|^2 - n(h_n - h_0)x_1 + \frac{(q-1)(q-2)}{q}\|\boldsymbol{v}\|^2 + \frac{4}{3}(q-1)^3t^4 + o_n(1) \quad (4.19)$$

uniformly over $\boldsymbol{v} \in \mathcal{H} \cap \boldsymbol{u}^\perp \cap B(0,R)$ and $t \in [-R, R]$.

ii. For small enough $\varepsilon > 0$ and large enough R, for n large enough, for any $\boldsymbol{v} \in \mathcal{H} \cap \boldsymbol{u}^\perp$, $t \in \mathbb{R} \setminus [-R, R]$ such that $\|n^{-1/4}t\boldsymbol{u} + n^{-1/2}\boldsymbol{v}\| \leqslant \varepsilon$,

$$nf_{\beta_n,h_n}\left(\boldsymbol{x} + n^{-1/4}t\boldsymbol{u} + n^{-1/2}\boldsymbol{v}\right) \geqslant nf_{\beta,h}(\boldsymbol{x}) - n\frac{\beta_n - \beta_0}{2}\|\boldsymbol{x}\|^2 - n(h_n - h_0)x_1 + \|\boldsymbol{v}\|^2/4 + t^4/2 \quad (4.20)$$

Proof We first apply Lemma 4.6: for $\boldsymbol{w} = n^{-1/4}t\boldsymbol{u} + n^{-1/2}\boldsymbol{v}$, we have

$$nf_{\beta_n,h_n}(\boldsymbol{x} + \boldsymbol{w}) = nf_{\beta,h}(\boldsymbol{x} + \boldsymbol{w}) - n\frac{\beta_n - \beta_0}{2}\|\boldsymbol{x}\|^2 - n(h_n - h_0)x_1 + o_n(1)$$

uniformly over $\boldsymbol{v} \in \mathcal{H} \cap \boldsymbol{u}^\perp \cap B(0,R)$ and $t \in [-R, R]$ as $\beta_n - \beta_0 = o(n^{-3/4})$ and $h_n - h_0 = o(n^{-3/4})$. Then, Lemma 4.10 yields

$$nf_{\beta,h}(\boldsymbol{x} + \boldsymbol{w}) = nf_{\beta,h}(\boldsymbol{x}) + \frac{(q-1)(q-2)}{q}\|\boldsymbol{v}\|^2 + \frac{4}{3}(q-1)^3t^4 + o_n(1)$$

uniformly over the same domain, and (4.19) follows.

Now we only assume that $\|\boldsymbol{w}\| \leqslant \varepsilon$. For small enough $\varepsilon > 0$ we have, after Lemma 4.10, the lower bound

$$nf_{\beta,h}(\boldsymbol{x} + \boldsymbol{w}) \geqslant nf_{\beta,h}(\boldsymbol{x}) + \frac{1}{2}\|\boldsymbol{v}\|^2 + t^4 \quad (4.21)$$

(note that, for $q = 2$, $\boldsymbol{v}$ is necessarily 0). Combining with Lemma 4.6 we obtain that, whenever (4.21) holds,

$$nf_{\beta_n,h_n}(\boldsymbol{x} + \boldsymbol{w}) \geqslant c_n(\boldsymbol{x}) + \|\boldsymbol{v}\|^2\left(\frac{1}{2} - \frac{\beta_n - \beta_0}{2}\right) - l_nt - m_nt^2 + t^4$$

where

$$\begin{aligned} c_n(\boldsymbol{x}) &= nf_{\beta,h}(\boldsymbol{x}) - n\frac{\beta_n - \beta_0}{2}\|\boldsymbol{x}\|^2 - n(h_n - h_0)x_1, \\ l_n &= n^{3/4}(\beta_n - \beta_0)\boldsymbol{x} \cdot \boldsymbol{u} - n^{3/4}(h_n - h_0)(q-1) \\ m_n &= n^{1/2}\frac{\beta_n - \beta_0}{2}\|\boldsymbol{u}\|^2 \end{aligned}$$

as $\boldsymbol{x}, \boldsymbol{u} \perp \boldsymbol{v}$ and $v_1 = 0$. Now we conclude: for any large n,

$$\|\boldsymbol{v}\|^2\left(\frac{1}{2} - \frac{\beta_n - \beta_0}{2}\right) \geqslant \frac{\|\boldsymbol{v}\|^2}{4}.$$

Similarly, as $l_n = o_n(1)$ and $m_n = o(n^{-1/4}) = o_n(1)$, for any large n and t large,

$$-l_nt - m_nt^2 + t^4 \geqslant \frac{t^4}{2}.$$

□

Finally we give the proof of Theorem 3.7:

Proof (Theorem 3.7). Here we define Π as the affine transformation such that

$$\Pi\left(\boldsymbol{x}+n^{-1/4}T\boldsymbol{u}+n^{-1/2}\boldsymbol{V}+\boldsymbol{z}\right)=T\boldsymbol{u}+\boldsymbol{V}+\boldsymbol{z}$$

for any $T\in\mathbb{R},\boldsymbol{V}\in\mathcal{H}\cap\boldsymbol{u}^\perp$ and $\boldsymbol{z}\in\mathcal{H}^\perp$. It is a consequence of Lemmas 4.1, 4.3 and (4.19) in Proposition 4.11 that, conditionally on $\boldsymbol{Z}=T\boldsymbol{u}+\boldsymbol{V}\in B(0,R)$, the variable $\boldsymbol{Z}$ converges in law towards the probability measure on $\mathcal{H}\cap B(0,R)$ with density proportional to

$$e^{-\frac{(q-1)(q-2)}{q}\|\boldsymbol{v}\|^2-\frac{4(q-1)^4}{3}t^4} \tag{4.22}$$

with respect to the Lebesgue measure on $\mathcal{H}\cap B(0,R)$. Here again, the variable $\boldsymbol{Z}$ is tight thanks to Lemmas 4.1, 4.3 and Proposition 4.11.

The probability measure on $t\boldsymbol{u}+\boldsymbol{v}\in\mathcal{H}$, $\boldsymbol{v}\perp\boldsymbol{u}$ with density (4.22) has a simple structure. It is clear that T and $\boldsymbol{V}$ are independent. The vector $\boldsymbol{V}$ is determined by

$$\tilde{\boldsymbol{V}}=(V_3,\dots,V_q)$$

which has a density proportional to

$$e^{-\frac{(q-1)(q-2)}{q}\|\tilde{\boldsymbol{v}}\|^2}\mathrm{d}v_3\cdots\mathrm{d}v_q \quad=\quad e^{-\frac{1}{2}{}^t\tilde{\boldsymbol{v}}H\tilde{\boldsymbol{v}}}\mathrm{d}v_3\cdots\mathrm{d}v_q$$

where

$$H \quad=\quad 2\frac{(q-1)(q-2)}{q}\left(A_{q-2}+I_{q-2}\right).$$

Thus $\tilde{\boldsymbol{V}}$ is the centered Gaussian vector with covariance matrix

$$H^{-1} \quad=\quad \frac{q}{2(q-1)(q-2)}\left(I_{q-2}-\frac{1}{q-1}A_{q-2}\right).$$

The covariance matrix for $\boldsymbol{V}$ is computed according to $V_1=0$ and $V_2=-\sum_{i=3}^q V_i$. □

4.4 Consequences on the random-cluster model.

Here we give the proofs of Corollary 3.8 and Proposition 3.9:

Proof (Corollary 3.8). When $q>2$ is an integer, at the critical point $(\beta,h)=(\beta_c,0)$ there are $q+1$ minimizers for the free energy $f_{\beta,h}$, which are, on the one hand, the symmetric state

$$\boldsymbol{x}^s=\left(\frac{1}{q},\cdots,\frac{1}{q}\right)$$

and on the other hand, the q permutation $\boldsymbol{x}^{a,i}$ of the asymmetric state

$$\boldsymbol{x}^{a,1}=\left(\frac{q-1}{q},\frac{1}{q(q-1)},\cdots,\frac{1}{q(q-1)}\right).$$

We prove now that the probability of having a giant component in $G(n,p_n,q)$ has the same limit as the probability

$$\mu_{\beta_n,h_n,n}\left(\frac{\boldsymbol{N}}{n}\notin B(\boldsymbol{x}^s,\varepsilon)\right) \tag{4.23}$$

for small enough $\varepsilon>0$, for β_n satisfying $p_n=1-\exp(-\beta_n/n)$ and $h_n=0$.

Indeed, let us fix a realization of the spins. Then we open edges between spins of equal color with probability p_n, resulting in a collection of q Erdös-Rényi random graphs $G(N_i,p_n,1)$ for $i=1,\dots,q$. It is known that a giant cluster appears in such a graph when $\lim_n N_ip_n>1$ (see for instance [14]). Yet, in the symmetric state one has $\lim_n N_ip_n=\beta_c/q<1$ as $q>2$, hence no giant component appears. In the asymmetric state $\boldsymbol{x}^{a,i}$ on the opposite, one has $\lim p_nN_i=\beta_c(q-1)/q>1$ thus a giant component emerges with conditional probability going to 1.

Finally, the quantity (4.23) is computed using Theorem 3.5 after we remark that

$$\beta_n=\beta_c+\frac{1}{n}\left(\gamma+\frac{\beta_c^2}{2}\right)+o\left(\frac{1}{n}\right).$$

□

Let us conclude on the computation of the partition function for the random-cluster model:
Proof (Proposition 3.9). We begin with a computation that permit to relate the partition function of the Curie-Weiss Potts model to that of the random-cluster model, defined at (3.8). Now we say that an edge configuration $\omega \in \{0,1\}^{E(K_n)}$ and a spin configurations $\sigma \in \{1,\dots,q\}^n$ are *compatible* when $\omega_e = 1 \Rightarrow \sigma_i = \sigma_j$, for all $e = \{i,j\} \in E(K_n)$, which we denote as $\omega \prec \sigma$. The factor $q^{C(\omega)}$ can be understood as the number of spin configurations σ that are compatible with ω. Hence:

$$\begin{aligned}
Z^{\mathrm{RC}}_{p,q,n} &= \sum_{\omega\in\{0,1\}^{E(K_n)}} \sum_{\sigma\in\{1,\dots,q\}^n:\omega\prec\sigma} \prod_{e\in K_n} p^{\omega_e}(1-p)^{1-\omega_e} \\
&= \sum_{\sigma\in\{1,\dots,q\}^n} \sum_{\omega\in\{0,1\}^{E(K_n)}:\omega\prec\sigma} \prod_{e\in K_n} p^{\omega_e}(1-p)^{1-\omega_e} \\
&= \sum_{\sigma\in\{1,\dots,q\}^n} \exp\left(-\frac{\beta}{n}\sum_{1\leqslant i<j\leqslant n}(1-\delta_{\sigma_i,\sigma_j})\right) \\
&= \sum_{\sigma\in\{1,\dots,q\}^n} \exp\left(-\frac{\beta}{2}(n-1)+\frac{\beta}{n}\sum_{1\leqslant i<j\leqslant n}\delta_{\sigma_i,\sigma_j}\right) \\
&= Z_{\beta,0,n}\exp\left(-\frac{\beta}{2}(n-1)\right). \qquad (4.24)
\end{aligned}$$

for β such that $p = 1-\exp(-\beta/n)$. Remains to determine the asymptotics of $Z_{\beta_n,0,n}$ for $\beta < \beta_c$. Thanks to the assumption $\beta < \beta_c$ the minimizer of the free energy is unique and symmetric:

$$\boldsymbol{x}^s = \left(\frac{1}{q},\cdots,\frac{1}{q}\right).$$

This implies $\boldsymbol{d}_n = 0$ (see Remark 3.2), thus $\boldsymbol{N} = n\boldsymbol{x}^s + n^{1/2}\boldsymbol{W}$. Equation (4.13), in the limit $R\to\infty$, gives

$$Z_{\beta_n,0,n} = (1+o_n(1))A_\beta(\boldsymbol{x}^s)e^{-nf_{\beta_n,0}(\boldsymbol{x}^s)}\int_{\mathcal{H}} e^{-\frac{1}{2}\boldsymbol{Q}_{\boldsymbol{x}^s,\beta}(\boldsymbol{w})}\mathrm{d}w_2\dots\mathrm{d}w_q \qquad (4.25)$$

as $\int_P \mathrm{d}w_2\dots\mathrm{d}w_q = 1$ for $\boldsymbol{w} = (-\sum_{i=2}^q w_i, w_2,\dots,w_q)$. According to (4.17) one has

$$A_\beta(\boldsymbol{x}^s)\int_{\mathcal{H}} e^{-\frac{1}{2}\boldsymbol{Q}_{\boldsymbol{x}^s,\beta}(\boldsymbol{w})}\mathrm{d}w_2\dots\mathrm{d}w_q = e^{-\frac{\beta}{2}}\left(1-\frac{\beta}{q}\right)^{-\frac{q-1}{2}}.$$

On the other hand, the free energy is easily computed:

$$f_{\beta_n,0}(\boldsymbol{x}^s) = \log\frac{1}{q}-\frac{\beta_n}{2q}$$

leading to

$$Z_{\beta_n,0,n} = (1+o_n(1))\left(1-\frac{\beta}{q}\right)^{-\frac{q-1}{2}} q^n\exp\left(\frac{n\beta_n}{2q}-\frac{\beta}{2}\right).$$

Then (4.24) for $p_n = 1-\exp(-\beta_n/n)$ gives:

$$\begin{aligned}
Z^{\mathrm{RC}}_{p_n,q,n} &= Z_{\beta_n,0,n}\exp\left(-\frac{\beta_n}{2}(n-1)\right) \\
&= (1+o_n(1))\left(1-\frac{\beta}{q}\right)^{-\frac{q-1}{2}} q^n\exp\left(-\frac{n\beta_n}{2}\left(\frac{q-1}{q}\right)\right)
\end{aligned}$$

and the proof is over as $p_n = \beta/n+\gamma/n^2+o(1/n^2)$ implies $\beta_n = \beta+(\gamma+\beta^2/2)/n+o(1/n)$. □

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