Some remarks on analytic continuation and embeddings in weighted backward shift invariant subspaces

Andreas Hartmann

ANALYTIC CONTINUATION AND EMBEDDINGS IN WEIGHTED BACKWARD SHIFT INVARIANT SUBSPACES

ANDREAS HARTMANN

ABSTRACT. By a famous result, functions in backward shift invariant subspaces in Hardy spaces are characterized by the fact that they admit a pseudocontinuation a.e. on $\mathbb{T}$. More can be said if the spectrum of the associated inner function has holes on $\mathbb{T}$. Then the functions of the invariant subspaces even extend analytically through these holes. We will discuss the situation in weighted backward shift invariant subspaces. The results on analytic continuation will be applied to consider some embeddings of weighted invariant subspaces into their unweighted companions. Such weighted versions of invariant subspaces appear naturally in the context of Toeplitz operators. A connection between the spectrum of the inner function and the approximate point spectrum of the backward shift in the weighted situation is established in the spirit of results by Aleman, Richter and Ross.

1. INTRODUCTION

Backward shift invariant subspaces have shown to be of great interest in many domains in complex analysis and Operator Theory. In H^2, the classical Hardy space of holomorphic functions on the unit disk $\mathbb{D}$ satisfying

$$\|f\|_2^2 := \sup_{0<r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{it})|^2 dt < \infty,$$

they are given by $H^2 \ominus IH^2$, where I is an inner function, that is a bounded analytic function in $\mathbb{D}$ the boundary values of which are in modulus equal to 1 a.e. on $\mathbb{T}$. Another way of writing the model spaces is

$$K_I^2 = H^2 \cap I\overline{H_0^2},$$

where $H_0^2 = zH^2$ is the subspace of functions in H^2 vanishing in 0. The bar sign means complex conjugation here. This second writing $K_I^2 = H^2 \cap I\overline{H_0^2}$ does not appeal to the Hilbert space structure and thus generalizes to H^p (which is defined as H^2 but replacing the integration power 2 by $p \in (0, \infty)$; it should be noted that for $p \in (0,1)$ the expression $\|f\|_p^p$ defines a metric; for $p = \infty$, H^∞ is the Banach space of bounded analytic functions on $\mathbb{D}$ with obvious norm). When $p = 2$, then these spaces are also called model spaces. Model spaces have attracted a lot of attention of course in operator theory, initially in the function model of Nagy and Foias, but then also in perturbation theory with Clark's seminal paper on rank one perturbations of the compressed shift on K_I^2. As a result of Clark the Cauchy transform allows to identify K_I^2 with

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$L^2(\sigma_\alpha)$ where σ_α is a so-called Clark measure that one can deduce from I. Clark's motivation was in fact to consider completeness problems in model spaces K_I^2. In a series of papers, Aleksandrov and Poltoratski were interested especially in the behaviour of the Cauchy transform when $p \neq 2$.

Another interest in backward shift invariant subspaces concerns embedding questions, especially when K_I^p embeds into some $L^p(\mu)$. Those questions were investigated for instance by Aleksandrov, Treil, Volberg and many others (see for instance [TV96] for some results). Here we will in fact be interested in the different situation when the weight is not on L^p but on K_I^p.

A very important result in connection with K_I^p-space is that of Douglas, Shapiro and Shields ([DSS70], see also [CR00, Theorem 1.0.5]). They have in fact characterized K_I^p as the space of functions in H^p that admit a so-called pseudocontinuation. Recall that a function holomorphic in $\mathbb{D}_e := \hat{\mathbb{C}} \setminus \operatorname{clos}\mathbb{D}$ — we will use $\operatorname{clos} E$ to designate the closure of a set E in order to preserve the bar-sign for complex conjugation — is a pseudocontinuation of a function f meromorphic in $\mathbb{D}$ if ψ vanishes at ∞ and the outer nontangential limits of ψ on $\mathbb{T}$ coincide with the inner nontangential limits of f on $\mathbb{T}$ in almost every point of $\mathbb{T}$. Note that $f \in K_I^2 = H^2 \cap I\overline{H_0^2}$ implies that $f = I\overline{\psi}$ with $\psi \in H_0^2$. Then the meromorphic function f/I equals $\overline{\psi}$ a.e. $\mathbb{T}$, and writing $\psi(z) = \sum_{n\geq 1} b_n z^n$, it is clear that $\tilde{\psi}(z) := \sum_{n\geq 1} \overline{b_n}/z^n$ is a holomorphic function in $\mathbb{D}_e$, vanishing at ∞, and being equal to f/I almost everywhere on $\mathbb{T}$ (in fact, $\tilde{\psi} \in H^2(\mathbb{D}_e)$).

Note that there are functions analytic on $\mathbb{C}$ that do not admit a pseudocontinuation. An example of such a function is $f(z) = e^z$ which has an essential singularity at infinity.

On the other hand, there are of course pseudocontinuations that are not analytic continuations. A result by Moeller [Mo62] states that outside the spectrum of I, $\sigma(I) = \{\lambda \in \operatorname{clos}\mathbb{D} : \liminf_{z\to\lambda} I(z) = 0\}$, which is a closed set, every function $f \in K_I^2$ extends analytically through the circle. It is not difficult to construct inner functions I for which $\sigma(I) \cap \mathbb{T} = \mathbb{T}$. Take for instance for I the Blaschke product associated with the sequence $\Lambda = \{(1 - 1/n^2)e^{in}\}_n$, the zeros of which accumulate at every point on $\mathbb{T}$.

The problem we are interested in is the case of a weighted backward shift invariant subspace. Let I be any inner function, and g an outer function in H^p, $1 < p < \infty$. Set

$$K_I^p(|g|^p) = H^p(|g|^p) \cap I\overline{H_0^p(|g|^p)}.$$

Here

$$\begin{aligned} H^p(|g|^p) = \{f \in \operatorname{Hol}(\mathbb{D}) : \|f\|_{|g|^p}^p &:= \sup_{0<r<1} \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{it})|^p |g(re^{it})|^p dt \\ &= \int_{-\pi}^{\pi} |f(e^{it})|^p |g(e^{it})|^p dt < \infty\}. \end{aligned}$$

Clearly $H^p(|g|^p) = \{f \in \operatorname{Hol}(\mathbb{D}) : fg \in H^p\}$, and $f \longmapsto fg$ induces an isometry from $H^p(|g|^p)$ onto H^p. Such spaces are not artificial. They appear naturally in the context of Toeplitz operators. Indeed, if $\varphi = \overline{Ig}/g$, is a unimodular symbol, then $\ker T_\varphi = gK_I^2(|g|^2)$ (see [HS03]). Here T_φ is defined in the usual way by $T_\varphi f = P_+(\varphi f)$, P_+ being the standard Riesz projection on $L^p(\mathbb{T})$: $\sum_{n\in\mathbb{Z}} a_n\zeta^n \longmapsto \sum_{n\geq 0} a_n\zeta^n$, $\zeta \in \mathbb{T}$. Note that whenever $0 \neq f \in \ker T_\varphi$, where φ is unimodular and $f = Jg$ is the inner-outer factorization of f, then there exists an inner function I such that $\varphi = \overline{Ig}/g$.

The representation $\ker T_\varphi = gK_I^p(|g|^p)$ is particularly interesting when g is the extremal function of $\ker T_\varphi$. Then we know from a result by Hitt [Hi88] (see also [Sa94] for a de Branges-Rovnyak spaces approach to Hitt's result) that when $p = 2$, $\ker T_\varphi = gK_I^2$, and that g is an isometric divisor on $\ker T_\varphi = gK_I^2$ (or g is an isometric multiplier on K_I^2). In this situation we thus have $K_I^2(|g|^2) = K_I^2$. Note, that for $p \neq 2$, if g is extremal for $gK_I^p(|g|^p)$, then $K_I^p(|g|^p)$ can still be imbedded into K_I^2 when $p > 2$ and in K_I^p when $p \in (1,2)$ (see [HS03], where it is also shown that these imbeddings can be strict). In these situations when considering questions concerning pseudocontinuation and analytic continuation, we can carry over to $K_I^p(|g|^2)$ everything we know about K_I^2 or K_I^p (which is the same concerning these continuation matters).

However, in general the extremal function is not easily detectable (explicit examples of extremal functions were given in [HS03]), in that we cannot determine it, or for a given g it is not a simple matter to check whether it is extremal or not. So the first question that we would like to discuss is under which conditions on g and I, we can still say something about analytic continuation of functions in $K_I^p(|g|^p)$. Our main result is that under a local integrability condition of $1/g$ on a closed arc not meeting the spectrum of I it is possible to extend every $K_I^p(|g|^p)$ function through such an arc. The integrability condition is realized if for example $|g|^p$ is an (A_p) weight (but in this situation the analytic continuation turns out to be a simple consequence of Hölder's inequality and Moeller's original result, see Proposition 3 and comments thereafter).

In connection with analytic continuation under growth conditions, another important result can be mentioned. Beurling (see [Be72]) proved that under some integral condition of a weight w defined on a square Q, every function holomorphic on the upper and the lower half of the square and which is bounded by a constant times $1/w$ extends analytically to the whole square. Our result is different since we do not consider generic functions holomorphic in both halfs of the square but admitting already a certain type of pseudocontinuation. This allows to weaken the condition on the weight under which the analytic continuation is possible.

One could also ask when $K_I^p(|g|^p)$ still embeds into H^p, or even in H^r, $r < p$, in other words, when $K_I^p(|g|^p) \subset K_I^p$ or $K_I^p(|g|^p) \subset K_I^r$ (note that $K_I^p \cap H^\infty \subset K_I^p(|g|^p)$, which in particular gives $K_I^p \subset K_I^p(|g|^p)$ whenever g is bounded, so that in such a situation the preceding inclusion is in fact an equality). We will discuss some examples in this direction related to our main result.

Naturally related to the question of analytic continuation is the spectrum of the restriction of the backward shift operator to $K_I^p(|g|^p)$ (see [ARR98]). As was done in Moeller's paper, we will explore these relations in the proof of our main theorem on analytic continuations.

Finally we mention a paper by Dyakonov ([Dy96]). He discussed Bernstein type inequalities in kernels of Toeplitz operators which we know from our previous discussions are closely related with weighted spaces $K_I^p(|g|^p)$. More precisely, he discussed the regularity of functions in $\ker T_\varphi$ depending on the smoothness of the symbol φ.

The paper is organized as follows. In the next section we will discuss the analytic continuation when the spectrum of I is far from points where g vanishes essentially. We will also establish a link with the spectrum of the backward shift on $K_I^p(|g|^p)$ in this situation. As a corollary we deduce that in certain situations one can get an embedding of $K_I^p(|g|^p)$ into its unweighted companion. A simple situation is discussed when $K_I^p(|g|^p)$ can be embedded into a bigger K_I^r ($1 < r < p$), and so still guaranteeing the analytic continuation outside the spectrum of I. Section 3 is different in flavour. We will focus on the embedding problem by discussing some examples when $K_i^2 I(|g|^2)$ does not embed into K_I^2. It turns out that in the examples considered the analytic continuation is like in the unweighted case. Also, in these examples the spectrum of I comes close to points where g vanishes essentially.

2. Results when $\sigma(I)$ is far from the points where g vanishes

We start with a simple example. Let I be arbitrary with $-1 \notin \sigma(I)$, and let $g(z) = 1 + z$, so that $\sigma(I)$ is far from the only point where g vanishes. We know that $\ker T_{\frac{\overline{Ig}}{g}} = gK_I^p(|g|^p)$ (note that this is a so-called nearly invariant subspace). Let us compute this kernel. We first observe that $\overline{\frac{1+z}{1+z}} = \overline{z}$. Hence $T_{\frac{\overline{Ig}}{g}} = T_{\overline{zI}}$, the kernel of which is known to be K_{zI}^p. So

$$gK_I^p(|g|^p) = K_{zI}^p.$$

The space on the right hand side contains the constant functions. Hence $1/g \in K_I^p(|g|^p)$ (observe that $1/g = I\overline{\psi}$ with $\psi(z) = Iz/(1+z) = Iz/g \in H_0^p(|g|^p)$). In particular, $K_I^p(|g|^p)$ contains the function $1/g$ which is badly behaved in -1, and thus cannot extend analytically through -1.

This observation can be made more generally as stated in the following result.

Proposition 1. *Let g be an outer function in H^p. If $\ker T_{\overline{g}/g} \neq \{0\}$ contains an* inner *function, then $1/g \in K_I^p(|g|^2)$ for* every *inner function I.*

Before proving this simple result, we will do a certain number of comments:

(1) The example that we discussed above corresponds obviously to the situation when the inner function contained in the kernel of $T_{\overline{g}/g}$ is identically equal to 1.
(2) The proposition again indicates a way of finding examples of g and I such that $K_I^p(|g|^2)$ contains functions that cannot be analytically continued through points where g is "small".
(3) Suppose for the next two remarks that $p = 2$.
    - The claim that the kernel of $T_{\overline{g}/g}$ contains an inner function implies in particular that this Toeplitz operator is not injective and so g^2 is not rigid in H^1 (see [Sa95, X-2]), which means that it is not uniquely determined — up to a real multiple — by its argument (or equivalently, its normalized version $g^2/\|g^2\|_1$ is not exposed in the unit ball of H^1).
    - It is clear that if the kernel of a Toeplitz operator is not reduced to $\{0\}$ — or equivalently (since $p = 2$) g^2 is not rigid — then it contains an *outer* function (just divide out the inner factor of any non zero function contained in the kernel). However, Toeplitz operators with non trivial kernels containing no inner functions can be easily constructed. One could appeal to Hitt [Hi88], Hayashi [Hay90] and Sarason [Sa94]: the first states that every nearly invariant subspace is of the form gK_I^2 and g is extremal thus multiplying isometrically on K_I^2, the second tells us how g has to be in order that gK_I^2 is in particular the kernel of a Toeplitz operator (the symbol being $\overline{Ig}/g$), and the third one gives a general form of g insuring the extremality (or the isometric multiplication property). This allows to construct a kernel with the desired

properties. However we can short-circuit these results and take $T_{\overline{z g_0}/g_0} = T_{\overline{z}} T_{\overline{g_0}/g_0}$, where $g_0(z) = (1-z)^\alpha$ and $\alpha \in (0, 1/2)$. The Toeplitz operator $T_{\overline{g_0}/g_0}$ is invertible ($|g_0|^2$ satisfies the Muckenhoupt (A_2) condition — see Section 3 for more discussions on invertibility of Toeplitz operators), and $(T_{\overline{g_0}/g_0})^{-1} = g_0 P_+ \frac{1}{\overline{g_0}}$ [Ro77] so that the kernel of $T_{\overline{z g_0}/g_0}$ is given by the preimage under $T_{\overline{g_0}/g_0}$ of the constants (which define the kernel of $T_{\overline{z}}$). Since $g_0 P_+(c/\overline{g_0}) = c g_0/\overline{g_0(0)}$, c being any complex number, we have $\ker T_{\overline{z g_0}/g_0} = \mathbb{C} g_0$ which does not contain any inner function.

(4) See [Ka96] for a discussion of the intersection $I_g := H^p(|g|^p) \cap \overline{H_0^p(|g|^p)}$. Theorem 3 of that paper states that for points in the spectrum of the inverse of the backward shift — which is related with the complement of those points where every function in I_g continues analytically — there always exist functions with singularities in such a point.

Proof of Proposition 1. If $J \in \ker T_{\overline{g}/g}$, then $\overline{g}/g = \overline{J\psi}$ where $\psi \in H_0^p$. Since the functions appearing on both sides of the equality are of modulus 1, the function $\Theta = J\psi$ is inner and vanishes in 0. So

$$T_{\overline{Ig/g}} = T_{\overline{I\Theta}}.$$

Hence

$$K_I^p(|g|^p) = \frac{1}{g} \ker T_{\overline{I\Theta}} = \frac{1}{g} K_{I\Theta}^p.$$

Since $\Theta(0) = 0$, we have $1 \in K_{I\Theta}^p$, and so $1/g \in K_I^p(|g|^2)$. ■

One can also observe that if the inner function J is in $\ker T_{\overline{g}/g}$ then $T_{\overline{Jg/g}} 1 = 0$, and hence $1 \in \ker T_{\overline{Jg/g}} = g K_J^p(|g|^2)$ and $1/g \in K_J^p(|g|^2)$, which shows that with this simpler argument the proposition holds for $I = J$. Yet, our proof above allows to choose for I any arbitrary inner function and not necessarily that contained in $\ker T_{\overline{g}/g}$.

So, without any condition on g, we cannot hope for reasonable results. In the above example, when $p = 2$, then the function $g^2(z) = (1+z)^2$ is in fact not rigid (for instance the argument of $(1+z)^2$ is the same as that of z). Recall that rigidity of g^2 is also characterized by the fact that $T_{\overline{g}/g}$ is injective (see [Sa95, X-2]). Here $T_{\overline{g}/g} = T_{\overline{z}}$ the kernel of which is $\mathbb{C}$. From this it can also be deduced that g^2 is rigid if and only if $H^p(|g|^p) \cap \overline{H^p(|g|^p)} = \{0\}$ which indicates again that rigidity should be assumed if we want to have $K_I^p(|g|^p)$ reasonably defined.

A stronger condition than rigidity (at least when $p = 2$) is that of a Muckenhoupt weight. Let us recall the Muckenhoupt (A_p) condition: for general $1 < p < \infty$ a weight w satisfies the (A_p) condition if

$$B := \sup_{I \text{ subarc of } \mathbb{T}} \left\{ \frac{1}{|I|} \int_I w(x)dx \times \left(\frac{1}{|I|} \int_I w^{-1/(p-1)}(x)dx \right)^{p-1} \right\} < \infty.$$

When $p = 2$, it is known that this condition is equivalent to the so-called Helson-Szegő condition. The Muckenhoupt condition will play some rôle in the results to come. However, our main theorem on analytic continuation (Theorem 1) works under a weaker local integrability condition which follows for instance from the Muckenhoupt condition.

Another observation can be made now. We have already mentioned that the rigidity of g^2 in H^1 is equivalent to the injectivity of $T_{\overline{g}/g}$, when g is outer. It is also clear that $T_{g/\overline{g}}$ is *always* injectif so that when g^2 is rigid, the operator $T_{\overline{g}/g}$ is injectif with dense range. On the other hand, by a result of Devinatz and Widom (see e.g. [Ni02, Theorem B4.3.1]), the invertibility of $T_{\overline{g}/g}$, where g is outer, is equivalent to $|g|^2$ being (A_2). So the difference between rigidity and (A_2) is the surjectivity (in fact the closedness of the range) of the corresponding Toeplitz operator. A criterion for surjectivity of non-injective Toeplitz operators can be found in [HSS04]. It appeals to a parametrization which was earlier used by Hayashi [Hay90] to characterize kernels of Toeplitz operators among general nearly invariant subspaces. Rigid functions do appear in the characterization of Hayashi.

As a consequence of our main theorem (see Remark 1) analytic continuation can be expected on arcs not meeting the spectrum of I when $|g|^p$ is (A_p). However the (A_p) condition cannot be expected to be necessary since it is a global condition whereas continuation depends on the local behaviour of I and g. We will even give an example of a non-rigid function g (hence not satisfying the (A_p) condition) for which analytic continuation is always possible in certain points of $\mathbb{T}$ where g vanishes essentially.

Closely connected with backward shift invariant subspaces is the spectrum of the backward shift operator on the space under consideration. The following result follows from [ARR98, Theorem 1.9]: Let B be the backward shift on $H^p(|g|^p)$, defined by $Bf(z) = (f - f(0))/z$. Clearly, $K_I^p(|g|^p)$ is invariant with respect to B whenever I is inner. Then, $\sigma(B|K_I^p(|g|^p)) = \sigma_{ap}(B|K_I^p(|g|^p))$, where $\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : \exists (f_n)_n$ with $\|f_n\| = 1$ and $(\lambda - T)f_n \to 0\}$ denotes the approximate point spectrum of T, and this spectrum is equal to

$$\mathbb{T} \setminus \{1/\zeta \in \mathbb{T} : \text{ every } f \in K_I^p(|g|^p) \text{ extends analytically in a neighbourhood of } \zeta\}.$$

We would like to establish a link between this set and $\sigma(I)$. To this end we will adapt the proof of the unweighted case [Mo62, Theorem 2.3] to our situation. As in the unweighted situation — provided the Muckenhoupt condition holds — the approximate spectrum of $B|K_I^p(|g|^p)$ on $\mathbb{T}$ contains the conjugated spectrum of I. We will see later that the containment in the following proposition actually is an equality.

Proposition 2. *Let g be outer in H^p such that $|g|^p$ is a Muckenhoupt (A_p)-weight. Let I be an inner function with spectrum $\sigma(I) = \{\lambda \in \operatorname{clos}\mathbb{D} : \liminf_{z\to\lambda} I(z) = 0\}$. Then $\overline{\sigma(I)} \subset \sigma_{ap}(B|K_I^p(|g|^p))$.*

Proof. It is clear that when $\lambda \in \mathbb{D} \cap \sigma(I)$, then λ is a zero of I and $k_\lambda^I = k_\lambda$. It is a general fact that $Bk_\lambda = \overline{\lambda} k_\lambda$. And since $k_\lambda \in K_I^p \cap H^\infty \subset K_I^p(|g|^p)$, we see that $\overline{\lambda}$ is an eigenvalue, so it is in the point spectrum of $B|K_I^p(|g|^p)$ and then also in the approximate point spectrum.

So, let us consider $\lambda \in \mathbb{T}$. Take such a $\lambda \in \mathbb{T}$ with $\liminf_{z\to\overline{\lambda}, z\in\mathbb{D}} |I(z)| = 0$. We want to prove that $\lambda \in \sigma_{ap}(B|K_I^p(|g|^p))$. To this end, let $\lambda_n \to \lambda$ a sequence of $\lambda_n \in \mathbb{D}$ with $I(\overline{\lambda_n}) \to 0$.

Clearly

$$c_nk_{\overline{\lambda_n}}(z)=\frac{c_n}{1-\lambda_n z}=c_n\frac{1-\overline{I(\overline{\lambda_n})}I(z)}{1-\lambda_n z}+c_n\frac{\overline{I(\overline{\lambda_n})}I(z)}{1-\lambda_n z}=\underbrace{c_nk^I_{\overline{\lambda_n}}(z)}_{=:l_n(z)}+\underbrace{c_n\overline{I(\overline{\lambda_n})}I(z)k_{\overline{\lambda_n}}(z)}_{=:r_n(z)},$$

where c_n is chosen so that $\|c_nk_{\overline{\lambda_n}}\|_{|g|^p}=1$. Clearly $l_n=c_nP_Ik_{\overline{\lambda_n}}$ as a projected reproducing kernel is in $K^p_I\cap H^\infty$ which is contained in $K^p_I(|g|^p)$, and $r_n\in I(H^p\cap H^\infty)\subset IH^p(|g|^p)$. Since $|g|^p$ is Muckenhoupt (A_p), the Riesz projection P_+ is continuous on $L^p(|g|^p)$ and so also $P_I=IP_-\overline{I}$. As in the unweighted case $K^p_I(|g|^p)=P_IH^p(|g|^p)$ and $IH^p(|g|^p)=\ker P_I|H^p(|g|^p)$, so that the norm of $c_nk_{\overline{\lambda_n}}$ in $H^p(|g|^p)$ is comparable to the sum of the norms of l_n and r_n:

$$1=\|c_nk_{\overline{\lambda_n}}\|_{|g|^p}\simeq\|l_n\|_{|g|^p}+\|r_n\|_{|g|^p}=\|l_n\|_{|g|^p}+|I(\overline{\lambda_n})|.$$

(For $p=2$, the boundedness of the projection P_I means that the angle between $K^2_I(|g|^2)$ and $IH^2(|g|^2)$ is strictly positive). Since $I(\overline{\lambda_n})$ goes to zero, this implies that the norms $\|l_n\|_{|g|^p}$ are comparable to a strictly positive constant. Now,

$$\|(B-\lambda)c_nk_{\overline{\lambda_n}}\|_{|g|^p}=\|c_n(\lambda_n-\lambda)k_{\overline{\lambda_n}}\|_{|g|^p}=|\lambda-\lambda_n|,$$

and hence

$$\begin{aligned}\|(B-\lambda)l_n\|_{|g|^p}&=\|(B-\lambda)c_nk_{\overline{\lambda_n}}-(B-\lambda)r_n\|_{|g|^p}\\&\leq|\lambda-\lambda_n|+\|B-\lambda\|_{H^p(|g|^p)\to H^p(|g|^p)}|I(\overline{\lambda_n})|\\&\leq|\lambda-\lambda_n|+(\|B\|_{H^p(|g|^p)\to H^p(|g|^p)}+|\lambda|)|I(\overline{\lambda_n})|,\end{aligned}$$

which tends to zero, while $\|l_n\|_{|g|^p}$ is uniformly bounded away from zero. So, $\lambda\in\sigma_{ap}(B|K^p_I(|g|^p)$. ■

We now come to our main theorem.

Theorem 1. *Let g be an outer function in H^p, $1<p<\infty$ and I an inner function with associated spectrum $\sigma(I)$. Let Γ be a closed arc in $\mathbb{T}$ not meeting $\sigma(I)$. If there exists $s>q$, $\frac{1}{p}+\frac{1}{q}=1$, with $1/g\in L^s(\Gamma)$, then every function $f\in K^p_I(|g|^p)$ extends analytically through Γ.*

Remark 1. *It is known (see e.g.* [Mu72]*) that when $|g|^p\in(A_p)$, $1<p<\infty$, then there exists $r_0\in(1,p)$ such that $|g|^p\in(A_r)$ for every $r>r_0$. Take $r\in(r_0,p)$. Then in particular $1/g\in L^s$, where $\frac{1}{r}+\frac{1}{s}=1$. Since $r<p$ we have $s>q$. which allows to conclude that in this situation $1/g\in L^s(\Gamma)$ for every $\Gamma\subset\mathbb{T}$ (s independant of Γ).*

We promised earlier an example of a non-rigid function g for which analytic continuation of K^p_I-functions *is* possible in certain points where g vanishes.

Example. For $\alpha\in(0,1/2)$, let $g(z)=(1+z)(1-z)^\alpha$. Clearly g is an outer function vanishing essentially in 1 and -1. Set $h(z)=z(1-z)^{2\alpha}$, then by similar arguments as those employed in the introducing example to this section one can check that $\arg g^2=\arg h$ a.e. on $\mathbb{T}$. Hence g is not rigid (it is the “big” zero in -1 which is responsible for non-rigidity). On the other hand, the zero in $+1$ is “small” in the sense that g satisfies the local integrability condition in a neighbourhood of 1 as required in the theorem, so that whenever I has its spectrum far from 1, then every $K^2_I(|g|^2)$-function can be analytically continued through suitable arcs around 1.

This second example can be pushed a little bit further. In the spirit of Proposition 1 we check that (even) when the spectrum of an inner function I does not meet -1, there are functions in $K^p_I(|g|^p)$ that are badly behaved in -1. Let again $g_0(z) = (1-z)^\alpha$. Then

$$\frac{\overline{g(z)}}{g(z)} = \frac{\overline{(1+z)(1-z)^\alpha}}{(1+z)(1-z)^\alpha} = \overline{z}\frac{\overline{g_0(z)}}{g_0(z)}.$$

As already explained, for every inner function I, we have $\ker T_{\overline{Ig}/g} = gK^p_I(|g|^p)$, so that we are interested in the kernel $\ker T_{\overline{Ig}/g}$. We have $T_{\overline{Ig}/g}f = 0$ when $f = Iu$ and $u \in \ker T_{\overline{g}/g} = \ker T_{\overline{zg_0}/g_0} = \mathbb{C}g_0$ (see the discussion just before the proof of Proposition 1). Hence the function defined by

$$F(z) = \frac{f(z)}{g(z)} = I(z)\frac{g_0(z)}{g(z)} = \frac{I(z)}{1+z}$$

is in $K^p_I(|g|^p)$ and it is badly behaved in -1 when the spectrum of I does not meet -1 (but not only).

The preceding discussions motivate the following question: does rigidity of g suffice to get analytic continuation for $K^p_I(|g|^2)$-function whenever $\sigma(I)$ is far from zeros of g?

Proof of Theorem 1. Take an arc as in the theorem. Since $\sigma(I)$ is closed, the distance between $\sigma(I)$ and Γ is strictly positif. Then there is a neigbourhood of Γ intersected with $\mathbb{D}$ where $|I(z)| \geq \delta > 0$. It is clear that in this neighbourhood we are far away from the spectrum of I. Thus I extends analytically through Γ. For what follows we will call the endpoints of this arc ζ_1 and ζ_2. We would like to know whether every function in $K^p_I(|g|^p)$ extends also analytically through Γ.

We adapt an argument by Moeller based on Morera's theorem. Let us first introduce some notation (see Figure 1).

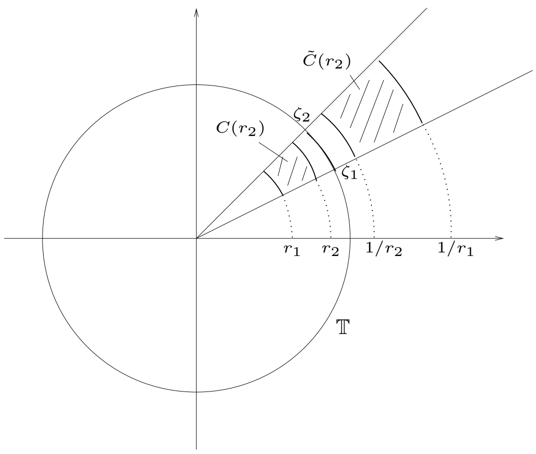

Figure 1: The regions $C(r_2)$ and $\tilde{C}(r_2)$

For suitable $r_0 \in (0,1)$ let $r_0 < r_1 < r_2 < 1$. Using Moeller's notation, we call $C(r_2)$ the boundary of a circular sector whose vertices are $\{r_k\zeta_l\}_{k,l=1,2}$ (inside $\mathbb{D}$) and $\tilde{C}(r_2)$ is the boundary of a circular sector with vertices $\{r_k^{-1}\zeta_l\}_{k,l=1,2}$ (outside $\mathbb{D}$). These sectors are thus in a sense symmetric with respect to $\mathbb{T}$ and when r_2 goes to 1, then at the limit they will form a circular sector with vertices $\{r_1\zeta_l\}_{l=1,2}$ and $\{r_1^{-1}\zeta_l\}_{l=1,2}$ the interior of which contains in particular Γ. For the construction to come, we need to assume that r_1 is chosen in such a way that $1/I$ is analytic (and bounded) in $C(r_2)$ for every $r_1 < r_2 < 1$, which is of course possible. We will also assume that f admits boundary values in ζ_1 and ζ_2. Since f is in N^+ it has boundary values a.e., and so the previous requirement is not difficult to meet (by possibly moving the endpoints ζ_1, ζ_2 if necessary).

Take now $f \in K_I^p(|g|^p) = H^p(|g|^p) \cap I\overline{H_0^p(|g|^p)}$, so that $f = I\overline{\psi}$, where $\psi \in H_0^p(|g|^p)$ can be written as $\psi(z) = \sum_{n\geq 1} c_n z^n$, $z \in \mathbb{D}$. The function $\tilde{\psi}$ defined by

$$\tilde{\psi}(z) = \sum_{n\geq 1} \overline{c_n}\frac{1}{z^n}, \quad z \in \mathbb{D}_e := \hat{C} \setminus \operatorname{clos}\mathbb{D},$$

(the tilde-sign does not mean harmonic conjugation here) is the pseudocontinuation of the meromorphic function f/I. Note that f/I is even analytic in every $C(r_2)$, $r_1 < r_2 < 1$. Since $f \in K_I^p(|g|^p)$, we in particular have

$$\sup_{r<1}\int_\Gamma |f(re^{it})|^p|g(re^{it})|^p\frac{dt}{2\pi} < \infty.$$

Then, since by construction $1/I$ is analytic and bounded on $C(r_2)$, $r_0 < r_1 < r_2 < 1$, we also have

$$M_1 := \sup_{r_1<r<1}\int_\Gamma |\frac{f(re^{it})}{I(re^{it})}|^p|g(re^{it})|^p\frac{dt}{2\pi} < \infty.$$

On the other hand, for $r_1 < r < 1$, we have

$$\begin{aligned}\int_\Gamma |\tilde{\psi}\left(\frac{e^{it}}{r}\right)|^p|g(re^{it})|^p\frac{dt}{2\pi} &= \int_\Gamma |\psi(re^{it})|^p|g(re^{it})|^p\frac{dt}{2\pi}\\ &\leq \sup_{r_1<\rho<1}\int_\Gamma |\psi(\rho e^{it})|^p|g(\rho e^{it})|^p\frac{dt}{2\pi} =: M_2 < \infty\end{aligned}$$

since $\psi \in H_0^p(|g|^p)$. We will now use a version of Lebesgue's monotone convergence theorem. Recall that $\tilde{\psi}$ is the pseudocontinuation of f/I, so that a.e. on $\mathbb{T}$ and in particular on Γ,

$$\frac{f(re^{it})}{I(re^{it})} - \tilde{\psi}(\frac{e^{it}}{r}) \to 0, \quad \text{when } r \to 1^-. \tag{1}$$

Let us prove that there exists $\varepsilon > 0$ and $M > 0$ such that for every $r_1 < r < 1$,

$$\int_\Gamma \left|\frac{f(re^{it})}{I(re^{it})} - \tilde{\psi}(\frac{e^{it}}{r})\right|^{1+\varepsilon}\frac{dt}{2\pi} \leq M. \tag{2}$$

Note first that for $r_1 < r < 1$

$$\begin{aligned}
&\int_\Gamma \left|\frac{f(re^{it})}{I(re^{it})} - \tilde{\psi}(\frac{e^{it}}{r})\right|^p |g(re^{it})|^p \frac{dt}{2\pi} \\
&\le c_p \left(\int_\Gamma \left|\frac{f(re^{it})}{I(re^{it})}\right|^p |g(re^{it})|^p \frac{dt}{2\pi} + \int_\Gamma \left|\tilde{\psi}(\frac{e^{it}}{r})\right|^p |g(re^{it})|^p \frac{dt}{2\pi}\right) \\
&\le c_p(M_1 + M_2) =: M'.
\end{aligned}$$

From this and the condition $1/g \in L^s(\Gamma)$ we will deduce via a simple Hölder inequality our estimate (2).

By assumption there is $s > q$ with $1/g \in L^s(\Gamma)$. Then there exists $\varepsilon > 0$ such that $s > p\frac{1+\varepsilon}{p-(1+\varepsilon)} > q = p/(p-1)$. Hence $L^s(\Gamma) \subset L^{p\frac{1+\varepsilon}{p-(1+\varepsilon)}}(\Gamma)$ and

$$\int_\Gamma \left(\frac{1}{|g|}\right)^{p\frac{1+\varepsilon}{p-(1+\varepsilon)}} dm \le c \int_\Gamma \frac{1}{|g|^s} dm < \infty \tag{3}$$

Then by Hölder's inequality, assuming also $p/(1+\varepsilon) > 1$,

$$\begin{aligned}
&\int_\Gamma \left|\frac{f(re^{it})}{I(re^{it})} - \tilde{\psi}(\frac{e^{it}}{r})\right|^{1+\varepsilon} \frac{dt}{2\pi} = \int_\Gamma \left|\frac{f(re^{it})}{I(re^{it})} - \tilde{\psi}(\frac{e^{it}}{r})\right|^{1+\varepsilon} \frac{|g(re^{it})|^{1+\varepsilon}}{|g(re^{it})|^{1+\varepsilon}} \frac{dt}{2\pi} \\
&\le \left\{\int_\Gamma \left|\frac{f(re^{it})}{I(re^{it})} - \tilde{\psi}(\frac{e^{it}}{r})\right|^p |g(re^{it})|^p \frac{dt}{2\pi}\right\}^{(1+\varepsilon)/p} \left\{\int_\Gamma \left(\frac{1}{|g(re^{it})|^{1+\varepsilon}}\right)^{p/(p-(1+\varepsilon))} \frac{dt}{2\pi}\right\}^{(p-(1+\varepsilon))/p}
\end{aligned}$$

The first factor in this product is uniformly bounded by our previous discussions. Consider the second factor. Recall that g is outer (hence $1/g$ is in the Smirnov class) so that it is sufficient to check whether the second factor is bounded for $r = 1$, and this follows from (3). We have thus proved (2).

By standard arguments based on Lebesgue's dominated convergence theorem and Tchebychev's inequality we get that (2) together with (1) imply that

$$\int_\Gamma \frac{f(re^{it})}{I(re^{it})} - \tilde{\psi}(\frac{e^{it}}{r})\frac{dt}{2\pi} \to 0.$$

From this point on we can repeat Moeller's proof [Mo62, Lemma 2.2], which is based on Morera's theorem, and which uses the fact that when $r_2 \to 1^-$, then the regions $C(r_2)$ and $\tilde{C}(r_2)$ fusion to a big angular sector, where the arcs $\Gamma_{r_2} := \{r_2\zeta : \zeta \in \Gamma\}$ and Γ_{1/r_2} are oriented in an opposite direction and the difference of the integrals of our fonctions f/I and ψ on these two arcs tends to zero.

This implies that f/I extends analytically through Γ. ■

The theorem together with Proposition 2 and Remark 1 allow us to obtain the following result.

Corollary 1. *Let g be outer in H^p such that $|g|^p$ is a Muckenhoupt (A_p) weight. Let I be an inner function with spectrum $\sigma(I) = \{\lambda \in \operatorname{clos}\mathbb{D} : \liminf_{z\to\lambda} I(z) = 0\}$. Then $\overline{\sigma(I)} = \sigma_{ap}(B|K_I^p(|g|^p)$.*

Proof. The inclusion $\overline{\sigma(I)} \subset \sigma_{ap}(B|K_I^p(|g|^p))$ has been discussed in Proposition 2.

For the reverse inclusion, suppose that $\overline{\lambda} \notin \sigma(I)$. Note that $\overline{\sigma(I)} \cap \mathbb{D} = \sigma_p(B|K_I^p(|g|^p)) = \sigma_{ap}(B|K_I^p(|g|^p)) = \sigma(B|K_I^p(|g|^p))$ so that it is sufficient to consider the case $\lambda \in \mathbb{T}$. Since $\sigma(I)$ is closed, there is an arc not meeting $\sigma(I)$ and containing $\overline{\lambda}$. By Theorem 1 and Remark 1, every $f \in K_I^p(|g|^p)$ extends analytically through this arc, and thus in particular through $\overline{\lambda}$. So, by [ARR98, Theorem 1.9], $\overline{\lambda}$ cannot be in $\sigma_{ap}(B|K_I^p(|g|^p))$ (neither in $\sigma(B|K_I^p(|g|^p))$). ■

Another simple consequence of Theorem 1 concerns embeddings.

Corollary 2. *Let I be an inner function with spectrum $\sigma(I)$. If $\Gamma \subset \mathbb{T}$ is a closed arc not meeting $\sigma(I)$ and if g is an outer function in H^p such that $|g| \geq \delta$ on $\mathbb{T} \setminus \Gamma$ for some constant $\delta > 0$ and $1/g \in L^s(\Gamma)$, $s > q$, $\frac{1}{p} + \frac{1}{q} = 1$. Then $K_I^p(|g|^p) \subset K_I^p$. If moreover g is bounded, then the last inclusion is an equality.*

Remark 2. *1) Suppose $p = 2$. By Hitt's result* [Hi88], *when g is the extremal function of a nearly invariant subspace $M \subset H^2$, then there exists an inner function I such that $M = gK_I^2$, and g is an isometric multiplier on K_I^2 so that $K_I^2 = K_I^2(|g|^2)$. With the corollary we can construct examples of spaces of infinite dimension where the latter identity holds without g being extremal. Recall from* [HS03, Lemma 3] *that a function g is extremal for $gK_I^2(|g|^2)$ if $\int f|g|^2 dm = f(0)$ for every function $f \in K_I^2(|g|^2)$. The following example is constructed in the spirit of the example in* [HS03, p.356]. *Fix $\alpha \in (0, 1/2)$. Let $\gamma(z) = (1-z)^\alpha$ and let g be an outer function in H^2 such that $|g|^2 = \operatorname{Re}\gamma$ a.e. on $\mathbb{T}$ (such a function clearly exists). Let now $I = B_\Lambda$ be a Blaschke product with $0 \in \Lambda$. If Λ accumulates to points outside 1, then the corollary shows that $K_I^2 = K_I^2(|g|^2)$. Let us check that g is not extremal. To this end we compute $\int k_\lambda |g|^2 dm$ for $\lambda \in \Lambda$ (recall that for $\lambda \in \Lambda$, $k_\lambda \in K_I^2 = K_I^2(|g|^2)$):*

$$\begin{aligned} \int k_\lambda |g|^2 dm &= \int k_\lambda \operatorname{Re}\gamma dm = \frac{1}{2}\left(\int k_\lambda \gamma dm + \int k_\lambda \overline{\gamma} dm\right) = \frac{1}{2} k_\lambda(0)\gamma(0) + \frac{1}{2}\langle k_\lambda, \gamma\rangle \\ &= \frac{1}{2}(1 + \overline{(1-\lambda)^\alpha}) \end{aligned}$$

which is different from $k_\lambda(0) = 1$ (except when $\lambda = 0$). Hence g is not extremal.

2) Observe that **if** *$K_I^p(|g|^p) \subset K_I^p$, then the analytic continuation is of course a simple consequence of that in K_I^p (and hence of Moeller's result). And since $K_I^p(|g|^2)$ always contains k_λ^I which continues only outside $\sigma(I)$, one cannot hope for a better result. Note also that the inclusion $K_I^p(|g|^p) \subset K_I^p$ is a kind of reverse inclusion to those occuring in the context of Carleson measures. Indeed, the problem of knowing when $K_I^p \subset L^p(\mu)$ continues attracting a lot of attention (and the reverse situation to ours would correspond to $d\mu = |g|^2 dm$). Such a measure μ is called a Carleson measure for K_I^p, and it is notoriously difficult to describe these in the general situation (see* [TV96] *for some results; when I is a so-called one-component inner function I, a geometric characterization is available).*

Proof of Corollary 2. Pick $f \in K_I^p(|g|^2)$. We only have to prove that $f \in L^p$. By Theorem 1, f continues analytically through I and so f is bounded on I. On the other hand

$$\int_{\mathbb{T}\setminus\Gamma} |f|^p dm \leq \frac{1}{\delta^p} \int_{\mathbb{T}\setminus\Gamma} |f|^p |g|^p dm < \infty.$$

∎

It is clear that the corollary is still valid when Γ is replaced by a finite union of intervals. However, in the next section we will see that it is no longer valid when Γ is replaced by an infinite union of intervals under a yet weaker integrability condition on $1/g$ (see Proposition 5).

A final simple observation concerning the local integrability condition $1/g \in L^s(\Gamma)$, $s > q$: if it is replaced by the global condition $1/g \in L^s(\mathbb{T})$, then we have an embedding into a bigger backward shift invariant subspace:

Proposition 3. *Let* $1 < p < \infty$ *and* $1/p + 1/q = 1$. *If there exists* $s > q$ *such that* $1/g \in L^s(\mathbb{T})$, *then for* r *with* $1/r = 1/p + 1/s$ *we have* $L^p(|g|^p) \subset L^r$.

Proof. This is a simple application of Young's (or Hölder's) inequality:

$$\int |f|^r dm = \int |fg|^r \frac{1}{|g|^r} dm \leq \left(\int |fg|^p dm \right)^{r/p} \left(\int \frac{1}{|g|^s} dm \right)^{r/s} < \infty.$$

∎

Under the assumptions of the proposition we of course also have $K_I^p(|g|^p) \subset K_I^r$. In particular, under the assumption of the proposition, Moeller's theorem then shows that every function $f \in K_I^p(|g|^p)$ extends analytically outside $\sigma(I)$.

Another observation is that when $|g|^p \in (A_p)$, then by Remark 1, we have $1/g \in L^s$ for some $s > q$ and so the assumptions of the proposition are met, and again $K_I^p(|g|^p)$ embeds into K_I^r.

3. Examples when $K_I^2(|g|^2)$ does not embed into K_I^2

Here we will discuss some examples when $K_I^2(|g|^2)$ does not embed into K_I^2 even when $|g|^2$ satisfies some regularity condition. The first example is when $|g|^2$ is (A_2). The second example, discussed in Proposition 5, is a kind of counterpart to Corollary 2. In both examples the spectrum of I comes close to the points where g vanishes essentially.

Before entering into the discussion of our examples, we give a result in connection with invertibility of Toeplitz operators. For an outer function $g \in H^p$, the Toeplitz operator $T_{\overline{g}/g}$ is invertible if and only if $|g|^p$ is an (A_p) weight (we have already mentioned the result by Devinatz and Widom for the case $p = 2$, see e.g. [Ni02, Theorem B4.3.1]; for general p, see [Ro77]). If this is the case, the inverse of $T_{\overline{g}/g}$ is defined by $A = gP_+\frac{1}{\overline{g}}$ (see [Ro77]). Then, the operator $A_0 = P_+\frac{1}{\overline{g}}$ is an isomorphism of H^p onto $H^p(|g|^p)$.

Lemma 1. *Suppose* $|g|^p$ *is an* (A_p) *weight and* I *an inner function. Then* $A_0 = P_+\frac{1}{\overline{g}}$ *is an isomorphism of* K_I^p *onto* $K_I^p(|g|^p)$. *Also, for every* $\lambda \in \mathbb{D}$ *we have*

$$A_0 k_\lambda = \frac{k_\lambda(\mu)}{\overline{g(\lambda)}}. \tag{4}$$

Proof. Let us first discuss the action of A_0 on the reproducing kernels:

$$A_0 k_\lambda(\mu) = (P_+ \frac{k_\lambda}{\overline{g}})(\mu) = \langle \frac{k_\lambda}{\overline{g}}, k_\mu \rangle = \langle k_\lambda, \frac{k_\mu}{g} \rangle = \overline{\left(\frac{k_\mu(\lambda)}{g(\lambda)} \right)} = \frac{k_\lambda(\mu)}{\overline{g(\lambda)}},$$

so that $A_0 k_\lambda = k_\lambda / \overline{g(\lambda)}$.

Note that from this we can deduce the inclusion $A_0 K_I^p \subset K_I^p(|g|^p)$ when I is a Blaschke product with simple zeros Λ, since in that case $k_\lambda^I = k_\lambda$, $\lambda \in \Lambda$, span the space K_I^p, and $k_\lambda = k_\lambda^I \in H^p(|g|^p) \cap I\overline{H_0^p(|g|^p)} = K_I^p(|g|^p)$.

For general inner functions I we need a different argument. Recall that $|g|^p$ is an (A_p) weight. So, the function $G := 1/g$ is in H^q, $1/p + 1/q = 1$. Taking Fejér polynomials G_N of G, we get $G_N \to G$ in H^q (e.g. [Ni02, A3.3.4]), $G_N \in H^\infty$. Then $A_N := P_+ \overline{G}_N$ is a finite linear combination of composed backward shifts thus leaving K_I^q invariant, so that $P_+ \overline{G}_N k_\lambda^I \in K_I^q$ for every $\lambda \in \mathbb{D}$. Now, since $\overline{G}_N k_\lambda^I \to \overline{G} k_\lambda^I$ in H^q, we obtain $A_N k_\lambda^I \to A_0 k_\lambda^I$ which is thus in K_I^q. On the other hand, since $k_\lambda^I \in H^\infty \subset H^p$, we also have $A_0 k_\lambda^I \in H^p(|g|^p)$. Hence $A_0 k_\lambda^I \in K_I^q \cap H^p(|g|^p) \subset K_I^p(|g|^p)$. Note that the projected reproducing kernels k_λ^I, $\lambda \in \mathbb{D}$, generate a dense subspace in K_I^p, so $A_0 K_I^p \subset K_I^p(|g|^p)$.

Let us prove that A_0 is from K_I^p onto $K_I^p(|g|^p)$. To this end, let $h \in K_I^p(|g|^p)$, then $gh \in gK_I^p(|g|^p) = \ker T_{\overline{Ig}/g}$, and since $T_{\overline{Ig}/g} = T_{\overline{I}} T_{\overline{g}/g}$ with $T_{\overline{g}/g}$ invertible, we have $gh \in \ker T_{\overline{Ig}/g}$ if and only if $T_{\overline{g}/g}(gh) \in \ker T_{\overline{I}} = K_I^2$. And so $gh \in T_{\overline{g}/g}^{-1} K_I^2 = gP_+ \frac{1}{\overline{g}} K_I^p$, from where we get $h \in P_+ \frac{1}{\overline{g}} K_I^p = A_0 K_I^p$. ■

We refer the reader to [Dy08], in particular Proposition 1.3, for some discussions of the action of $T_{\overline{G}}$ on K_I^p spaces.

Proposition 4. *There exists an outer function g in H^2 with $|g|^2$ Muckenhoupt (A_2), and an inner function I such that $K_I^2(|g|^2) \not\subset K_I^2$.*

Proof. Take $g(z) = (1-z)^\alpha$ with $\alpha \in (0, 1/2)$. Then $|g|^2$ is (A_2). Let also $I = B_\Lambda$ where $\Lambda = \{1 - 1/2^n\}_n$. In this situation, $\sigma(I) \cap \mathbb{T} = \{1\}$, which corresponds to the point where g vanishes. Clearly, since Λ is an interpolating sequence, the sequence $\{k_{\lambda_n}/\|k_{\lambda_n}\|_2\}_n$ is a normalized unconditional basis in K_I^2. This means that we can write $K_I^2 = l^2(\frac{k_{\lambda_n}}{\|k_{\lambda_n}\|_2})$ meaning that $f \in K_I^2$ if and only if

$$f = \sum_{n \geq 1} \alpha_n \frac{k_{\lambda_n}}{\|k_{\lambda_n}\|_2}$$

with $\sum_{n \geq 1} |\alpha_n|^2 < \infty$ (the last sum defines the square of an equivalent norm in K_I^2).

So, since $|g|^2$ is Muckenhoupt (A_2), we get from (4)

$$\{A_0(k_{\lambda_n}/\|k_{\lambda_n}\|_2)\}_n = \{\frac{k_{\lambda_n}}{\overline{g(\lambda_n)}\|k_{\lambda_n}\|_2}\}_n,$$

and $\{k_{\lambda_n}/(\overline{g(\lambda_n)}\|k_{\lambda_n}\|_2)\}_n$ is thus an unconditional basis in $K_I^2(|g|^2)$ (almost normalized in the sense that $\|A_0(k_{\lambda_n}/\|k_{\lambda_n}\|_2)\|_{|g|^2}$ is comparable to a constant independant of n). Hence for every sequence $\alpha = (\alpha_n)_n$ with $\sum_{n\geq 1} |\alpha_n^2| < \infty$, we have

$$f_\alpha := \sum_{n\geq 1} \alpha_n \frac{k_{\lambda_n}}{\overline{g(\lambda_n)}\|k_{\lambda_n}\|_2} \in K_I^2(|g|^2).$$

Now, in order to construct a function in $K_I^2(|g|^2)$ which is not in K_I^2, it suffices to choose $(\alpha_n)_n$ in such a way that

$$\sum_{n\geq 1} |\alpha_n|^2 < +\infty,$$

so that $f_\alpha \in K_I^2(|g|^2)$, and

$$\sum_{n\geq 1} \left| \frac{\alpha_n}{\overline{g(\lambda_n)}} \right|^2 = +\infty,$$

so that $f_\alpha \notin K_I^2$. Recall that $\lambda_n = 1 - 1/2^n$ and $g(z) = (1-z)^\alpha$. Hence $|g(\lambda_n)|^2 = 2^{-2n\alpha}$. Now, taking e.g. $\alpha_n = 1/n$ the first of the above two sums converges, and $|\alpha_n/\overline{g(\lambda_n)}|^2 = 2^{2n\alpha}/n^2$ which does even not converge to zero. ∎

According to Proposition 3, the function f_α constructed in the proof is in some K_I^r for a suitable $r \in (1,2)$ (this can also be checked directly by choosing $r \in (1,2)$ in such a way that $(\frac{\alpha_n}{g(\lambda_n)} \frac{\|k_{\lambda_n}\|_r}{\|k_{\lambda_n}\|_2})_n \in l^r$, which is possible).

In view of Corollary 2, we will discuss another situation. In that corollary we obtained that when g is uniformly bounded away from zero on $\mathbb{T} \setminus \Gamma$ where Γ is an arc on which $1/g$ is s-integrable, $s > q$, and Γ not meeting $\sigma(I)$, then $K_I^p(|g|^p)$ embeds into K_I^p . We will now be interested in the situation when the arc Γ is replaced by an infinite union of arcs. Our example is constructed for $p = 2$. Then under the weaker assumption $1/g \in L^s$, $s < 2$, the embedding turns out to be false in general.

Proposition 5. *There exists an inner function I, a sequence of disjoint closed arcs $(\Gamma_n)_n$ in $\mathbb{T}$ not meeting the spectrum of I, such that for every $s < 2$ there is an outer function $g \in H^2$ with $1/g \in L^s(\Gamma)$ and $|1/g| \geq \delta$ on $\mathbb{T} \setminus \Gamma$, where $\Gamma = \bigcup_n \Gamma_n$, but $K_I^2(|g|^2) \not\subset K_I^2$.*

Proof. Let $I(z) = \exp \frac{z+1}{z-1}$ which is a singular inner function the associated measure of which is supported on $\{1\}$ (which is equal to $\sigma(I)$). As in the preceding proposition we choose $\Lambda = \{\lambda_n\}_n = \{1 - 1/2^n\}_n$, which is an interpolating sequence in H^2. Moreover, $I(\lambda_n) \to 0$ when $n \to \infty$, so that Λ is also an interpolating sequence for K_I^2 (see [HNP81] or [Ni02, D4.4.9 (8)]). Set $f(z) = (1 - I(z))/(1 - z)$, then

$$|f(\lambda_n)|^2 = \left| \frac{1 - I(\lambda_n)}{1 - \lambda_n} \right|^2 = \frac{1 - e^{1-2^{n+1}}}{1/2^{2n}} \simeq 2^{2n}$$

So, $\sum_n (1 - |\lambda_n|^2)|f(\lambda_n)|^2 \simeq \sum_n 2^n = +\infty$, and f cannot be in H^2 neither in K_I^2.

Still f can be written $f = I\overline{\psi}$ with $\psi \in N^+$ (the Smirnov class) and $\psi(0) = 0$ (more precisely $\psi(z) = zf(z)$). It remains to choose g suitably so that $|f|^2$ is integrable against $|g|^2$.

For this construction we consider the argument of I:

$$\varphi(t) := \arg I(e^{it}) = \operatorname{Im} \frac{1+e^{it}}{1-e^{it}} = \frac{1}{i}\frac{e^{-it}-e^{it}}{|e^{it}-1|^2} = -\frac{\cos t}{1-\cos t}$$

Observe that $\varphi'(t) = \sin t/(1-\cos t)^2$ so that φ is strictly increasing on $(0,\pi)$. Now we consider the intervalls $M_k = [2^{-(k+1)}, 2^{-k})$. We check that on these intervals the function φ increases more than 2π (k sufficiently big). For this let $t \in (0,\pi/2)$, then there exists $\xi_t \in (t, 2t)$ such that

$$|\varphi(2t)-\varphi(t)| = t\frac{\sin \xi_t}{(1-\cos\xi_t)^2} \simeq t\frac{\xi_t}{\xi_t^4/4} \simeq \frac{1}{t^2}$$

(note that $t \le \xi_t \le 2t$). Since the last expression tends to infinity when $t \to 0^+$ there exists an $N \in \mathbb{N}$ such that for every $n \ge N$, $\varphi(2^{-k}) - \varphi(2^{-(k+1)}) \ge 2\pi$. Hence, by the intermediate value theorem, for every $n \ge N$, there exists $t_k \in M_k$ such that $\varphi(t_k) = 0[2\pi]$ and hence $I(e^{it_k}) = 1$ and $f(e^{it_k}) = 0$. Since $t \longmapsto I(e^{it})$ depends continuously on t outside 0, there exists $\delta_k > 0$ such that $|f(e^{it})| \le 1$ for $t \in F_k := [t_k - \delta_k, t_k + \delta_k]$. We will set $E_k = \{e^{it} : t \in F_k\}$ and $\Gamma_k^+ := \{e^{it} : t \in [t_k + \delta_k, t_{k-1} - \delta_{k-1})\} \subset M_k \cup M_{k-1}$ (we can suppose that δ_k is sufficiently small so that Γ_k^+ is non void). We will also use the symmetric arc (with respect to the real axis): $\Gamma_k^- := \overline{\Gamma_k^+}$ and $\Gamma := \Gamma_k^+ \cup \Gamma_k^-$. Then $|\Gamma_k^\pm| \lesssim 1/2^k$. Now for $s < 2$ pick $r \in (s, 2)$ and let $\alpha_k = 1/2^{k/r}$. Define g to be the outer function in H^2 such that

$$|g| = \omega := \sum_k \alpha_k \chi_k + \chi_{\mathbb{T}\setminus\Gamma} \quad \text{a.e. } \mathbb{T},$$

where $\chi_k = \chi_{\Gamma_k}$ is the characteristic function of the set Γ_k and $\Gamma = \bigcup_n \Gamma_n$. The function ω is bounded and $\log$-integrable:

$$\int_{\mathbb{T}} |\log w| dm = \sum_k |\Gamma_k||\log \alpha_k| \lesssim \sum_k \frac{2}{2^k}\frac{k}{r}\log 2 < \infty.$$

Hence g is in $H^\infty \subset H^2$. We check that $1/g \in H^s$.

$$\int_{\mathbb{T}} \frac{1}{|g|^s} dm = |\mathbb{T}\setminus\Gamma| + \sum |\Gamma_k|(2^{k/r})^s \lesssim 1 + 2\sum 2^{k(s/r-1)}$$

which converges since $s < r$. Now

$$\begin{aligned}
\int_{\mathbb{T}} |f|^2|g|^2 &= \int_{\mathbb{T}\setminus\Gamma} \left|\frac{1-I}{1-z}\right|^2 dm + \sum \alpha_k^2 \int_{\Gamma_k} \left|\frac{1-I}{1-z}\right|^2 dm \\
&\lesssim 1 + \sum \frac{8}{(2^{k/r})^2}\frac{1}{2^k}2^{2k} \\
&\lesssim 1 + 8\sum 2^{k(2-1-2/r)}
\end{aligned}$$

which converges since $r < 2$ (note that for the estimate on $\mathbb{T}\setminus\Gamma$, we have used the fact that on E_k the function f is bounded by 1, and on the remaining closed arc joining $t_1 + \delta_1$ to $2\pi - (t_1 + \delta_t)$ it is continuous. ■

Observe that the function g constructed in the previous proof is big (equal to 1) on small sets coming arbitrarily close to 1, and tending to zero on the remaining sets when $e^{it} \to 1$. In particular, such a function cannot satisfy the Muckenhoupt condition.

The examples in the preceding two propositions show that it is not always possible to embed $K_I^2(|g|^2)$ into K_I^2 under different conditions on $|g|$. However in both cases we have the global integrability condition that appeared in Proposition 3, so that in these cases we can embed $K_I^p(|g|^p)$ into a K_I^r for a suitable $r > 1$.

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