

GLOBAL EXISTENCE OF SOLUTIONS FOR THE COUPLED VLASOV AND NAVIER-STOKES EQUATIONS

LAURENT BOUDIN, LAURENT DESVILLETES, CÉLINE GRANDMONT, AND AYMAN MOUSSA

ABSTRACT. In this work, we obtain a result of global existence for weak solutions of the three-dimensional incompressible Vlasov-Navier-Stokes equations, the coupling being done through a drag force which is linear with respect to the relative velocity of fluid and particles.

1. INTRODUCTION

When we consider a spray, that is a dispersed phase of particles (or droplets) moving inside a continuous fluid, we can choose between several models. One of them, first introduced by Williams in the framework of the combustion theory [14] (see also [3]), consists in coupling a kinetic equation with fluid mechanics equations. In such models, the fluid is described through macroscopic quantities (density, velocity, for instance), whereas the particles are represented by a probability density function. We assume that the spray is thin [13], i.e. the kinetic and fluid equations are coupled through a drag term (which depends on both the fluid macroscopic quantities, and the probability density function and its variables), but not through the volume fraction occupied by the fluid, as in the case of the so-called thick sprays.

In this work, we consider the coupling of the three-dimensional incompressible Navier-Stokes equations with a Vlasov-type equation. This system appears in various situations, for instance, when one wants to model the transport, with no diffusive effects, of an aerosol in a Newtonian, viscous, incompressible airflow, inside the human upper airways [5, 1]. We here aim to prove global existence of weak solutions for the three-dimensional Vlasov-Navier-Stokes equations, assuming periodicity with respect to the space variable.

The same kind of model has been studied in [9] where the author considered, in all dimensions, the unsteady Stokes equations coupled with the Vlasov equation in a bounded domain with reflexion boundary conditions. The main differences between the present work and [9] are the following ones.

- We do not neglect the convection term and consider the Navier-Stokes equations.
- Our proof only works in dimension smaller or equal than 3 (because of the use of Sobolev inequalities).
- Our proof is constructive and does not rely on a fixed point argument of Schauder type: we build a sequence of approximated solutions that converges towards a solution of the coupled problem.

Note that Goudon, Jabin and Vasseur [7, 8] studied the hydrodynamic limit of the Vlasov-Navier-Stokes system, for some particular regimes of the dispersed phase. In [11], Mellet and Vasseur proved the existence of global weak solutions to the compressible Vlasov-Fokker-Planck-Navier-Stokes equation (hence including diffusive effects for the particles and compressibility effects for the fluid), for absorption or reflection boundary conditions, and in [12], they investigate some asymptotic regimes of this system.

Local existence of solutions in the case of the compressible Vlasov-Euler equation was proved by Baranger and Desvillettes [2]. One can find, in that same article, references on previous works about fluid-kinetic coupling, in particular, when the fluid is modelled by one single equation [4, 6]. The extension of this result to moderately thick sprays (that is, sprays in which the collisions between droplets has been taken into account) has been done by Mathiaud [10].

We present our model in the next section, along with our main results. Those are proven in Sec. 3.

2. PRESENTATION OF THE MODEL

Let us consider a constant $T > 0$ and set $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$. We investigate the interaction of a (monodispersed) spray of particles with a Newtonian, viscous, and incompressible fluid. The fluid is commonly described by macroscopic quantities, such as its density, here assumed to be constant (equal to 1), its velocity $\mathbf{u}(t, \mathbf{x})$ and pressure $p(t, \mathbf{x})$ at time $t \in [0, T]$ and position $\mathbf{x} \in \mathbb{T}^3$. The evolution of (\mathbf{u}, p) is here governed by the incompressible Navier-Stokes equations. The particles are described by a density function f in the phase space which solves a Vlasov-like equation. The quantity $f(t, \mathbf{x}, \mathbf{v})$ is the density of particles located in $\mathbf{x} \in \mathbb{T}^3$ at time t which have the velocity $\mathbf{v} \in \mathbb{R}^3$. Equations for the fluid and the spray are coupled through a drag force, which depends on the relative velocity of the fluid and the particles, namely $\mathbf{u} - \mathbf{v}$ and on the density function f . More precisely, f , \mathbf{u} and p satisfy the following system

$$\begin{aligned} (1) \quad & \partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \nabla_{\mathbf{v}} \cdot [(\mathbf{u} - \mathbf{v})f] = 0 \quad \text{in } (0, T) \times \mathbb{T}^3 \times \mathbb{R}^3, \\ (2) \quad & \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_{\mathbf{x}}) \mathbf{u} + \nabla_{\mathbf{x}} p - \Delta_{\mathbf{x}} \mathbf{u} = - \int_{\mathbb{R}^3} f(t, \mathbf{x}, \mathbf{v})(\mathbf{u}(t, \mathbf{x}) - \mathbf{v}) \, d\mathbf{v} + \mathbf{F}_{\text{ext}} \quad \text{in } (0, T) \times \mathbb{T}^3, \\ (3) \quad & \nabla_{\mathbf{x}} \cdot \mathbf{u} = 0 \quad \text{in } (0, T) \times \mathbb{T}^3, \end{aligned}$$

where \mathbf{F}_{ext} is an external force applied to the fluid that is periodic in space. The system has to be completed by initial conditions for \mathbf{u} and f , which are

$$\begin{aligned} (4) \quad & f(0, \mathbf{x}, \mathbf{v}) = f_{\text{in}}(\mathbf{x}, \mathbf{v}), \quad \mathbf{x} \in \mathbb{T}^3, \quad \mathbf{v} \in \mathbb{R}^3, \\ (5) \quad & \mathbf{u}(0, \mathbf{x}) = \mathbf{u}_{\text{in}}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{T}^3, \end{aligned}$$

where $f_{\text{in}} \geq 0$ and \mathbf{u}_{in} are given.

Remark 1. *Note that we here make the thin spray assumption. First, there is no interaction (collisions or coalescences) between particles, which implies that (1) is linear with respect to f . Secondly, the drag force is the only coupling phenomenon between the spray and the fluid and appears in both (1) and (2): the volume fraction of the spray remains negligible with respect to the fluid volume fraction, and is not taken into account explicitly.*

If we assume that all quantities are smooth enough, system (1)–(5) satisfies, at least formally, an energy equality. It is obtained in the following way. We first multiply the Navier-Stokes equations (2) by \mathbf{u} and integrate over \mathbb{T}^3 . Then we multiply the Vlasov equation (1) by $|\mathbf{v}|^2/2$ and integrate over $\mathbb{T}^3 \times \mathbb{R}^3$. Finally, we add both previous equalities and obtain, after integrations by parts,

$$(6) \quad \frac{1}{2} \frac{d}{dt} \|\mathbf{u}\|_{L^2(\mathbb{T}^3)}^2 + \frac{1}{2} \frac{d}{dt} \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f(t, \mathbf{x}, \mathbf{v}) |\mathbf{v}|^2 + \|\nabla_{\mathbf{x}} \mathbf{u}\|_{L^2(\mathbb{T}^3)}^2 + \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f(\mathbf{u} - \mathbf{v})^2 = \int_{\mathbb{T}^3} \mathbf{F}_{\text{ext}} \cdot \mathbf{u}.$$

In the previous equation, the first two terms represent the total kinetic energy of the coupled system, whereas the next two ones represent the energy dissipation coming from, respectively, the fluid itself and the drag force. Note that, since f_{in} is nonnegative, the density function f and consequently $\iint_{\mathbb{T}^3 \times \mathbb{R}^3} f(\mathbf{u} - \mathbf{v})^2$ also remain positive.

Remark 2. *When no external force is applied, the global energy $\frac{1}{2} \|\mathbf{u}(t)\|_{L^2(\mathbb{T}^3)}^2 + \frac{1}{2} \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f(t, \mathbf{x}, \mathbf{v}) |\mathbf{v}|^2$ of the system is decreasing. We can also point out that, in the same case, the global momentum of the system is conserved.*

In the sequel, we shall use the

Assumption 1.

$$(7) \quad f_{\text{in}} \in L^\infty(\mathbb{T}^3 \times \mathbb{R}^3), \quad \int_{\mathbb{R}^3} |\mathbf{v}|^2 f_{\text{in}} \, d\mathbf{v} \in L^\infty((0, T) \times \mathbb{T}^3), \quad \mathbf{u}_{\text{in}} \in L^2(\mathbb{T}^3), \quad \mathbf{F}_{\text{ext}} \in L^2(0, T; L^2(\mathbb{T}^3)).$$

Under this assumption, the energy equality gives our functional framework. More precisely, we set

$$\begin{aligned}\mathcal{H} &= \{ \mathbf{w} \in L^2(\mathbb{T}^3) \mid \nabla_{\mathbf{x}} \cdot \mathbf{w} = 0 \}, \\ \mathcal{V} &= \{ \mathbf{w} \in H^1(\mathbb{T}^3) \mid \nabla_{\mathbf{x}} \cdot \mathbf{w} = 0 \}.\end{aligned}$$

Definition 1. We say that (\mathbf{u}, f) is a weak solution of (1)–(5) if the following conditions are satisfied:

- $\mathbf{u} \in L^\infty(0, T; \mathcal{H}) \cap L^2(0, T; \mathcal{V}) \cap \mathcal{C}^0([0, T]; \mathcal{V}')$,
- $f(t, \mathbf{x}, \mathbf{v}) \geq 0$, for any $(t, \mathbf{x}, \mathbf{v}) \in (0, T) \times \mathbb{T}^3 \times \mathbb{R}^3$,
- $f \in L^\infty(0, T; L^\infty(\mathbb{T}^3 \times \mathbb{R}^3) \cap L^1(\mathbb{T}^3 \times \mathbb{R}^3))$,
- $f|\mathbf{v}|^2 \in L^\infty(0, T; L^1(\mathbb{T}^3 \times \mathbb{R}^3))$,
- for all $\phi \in \mathcal{C}^1([0, T] \times \mathbb{T}^3 \times \mathbb{R}^3)$ with compact support in \mathbf{v} , such that $\phi(T, \cdot, \cdot) = 0$,

$$(8) \quad - \int_0^T \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f [\partial_t \phi + \mathbf{v} \cdot \nabla_{\mathbf{x}} \phi + (\mathbf{u} - \mathbf{v}) \cdot \nabla_{\mathbf{v}} \phi] \, d\mathbf{x} \, d\mathbf{v} \, ds = \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f_{\text{in}} \phi(0, \cdot, \cdot) \, d\mathbf{x} \, d\mathbf{v},$$

- for all $\psi \in \mathcal{C}^1([0, T] \times \mathbb{T}^3)$ such that $\nabla_{\mathbf{x}} \cdot \psi = 0$, for a. e. t ,

$$(9) \quad \begin{aligned} & \int_{\mathbb{T}^3} \mathbf{u}(t) \cdot \psi(t) \, d\mathbf{x} - \int_0^t \int_{\mathbb{T}^3} \mathbf{u} \cdot \partial_t \psi \, d\mathbf{x} \, ds + \int_0^t \int_{\mathbb{T}^3} (\mathbf{u} \cdot \nabla_{\mathbf{x}}) \mathbf{u} \cdot \psi \, d\mathbf{x} \, ds + \int_0^t \int_{\mathbb{T}^3} \nabla_{\mathbf{x}} \mathbf{u} : \nabla_{\mathbf{x}} \psi \, d\mathbf{x} \, ds \\ &= - \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f(\mathbf{u} - \mathbf{v}) \cdot \psi \, d\mathbf{x} \, d\mathbf{v} \, ds + \int_0^t \int_{\mathbb{T}^3} \mathbf{F}_{\text{ext}} \cdot \psi \, d\mathbf{x} \, ds + \int_{\mathbb{T}^3} \mathbf{u}_{\text{in}} \cdot \psi(0, \cdot) \, d\mathbf{x}. \end{aligned}$$

In the following, the moments of f are denoted

$$\begin{aligned} m_\alpha f(t, \mathbf{x}) &= \int_{\mathbb{R}^3} f(t, \mathbf{x}, \mathbf{v}) |\mathbf{v}|^\alpha \, d\mathbf{v}, \\ M_\alpha f(t) &= \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f(t, \mathbf{x}, \mathbf{v}) |\mathbf{v}|^\alpha \, d\mathbf{x} \, d\mathbf{v}, \end{aligned}$$

for any $t \in [0, T]$, $\mathbf{x} \in \mathbb{T}^3$ and $\alpha \geq 0$. Note that

$$M_\alpha f(t) = \int_{\mathbb{T}^3} m_\alpha f(t, \mathbf{x}) \, d\mathbf{x}.$$

The main result of this work is hereby stated.

Theorem 1. Let $T > 0$. Under Assumption 1, there exists at least one weak solution (f, \mathbf{u}) to (1)–(5) on $(0, T)$ in the sense of Definition 1. Moreover, this solution satisfies the energy equality (6), and the L^∞ bound

$$(10) \quad \|f\|_{L^\infty((0, T) \times \mathbb{T}^3 \times \mathbb{R}^3)} \leq e^{3T} \|f_{\text{in}}\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)}.$$

Proof of Theorem 1. The proof of Thm. 1 is split into three main steps.

In the first one, we build a sequence of approximated solutions where the convection velocity of the Navier-Stokes equations and the fluid velocity appearing in the Vlasov equation are regularized. We also cut off the high particle velocities in the drag force applied by the spray on the flow. In order to prove the existence of a solution to this regularized but still coupled system, we build a sequence of uncoupled problems where we first solve the kinetic part, then the fluid part. We prove that the sequence of solutions of these regularized uncoupled systems converges towards the solution of the regularized system. This is the object of Prop. 2 below.

In a second step, we prove that the approximated solutions satisfy uniform bounds (with respect to the regularization) on a certain time interval allowing to pass to the limit when the parameter of regularization tends to 0. This is the object of Prop. 1 below.

We eventually prove that the local (in time) solution obtained in the previous step can in fact be extended to the whole interval $[0, T]$, thanks to the energy estimate.

2.1. The regularized system. First we define the regularized system. Let $\varepsilon > 0$ and define θ_ε a mollifier such that $\theta_\varepsilon = \varepsilon^3 \theta(\mathbf{x}/\varepsilon)$ with $\theta \in C^\infty(\mathbb{R}^3)$, $\theta \geq 0$, and $\int_{\mathbb{R}^3} \theta = 1$. Moreover, we introduce $\gamma_\varepsilon \in C^\infty(\mathbb{R}^3)$ whose support is included in the ball $B(0, 1/\varepsilon)$, $0 \leq \gamma_\varepsilon \leq 1$, $\gamma_\varepsilon = 1$ on $B(0, 1/2\varepsilon)$ and such that $\gamma_\varepsilon(\mathbf{v}) \rightarrow 1$ as ε goes to zero. We then consider the following regularization of the initial problem:

$$(11) \quad \partial_t f_\varepsilon + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_\varepsilon + \nabla_{\mathbf{v}} \cdot [(\theta_\varepsilon \star \mathbf{u}_\varepsilon - \mathbf{v}) f_\varepsilon] = 0 \quad \text{in } (0, T) \times \mathbb{T}^3 \times \mathbb{R}^3,$$

$$(12) \quad \begin{aligned} \partial_t \mathbf{u}_\varepsilon + ((\theta_\varepsilon \star \mathbf{u}_\varepsilon) \cdot \nabla_{\mathbf{x}}) \mathbf{u}_\varepsilon + \nabla_{\mathbf{x}} p_\varepsilon - \Delta_{\mathbf{x}} \mathbf{u}_\varepsilon \\ = - \int_{\mathbb{R}^3} f_\varepsilon(t, \mathbf{x}, \mathbf{v}) (\mathbf{u}_\varepsilon(t, \mathbf{x}) - \mathbf{v}) \gamma_\varepsilon(\mathbf{v}) \, d\mathbf{v} + \mathbf{F}_{\text{ext}}^\varepsilon \quad \text{in } (0, T) \times \mathbb{T}^3, \end{aligned}$$

$$(13) \quad \nabla_{\mathbf{x}} \cdot \mathbf{u}_\varepsilon = 0 \quad \text{in } (0, T) \times \mathbb{T}^3,$$

where $\mathbf{F}_{\text{ext}}^\varepsilon$ is a space regularization of the force \mathbf{F}_{ext} .

We complete the system by initial conditions for \mathbf{u}_ε and f_ε :

$$(14) \quad f_\varepsilon(0, \mathbf{x}, \mathbf{v}) = f_{\text{in}}^\varepsilon(\mathbf{x}, \mathbf{v}), \quad \mathbf{x} \in \mathbb{T}^3, \quad \mathbf{v} \in \mathbb{R}^3,$$

$$(15) \quad \mathbf{u}_\varepsilon(0, \mathbf{x}) = \mathbf{u}_{\text{in}}^\varepsilon(\mathbf{x}), \quad \mathbf{x} \in \mathbb{T}^3,$$

where $f_{\text{in}}^\varepsilon$ and $\mathbf{u}_{\text{in}}^\varepsilon$ are C^∞ approximations of f_{in} and \mathbf{u}_{in} such that $(f_{\text{in}}^\varepsilon)$ converges to f_{in} strongly in $L^p(\mathbb{T}^3 \times \mathbb{R}^3)$, for all $p < \infty$, and weakly in $w^*-L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)$, $(M_2 f_{\text{in}}^\varepsilon)$ is uniformly bounded with respect to ε , and $(\mathbf{u}_{\text{in}}^\varepsilon)$ strongly converges to \mathbf{u}_{in} in $L^2(\mathbb{T}^3)$. We can also assume, without any loss of generality, that $f_{\text{in}}^\varepsilon$ has a compact support in \mathbf{v} .

We prove that there exists a solution $(\mathbf{u}_\varepsilon, f_\varepsilon)$ of (11)–(15) in $(L^2(0, T; H^1(\mathbb{T}^3)) \cap L^\infty(0, T; L^2(\mathbb{T}^3))) \times L^\infty(0, T; L^\infty(\mathbb{T}^3 \times \mathbb{R}^3))$ such that $\partial_t \mathbf{u}_\varepsilon \in L^2(0, T; L^2(\mathbb{T}^3))$, and satisfying the following variational formulation:

- for any $\phi \in C^1([0, T] \times \mathbb{T}^3 \times \mathbb{R}^3)$, for a. e. t ,

$$(16) \quad \begin{aligned} \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon(t) \phi(t, \cdot, \cdot) \, d\mathbf{x} \, d\mathbf{v} - \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon [\partial_t \phi + \mathbf{v} \cdot \nabla_{\mathbf{x}} \phi + (\theta_\varepsilon \star \mathbf{u}_\varepsilon - \mathbf{v}) \cdot \nabla_{\mathbf{v}} \phi] \, d\mathbf{x} \, d\mathbf{v} \, ds \\ = \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f_{\text{in}}^\varepsilon \phi(0, \cdot, \cdot) \, d\mathbf{x} \, d\mathbf{v}, \end{aligned}$$

- for all $\psi \in C^1([0, T] \times \mathbb{T}^3)$ such that $\nabla_{\mathbf{x}} \cdot \psi = 0$, for a. e. t ,

$$(17) \quad \begin{aligned} \int_0^t \int_{\mathbb{T}^3} \partial_t \mathbf{u}_\varepsilon \cdot \psi \, d\mathbf{x} \, ds + \int_0^t \int_{\mathbb{T}^3} ((\theta_\varepsilon \star \mathbf{u}_\varepsilon) \cdot \nabla_{\mathbf{x}}) \mathbf{u}_\varepsilon \psi \, d\mathbf{x} \, ds + \int_0^t \int_{\mathbb{T}^3} \nabla_{\mathbf{x}} \mathbf{u}_\varepsilon : \nabla_{\mathbf{x}} \psi \, d\mathbf{x} \, ds \\ = - \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon (\mathbf{u}_\varepsilon - \mathbf{v}) \gamma_\varepsilon(\mathbf{v}) \cdot \psi \, d\mathbf{x} \, d\mathbf{v} \, ds + \int_0^t \int_{\mathbb{T}^3} \mathbf{F}_{\text{ext}}^\varepsilon \cdot \psi \, d\mathbf{x} \, ds. \end{aligned}$$

We also obtain the following proposition, which is crucial to get the asymptotics $\varepsilon \rightarrow 0$ and obtain the existence of at least one weak solution to the original problem (1)–(3).

Proposition 1. *There exists a time $T^* \in (0, T]$, only depending on T and both quantities $\|\mathbf{u}_{\text{in}}\|_{L^2(\mathbb{T}^3)}^2 + |M_2 f_{\text{in}}| + \|\mathbf{F}_{\text{ext}}\|_{L^2(0, T; L^2(\mathbb{T}^3))}^2$ and $\|f_{\text{in}}\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)}$, such that*

$$(18) \quad \begin{aligned} \|\mathbf{u}_\varepsilon\|_{L^\infty(0, T^*; L^2(\mathbb{T}^3))}^2 + \|\nabla_{\mathbf{x}} \mathbf{u}_\varepsilon\|_{L^2(0, T^*; L^2(\mathbb{T}^3))}^2 + \|M_2 f_\varepsilon\|_{L^\infty(0, T^*)} \\ \leq C(T, T^*, \|\mathbf{u}_{\text{in}}\|_{L^2(\mathbb{T}^3)}^2 + |M_2 f_{\text{in}}| + \|\mathbf{F}_{\text{ext}}\|_{L^2(0, T; L^2(\mathbb{T}^3))}^2, \|f_{\text{in}}\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)}). \end{aligned}$$

Here $C(A, B, \dots)$ denotes a constant depending on A, B, \dots

Moreover, we have

$$(19) \quad \|f_\varepsilon\|_{L^\infty(0, T) \times \mathbb{T}^3 \times \mathbb{R}^3} \leq e^{3T} \|f_{\text{in}}\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)}.$$

Finally, there exists $(f, \mathbf{u}) \in L^\infty((0, T^*) \times \mathbb{T}^3 \times \mathbb{R}^3) \times (L^\infty(0, T^*; L^2(\mathbb{T}^3)) \cap L^2(0, T^*; H^1(\mathbb{T}^3)) \times C^0([0, T^*]; \mathcal{V}')$ such that, up to a subsequence, (f_ε) weakly converges to f in $w^*-L^\infty((0, T^*) \times \mathbb{T}^3 \times \mathbb{R}^3)$ and (\mathbf{u}_ε) weakly converges to \mathbf{u} in $w-L^2(0, T^*; w-H^1(\mathbb{T}^3))$ and in $w^*-L^\infty(0, T^*; w-L^2(\mathbb{T}^3))$. Moreover, (\mathbf{u}_ε) strongly converges to \mathbf{u} in $L^2(0, T^*; L^2(\mathbb{T}^3))$.

Finally, in order to prove the existence of a solution to the regularized system (11)–(15), we have to use another system, which is defined in the next subsection.

2.2. The regularized and uncoupled system. For a fixed $\varepsilon > 0$, let us consider the following problem, for any $n \in \mathbb{N}$, defined by $\mathbf{u}_\varepsilon^0 = \mathbf{u}_{\text{in}}^\varepsilon$ and

$$(20) \quad \partial_t f_\varepsilon^n + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_\varepsilon^n + \nabla_{\mathbf{v}} \cdot [(\theta_\varepsilon \star \mathbf{u}_\varepsilon^n - \mathbf{v}) f_\varepsilon^n] = 0 \quad \text{in } (0, T) \times \mathbb{T}^3 \times \mathbb{R}^3,$$

$$(21) \quad \begin{aligned} \partial_t \mathbf{u}_\varepsilon^{n+1} + ((\theta_\varepsilon \star \mathbf{u}_\varepsilon^{n+1}) \cdot \nabla_{\mathbf{x}}) \mathbf{u}_\varepsilon^{n+1} + \nabla_{\mathbf{x}} p_\varepsilon^{n+1} - \Delta_{\mathbf{x}} \mathbf{u}_\varepsilon^{n+1} \\ = - \int_{\mathbb{R}^3} f_\varepsilon^n(t, \mathbf{x}, \mathbf{v}) (\mathbf{u}_\varepsilon^n(t, \mathbf{x}) - \mathbf{v}) \gamma_\varepsilon(\mathbf{v}) \, d\mathbf{v} + \mathbf{F}_{\text{ext}}^\varepsilon \quad \text{in } (0, T) \times \mathbb{T}^3, \end{aligned}$$

$$(22) \quad \nabla_{\mathbf{x}} \cdot \mathbf{u}_\varepsilon^{n+1} = 0 \quad \text{in } (0, T) \times \mathbb{T}^3,$$

supplemented with initial conditions (14)–(15).

For the sake of simplicity, we shall drop the subscript ε , and denote, for instance, f_ε^n or \mathbf{u}_ε^n by f^n or \mathbf{u}^n .

Proposition 2. *For any $n \in \mathbb{N}$, there exists a unique solution (f^n, \mathbf{u}^n) of (20)–(22), supplemented with initial conditions (14)–(15), in $C^1([0, T]; C^\infty(\mathbb{T}^3 \times \mathbb{R}^3)) \times (H^1(0, T; L^2(\mathbb{T}^3)) \cap L^2(0, T; H^1(\mathbb{T}^3)))$.*

Moreover it satisfies the following estimates:

$$(23) \quad \|f^n\|_{L^\infty((0, T) \times \mathbb{T}^3 \times \mathbb{R}^3)} \leq e^{3T} \|f_{\text{in}}\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)},$$

$$(24) \quad \|\mathbf{u}^n\|_{L^\infty(0, T; L^2(\mathbb{T}^3)) \cap L^2(0, T; H^1(\mathbb{T}^3))} \leq C(\varepsilon),$$

$$(25) \quad \|\partial_t \mathbf{u}^n\|_{L^2(0, T; L^2(\mathbb{T}^3))} \leq C(\varepsilon),$$

where $C(\varepsilon)$ does not depend on n . Furthermore, the sequence (f^n, \mathbf{u}^n) strongly converges in $L^\infty((0, T) \times \mathbb{T}^3 \times \mathbb{R}^3) \times L^\infty(0, T; L^2(\mathbb{T}^3))$ towards a variational solution of (11) – (15) [that is, satisfying (16), (17)].

3. PROOF OF THE MAIN RESULTS

We first study the properties of the sequence $(f^n, \mathbf{u}^n)_{n \in \mathbb{N}}$ defined by (20)–(22), and derive bounds needed for the asymptotics $n \rightarrow +\infty$. We can then obtain the existence of $(f_\varepsilon, \mathbf{u}_\varepsilon)_{\varepsilon > 0}$ solution of the regularized problems (11)–(15), that is Proposition 2.

3.1. Proof of Proposition 2.

3.1.1. Existence and uniqueness of (f^n, \mathbf{u}^n) . First we focus on the kinetic equation (20). For any n , let us assume that \mathbf{u}^n is given and belongs to $H^1(0, T; L^2(\mathbb{T}^3)) \cap L^2(0, T; H^1(\mathbb{T}^3))$. Since we have regularized the fluid velocity appearing in the kinetic equation, the unique solution f^n of (20) can be defined thanks to the method of characteristics.

Indeed, $g^n = e^{-3t} f^n$ satisfies the transport equation:

$$\partial_t g^n + \mathbf{v} \cdot \nabla_{\mathbf{x}} g^n + (\theta_\varepsilon \star \mathbf{u}^n - \mathbf{v}) \cdot \nabla_{\mathbf{v}} g^n = 0.$$

The solution of the previous equation can be written in terms of characteristics. As a matter of fact, let us consider the solutions of the following system

$$(26) \quad \frac{d\mathbf{x}^n}{dt}(t) = \mathbf{v}^n(t),$$

$$(27) \quad \frac{d\mathbf{v}^n}{dt}(t) = (\theta_\varepsilon \star \mathbf{u}^n)(t, \mathbf{x}^n(t)) - \mathbf{v}^n(t),$$

with initial conditions $\mathbf{x}^n(0) = \mathbf{x}$ and $\mathbf{v}^n(0) = \mathbf{v}$, and set $\chi^n(t, \mathbf{x}, \mathbf{v}) = (\mathbf{x}^n(t), \mathbf{v}^n(t))$ for any $(t, \mathbf{x}, \mathbf{v})$. Then we have

$$(28) \quad g^n(t, \mathbf{x}, \mathbf{v}) = f_{\text{in}}^\varepsilon(\chi^n(t, \mathbf{x}, \mathbf{v})), \quad \forall t, \mathbf{x}, \mathbf{v}.$$

Since $\mathbf{u}^n \in H^1(0, T; L^2(\mathbb{T}^3))$, $\theta \star \mathbf{u}^n \in C^0([0, T]; C^\infty(\mathbb{T}^3 \times \mathbb{R}^3))$, and consequently f^n lies in $C^1([0, T]; C^\infty(\mathbb{T}^3 \times \mathbb{R}^3))$.

Next we focus on the modified Navier-Stokes equation (21).

Assume that (f^n, \mathbf{u}^n) is given in $L^\infty((0, T) \times \mathbb{T}^3 \times \mathbb{R}^3) \times L^\infty(0, T; L^2(\mathbb{T}^3))$. Since we cut off the high particle velocities, the drag force $\int_{\mathbb{R}^3} f^n(\mathbf{v} - \mathbf{u}^n) \gamma(\mathbf{v}) d\mathbf{v}$ at least belongs to $L^2(0, T; L^2(\mathbb{T}^3))$. Applying standard results about the Navier-Stokes system (with a mollified convection term), one can prove the existence and uniqueness of $\mathbf{u}^{n+1} \in L^\infty(0, T; L^2(\mathbb{T}^3)) \cap L^2(0, T; H^1(\mathbb{T}^3))$ solving (21).

Note that the uniqueness relies on the fact that the convection velocity has been regularized. Moreover, since the convection velocity has been regularized and the initial velocity is smooth, one can check that $\partial_t \mathbf{u}^n \in L^2(0, T; L^2(\mathbb{T}^3))$. By a simple induction argument, these properties hold for all $n \in \mathbb{N}$.

In the next subsection, we derive uniform estimates in these spaces with respect to n satisfied by the sequence (f^n, \mathbf{u}^n) .

3.1.2. Uniform estimates for the sequence (f^n, \mathbf{u}^n) . Using (28), we immediately obtain (23). Note that this estimate does not depend on the fluid velocity \mathbf{u}^n .

To obtain the uniform bounds on $(\mathbf{u}^n)_{n \in \mathbb{N}}$, let us multiply (21) by \mathbf{u}^{n+1} and integrate over \mathbb{T}^3 . Since the fluid is incompressible and the problem periodic in the space variable, we have

$$\int_{\mathbb{T}^3} ((\theta_\varepsilon \star \mathbf{u}^{n+1}) \cdot \nabla_{\mathbf{x}}) \mathbf{u}^{n+1} \cdot \mathbf{u}^{n+1} d\mathbf{x} = 0.$$

Moreover, thanks to (23) and the use the cutoff function γ_ε ,

$$\left| \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f^n(\mathbf{v} - \mathbf{u}^n) \cdot \mathbf{u}^{n+1} \gamma_\varepsilon(\mathbf{v}) d\mathbf{x} d\mathbf{v} \right| \leq C(\varepsilon) \left(1 + \|\mathbf{u}^n\|_{L^2(\mathbb{T}^3)}^2 + \|\mathbf{u}^{n+1}\|_{L^2(\mathbb{T}^3)}^2 \right).$$

Consequently, using the periodicity, the incompressibility and the fact that

$$\left| \int_{\mathbb{T}^3} \mathbf{F}_{\text{ext}}^\varepsilon \cdot \mathbf{u}^{n+1} d\mathbf{x} \right| \leq \frac{1}{2} \|\mathbf{u}^{n+1}\|_{L^2(\mathbb{T}^3)}^2 + \frac{1}{2} \|\mathbf{F}_{\text{ext}}^\varepsilon\|_{L^2(\mathbb{T}^3)}^2,$$

we get

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{u}^{n+1}\|_{L^2(\mathbb{T}^3)}^2 + \|\nabla_{\mathbf{x}} \mathbf{u}^{n+1}\|_{L^2(\mathbb{T}^3)}^2 \leq C(\varepsilon) \left(1 + \|\mathbf{u}^n\|_{L^2(\mathbb{T}^3)}^2 + \|\mathbf{u}^{n+1}\|_{L^2(\mathbb{T}^3)}^2 \right).$$

This implies that $(\mathbf{u}^n)_{n \in \mathbb{N}}$ is uniformly bounded in $L^\infty(0, T; L^2(\mathbb{T}^3)) \cap L^2(0, T; H^1(\mathbb{T}^3))$ (see Lemma 3 in Appendix A).

Estimate (25) is obtained by multiplying (21) by $\partial_t \mathbf{u}^{n+1}$. Indeed, integrating over \mathbb{T}^3 , using estimate (24) on (\mathbf{u}^n) , the fact that $\int_{\mathbb{R}^3} f^n(\mathbf{v} - \mathbf{u}^n) \gamma(\mathbf{v}) d\mathbf{v}$ is uniformly bounded in $L^2(0, T; L^2(\mathbb{T}^3))$ with respect to n and the fact that the convection velocity has been regularized, we deduce the uniform bound of $(\partial_t \mathbf{u}^n)_{n \in \mathbb{N}}$ in $L^2(0, T; L^2(\mathbb{T}^3))$.

Remark 3. One could also obtain a uniform (in n) bound of (f^n) , for instance, in $W^{1,\infty}((0, T) \times \mathbb{T}^3 \times \mathbb{R}^3)$.

3.1.3. *Convergence.* We now prove that the sequence (f^n, \mathbf{u}^n) is convergent in $L^\infty((0, T) \times \mathbb{T}^3 \times \mathbb{R}^3) \times L^\infty(0, T; L^2(\mathbb{T}^3))$. To that purpose, we study the differences $\mathbf{w}^{n+1} = \mathbf{u}^{n+1} - \mathbf{u}^n$ and $f^{n+1} - f^n$, for any $n \in \mathbb{N}$.

The function \mathbf{w}^{n+1} satisfies the following weak formulation:

$$\begin{aligned} \int_{\mathbb{T}^3} \partial_t \mathbf{w}^{n+1} \cdot \boldsymbol{\psi} + \int_{\mathbb{T}^3} ((\theta_\varepsilon \star \mathbf{w}^{n+1}) \cdot \nabla_{\mathbf{x}}) \mathbf{u}^{n+1} \cdot \boldsymbol{\psi} + \int_{\mathbb{T}^3} ((\theta_\varepsilon \star \mathbf{u}^n) \cdot \nabla_{\mathbf{x}}) \mathbf{w}^{n+1} \cdot \boldsymbol{\psi} + \int_{\mathbb{T}^3} \nabla_{\mathbf{x}} \mathbf{w}^{n+1} : \nabla_{\mathbf{x}} \boldsymbol{\psi} \\ = \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f^n(\mathbf{u}^n - \mathbf{v}) \gamma(\mathbf{v}) \cdot \boldsymbol{\psi} - \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f^{n-1}(\mathbf{u}^{n-1} - \mathbf{v}) \gamma(\mathbf{v}) \cdot \boldsymbol{\psi}, \end{aligned}$$

for any $\boldsymbol{\psi} \in L^2(0, T; H^1(\mathbb{T}^3))$ such that $\nabla_{\mathbf{x}} \cdot \boldsymbol{\psi} = 0$.

The function $\boldsymbol{\psi} = \mathbf{w}^{n+1}$ is then an admissible test function. The only terms which we must take care of are the nonlinear ones: the convection one

$$T_1 = \int_{\mathbb{T}^3} ((\theta_\varepsilon \star \mathbf{w}^{n+1}) \cdot \nabla_{\mathbf{x}}) \mathbf{u}^{n+1} \cdot \mathbf{w}^{n+1} + \int_{\mathbb{T}^3} ((\theta_\varepsilon \star \mathbf{u}^n) \cdot \nabla_{\mathbf{x}}) \mathbf{w}^{n+1} \cdot \mathbf{w}^{n+1},$$

and the drag force one

$$T_2 = \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f^n(\mathbf{u}^n - \mathbf{v}) \gamma(\mathbf{v}) \cdot \mathbf{w}^{n+1} - \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f^{n-1}(\mathbf{u}^{n-1} - \mathbf{v}) \gamma(\mathbf{v}) \cdot \mathbf{w}^{n+1}.$$

Thanks to the space periodicity and the fluid incompressibility, we have

$$\int_{\mathbb{T}^3} ((\theta \star_\varepsilon \mathbf{w}^{n+1}) \cdot \nabla_{\mathbf{x}}) \mathbf{u}^{n+1} \cdot \mathbf{w}^{n+1} = - \int_{\mathbb{T}^3} ((\theta \star \mathbf{w}^{n+1}) \cdot \nabla_{\mathbf{x}}) \mathbf{w}^{n+1} \cdot \mathbf{u}^{n+1},$$

and

$$\int_{\mathbb{T}^3} ((\theta \star_\varepsilon \mathbf{u}^n) \cdot \nabla_{\mathbf{x}}) \mathbf{w}^{n+1} \cdot \mathbf{w}^{n+1} = 0.$$

Consequently, using (24), we successively get

$$\begin{aligned} |T_1| &\leq \|\theta_\varepsilon \star \mathbf{w}^{n+1}\|_{L^\infty(\mathbb{T}^3)} \|\mathbf{u}^{n+1}\|_{L^2(\mathbb{T}^3)} \|\nabla_{\mathbf{x}} \mathbf{w}^{n+1}\|_{L^2(\mathbb{T}^3)}, \\ &\leq C(\varepsilon) \|\mathbf{w}^{n+1}\|_{L^2(\mathbb{T}^3)} \|\nabla_{\mathbf{x}} \mathbf{w}^{n+1}\|_{L^2(\mathbb{T}^3)}. \end{aligned}$$

Finally, we have

$$|T_1| \leq C(\varepsilon) \|\mathbf{w}^{n+1}\|_{L^2(\mathbb{T}^3)}^2 + \frac{1}{2} \|\nabla_{\mathbf{x}} \mathbf{w}^{n+1}\|_{L^2(\mathbb{T}^3)}^2.$$

We now take care of T_2 :

$$T_2 = \iint_{\mathbb{T}^3 \times \mathbb{R}^3} (f^n - f^{n-1})(\mathbf{u}^n - \mathbf{v}) \gamma_\varepsilon(\mathbf{v}) \cdot \mathbf{w}^{n+1} - \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f^{n-1}(\mathbf{u}^n - \mathbf{u}^{n-1}) \gamma_\varepsilon(\mathbf{v}) \cdot \mathbf{w}^{n+1}.$$

Taking into account the cutoff function and (23)–(24), we get

$$\begin{aligned} |T_2| &\leq C(\varepsilon) (\|f^n - f^{n-1}\|_{L^\infty((0, T) \times \mathbb{T}^3 \times \mathbb{R}^3)} \|\mathbf{w}^{n+1}\|_{L^2(\mathbb{T}^3)} + \|\mathbf{w}^{n+1}\|_{L^2(\mathbb{T}^3)} \|\mathbf{w}^n\|_{L^2(\mathbb{T}^3)}) \\ (29) \quad &\leq C(\varepsilon) \left(\|f^n - f^{n-1}\|_{L^\infty((0, T) \times \mathbb{T}^3 \times \mathbb{R}^3)}^2 + \|\mathbf{w}^{n+1}\|_{L^2(\mathbb{T}^3)}^2 + \|\mathbf{w}^n\|_{L^2(\mathbb{T}^3)}^2 \right). \end{aligned}$$

The quantity $\|f^n - f^{n-1}\|_{L^\infty((0, T) \times \mathbb{T}^3 \times \mathbb{R}^3)}^2$ has now to be estimated. We recall that, for all $n \in \mathbb{N}$ and $(t, \mathbf{x}, \mathbf{v})$, $f^n(t, \mathbf{x}, \mathbf{v}) = e^{3t} f_{\text{in}}^\varepsilon(\boldsymbol{\chi}^n(t, \mathbf{x}, \mathbf{v}))$ and that $f_{\text{in}}^\varepsilon$ has been mollified. Consequently, we have

$$(30) \quad \|f^n - f^{n-1}\|_{L^\infty((0, T) \times \mathbb{T}^3 \times \mathbb{R}^3)} \leq C(\varepsilon, T) \|\boldsymbol{\chi}^n - \boldsymbol{\chi}^{n-1}\|_{L^\infty((0, T) \times \mathbb{T}^3 \times \mathbb{R}^3)}.$$

Definition (26)–(27) of the characteristics $\boldsymbol{\chi}^n = (\mathbf{x}^n, \mathbf{v}^n)$ and estimate (24) imply that, for any t ,

$$\begin{aligned} \|(\boldsymbol{\chi}^n - \boldsymbol{\chi}^{n-1})(t)\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)} &\leq C(\varepsilon) \left(\int_0^t \|(\theta_\varepsilon \star \mathbf{u}^n - \theta_\varepsilon \star \mathbf{u}^{n-1})(s)\|_{L^\infty(\mathbb{T}^3)} ds \right. \\ &\quad \left. + \int_0^t (1 + \|(\theta_\varepsilon \star \mathbf{u}^n)(s)\|_{W^{1, \infty}(\mathbb{T}^3)}) \|(\boldsymbol{\chi}^n - \boldsymbol{\chi}^{n-1})(s)\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)} ds \right). \end{aligned}$$

Thus, for any t , we can write

$$\|(\boldsymbol{\chi}^n - \boldsymbol{\chi}^{n-1})(t)\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)} \leq C(\varepsilon) \left(\int_0^t \|\mathbf{w}^n(s)\|_{L^2(\mathbb{T}^3)} ds + \int_0^t \|(\boldsymbol{\chi}^n - \boldsymbol{\chi}^{n-1})(s)\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)} ds \right).$$

Using Gronwall's lemma, we obtain, for any $t \leq T$,

$$(31) \quad \|(\boldsymbol{\chi}^n - \boldsymbol{\chi}^{n-1})(t)\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)} \leq C(\varepsilon, T) \int_0^t \|\mathbf{w}^n(s)\|_{L^2(\mathbb{T}^3)} ds.$$

Consequently, with (30) and (31), (29) becomes

$$|T_2| \leq C(\varepsilon, T) \left(\int_0^t \|\mathbf{w}^n\|_{L^2(\mathbb{T}^3)}^2 ds + \|\mathbf{w}^{n+1}\|_{L^2(\mathbb{T}^3)}^2 + \|\mathbf{w}^n\|_{L^2(\mathbb{T}^3)}^2 \right).$$

We finally obtain

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{w}^{n+1}\|_{L^2(\mathbb{T}^3)}^2 + \frac{1}{2} \|\nabla_{\mathbf{x}} \mathbf{w}^{n+1}\|_{L^2(\mathbb{T}^3)}^2 \leq C(\varepsilon, T) \left(\int_0^t \|\mathbf{w}^n(s)\|_{L^2(\mathbb{T}^3)}^2 ds + \|\mathbf{w}^{n+1}\|_{L^2(\mathbb{T}^3)}^2 + \|\mathbf{w}^n\|_{L^2(\mathbb{T}^3)}^2 \right).$$

Therefore, integrating the previous inequality with respect to t , and using

$$\int_0^t \int_0^s \|\mathbf{w}^n(r)\|_{L^2(\mathbb{T}^3)}^2 dr ds \leq T \int_0^t \|\mathbf{w}^n(r)\|_{L^2(\mathbb{T}^3)}^2 dr,$$

we obtain, for any t , $0 \leq t \leq T$,

$$\|\mathbf{w}^{n+1}(t)\|_{L^2(\mathbb{T}^3)}^2 \leq C(\varepsilon, T) \left(\int_0^t \|\mathbf{w}^n(s)\|_{L^2(\mathbb{T}^3)}^2 ds + \int_0^t \|\mathbf{w}^{n+1}(s)\|_{L^2(\mathbb{T}^3)}^2 ds \right).$$

Lemma 3 then implies that there exists a constant $K(\varepsilon, T) > 0$, which does not depend on n , such that

$$\|\mathbf{w}^n\|_{L^\infty(0, T; L^2(\mathbb{T}^3))}^2 \leq \frac{K^n T^n}{n!}.$$

Consequently, the sequence $(\mathbf{u}^n)_{n \in \mathbb{N}}$ converges in $L^\infty(0, T; L^2(\mathbb{T}^3))$. That also implies, thanks to (30) and (31), that $(f^n)_{n \in \mathbb{N}}$ converges in $L^\infty((0, T) \times \mathbb{T}^3 \times \mathbb{R}^3)$.

Remark 4. *Note that, at this stage, to study the regularized and uncoupled system, we used the high particle velocity cutoff function to control the force applied by the spray on the fluid. Moreover, the space regularization of the fluid velocity in the Vlasov equation enables to apply the method of characteristics and obtain a strong solution (this will also be true at the limit as $n \rightarrow \infty$ for any fixed $\varepsilon > 0$). Finally, the space regularization of the convection fluid velocity simplifies the study of the fluid equations by making the convection term easier to deal with.*

3.2. Asymptotics with respect to n . As $\varepsilon > 0$ is still fixed, we can pass to the limit when n goes to infinity. We already know that $(\mathbf{u}_\varepsilon^n)_n$ converges to \mathbf{u}_ε strongly in $L^\infty(0, T; L^2(\mathbb{T}^3))$ and, thanks to (24), weakly in $L^2(0, T; H^1(\mathbb{T}^3))$, and that $(f_\varepsilon^n)_n$ strongly converges to f_ε in $L^\infty((0, T) \times \mathbb{T}^3 \times \mathbb{R}^3)$. Thus, since f_ε^n belongs to $\mathcal{C}^0([0, T] \times \mathbb{T}^3 \times \mathbb{R}^3)$, so is f_ε .

These convergences enable us to pass to the limit in n in the following weak formulation, for all t , for all $\phi \in \mathcal{C}^1([0, T] \times \mathbb{T}^3 \times \mathbb{R}^3)$:

$$(32) \quad \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon^n(t) \phi(t) d\mathbf{x} d\mathbf{v} - \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon^n [\partial_t \phi + \mathbf{v} \cdot \nabla_{\mathbf{x}} \phi + (\theta_\varepsilon \star \mathbf{u}_\varepsilon^n - \mathbf{v}) \cdot \nabla_{\mathbf{v}} \phi] d\mathbf{x} d\mathbf{v} ds = \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon^\varepsilon \phi(0) d\mathbf{x} d\mathbf{v}$$

Remark 5. *Note that here the test functions are not necessarily with compact support in \mathbf{v} . Indeed, since $(\boldsymbol{\chi}_\varepsilon^n)_n$ converges to $\boldsymbol{\chi}_\varepsilon$ in $L^\infty((0, T) \times \mathbb{T}^3 \times \mathbb{R}^3)$ when n goes to $+\infty$, we check*

$$(33) \quad f_\varepsilon(t, \mathbf{x}, \mathbf{v}) = e^{3t} f_\varepsilon^\varepsilon(\boldsymbol{\chi}_\varepsilon(t, \mathbf{x}, \mathbf{v})).$$

Since we assumed that $f_\varepsilon^\varepsilon$ has been chosen with a compact support in \mathbf{v} , it is also the case of $f_\varepsilon(t, \mathbf{x}, \cdot)$ for all $(t, \mathbf{x}) \in [0, T] \times \mathbb{T}^3$.

Note that here the solution f_ε is in fact a strong solution.

Furthermore, for all $\psi \in C^1([0, T] \times \mathbb{T}^3)$ such that $\nabla_{\mathbf{x}} \cdot \psi = 0$, for a. e. t ,

$$\begin{aligned} \int_0^t \int_{\mathbb{T}^3} \partial_t \mathbf{u}_\varepsilon^{n+1} \cdot \psi \, dx \, ds + \int_0^t \int_{\mathbb{T}^3} ((\theta_\varepsilon \star \mathbf{u}_\varepsilon^{n+1}) \cdot \nabla_{\mathbf{x}}) \mathbf{u}_\varepsilon^{n+1} \psi \, dx \, ds + \int_0^t \int_{\mathbb{T}^3} \nabla_{\mathbf{x}} \mathbf{u}_\varepsilon^{n+1} : \nabla_{\mathbf{x}} \psi \, dx \, ds \\ = - \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon^n(\mathbf{u}_\varepsilon^n - \mathbf{v}) \gamma_\varepsilon(\mathbf{v}) \cdot \psi \, dx \, dv \, ds + \int_0^t \int_{\mathbb{T}^3} \mathbf{F}_{\text{ext}}^\varepsilon \cdot \psi \, dx \, ds. \end{aligned}$$

This yields (16) and (17). The limit function $(f_\varepsilon, \mathbf{u}_\varepsilon)$ thus solves the (variational formulation of the) regularized system (11)–(15).

This ends the proof of Proposition 2.

We now derive uniform bounds in ε for $(f_\varepsilon, \mathbf{u}_\varepsilon)$. In a first step, these estimates only hold locally in time (but they only depend on $T, \|\mathbf{u}_{\text{in}}\|_{L^2(\mathbb{T}^3)}^2 + |M_2 f_{\text{in}}| + \|\mathbf{F}_{\text{ext}}\|_{L^2(0, T; L^2(\mathbb{T}^3))}^2$ and $\|f_{\text{in}}\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)}$: this will be of utmost importance in the last part of the proof of Thm. 1). This constitutes the

3.3. Proof of Proposition 1.

Uniform estimates for the sequence $(f_\varepsilon, \mathbf{u}_\varepsilon)$. The fact that $\|f_{\text{in}}^\varepsilon\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)} \leq \|f_{\text{in}}\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)}$ together with equality (33) implies (19). Therefore, (f_ε) is uniformly bounded in $L^\infty((0, T) \times \mathbb{T}^3 \times \mathbb{R}^3)$. Consequently, and thanks to Remark 5, $|\mathbf{v}|^2$ is an admissible test function in the variational formulation (16).

A similar argument to the one used in the proof of Proposition 2 ensures that f_ε at least belongs to $C^1([0, T] \times \mathbb{T}^3 \times \mathbb{R}^3)$. That allows us to write

$$(34) \quad \frac{d}{dt} M_2 f_\varepsilon + 2M_2 f_\varepsilon \leq 2 \int_{\mathbb{T}^3} |\theta_\varepsilon \star \mathbf{u}_\varepsilon(t, \mathbf{x})| m_1 f_\varepsilon(t, \mathbf{x}) \, dx.$$

To estimate $m_1 f_\varepsilon$ in terms of second order moments of f_ε , we need the following

Lemma 1. *Let $\beta > 0$ and g be a nonnegative function in $L^\infty((0, T) \times \mathbb{T}^3 \times \mathbb{R}^3)$, such that $m_\beta g(t, \mathbf{x}) < +\infty$, for a. e. (t, \mathbf{x}) . The following estimate holds for any $\alpha < \beta$:*

$$m_\alpha g(t, \mathbf{x}) \leq \left(\frac{4}{3} \pi \|g(t, \mathbf{x}, \cdot)\|_{L^\infty(\mathbb{R}^3)} + 1 \right) m_\beta g(t, \mathbf{x})^{\frac{\alpha+3}{\beta+3}}, \quad \text{a. e. } (t, \mathbf{x}).$$

the same kind of result can be found in [11] or [9].

Proof of Lemma 1. We write, for a. e. (t, \mathbf{x}) ,

$$\begin{aligned} m_\alpha g(t, \mathbf{x}) &= \int_{|\mathbf{v}| \leq R} |\mathbf{v}|^\alpha g + \int_{|\mathbf{v}| \geq R} |\mathbf{v}|^\alpha g \\ &\leq \frac{4}{3} \pi R^{\alpha+3} \|g(t, \mathbf{x}, \cdot)\|_{L^\infty(\mathbb{R}^3)} + \frac{1}{R^{\beta-\alpha}} m_\beta g(t, \mathbf{x}). \end{aligned}$$

By choosing $R = m_\beta g(t, \mathbf{x})^{\frac{1}{\beta+3}}$, we find the required result. This ends the proof of Lemma 1.

Applying Lemma 1, we obtain, thanks to (19),

$$0 \leq m_1 f_\varepsilon(t, \mathbf{x}) \leq C m_2 f_\varepsilon(t, \mathbf{x})^{4/5},$$

which yields

$$(35) \quad \|m_1 f_\varepsilon(t)\|_{L^{5/4}(\mathbb{T}^3)} \leq C M_2 f_\varepsilon(t)^{4/5},$$

where C only depends on T and $\|f_{\text{in}}\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)}$. This estimate, together with (34), gives

$$\begin{aligned} \frac{d}{dt} M_2 f_\varepsilon + 2M_2 f_\varepsilon &\leq \|\theta_\varepsilon \star \mathbf{u}_\varepsilon\|_{L^5(\mathbb{T}^3)} \|m_1 f_\varepsilon\|_{L^{5/4}(\mathbb{T}^3)} \\ &\leq C \|\mathbf{u}_\varepsilon\|_{L^5(\mathbb{T}^3)} (M_2 f_\varepsilon)^{4/5}. \end{aligned}$$

Setting $H(t) = M_2 f_\varepsilon(t)^{1/5}$, we obtain

$$H' + H \leq C \|\mathbf{u}_\varepsilon\|_{L^5(\mathbb{T}^3)}.$$

Since $M_2 f_\varepsilon \geq 0$, and assuming that $M_2 f_{\text{in}}^\varepsilon \leq C M_2 f_{\text{in}}$, we have, for a. e. t ,

$$H(t) \leq C \left(1 + \int_0^t \|\mathbf{u}_\varepsilon(s)\|_{L^5(\mathbb{T}^3)} ds \right),$$

and thus

$$(36) \quad M_2 f_\varepsilon(t) \leq C \left(1 + \int_0^t \|\mathbf{u}_\varepsilon(s)\|_{L^5(\mathbb{T}^3)} ds \right)^5,$$

where C only depends on the initial data.

Next, we take care of the fluid velocity. The function \mathbf{u}_ε is an admissible test function for the variational formulation (17). Indeed, by density, (17) remains valid for any $\psi \in L^2(0, T; H^1(\mathbb{T}^3))$. After integration by parts, we have

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{u}_\varepsilon\|_{L^2(\mathbb{T}^3)}^2 + \|\nabla_{\mathbf{x}} \mathbf{u}_\varepsilon\|_{L^2(\mathbb{T}^3)}^2 = \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon \mathbf{u}_\varepsilon \cdot (\mathbf{v} - \mathbf{u}_\varepsilon) \gamma_\varepsilon(\mathbf{v}) d\mathbf{x} d\mathbf{v} + \int_{\mathbb{T}^3} \mathbf{F}_{\text{ext}} \cdot \mathbf{u}_\varepsilon.$$

Since $\gamma_\varepsilon \geq 0$, that implies

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{u}_\varepsilon\|_{L^2(\mathbb{T}^3)}^2 + \|\nabla_{\mathbf{x}} \mathbf{u}_\varepsilon\|_{L^2(\mathbb{T}^3)}^2 &\leq \int_{\mathbb{T}^3} |\mathbf{u}_\varepsilon| m_1 f_\varepsilon d\mathbf{x} + \|\mathbf{F}_{\text{ext}}\|_{L^2(\mathbb{T}^3)} \|\mathbf{u}_\varepsilon\|_{L^2(\mathbb{T}^3)} \\ &\leq \|\mathbf{u}_\varepsilon\|_{L^5(\mathbb{T}^3)} \|m_1 f_\varepsilon\|_{L^{5/4}(\mathbb{T}^3)} + \|\mathbf{F}_{\text{ext}}\|_{L^2(\mathbb{T}^3)} \|\mathbf{u}_\varepsilon\|_{L^2(\mathbb{T}^3)}. \end{aligned}$$

Applying Young's inequality and remembering (35) and (36), we get

$$(37) \quad \frac{1}{2} \frac{d}{dt} \|\mathbf{u}_\varepsilon\|_{L^2(\mathbb{T}^3)}^2 + \|\nabla_{\mathbf{x}} \mathbf{u}_\varepsilon\|_{L^2(\mathbb{T}^3)}^2 \leq C \|\mathbf{u}_\varepsilon\|_{L^5(\mathbb{T}^3)} \left(1 + \int_0^t \|\mathbf{u}_\varepsilon\|_{L^5(\mathbb{T}^3)} ds \right)^4 + C \|\mathbf{F}_{\text{ext}}\|_{L^2(\mathbb{T}^3)}^2 + C \|\mathbf{u}_\varepsilon\|_{L^2(\mathbb{T}^3)}^2.$$

Next, using the continuous injection of $H^1(\mathbb{T}^3)$ in $L^5(\mathbb{T}^3)$ (in dimension less or equal than 3), we obtain

$$(38) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{u}_\varepsilon\|_{L^2(\mathbb{T}^3)}^2 + \|\nabla_{\mathbf{x}} \mathbf{u}_\varepsilon\|_{L^2(\mathbb{T}^3)}^2 &\leq \frac{1}{2} \|\nabla_{\mathbf{x}} \mathbf{u}_\varepsilon\|_{L^2(\mathbb{T}^3)}^2 + C \|\mathbf{u}_\varepsilon\|_{L^2(\mathbb{T}^3)}^2 + C \\ &+ C \left(\int_0^t (\|\nabla_{\mathbf{x}} \mathbf{u}_\varepsilon\|_{L^2(\mathbb{T}^3)} + \|\mathbf{u}_\varepsilon\|_{L^2(\mathbb{T}^3)}) ds \right)^8 + C \|\mathbf{F}_{\text{ext}}\|_{L^2(\mathbb{T}^3)}^2. \end{aligned}$$

Using

$$\int_0^t \int_0^s \|\mathbf{u}_\varepsilon(r)\|_{L^2(\mathbb{T}^3)}^8 dr ds \leq T \int_0^t \|\mathbf{u}_\varepsilon(r)\|_{L^2(\mathbb{T}^3)}^8 dr,$$

and Hölder's inequality, we integrate (38) to obtain

$$\begin{aligned} \frac{1}{2} \|\mathbf{u}_\varepsilon\|_{L^2(\mathbb{T}^3)}^2(t) + \frac{1}{4} \int_0^t \|\nabla_{\mathbf{x}} \mathbf{u}_\varepsilon\|_{L^2(\mathbb{T}^3)}^2 ds &\leq \frac{1}{4} \int_0^t \|\mathbf{u}_\varepsilon\|_{L^2(\mathbb{T}^3)}^2 ds + \frac{1}{2} \|\mathbf{u}_{\text{in}}^\varepsilon\|_{L^2(\mathbb{T}^3)}^2 + C \\ &+ C \int_0^t \left(\int_0^s \|\nabla_{\mathbf{x}} \mathbf{u}_\varepsilon\|_{L^2(\mathbb{T}^3)}^2 ds \right)^4 ds + C \int_0^t \|\mathbf{u}_\varepsilon\|_{L^2(\mathbb{T}^3)}^8 ds \\ &+ C \int_0^t \|\mathbf{F}_{\text{ext}}\|_{L^2(\mathbb{T}^3)}^2 ds. \end{aligned}$$

A nonlinear Gronwall lemma thus implies that there exist a time $T^* > 0$ and a constant C such that, for almost every $t \leq T^*$,

$$(39) \quad \|\mathbf{u}_\varepsilon(t)\|_{L^2(\mathbb{T}^3)}^2 + \int_0^t \|\nabla_{\mathbf{x}} \mathbf{u}_\varepsilon(s)\|_{L^2(\mathbb{T}^3)}^2 ds \leq C.$$

We recall here that

$$C := C(T, T^*, \|\mathbf{u}_{\text{in}}\|_{L^2(\mathbb{T}^3)}^2 + |M_2 f_{\text{in}}| + \|\mathbf{F}_{\text{ext}}\|_{L^2(0, T; L^2(\mathbb{T}^3))}^2, \|f_{\text{in}}\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)}).$$

Finally, estimates (36) and (39) imply that

$$(40) \quad M_2 f_\varepsilon(t) \leq C, \quad \text{a. e. } t \leq T^*.$$

3.4. Asymptotics with respect to ε . In order to pass to the limit in the weak formulations (16)–(17), we need to have some compactness properties for the sequence $(f_\varepsilon, \mathbf{u}_\varepsilon)_{\varepsilon>0}$. In fact, proving that $(\mathbf{u}_\varepsilon)_{\varepsilon>0}$ is compact in $L^p(0, T; L^2(\mathbb{T}^3))$ for some p , is enough for the asymptotics $\varepsilon \rightarrow 0$. From (17), it is easy to check that $(\partial_t \mathbf{u}_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^{3/2}(0, T^*; \mathcal{V}')$. As a matter of fact, once again, the only two terms we have to take care of are the convection and the drag force ones.

For the convection term, we use standard arguments:

$$\int_{\mathbb{T}^3} ((\theta_\varepsilon \star \mathbf{u}_\varepsilon) \cdot \nabla_{\mathbf{x}}) \mathbf{u}_\varepsilon \cdot \boldsymbol{\psi} = - \int_{\mathbb{T}^3} ((\theta_\varepsilon \star \mathbf{u}_\varepsilon) \cdot \nabla_{\mathbf{x}}) \boldsymbol{\psi} \cdot \mathbf{u}_\varepsilon.$$

But $(\mathbf{u}_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^\infty(0, T^*; L^2(\mathbb{T}^3)) \cap L^2(0, T^*; H^1(\mathbb{T}^3))$, which is continuously embedded in $L^3(0, T^*; L^4(\mathbb{T}^3))$. Thus

$$\left| \int_0^{T^*} \int_{\mathbb{T}^3} ((\theta_\varepsilon \star \mathbf{u}_\varepsilon) \cdot \nabla_{\mathbf{x}}) \mathbf{u}_\varepsilon \cdot \boldsymbol{\psi} \right| \leq C \|\nabla_{\mathbf{x}} \boldsymbol{\psi}\|_{L^3(0, T^*; L^2(\mathbb{T}^3))}.$$

Thus the form

$$\boldsymbol{\psi} \mapsto \int_0^{T^*} \int_{\mathbb{T}^3} ((\theta_\varepsilon \star \mathbf{u}_\varepsilon) \cdot \nabla_{\mathbf{x}}) \mathbf{u}_\varepsilon \cdot \boldsymbol{\psi}$$

is bounded in $L^{3/2}(0, T^*; \mathcal{V}')$.

The term coming from the drag force can be estimated as follows:

$$\begin{aligned} \left| \int_0^{T^*} \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon(\mathbf{u}_\varepsilon - \mathbf{v}) \gamma_\varepsilon(\mathbf{v}) \boldsymbol{\psi} \right| &\leq C \|\mathbf{u}_\varepsilon\|_{L^2(0, T^*; L^6(\mathbb{T}^3))} \|\boldsymbol{\psi}\|_{L^2(0, T^*; L^6(\mathbb{T}^3))} \|m_0 f_\varepsilon\|_{L^\infty(0, T^*; L^{3/2}(\mathbb{T}^3))} \\ &\quad + \|\boldsymbol{\psi}\|_{L^2(0, T^*; L^5(\mathbb{T}^3))} \|m_1 f_\varepsilon\|_{L^2(0, T^*; L^{5/4}(\mathbb{T}^3))}. \end{aligned}$$

But, thanks to Lemma 1, we have

$$m_0 f_\varepsilon \leq C(m_2 f_\varepsilon)^{3/5}, \quad m_1 f_\varepsilon \leq C(m_2 f_\varepsilon)^{4/5},$$

and we proved that $\|M_2 f_\varepsilon\|_{L^\infty(0, T^*)} \leq C$.

Therefore, the sequence $(m_0 f_\varepsilon)_{\varepsilon>0}$ and $(m_1 f_\varepsilon)_{\varepsilon>0}$ are respectively bounded in $L^\infty(0, T^*; L^{5/3}(\mathbb{T}^3 \times \mathbb{R}^3))$ and $L^\infty(0, T^*; L^{5/4}(\mathbb{T}^3 \times \mathbb{R}^3))$.

Moreover, (\mathbf{u}_ε) is bounded in $L^2(0, T^*; H^1(\mathbb{T}^3))$ and thus in $L^2(0, T^*; L^6(\mathbb{T}^3))$.

Eventually, we can write

$$\left| \int_0^{T^*} \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon(\mathbf{u}_\varepsilon - \mathbf{v}) \gamma_\varepsilon(\mathbf{v}) \boldsymbol{\psi} \right| \leq C \|\boldsymbol{\psi}\|_{L^2(0, T^*; L^6(\mathbb{T}^3))} \leq C \|\boldsymbol{\psi}\|_{L^2(0, T^*; H^1(\mathbb{T}^3))},$$

and the drag force $\left(\int_{\mathbb{R}^3} f_\varepsilon(\mathbf{u}_\varepsilon - \mathbf{v}) \gamma_\varepsilon(\mathbf{v}) d\mathbf{v} \right)_{\varepsilon>0}$ is uniformly bounded in $L^2(0, T^*; \mathcal{V}')$.

That ensures that $(\partial_t \mathbf{u}_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^{3/2}(0, T^*; \mathcal{V}')$.

Thanks to those bounds, there exists $(f, \mathbf{u}) \in L^\infty((0, T^*) \times \mathbb{T}^3 \times \mathbb{R}^3) \times (L^\infty(0, T^*; L^2(\mathbb{T}^3)) \cap L^2(0, T^*; H^1(\mathbb{T}^3))) \times \mathcal{C}^0([0, T^*]; \mathcal{V}')$ such that, up to a subsequence, (f_ε) weakly converges to f in $w^*-L^\infty((0, T^*) \times \mathbb{T}^3 \times \mathbb{R}^3)$ and (\mathbf{u}_ε) weakly converges to \mathbf{u} in $w-L^2(0, T^*; w-H^1(\mathbb{T}^3))$ and in $w^*-L^\infty(0, T^*; w-L^2(\mathbb{T}^3))$. Thanks to Aubin's compactness lemma, since $(\partial_t \mathbf{u}_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^{3/2}(0, T^*; \mathcal{V}')$, we can state that (\mathbf{u}_ε) strongly converges to \mathbf{u} in $L^2(0, T^*; L^2(\mathbb{T}^3))$.

These convergences allow us to pass to the limit in the following weak formulations:

$$- \int_0^{T^*} \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon [\partial_t \phi + \mathbf{v} \cdot \nabla_{\mathbf{x}} \phi + (\theta_\varepsilon \star \mathbf{u}_\varepsilon - \mathbf{v}) \cdot \nabla_{\mathbf{v}} \phi] \, d\mathbf{x} \, d\mathbf{v} \, ds = \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f_{\text{in}}^\varepsilon \phi(0) \, d\mathbf{x} \, d\mathbf{v}, \quad ,$$

for all $\phi \in \mathcal{C}^1([0, T^*] \times \mathbb{T}^3 \times \mathbb{R}^3)$ with compact support in \mathbf{v} and such that $\phi(T^*) = 0$, and

$$\begin{aligned} - \int_{\mathbb{T}^3} \mathbf{u}_\varepsilon(t) \cdot \boldsymbol{\psi}(t) \, d\mathbf{x} \, ds + \int_0^t \int_{\mathbb{T}^3} ((\theta_\varepsilon \star \mathbf{u}_\varepsilon) \cdot \nabla_{\mathbf{x}}) \mathbf{u}_\varepsilon \cdot \boldsymbol{\psi} \, d\mathbf{x} \, ds + \int_0^t \int_{\mathbb{T}^3} \nabla_{\mathbf{x}} \mathbf{u}_\varepsilon : \nabla_{\mathbf{x}} \boldsymbol{\psi} \, d\mathbf{x} \, ds \\ = - \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon (\mathbf{u}_\varepsilon - \mathbf{v}) \gamma_\varepsilon(\mathbf{v}) \cdot \boldsymbol{\psi} \, d\mathbf{x} \, d\mathbf{v} \, ds + \int_0^t \int_{\mathbb{T}^3} \mathbf{F}_{\text{ext}}^\varepsilon \cdot \boldsymbol{\psi} \, d\mathbf{x} \, ds, \quad \text{a. e. } t \leq T^*, \end{aligned}$$

for all $\boldsymbol{\psi} \in \mathcal{C}^1([0, T^*] \times \mathbb{T}^3)$ such that $\nabla_{\mathbf{x}} \cdot \boldsymbol{\psi} = 0$.

We then obtain a solution satisfying (8)–(9).

This ends the proof of Proposition 1.

3.5. Global existence. We just derived the existence locally in time of a weak solution (f, \mathbf{u}) . In this section, we prove that this solution is, in fact, global in time, which constitutes the last step of the proof of Thm. 1. We start with the

Lemma 2. *The solution (f, \mathbf{u}) obtained above satisfies*

$$(41) \quad \begin{aligned} \frac{1}{2} M_2 f(t) + \frac{1}{2} \|\mathbf{u}(t)\|_{L^2(\mathbb{T}^3)}^2 + \int_0^t \|\nabla_{\mathbf{x}} \mathbf{u}(s)\|_{L^2(\mathbb{T}^3)}^2 \, ds + \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f |\mathbf{u} - \mathbf{v}|^2 \, d\mathbf{x} \, d\mathbf{v} \, ds \\ \leq (1 + e^T + T e^T) \left(\frac{1}{2} \|\mathbf{u}_{\text{in}}\|_{L^2(\mathbb{T}^3)}^2 + \frac{1}{2} |M_2 f_{\text{in}}| + \frac{1}{2} \|\mathbf{F}_{\text{ext}}\|_{L^2(0, T; L^2(\mathbb{T}^3))}^2 \right), \quad \text{a. e. } t \leq T^*, \end{aligned}$$

and

$$\|f\|_{L^\infty((0, T^*) \times \mathbb{T}^3 \times \mathbb{R}^3)} \leq e^{3T} \|f_{\text{in}}\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)}.$$

Proof of Lemma 2. We first note that the second inequality is directly obtained from the definition of f , as the weak limit of (f_ε) for the $w^*-L^\infty((0, T^*) \times \mathbb{T}^3 \times \mathbb{R}^3)$ topology. For the first inequality, we shall study the energy estimate satisfied by $(f_\varepsilon, \mathbf{u}_\varepsilon)$ obtained by choosing once again $\phi = |\mathbf{v}|^2/2$ as a test function in (16), and \mathbf{u}_ε as a test function in (17):

$$(42) \quad \begin{aligned} \frac{1}{2} M_2 f_\varepsilon(t) + \frac{1}{2} \|\mathbf{u}_\varepsilon(t)\|_{L^2(\mathbb{T}^3)}^2 \\ + \int_0^t \|\nabla_{\mathbf{x}} \mathbf{u}_\varepsilon(s)\|_{L^2(\mathbb{T}^3)}^2 \, ds + \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon |\mathbf{u}_\varepsilon - \mathbf{v}|^2 \, d\mathbf{x} \, d\mathbf{v} \, ds \\ = \frac{1}{2} M_2 f_{\text{in}}^\varepsilon + \frac{1}{2} \|\mathbf{u}_{\text{in}}^\varepsilon\|_{L^2(\mathbb{T}^3)}^2 + \int_0^t \int_{\mathbb{T}^3} \mathbf{F}_{\text{ext}}^\varepsilon \cdot \mathbf{u}_\varepsilon \, d\mathbf{x} \, ds + R_\varepsilon(t), \end{aligned}$$

where

$$\begin{aligned} R_\varepsilon(t) &= \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon |\mathbf{u}_\varepsilon|^2 (1 - \gamma_\varepsilon(\mathbf{v})) \, d\mathbf{x} \, d\mathbf{v} \, ds \\ &\quad - \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon \mathbf{u}_\varepsilon \cdot \mathbf{v} (1 - \gamma_\varepsilon(\mathbf{v})) \, d\mathbf{x} \, d\mathbf{v} \, ds \\ &\quad + \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon (\mathbf{u}_\varepsilon - \theta_\varepsilon \star \mathbf{u}_\varepsilon) \cdot \mathbf{v} \, d\mathbf{x} \, d\mathbf{v} \, ds. \end{aligned}$$

Let R_ε^i , $i = 1, 2, 3$, denote each of the three terms appearing in R_ε and $h_\varepsilon = f_\varepsilon(1 - \gamma_\varepsilon)$.

We first have, for a. e. t ,

$$0 \leq R_\varepsilon^1(t) \leq \|m_0 h^\varepsilon\|_{L^\infty(0, T^*; L^{3/2}(\mathbb{T}^3))} \|\mathbf{u}_\varepsilon\|_{L^2(0, T^*; L^6(\mathbb{T}^3))}^2.$$

Lemma 1 implies that

$$\|m_0 h_\varepsilon(t)\|_{L^{3/2}(\mathbb{T}^3)} \leq C M_{3/2} h_\varepsilon(t)^{3/2},$$

and

$$\begin{aligned} M_{3/2} h_\varepsilon(t) &= \iint_{\mathbb{T}^3 \times \mathbb{R}^3} |\mathbf{v}|^{3/2} f_\varepsilon (1 - \gamma_\varepsilon) \\ &\leq \iint_{\mathbb{T}^3 \times \{|\mathbf{v}| \geq \frac{1}{2} \varepsilon\}} |\mathbf{v}|^{3/2} f_\varepsilon \\ &\leq \sqrt{2\varepsilon} M_2 f_\varepsilon(t). \end{aligned}$$

Consequently, from the uniform bound obtained on $(M_2 f_\varepsilon)_{\varepsilon > 0}$ in $L^\infty(0, T^*)$, we deduce that the sequence $(\|m_0 h^\varepsilon\|_{L^\infty(0, T^*; L^{3/2}(\mathbb{T}^3))})$ converges to 0 when ε goes to 0.

Finally, since $(\mathbf{u}_\varepsilon)_{\varepsilon > 0}$ is uniformly bounded in $L^2(0, T^*; H^1(\mathbb{T}^3)) \hookrightarrow L^2(0, T^*; L^6(\mathbb{T}^3))$, we have proven that (R_ε^1) goes to 0 when $\varepsilon \rightarrow 0$.

To treat the second term R_ε^2 , we proceed in the same way. For a. e. t , we have

$$0 \leq R_\varepsilon^2(t) \leq \|m_1 h^\varepsilon\|_{L^\infty(0, T^*; L^{6/5}(\mathbb{T}^3))} \|\mathbf{u}_\varepsilon\|_{L^2(0, T^*; L^6(\mathbb{T}^3))}.$$

Lemma 1 implies that

$$\|m_1 h_\varepsilon\|_{L^{5/6}(\mathbb{T}^3)}(t) \leq C M_{9/5} h_\varepsilon(t)^{5/6},$$

and

$$\begin{aligned} M_{9/5} h_\varepsilon(t) &= \iint_{\mathbb{T}^3 \times \mathbb{R}^3} |\mathbf{v}|^{9/5} f_\varepsilon (1 - \gamma_\varepsilon) \\ &\leq \iint_{\mathbb{T}^3 \times \{|\mathbf{v}| \geq \frac{1}{2} \varepsilon\}} |\mathbf{v}|^{9/5} f_\varepsilon \\ &\leq (2\varepsilon)^{1/5} M_2 f_\varepsilon(t). \end{aligned}$$

Consequently, from the uniform bound obtained on $(M_2 f_\varepsilon)_{\varepsilon > 0}$ in $L^\infty(0, T^*)$, we deduce that the sequence $(\|m_1 h^\varepsilon\|_{L^\infty(0, T^*; L^{5/6}(\mathbb{T}^3))})$ converges to 0 when ε goes to 0.

Finally, since $(\mathbf{u}_\varepsilon)_{\varepsilon > 0}$ is uniformly bounded in $L^2(0, T^*; H^1(\mathbb{T}^3)) \hookrightarrow L^2(0, T^*; L^6(\mathbb{T}^3))$, we have proven that (R_ε^2) goes to 0 when $\varepsilon \rightarrow 0$.

We now take care of the last term R_ε^3 . Let us write, for any t ,

$$\begin{aligned} R_\varepsilon^3(t) &= \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon(\mathbf{u}_\varepsilon - \theta_\varepsilon \star \mathbf{u}_\varepsilon) \cdot \mathbf{v} \gamma_\mu(\mathbf{v}) \, d\mathbf{x} \, d\mathbf{v} \, ds \\ &\quad + \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon(\mathbf{u}_\varepsilon - \theta_\varepsilon \star \mathbf{u}_\varepsilon) \cdot \mathbf{v} (1 - \gamma_\mu(\mathbf{v})) \, d\mathbf{x} \, d\mathbf{v} \, ds, \end{aligned}$$

where $\mu > 0$.

From arguments similar to those used for the second term, we can check that

$$0 \leq \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon(\mathbf{u}_\varepsilon - \theta_\varepsilon \star \mathbf{u}_\varepsilon) \cdot \mathbf{v} (1 - \gamma_\mu(\mathbf{v})) \, d\mathbf{x} \, d\mathbf{v} \, ds \leq C (2\mu)^{1/5},$$

where C depends neither on ε nor on μ .

Furthermore, since $(f_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^\infty((0, T^*) \times \mathbb{T}^3 \times \mathbb{R}^3)$, we have, for a. e. t ,

$$0 \leq \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon(\mathbf{u}_\varepsilon - \theta_\varepsilon \star \mathbf{u}_\varepsilon) \cdot \mathbf{v} \gamma_\mu(\mathbf{v}) \, d\mathbf{x} \, d\mathbf{v} \, ds \leq C \|\mathbf{u}_\varepsilon - \theta_\varepsilon \star \mathbf{u}_\varepsilon\|_{L^1(0, T^*; L^1(\mathbb{T}^3))}.$$

Consequently, since \mathbf{u}_ε strongly converges in $L^2(0, T^*; L^2(\mathbb{T}^3))$, the sequence

$$\left(\int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon(\mathbf{u}_\varepsilon - \theta_\varepsilon \star \mathbf{u}_\varepsilon) \cdot \mathbf{v} \gamma_\mu(\mathbf{v}) \, d\mathbf{x} \, d\mathbf{v} \, ds \right)_{\varepsilon>0}$$

goes to 0 when $\varepsilon \rightarrow 0$. That ensures that (R_ε^3) goes to 0 when $\varepsilon \rightarrow 0$.

Eventually we have proven that (R_ε) converges to 0.

For the force term, we note that, by Young's inequality,

$$\left| \int_0^t \int_{\mathbb{T}^3} \mathbf{F}_{\text{ext}}^\varepsilon \cdot \mathbf{u}_\varepsilon \right| \leq \frac{1}{2} \int_0^t \|\mathbf{u}_\varepsilon\|_{L^2(\mathbb{T}^3)}^2 \, ds + \frac{1}{2} \int_0^t \|\mathbf{F}_{\text{ext}}^\varepsilon\|_{L^2(\mathbb{T}^3)}^2 \, ds.$$

Equality (42) implies that

$$\begin{aligned} \frac{1}{2} M_2 f_\varepsilon(t) + \frac{1}{2} \|\mathbf{u}_\varepsilon(t)\|_{L^2(\mathbb{T}^3)}^2 + \int_0^t \|\nabla_{\mathbf{x}} \mathbf{u}_\varepsilon(s)\|_{L^2(\mathbb{T}^3)}^2 \, ds + \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon |\mathbf{u}_\varepsilon - \mathbf{v}|^2 \, d\mathbf{x} \, d\mathbf{v} \, ds \\ \leq \frac{1}{2} M_2 f_{\text{in}}^\varepsilon + \frac{1}{2} \|\mathbf{u}_{\text{in}}^\varepsilon\|_{L^2(\mathbb{T}^3)}^2 + \frac{1}{2} \int_0^t \|\mathbf{u}_\varepsilon(s)\|_{L^2(\mathbb{T}^3)}^2 \, ds + \frac{1}{2} \int_0^t \|\mathbf{F}_{\text{ext}}^\varepsilon(s)\|_{L^2(\mathbb{T}^3)}^2 \, ds. \end{aligned}$$

We then apply Gronwall's lemma to get

$$\begin{aligned} \frac{1}{2} M_2 f_\varepsilon(t) + \frac{1}{2} \|\mathbf{u}_\varepsilon(t)\|_{L^2(\mathbb{T}^3)}^2 + \int_0^t \|\nabla_{\mathbf{x}} \mathbf{u}_\varepsilon(s)\|_{L^2(\mathbb{T}^3)}^2 \, ds + \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon |\mathbf{u}_\varepsilon - \mathbf{v}|^2 \, d\mathbf{x} \, d\mathbf{v} \, ds \\ \leq (1 + e^t) \left(\frac{1}{2} \|\mathbf{u}_{\text{in}}^\varepsilon\|_{L^2(\mathbb{T}^3)}^2 + \frac{1}{2} |M_2 f_{\text{in}}^\varepsilon| \right) + \frac{1}{2} (1 + t e^t) \int_0^t \|\mathbf{F}_{\text{ext}}^\varepsilon\|_{L^2(0, T; L^2(\mathbb{T}^3))}^2 \, ds, \end{aligned}$$

and let $\varepsilon \rightarrow 0$ in the equation. It is easy to check that

$$M_2 f(t) \leq \liminf_{\mu \rightarrow 0} M_2(f \gamma_\mu)(t),$$

and

$$\|M_2(f \gamma_\mu)\|_{L^\infty(0, T^*)} \leq \liminf_{\varepsilon \rightarrow 0} \|M_2(f_\varepsilon \gamma_\mu)\|_{L^\infty(0, T^*)} \leq \liminf_{\varepsilon \rightarrow 0} \|M_2 f_\varepsilon\|_{L^\infty(0, T^*)}.$$

For the remaining terms, the weak convergences of $(f_\varepsilon, \mathbf{u}_\varepsilon)_{\varepsilon>0}$ and the strong convergence of $(\mathbf{u}_\varepsilon)_{\varepsilon>0}$ enable us to obtain (41).

This ends the proof of Lemma 2.

Thanks to this lemma, we can now end the proof of Thm. 1.

3.6. Extension of the solution. Let $T > 0$ fixed, and consider a set of admissible initial data and external force. Let us choose \mathcal{K} such that

$$(1 + e^T + T e^T) \left(\frac{1}{2} \|\mathbf{u}_{\text{in}}\|_{L^2(\mathbb{T}^3)}^2 + \frac{1}{2} |M_2 f_{\text{in}}| + \frac{1}{2} \|\mathbf{F}_{\text{ext}}\|_{L^2(0,T;L^2(\mathbb{T}^3))}^2 \right) \leq \mathcal{K},$$

$$e^{3T} \|f_{\text{in}}\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)} \leq \mathcal{K}.$$

Then all the constants appearing in the proof of the local estimates can be bounded by a constant $C_{T,\mathcal{K}}$ only depending on T and \mathcal{K} . Moreover, because of the choice of \mathcal{K} , the external force obviously satisfies

$$\frac{1}{2} \|\mathbf{F}_{\text{ext}}\|_{L^2(0,T;L^2(\mathbb{T}^3))}^2 \leq \mathcal{K}$$

and the initial data satisfy

$$\frac{1}{2} \|\mathbf{u}_{\text{in}}\|_{L^2(\mathbb{T}^3)}^2 \leq \mathcal{K},$$

$$\frac{1}{2} |M_2 f_{\text{in}}| \leq \mathcal{K},$$

$$\|f_{\text{in}}\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)} \leq \mathcal{K}.$$

Consequently, T^* (defined in Prop. 1) can be lower-bounded by a nonnegative time $\tau_{T,\mathcal{K}} > 0$, only depending on T and \mathcal{K} . From Lemma 2, we can then check that, for almost every $t \leq T^*$,

$$\frac{1}{2} \|\mathbf{u}(t)\|_{L^2(\mathbb{T}^3)}^2 \leq \mathcal{K},$$

$$\frac{1}{2} |M_2 f(t)| \leq \mathcal{K},$$

$$\|f(t)\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)} \leq \mathcal{K}.$$

Moreover, we have

$$\frac{1}{2} \int_t^T \|\mathbf{F}_{\text{ext}}(s)\|_{L^2(\mathbb{T}^3)}^2 ds \leq \mathcal{K}, \quad \text{a.e. } t.$$

That ensures that, starting again from time T^* (or $T^* - \sigma$, $\sigma > 0$ small enough), and applying once again Prop. 1, we can extend the solution (f, \mathbf{u}) on $(T^*, T^* + \tau_{T,\mathcal{K}})$. We can then iterate this process till we reach T .

Estimate (6) is obtained by letting $\varepsilon \rightarrow 0$ in estimate (42).

That ends the proof of Thm. 1.

APPENDIX A. GRONWALL LEMMA

In this section, we state and prove the Gronwall lemma we used several times in this work.

Lemma 3. *Let $T > 0$ and consider a sequence $(a_n)_{n \in \mathbb{N}}$ of nonnegative continuous functions on $[0, T]$. Assume that (a_n) satisfies, for any n ,*

$$(43) \quad a_{n+1}(t) \leq A + B \int_0^t a_n(s) ds + C \int_0^t a_{n+1}(s) ds, \quad 0 \leq t \leq T,$$

where A , B and C are nonnegative constants.

If $A = 0$, there exists a constant $K \geq 0$ such that

$$(44) \quad a_n(t) \leq \frac{K^n t^n}{n!}, \quad 0 \leq t \leq T, \quad n \in \mathbb{N}.$$

If $A > 0$, there exists a constant $K \geq 0$ depending on A, B, C such that

$$(45) \quad a_n(t) \leq K \exp(Kt), \quad 0 \leq t \leq T, \quad n \in \mathbb{N}.$$

Proof of Lemma 3: If both $A = C = 0$, we choose

$$K = \max \left(B, \max_{[0, T]} a_0 \right),$$

and the induction is immediate to get (44).

If $A > 0$ and $C = 0$, we choose

$$K = \max \left(A, B, \max_{[0, T]} a_0 \right),$$

and the induction is once again immediate to get (45).

If $C > 0$, we can apply one of the two previous cases to the sequence (α_n) defined, for any n , by

$$\alpha_n(t) = \frac{d}{dt} \left(\left[\int_0^t a_n(s) ds \right] \exp(-Ct) \right),$$

and obtain (44) or (45).

That ends the proof of Lemma 3.

ACKNOWLEDGEMENTS

This work has been partially supported by the French national research agency (ANR) grant # 0003205 *Transport et transfert via le système pulmonaire humain*. L. Boudin wishes to thank the INRIA research center of Paris – Rocquencourt, and the Reo project-team for the temporary researcher position (*délégation*) he got there.

REFERENCES

- [1] C. Baranger, L. Boudin, P.-E. Jabin, and S. Mancini. A modeling of biospray for the upper airways. In *CEMRACS 2004—mathematics and applications to biology and medicine*, volume 14 of *ESAIM Proc.*, pages 41–47 (electronic). EDP Sci., Les Ulis, 2005.
- [2] C. Baranger and L. Desvillettes. Coupling Euler and Vlasov equations in the context of sprays: the local-in-time, classical solutions. *J. Hyperbolic Differ. Equ.*, 3(1):1–26, 2006.
- [3] R. Caflisch and G. C. Papanicolaou. Dynamic theory of suspensions with Brownian effects. *SIAM J. Appl. Math.*, 43(4):885–906, 1983.
- [4] K. Domelevo and J.-M. Roquejoffre. Existence and stability of travelling wave solutions in a kinetic model of two-phase flows. *Commun. Partial Differ. Equations*, 24(1-2):61–108, 1999.
- [5] T. Gemci, T. E. Corcoran, and N. Chigier. A numerical and experimental study of spray dynamics in a simple throat model. *Aerosol Sci. Technol.*, 36:18–38, 2002.
- [6] T. Goudon. Asymptotic problems for a kinetic model of two-phase flow. *Proc. R. Soc. Edinb., Sect. A, Math.*, 131(6):1371–1384, 2001.
- [7] T. Goudon, P.-E. Jabin, and A. Vasseur. Hydrodynamic limit for the Vlasov-Navier-Stokes equations. I. Light particles regime. *Indiana Univ. Math. J.*, 53(6):1495–1515, 2004.
- [8] T. Goudon, P.-E. Jabin, and A. Vasseur. Hydrodynamic limit for the Vlasov-Navier-Stokes equations. II. Fine particles regime. *Indiana Univ. Math. J.*, 53(6):1517–1536, 2004.
- [9] K. Hamdache. Global existence and large time behaviour of solutions for the Vlasov-Stokes equations. *Japan J. Indust. Appl. Math.*, 15(1):51–74, 1998.
- [10] J. Mathiaud. *Étude de systèmes de type gaz-particules*. PhD thesis, ENS Cachan, France, 2006.
- [11] A. Mellet and A. Vasseur. Global weak solutions for a Vlasov-Fokker-Planck/Navier-Stokes system of equations. *Math. Models Methods Appl. Sci.*, 17(7):1039–1063, 2007.
- [12] A. Mellet and A. Vasseur. On the barotropic compressible Navier-Stokes equations. *Comm. Partial Differential Equations*, 32(1-3):431–452, 2007.
- [13] P. J. O'Rourke. *Collective drop effects on vaporizing liquid sprays*. PhD thesis, Los Alamos National Laboratory, 1981.
- [14] F. A. Williams. *Combustion theory*. Benjamin Cummings, second edition edition, 1985.

L.B.: UPMC UNIV PARIS 06, UMR 7598, LABORATOIRE JACQUES-LOUIS LIONS, F-75005, PARIS, FRANCE

E-mail address: laurent.boudin@upmc.fr

L.D.: CMLA, ENS CACHAN, IUF & CNRS, PRES UNIVERSUD, 61 AV. DU PDT. WILSON, 94235 CACHAN CEDEX

E-mail address: laurent.desvillettes@cmla.ens-cachan.fr

C.G.: INRIA, CRI PARIS-ROCQUENCOURT, EPI REO, F-78153, LE CHESNAY CEDEX, FRANCE

E-mail address: celine.grandmont@inria.fr

A.M.: CMLA, ENS CACHAN & CNRS, PRES UNIVERSUD, 61 AV. DU PDT. WILSON, 94235 CACHAN CEDEX

E-mail address: ayman.moussa@cmla.ens-cachan.fr