SEMIDEFINITE PROGRAMMING FOR MIN-MAX PROBLEMS AND GAMES

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Abstract. We introduce two min-max problems: the first problem is to minimize the supremum of finitely many rational functions over a compact basic semi-algebraic set whereas the second problem is a 2-player zero-sum polynomial game in randomized strategies and with compact basic semi-algebraic pure strategy sets. It is proved that their optimal solution can be approximated by solving a hierarchy of semidefinite relaxations, in the spirit of the moment approach developed in Lasserre [19, 20]. This provides a unified approach and a class of algorithms to approximate all Nash equilibria and min-max strategies of many static and dynamic games. Each semidefinite relaxation can be solved in time which is polynomial in its input size and practice from global optimization suggests that very often few relaxations are needed for a good approximation (and sometimes even finite convergence). In many cases (e.g. for Nash equilibria) the error of a relaxation can be computed.

1. Introduction

This paper is concerned with effective computation (or approximation) of Nash equilibria for n-player games. To achieve this goal, we provide a numerical scheme which consists of a hierarchy of semidefinite programs whose associated sequence of optimal values converges (sometimes in finitely many steps) to the value of the game. When the convergence is finite and a sufficient condition is met, one may also compute an optimal strategy.

Background. Nash equilibrium, a central concept in game theory, is a profile of mixed strategies (a strategy for each player) such that each player is best-responding to the strategies of the opponents. To show existence of an equilibrium in randomized (mixed) strategies for n-player finite static games, Nash used Kakutanyi's (resp. Brouwer's) fixed point theorem in [27] (resp. [28]). Then Glicksberg [12] extended the proof in Nash [27] to compact-continuous euclidean games.

Computing a fixed point of a function is known to be PPAD-complete (the class of all search problems that are guaranteed to exist by means of a direct graph argument, introduced by Papadimitriou [30]). This may be understood from the fact that Brouwer's is a consequence of Sperner's lemma [38] which in turn can be proved by a direct graph argument (see Border [3]).

Computing optimal solutions for a 2-player zero-sum finite game reduces to solving a linear program (von-Neumann and Morgenstern [29]) and so can be done in polynomial time. For a long time it has been thought that the famous Lemke-Howson [21] algorithm to compute a Nash-equilibrium for a 2-player non-zero-sum finite game is efficient. Even if it has been extended to n-player games in

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Rosenmüller [36], the common belief in game theory is that the computational complexity of 2-player games should differ from that of 3-(or more) player games.

In 2001, Papadimitriou [31] wrote *"the complexity of finding a Nash equilibrium* [of a 2-player game] *is the most important concrete open problem on the boundary of* **P**" and he analyzes that *"because of the guaranteed existence of a solution, the problem is unlikely to be NP-hard; in fact it belongs to a class of problems between P and NP"* (referring to PPAD).

Since then, Savani and von Stengel [37] proved that the Lemke-Howson algorithm may be exponential for 2-player games. Daskalakis, Goldberg and Papadimitriou [5] proved that solving a 4-player game is PPAD-complete and conjectured that for 2-player games, finding a Nash equilibrium may be solved in polynomial time. The later PPAD-completeness result has been extended to 3-player games by Daskalakis and Papadimitriou [6] and by Chen and Deng [7]. Unfortunately, Cheng and Deng [8] showed that a similar PPAD-completeness result holds for 2-player games!

The surprising result of Deng and Chen [8] may perhaps be understood from the recent and elegant paper of McLenann and Tourky [24] where it is proved that Kakutanyi may be deduced from 2-player finite imitation games (or from a linear complementarity problem).

An imitation game is a 2-player game where the payoff matrix of player 2 (the imitator) is the identity. The game may be described by an $m \times m$ matrix $A = (a_{i,j})$ (the payoff function of player 1, the mover). Finding a Nash equilibrium is equivalent to a linear complementarity problem [24]:

$$\text{Find a } \beta \text{ in the unit simplex } \Delta^m \text{ such that:}$$
$$\{i : \beta_i = 0\} \cup \left\{ i : i \in \arg \max_{i \in \{1,\dots,m\}} \sum_{j=1}^{m} a_{i,j}\beta_j \right\} = \{1, ..., m\}.$$

One may prove existence of a solution which can be computed by a simple adaptation of the Lemke-Howson algorithm. So, McLenann and Tourky [24] provided an algorithm that computes approximate fixed points of an upper-hemicontinuous convex and compact correspondence F. Starting from any initial point x_1, define recursively $\{x_m\}$ and $\{y_m\}$. Pick $y_m \in F(x_m)$ arbitrarily and set $x_{m+1} = \sum_{j=1}^{m} \beta_j^m y_j$ where β^m is an equilibrium of the imitation game where the payoff of the mover is $a_{i,j} = -\|x_i - y_j\|^2$. Accumulation points of $\{x^m\}$ are fixed points for F.

A different approach is to view the set of Nash equilibria as the set of real nonnegative solutions to a system of polynomial equations. Methods of computational algebra (e.g. using Gröbner bases) can be applied as suggested and studied in e.g. Dutta [10], Lipton [22] and Sturmfels [39]. However, observe that in this algebraic approach one first computes *all* complex solutions to sort out all real nonnegative solutions afterwards.

In the class of polynomial games introduced by Dresher, Karlin and Shapley (1950), the strategy set S^i of each player i is a product of compact intervals and the payoff function is polynomial. When the game is zero-sum and $S^i = [0, 1]$, Parrilo [32] showed that finding an optimal solution is equivalent to solving a single semidefinite program. Then Shah and Parrilo [34] extended the methodology to discounted zero-sum stochastic games in which the transition is controlled by one player only. Finally, it is worth noticing recent algorithms designed to solve some specific classes of infinite games. For instance, Gürkan and Pang [13].

Contribution. In a first part we consider the problem **P** of minimizing the supremum of finitely many rational functions over a compact basic semi-algebraic set. In the spirit of the moment approach developed in Lasserre [19, 20], we define a hierarchy of semidefinite relaxations (in short SDP-relaxations) for which each SDP-relaxation is a SDP that can be solved in polynomial time and the monotone sequence of optimal values associated with the hierarchy converges to the optimal value of **P**. Sometimes the convergence is finite and a sufficient condition permits to detect whether a certain relaxation in the hierarchy is exact (i.e. provides the optimal value), and to extract optimal solutions. Next, we show that computing the min-max or a Nash equilibrium in mixed strategies for static games or dynamic absorbing games, reduces to solving problem **P** mentioned above. We extend Nash's result for finite games to a new class of games that we call Loomis's [23] games and show that finding a Nash equilibrium of a Loomis game also reduces to solving problem **P**. It is worth emphasizing that when the payoffs are linear then the hierarchy of SDP-relaxations reduces to the first one of the hierarchy, which in turn is a linear program. This is in support of the claim that the above methodology is a natural extension to the non linear case of the well-known LP-approach.

The approach may be used to solve imitation games. Combined with McLennan and Tourky's construction, it provides an algorithm for computing a fixed point of any upper-hemicontinuous convex and compact correspondence hence computing a Nash equilibrium for concave euclidean games [12].

The approach may also be used to compute minima of *team optimization* problems in which a continuous function $f(x) = f(x^1, ..., x^n)$ is to be minimized over a cartesian product of convex-compact sets $X = \prod_{i=1}^n X^i$. The theory of teams is a particular instance of N-player games. Conversely, computing Nash-equilibria may be viewed as a team optimization problem. The team model has been introduced in Marschak [25] and studied by many authors ([17, 26]). If the function to minimize is a supremum of finitely many rational functions and the compact sets X^i, $i = 1, ..., n$ are basic semi-algebraic sets then this is a particular instance of problem **P**. If the function is separately convex, one can combine the construction of McLennan and Tourky [24] described above and use our algorithm to solve the associated imitation game, where the correspondence F is defined as in N-player games: $F^i(x) = \arg\min_{y^i \in X^i} f(x^1, ..., x^{i-1}, y^i, x^{i+1}, ..., x^n)$ and $F(x) = \prod_{i=1}^n F^i(x)$. Because f is separately convex, finding a point in F can be done, in principle, efficiently.

In a second part, we consider general 2-player zero-sum polynomial games (whose action sets are basic compact semi-algebraic sets of $\mathbb{R}^n$ and the payoff function polynomial). We show that the value and optimal strategies can be approximated as closely as desired, again by solving a certain hierarchy of SDP-relaxations. This result is a multivariate extension of Parrilo's [32] result for the univariate case where one needs to solve a single semidefinite program (as opposed to a hierarchy). This approach may be extended to dynamic absorbing games with discounted rewards, and in the univariate case one can construct a polynomial time algorithm that combines a dichotomy on the value of the game with a semidefinite program. Note that in 2-player absorbing dynamic games, transitions are controlled by *both* players, and so our result extends those in Parrilo and Shah [34] where only one player controls the transition. A natural open question arises: how to adapt the techniques to approximate general non-zero-sum polynomial games?

Importantly, and in contrast with numerical algorithms that compute only one equilibrium, our moment approach allows to compute all Nash equilibria of a finite game (when that number is finite) and without computing all complex solutions as in the computational algebra algorithms described in Dutta [10], Lipton [22] and Sturmfels [39].

To conclude, if the rather negative computational complexity results ([37], [5], [6], [7], [8]) have conforted the game theory community with the idea that many game problems are computationally hard, on a more positive tone, our contribution provides a unified semidefinite programming approach to many game problems: it shows that optimal value and strategies can be approximated as closely as desired (and sometimes obtained exactly) by solving a hierarchy of semidefinite relaxations, very much in the spirit of the moment approach described in [19] for solving polynomial optimization problems (a particular instance of the generalized problem of moments [20]). Moreover, the algorithm is consistent with previous results [29] and [32] as it reduces to a linear program for finite zero-sum games and to a single semidefinite program for univariate infinite zero-sum games.

Finally, even if practice in global optimization seems to reveal that this approach is efficient, of course the size of the semidefinite relaxations grows rapidly with the initial problem size. Therefore, in view of the present status of public SDP solvers available, its application is limited to small to medium size problems so far. Two big challenges are to (a) detect in advance which relaxation in the hierarchy solves the problem up to a given tolerance, and (b) when a relaxation is exact, to determine whether its size is polynomial in the input size of the initial problem. These questions seem to be very difficult, maybe in the boundary of **P** = or not = to **NP**?

2. Notation and preliminary results

2.1. **Notation and definitions.** Let $\mathbb{R}[x]$ be the ring of real polynomials in the variables $x = (x_1, \ldots, x_n)$ and let $(X^\alpha)_{\alpha\in\mathbb{N}}$ be its canonical basis of monomials. Denote by $\Sigma[x] \subset \mathbb{R}[x]$ the subset (cone) of polynomials that are sums of squares (s.o.s.), and by $\mathbb{R}[x]_d$ the space of polynomials of degree at most d.

With $\mathbf{y} =: (y_\alpha) \subset \mathbb{R}$ being a sequence indexed in the canonical monomial basis (X^α), let $L_\mathbf{y} : \mathbb{R}[x] \to \mathbb{R}$ be the linear functional

$$f\,(=\sum_{\alpha\in\mathbb{N}^n} f_\alpha\, x^\alpha) \longmapsto \sum_{\alpha\in\mathbb{N}^n} f_\alpha\, y_\alpha, \quad f \in \mathbb{R}[x].$$

Moment matrix. Given $\mathbf{y} = (y_\alpha) \subset \mathbb{R}$, the *moment* matrix $M_d(\mathbf{y})$ of order d associated with $\mathbf{y}$, has its rows and columns indexed by (x^α) and its (α,β)-entry is defined by:

$$M_d(\mathbf{y})(\alpha,\beta) \,:=\, L_\mathbf{y}(x^{\alpha+\beta}) \,=\, y_{\alpha+\beta}, \qquad |\alpha|,\,|\beta| \,\leq d.$$

Localizing matrix. Similarly, given $\mathbf{y} = (y_\alpha) \subset \mathbb{R}$ and $\theta \in \mathbb{R}[x]\,(= \sum_\gamma \theta_\gamma x^\gamma)$, the *localizing* matrix $M_d(\theta\,\mathbf{y})$ of order d associated with $\mathbf{y}$ and θ, has its rows and columns indexed by (x^α) and its (α,β)-entry is defined by:

$$M_d(\theta\,\mathbf{y})(\alpha,\beta) \,:=\, L_\mathbf{y}(x^{\alpha+\beta}\theta(x)) \,=\, \sum_\gamma \theta_\gamma\, y_{\gamma+\alpha+\beta}, \qquad |\alpha|,\,|\beta| \,\leq d.$$

One says that $\mathbf{y} = (y_\alpha) \subset \mathbb{R}$ has a *representing* measure supported on $\mathbf{K}$ if there is some finite Borel measure μ on $\mathbf{K}$ such that

$$y_\alpha = \int_{\mathbf{K}} x^\alpha \, d\mu(x), \qquad \forall \, \alpha \in \mathbb{N}^n.$$

For later use, write

$$M_d(\mathbf{y}) = \sum_{\alpha \in \mathbb{N}^n} y_\alpha \, B_\alpha \tag{2.1}$$

$$M_d(\theta, \mathbf{y}) = \sum_{\alpha \in \mathbb{N}^n} y_\alpha \, B_\alpha^\theta, \tag{2.2}$$

for real symmetric matrices $(B_\alpha, B_\alpha^\theta)$ of appropriate dimensions.

Definition 2.1 (Putinar's property)**.** Let $(g_j)_{j=1}^m \subset \mathbb{R}[x]$. A basic closed semi algebraic set $\mathbf{K} := \{x \in \mathbb{R}^n : g_j(x) \geq 0, : j = 1, \ldots, m\}$ satisfies Putinar's property if there exists $u \in \mathbb{R}[x]$ such that $\{x \,:\, u(x) \geq 0\}$ is compact and

$$u = \sigma_0 + \sum_{j=1}^m \sigma_j \, g_j \tag{2.3}$$

for some $(u_j)_{j=0}^m \subset \Sigma[x]$. Equivalently, for some $M > 0$ the quadratic polynomial $x \mapsto M - \|x\|^2$ has Putinar's representation (2.3).

Obviously Putinar's property implies compactness of $\mathbf{K}$. However, notice that Putinar's property is not geometric but algebraic as it is related to the representation of $\mathbf{K}$ by the defining polynomials (g_j)'s. Putinar's property holds if e.g. the level set $\{x \,:\, g_j(x) \geq 0\}$ is compact for some j, or if all g_j are affine (in which case $\mathbf{K}$ is a polytope). In case it is not satisfied and if for some $M > 0$, $\|x\|^2 \leq M$ whenever $x \in \mathbf{K}$, then it suffices to add the redundant quadratic constraint $g_{m+1}(x) := M - \|x\|^2 \geq 0$ to the definition of $\mathbf{K}$. The importance of Putinar's property stems from the following result:

Theorem 2.2 (Putinar [33])**.** *Let $(g_j)_{j=1}^m \subset \mathbb{R}[x]$ and assume that*

$$\mathbf{K} := \{ \, x \in \mathbb{R}^n \;:\; g_j(x) \geq 0, \; j = 1, \ldots, m\}$$

satisfies Putinar's property.

(a) Let $f \in \mathbb{R}[x]$ be positive on $\mathbf{K}$. Then f can be written as u in (2.3).

(b) Let $\mathbf{y} = (y_\alpha)$. Then $\mathbf{y}$ has a representing measure on $\mathbf{K}$ if and only if

$$M_d(\mathbf{y}) \succeq 0, \quad M_d(g_j \, \mathbf{y}) \succeq 0, \qquad j = 1, \ldots, m; \quad d = 0, 1, \ldots \tag{2.4}$$

We also have:

Lemma 2.3. *Let $\mathbf{K} \subset \mathbb{R}^n$ be compact and let p, q continuous such that with $q > 0$ on $\mathbf{K}$. Let $M(\mathbf{K})$ be the set of finite Borel measures on $\mathbf{K}$ and let $P(\mathbf{K}) \subset M(\mathbf{K})$ be its subset of probability measures on $\mathbf{K}$. Then*

$$\min_{\mu \in P(\mathbf{K})} \frac{\int p \, d\mu}{\int q \, d\mu} = \min_{\varphi \in M(\mathbf{K})} \{ \int p \, d\varphi : \int q \, d\varphi = 1 \} \tag{2.5}$$

$$= \min_{\mu \in P(\mathbf{K})} \int \frac{p}{q} \, d\mu = \min_{x \in \mathbf{K}} : \frac{p(x)}{q(x)} \tag{2.6}$$

Proof. Let $\rho^* := \min_x\{p(x)/q(x) : x \in \mathbf{K}\}$. As $q > 0$ on $\mathbf{K}$,

$$\frac{\int p\,d\mu}{\int q\,d\mu} \geq \frac{\int (p/q)\,q\,d\mu}{\int q\,d\mu} \geq \rho^*.$$

Similarly, $\int (p/q)d\mu \geq \rho^* \int d\mu = \rho^*$. Other hand, with $x^* \in \mathbf{K}$ a global minimizer of p/q on $\mathbf{K}$, let $\mu := \delta_{x^*} \in P(\mathbf{K})$ be the Dirac measure at $x = x^*$. Then $\int p d\mu / \int q d\mu = p(x^*)/q(x^*) = \int (p/q) d\mu = \rho^*$, and therefore

$$\min_{\mu \in P(\mathbf{K})} \frac{\int p d\mu}{\int q d\mu} = \min_{\mu \in P(\mathbf{K})} \int \frac{p}{q} d\mu = \min_{x \in \mathbf{K}} : \frac{p(x)}{q(x)} = \rho^*.$$

Next, for every $\varphi \in M(\mathbf{K})$ with $\int q d\varphi = 1$, $\int p\,d\varphi \geq \int \rho^* q\,d\varphi = \rho^*$, and so $\min_{\varphi \in M(\mathbf{K})}\{\int p\,d\varphi : \int q\,d\varphi = 1\} \geq \rho^*$. Finally taking $\varphi := q(x^*)^{-1}\delta_{x^*}$ yields $\int q d\varphi = 1$ and $\int p\,d\varphi = p(x^*)/q(x^*) = \rho^*$.

Another way to see why this is true is throughout the following argument: the function $\mu \to \frac{\int p\,d\mu}{\int q\,d\mu}$ is quasi-concave so that the optimal value of the minimization problem may be achieved on the boundary. □

3. Minimizing a max of rational functions

Let $\mathbf{K} \subset \mathbb{R}^n$ be the basic semi-algebraic set

$$\mathbf{K} := \{x \in \mathbb{R}^n : g_j(x) \geq 0, \quad j = 1, \ldots, p\} \tag{3.1}$$

for some polynomials $(g_j) \subset \mathbb{R}[x]$, and let $f_i = p_i/q_i$ be rational functions, $i = 0, 1, \ldots, m$, with $p_i, q_i \in \mathbb{R}[x]$.

Consider the problem

$$\mathbf{P} : \quad \rho := \inf_x \{f_0(x) + \max_{i=1,\ldots,m} f_i(x) \;:\; x \in \mathbf{K}\}, \tag{3.2}$$

or, equivalently,

$$\mathbf{P} : \quad \rho = \inf_{x,z} \{f_0(x) + z : z \geq f_i(x),\; x \in \mathbf{K}\}. \tag{3.3}$$

Assumption 3.1. *$q_i > 0$ for all $x \in \mathbf{K}$ and every $i = 0, \ldots, m$.*

Assumption 3.2. *$\mathbf{K}$ satisfies Putinar's property.*

With $\mathbf{K} \subset \mathbb{R}^n$ as in (3.1), let $\widehat{\mathbf{K}} \subset \mathbb{R}^{n+1}$ be the basic semi algebraic set

$$\widehat{\mathbf{K}} := \{(x, z) \in \mathbb{R}^n \times \mathbb{R} \;:\; x \in \mathbf{K},\; z\,q_i(x) - p_i(x) \geq 0, i = 1, \ldots, m\} \tag{3.4}$$

and consider the new optimization problem

$$\mathcal{P} : \quad \hat{\rho} := \inf_\mu \{\int (p_0 + z\,q_0)\,d\mu : \int q_0\,d\mu = 1, \mu \in M(\widehat{\mathbf{K}})\} \tag{3.5}$$

where $M(\widehat{\mathbf{K}})$ is the set of finite Borel measures on $\widehat{\mathbf{K}}$.

Proposition 3.3. *Let $\mathbf{K} \subset \mathbb{R}^n$ be as in (3.1) and let Assumption 3.1 hold.*
If $\rho > -\infty$ then $\rho = \hat{\rho}$.

Proof. Let $x \in \mathbf{K}$ be such that $f_0(x) + \max_{i=1,\ldots,m} f_i(x) \leq \rho + \epsilon$ for $\epsilon > 0$ fixed, arbitrary. Let $z := \max_{i=1,\ldots,m} f_i(x)$ so that $(x, z) \in \widehat{\mathbf{K}}$ because $x \in \mathbf{K}$ and $q_i > 0$ on $\mathbf{K}$ for every $i = 1, \ldots, m$. Let $\mu := q_0(x)^{-1}\delta_{(x,z)}$, with $\delta_{(x,z)}$ being the Dirac measure at $(x, z) \in \widehat{\mathbf{K}}$. Then $\mu \in M(\widehat{\mathbf{K}})$ and $\int q_0 d\mu = 1$. In addition,

$\int(p_0 + zq_0)d\mu = p_0(x)/q_0(x) + z \leq \rho + \epsilon$. As $\epsilon > 0$ was arbitrary, it follows that $\hat{\rho} \leq \rho$.

On the other hand, let $\mu \in M(\widehat{\mathbf{K}})$ be such that $\int q_0 d\mu = 1$. As $p_0(x)/q_0(x) + \max_{i=1,\ldots,m} f_i(x) \geq \rho$ for all $x \in \mathbf{K}$, it follows that $p_0(x)/q_0(x) + z \geq \rho$ for all $(x, z) \in \widehat{\mathbf{K}}$. Equivalently, $p_0 + zq_0 \geq \rho q_0$ for all $(x, z) \in \widehat{\mathbf{K}}$ because $q_0 > 0$ on $\mathbf{K}$. Integrating with respect to $\mu \in M(\widehat{\mathbf{K}})$ yields $\int(p_0 + zq_0)d\mu \geq \rho \int q_0 d\mu = \rho$, which proves that $\hat{\rho} \geq \rho$, and so, $\hat{\rho} = \rho$, the desired result. $\square$

We next describe how to solve $\mathbf{P}$ via a hierarchy of semidefinite relaxations.

SDP-relaxations for solving P. If $\mathbf{K}$ is compact, and under Assumption 3.1, let

$$M_1 := \max_{i=1,\ldots,m} \left\{ \frac{\max\{|p_i(x)| : x \in \mathbf{K}\}}{\min\{q_i(x), x \in \mathbf{K}\}} \right\}, \tag{3.6}$$

and

$$M_2 := \min_{i=1,\ldots,m} \left\{ \frac{\min\{p_i(x) : x \in \mathbf{K}\}}{\max\{q_i(x) : x \in \mathbf{K}\}} \right\}. \tag{3.7}$$

Redefine the set $\widehat{\mathbf{K}}$ to be

$$\widehat{\mathbf{K}} := \{(x, z) \in \mathbb{R}^n \times \mathbb{R} : h_j(x, z) \geq 0, \quad j = 1, \ldots p + m + 2\} \tag{3.8}$$

with

$$\left\{ \begin{array}{llll} (x, z) \mapsto & h_j(x, z) & := g_j(x) & j = 1, \ldots, p \\ (x, z) \mapsto & h_j(x, z) & := z\, q_j(x) - p_j(x) & j = p + 1, \ldots, p + m \\ (x, z) \mapsto & h_j(x, z) & := M_1 - z & j = m + p + 1 \\ (x, z) \mapsto & h_j(x, z) & := z - M_2 & j = m + p + 2 \end{array} \right. . \tag{3.9}$$

Lemma 3.4. *Let $\mathbf{K} \subset \mathbb{R}^n$ be compact and let Assumptions 3.1, 3.2 hold. Then the set $\widehat{\mathbf{K}} \subset \mathbb{R}^{n+1}$ defined in (3.8) satisfies Putinar's property.*

Proof. By Assumption 3.2, $\mathbf{K}$ satisfies Putinar's property. Equivalently, the quadratic polynomial $x \mapsto M - \|x\|^2$ can be written in the form (2.3). Next,

$$(M_1 - z)(z - M_2) = (M_1 - M_2)\left[(z - M_2)^2(M_1 - z) + (M_1 - z)^2(z - M_2)\right],$$

and so consider quadratic polynomial

$$(x, z) \mapsto w(x, z) = M - \|x\|^2 + (M_1 - z)(z - M_2).$$

Obviously, its level set $\{x : w(x, z) \geq 0\} \subset \mathbb{R}^{n+1}$ is compact and moreover, w can be written in the form

$$w(x, z) = \sigma_0(x, z) + \sum_{j=1}^{p} \sigma_j(x, z)\, g_j(x) + \sum_{j=m+p+1}^{m+p+2} \sigma_j(x, z)\, h_j(x, z)$$

with $(\sigma_j) \subset \Sigma[x, z]$. Therefore $\widehat{\mathbf{K}}$ satisfies Putinar's property in Definition 2.1, the desired result. $\square$

We are now in position de define the hierarchy of semidefinite relaxations for solving $\mathbf{P}$. Let $\mathbf{y} = (y_\alpha)$ be a real sequence indexed in the monomial basis $(x^\beta z^k)$ of $\mathbb{R}[x, z]$ (with $\alpha = (\beta, k) \in \mathbb{N}^n \times \mathbb{N}$).

Let $h_0(x,z) := p_0(x)+zq_0(x)$, and let $v_j := \lceil(\deg h_j)/2\rceil$ for every $j=0,\ldots,m+p+2$. For $r \geq r_0 := \max\limits_{j=0,\ldots,p+m+1} v_j$, introduce the semidefinite program

$$\mathbf{Q}_r:\left\{\begin{array}{lll}\inf_{\mathbf{y}} & L_{\mathbf{y}}(h_0) & \\ \text{s.t.} & M_r(\mathbf{y}) & \succeq 0 \\ & M_{r-v_j}(h_j\,\mathbf{y}) & \succeq 0, \quad j=1,\ldots,m+p+2 \\ & L_y(q_0) & = 1\end{array}\right. \tag{3.10}$$

with optimal value denoted $\inf \mathbf{Q}_r$ (and $\min \mathbf{Q}_r$ if the infimum is attained).

Theorem 3.5. *Let $\mathbf{K} \subset \mathbb{R}^n$ be compact and as in (3.1). Let Assumptions 3.1, 3.2 hold. Let $\mathbf{Q}_r$ be the semidefinite program (3.10) with $(h_j) \subset \mathbb{R}[x,z]$ and M_1, M_2 defined in (3.9) and (3.6)-(3.7) respectively. Then:*

(a) $\inf \mathbf{Q}_r \uparrow \rho$ as $r \to \infty$.

(b) Let $\mathbf{y}^r$ be an optimal solution of the SDP-relaxation $\mathbf{Q}_r$ in (3.10). If

$$\operatorname{rank} M_r(\mathbf{y}^r) \;=\; \operatorname{rank} M_{r-r_0}(\mathbf{y}^r) \;=\; t \tag{3.11}$$

then one may extract t points $x^*(t) \in \mathbf{K}$, all global minimizers of $\mathbf{P}$.

For a proof the reader is referred to §7.1. To solve (3.10) one may use e.g. the Matlab based public software GloptiPoly 3 [15] dedicated to solve the generalized problem of moments described in [20]. It is an extension of GloptiPoly [14] previously dedicated to solve polynomial optimization problems. A procedure for extracting optimal solutions is implemented in Gloptipoly when the rank condition (3.11) is satisfied. For more details the interested reader is referred to [15] and `www.laas.fr/∼henrion/software/`.

Remark 3.6. If g_j is affine for every $j=1,\ldots,p$ and if p_j is affine and $q_j \equiv 1$ for every $j=0,\ldots,m$, then h_j is affine for every $j=0,\ldots,m$. In this case it suffices to solve the single semidefinite relaxation $\mathbf{Q}_1$ which is in fact a linear program. Indeed, for $r=1$, $\mathbf{y}=(y_0,(x,z),Y)$ and

$$M_1(\mathbf{y}) \;=\; \left[\begin{array}{ccc} y_0 & | & (x \quad z) \\ - & & - \\ \begin{pmatrix} x \\ z\end{pmatrix} & | & Y\end{array}\right].$$

Then (3.10) reads

$$\mathbf{Q}_1:\left\{\begin{array}{lll}\inf_{\mathbf{y}} & h_0(x) & \\ \text{s.t.} & M_1(\mathbf{y}) & \succeq 0 \\ & h_j(x,z) & \geq 0, \quad j=1,\ldots,m+p+2 \\ & y_0 & = 1\end{array}\right..$$

But as $v_j = 1$ for every j, $M_{1-1}(h_j\mathbf{y}) \succeq 0 \Leftrightarrow M_0(h_j\mathbf{y}) = L_{\mathbf{y}}(h_j) = h_j(x,z) \geq 0$, a linear constraint. Hence the constraint $M_1(\mathbf{y}) \succeq 0$ can be discarded as given any (x,z) one may always find Y such that $M_1(\mathbf{y}) \succeq 0$. Therefore, (3.10) is a linear program.

4. Applications to games

4.1. **Standard static games.** A finite game is a tuple $(N, \{S^i\}_{i=1,\ldots,N}, \{g^i\}_{i=1,\ldots,N})$ where $N \in \mathbb{N}$ is the set of players, S^i is the finite set of pure strategies of player i

and $g^i : \mathbf{S} \to \mathbb{R}$ is the payoff function of player i, where $\mathbf{S} := S^1 \times ... \times S^N$. The set

$$\Delta^i = \left\{ \left(p^i(s^i)\right)_{s^i \in S^i} : \quad p^i(s^i) \geq 0, \sum_{s^i \in S^i} p^i(s^i) = 1 \right\}$$

of probability distributions over S^i is called the set of mixed strategies of player i. Notice that Δ^i is a compact basic semi-algebraic set. If each player j chooses the mixed strategy $p^j(\cdot)$, the vector denoted $p = \left(p^1, ..., p^N\right) \in \mathbf{\Delta} = \Delta^1 \times ... \times \Delta^N$ is called the *profile* and the expected payoff of a player i is

$$g^i(p) = \sum_{s \in S} p^1(s^1) \times ... \times p^N(s^N) g^i(s).$$

For a player i, and a profile p, let p^{-i} be the profile of the other players except i: that is $p^{-i} = (p^1, ..., p^{i-1}, p^{i+1}, ..., p^N)$. Let $\mathbf{S}^{-i} = S^1 \times ... \times S^{i-1} \times S^{i+1} \times ... \times S^N$ and

$$g^i(s^i, p^{-i}) = \sum_{s^{-i} \in S^{-i}} p^1(s^1) \times ... \times p^{i-1}(s^N) \times p^{i+1}(s^N) \times ... \times p^N(s^N) g^i(s).$$

A profile p_0 is a Nash equilibrium (in mixed strategies) if and only for all $i \in N$ and all $s^i \in S^i$, $g^i(p_0) \geq g^i(s^i, p_0^{-i})$ or equivalently if

$$p_0 \in \arg\min_{p \in \mathbf{\Delta}} \max_{i \in N, s^i \in S^i} \left\{ g^i(s^i, p_0^{-i}) - g^i(p_0) \right\}$$

This min-max problem is a particular instance of problem $\mathbf{P}$ in (3.2). Assumption 3.1 and 3.2 are satisfied and so Theorem 3.5 applies. That is, by solving the hierarchy of SDP-relaxations (3.10), one can approximate the value of the game as closely as desired. In addition, if (3.11) is satisfied at some relaxation $\mathbf{Q}_r$, then one obtains an optimal strategy. Since the optimal value is zero, one knows when the algorithm should stop and if it does not stop, one has a bound on payoffs so that one knows which epsilon-equilibrium is reached.

Example 4.1. Consider the simple illustrative example of a 2×2 game with data

	s_1^2	s_2^1
s_1^1	(a, c)	$(0, 0)$
s_2^1	$(0, 0)$	(b, d)

for some scalars (a, b, c, d). Denote $x \in [0, 1]$ the probability for player 1 of playing s_1^1 and $y \in [0, 1]$ the probability for player 2 of playing s_1^2. Then one must solve

$$\min_{x,y} \max \left\{ \begin{array}{l} ax - axy - b(1-x)(1-y) \\ b(1-y) - axy - b(1-x)(1-y) \\ cx - cxy - d(1-x)(1-y) \\ d(1-x) - cxy - d(1-x)1-y) \end{array} \right. .$$

We have solved the hierarchy of semidefinite programs (3.10) with GloptiPoly 3 [15]. For instance, the moment matrix $M_1(\mathbf{y})$ of the first SDP-relaxation $\mathbf{Q}_1$ reads

$$M_1(\mathbf{y}) = \left[\begin{array}{cccc} y_0 & y_{100} & y_{010} & y_{001} \\ y_{100} & y_{200} & y_{010} & y_{001} \\ y_{010} & y_{110} & y_{020} & y_{011} \\ y_{001} & y_{101} & y_{011} & y_{002} \end{array} \right],$$

and $\mathbf{Q}_1$ reads

$$\mathbf{Q}_1 : \begin{cases} \inf\limits_{\mathbf{y}} & y_{001} \\ \text{s.t.} & M_1(\mathbf{y}) \succeq 0 \\ & y_{001} - ay_{100} + ay_{110} + b(y_0 - y_{100} - y_{010} + y_{110}) \geq 0 \\ & y_{001} - by_0 + by_{010} + ay_{110} + b(y_0 - y_{100} - y_{010} + y_{110}) \geq 0 \\ & y_{001} - cy_{100} + cy_{110} + d(y_0 - y_{100} - y_{010} + y_{110}) \geq 0 \\ & y_{00} - dy_0 + dy_{100} + cy_{110} + d(y_0 - y_{100} - y_{010} + y_{110}) \geq 0 \\ & y_0 \ = 1 \end{cases} .$$

With $(a, b, c, d) = (0.05, 0.82, 0.56, 0.76)$, solving $\mathbf{Q}_3$ yields the optimal value $3.93.10^{-11}$ and the three optimal solutions $(0, 0)$, $(1, 1)$ and $(0.57575, 0.94253)$. With randomly generated $a, b, c, d \in [0, 1]$ we also obtained a very good approximation of the global optimum 0 and 3 optimal solutions in most cases with $r = 3$ (i.e. with moments or order 6 only) and sometimes $r = 4$.

We have also solved 2-player non-zero-sum $p \times q$ games with randomly generated reward matrices $A, B \in \mathbb{R}^{p\times q}$ and $p, q \leq 5$. We could solve $(5, 2)$ and $(4, q)$ (with $q \leq 3$) games exactly with the 4th (sometimes 3rd) SDP-relaxation, i.e. $\inf \mathbf{Q}_4 = O(10^{-10}) \approx 0$ and one extracts an optimal solution[1]. However, the size is relatively large and one is close to the limit of present public SDP-solvers like SeDuMi. Indeed, for a 2-player $(5, 2)$ or $(4, 3)$ game, $\mathbf{Q}_3$ has 923 variables and $M_3(\mathbf{y}) \in \mathbb{R}^{84\times 84}$, whereas $\mathbf{Q}_4$ has 3002 variables and $M_4(\mathbf{y}) \in \mathbb{R}^{210\times 210}$. For a $(4, 4)$ game $\mathbf{Q}_3$ has 1715 variables and $M_3(\mathbf{y}) \in \mathbb{R}^{120\times 120}$ and $\mathbf{Q}_3$ is still solvable, whereas $\mathbf{Q}_4$ has 6434 variables and $M_4(\mathbf{y}) \in \mathbb{R}^{330\times 330}$.

Another important concept in game theory is the min-max payoff $\underline{v}_i$, also called the individually rational level of player i. It plays an important role is the famous folk theorem (Aumann and Shapley [2]). It is a min-max problem:

$$\underline{v}_i = \min_{p^{-i}\in\mathbf{\Delta}^{-i}} \max_{s^i\in S^i} g^i(s^i, p^{-i})$$

where $\mathbf{\Delta}^{-i} = \Delta^1 \times ... \times \Delta^{i-1} \times \Delta^{i+1} \times ... \times \Delta^N$. This problem is also a particular instance of problem $\mathbf{P}$ in (3.2). It seems more difficult to compute the min-max strategies compared to Nash equilibrium strategies because we do not know in advance the value of $\underline{v}_i$.

Note that in the case of two players, if the function $g^i(s^i, p^{-i})$ is linear in p then by remark 3.6 it suffices to solve the first relaxation $\mathbf{Q}_1$, a linear program.

4.2. **Loomis games.** Loomis [23] extended the min-max theorem of Von Neuman on zero-sum games to any rational fraction of two multilinear extensions. His model and result may be extended to N-player games.

Associates to each player $i \in N$ two functions $g^i : \mathbf{S} \to \mathbb{R}$ and $f^i : \mathbf{S} \to \mathbb{R}$ where $f^i > 0$. As above, their multilinear extensions to $\mathbf{\Delta}$ is also denoted by g^i and f^i.

Definition 4.2. Loomis game is an euclidean game. The (pure) strategy set of player i is Δ^i with payoff function $h^i(p) = \frac{g^i(p)}{f^i(p)}$ if the profile $p \in \mathbf{\Delta}$ is chosen.

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[1]In fact GloptiPoly 3 extracts *all* solutions because most SDP-solvers that one may call in GloptiPoly 3 (e.g. SeDuMi) use primal-dual interior points methods which find an optimal solution in the relative interior of the feasible set. In the present context of (3.10) this means that at an optimal solution $\mathbf{y}^*$, the moment matrix $M_r(\mathbf{y}^*)$ has maximum rank and its rank corresponds to the numbers of solutions.

Lemma 4.3 (Extension of Loomis [23] result)**.** *A Loomis game admits a (pure) Nash equilibrium.*

Proof. Note that each payoff function is quasi-concave in p^i (and also quasi-convex so that it is a quasi-linear function). Actually, if $h^i(p_1^i, p^{-i}) \geq \alpha$ and $h^i(p_2^i, p^{-i}) \geq \alpha$ then $\delta g^i(p_1^i, p^{-i}) \geq \delta\delta f^i(p_1^i, p^{-i})$, and $(1-\delta)g^i(p_1^i, p^{-i}) \geq (1-\delta)\delta f^i(p_1^i, p^{-i})$ so that

$$g^i(\delta p_1^i + (1-\delta)p_2^i, p^{-i}) \geq f^i(\delta p_1^i + (1-\delta)p_2^i, p^{-i})\alpha$$

hence $h^i(\delta p_1^i + (1-\delta)p_2^i, p^{-i}) \geq \alpha$. One may now apply Glicksberg's [12] theorem because the strategy sets are compact, convex, and the payoff functions are continuous. □

Corollary 4.4. $p_0 \in \mathbf{\Delta}$ *is a (pure) Nash equilibrium of a Loomis game if and only if*

$$p_0 \in \arg\min_{p\in\Delta} \max_{i\in N, s^i\in S^i} \left\{h^i(s^i, p^{-i}) - h^i(p)\right\}.$$

Proof. Clearly, $p_0 \in \mathbf{\Delta}$ is an equilibrium of the Loomis game if and only if

$$p_0 \in \arg\min_{p\in\Delta} \max_{i\in N, \widetilde{p}^i\in\Delta^i} \left\{\frac{g(\widetilde{p}^i, p^{-i})}{f^i(\widetilde{p}^i, p^{-i})} - \frac{g^i(p)}{f^i(p)}\right\}.$$

Using the quasi-linearity of the payoffs or Lemma 2.3, one deduces:

$$\max_{\widetilde{p}^i\in\Delta^i} \frac{g^i(\widetilde{p}^i, p^{-i})}{f^i(\widetilde{p}^i, p^{-i})} = \max_{s^i\in S^i} \frac{g^i(s^i, p^{-i})}{f^i(s^i, p^{-i})}$$

which is the desired result. □

Again, this problem is a particular instance of problem **P** in (3.2) and so can be solved via the hierarchy of semidefinite relaxations (3.10).

4.3. **Finite absorbing games.** This subclass of stochastic games have been introduced by Kohlberg [18]. A N-player finite absorbing games is defined as follows. As above, there are N finite sets $(S^1, ..., S^N)$. There are $2 \times N$-payoff functions $g^i : \mathbf{S} \to \mathbb{R}$ and $f^i : \mathbf{S} \to \mathbb{R}$ for each $i \in \{1, ..., N\}$ and a probability transition function $q : \mathbf{S} \to [0, 1]$.

The game is played in discrete time as follows. Inductively, at stage $t = 1, 2, ...$, players have to play simultaneously. A player i chooses at random an action $s_t^i \in S^i$. Then,

(i) with probability $1 - q\left(s_t^1, ..., s_t^N\right)$ the game is terminated and each player i gets at every stage $s \geq t$ the payoff $f^i\left(s_t^1, ..., s_t^N\right)$,and

(ii) with probability $q\left(s_t^1, ..., s_t^N\right)$ the game continues and the payoff of each player j at stage t is $g^i\left(a_t^1, ..., a_t^N\right)$.

We consider the λ-discounted game $G(\lambda)$ $(0 < \lambda < 1)$. If the payoff of player i at stage t is $r^i(t)$ then its λ-discounted payoff in the game is $\sum_{t=1}^{\infty} \lambda(1-\lambda)^{t-1} r^i(t)$. Hence, a player is optimizing his expected λ-discounted payoff.

Let $\widetilde{g}^i = g^i \times q$ and $\widetilde{f}^i = f^i \times (1-q)$ and extend $\widetilde{g}^i$, $\widetilde{f}^i$ and q multilinearly to $\mathbf{\Delta}$.

Lemma 4.5. *Stationary Nash-equilibria exists.* $p_0 \in \Delta$ *is a stationary equilibrium with a corresponding payoff vector* $w = \left(w^1, ..., w^N\right) \in \mathbb{R}^N$ *if and only if for every*

$i \in N$:

$$\begin{aligned} w_i &= \max_{s^i \in S^i} \left(\lambda \widetilde{g}^i(s^i, p_0^{-i}) + (1-\lambda) q(s^i, p_0^{-i}) w_i + \widetilde{f}^i(s^i, p_0^{-i}) \right) \\ &= \max_{p^i \in \Delta^i} \left(\lambda \widetilde{g}^i(p^i, p_0^{-i}) + (1-\lambda) q(p^i, p_0^{-i}) w_i + \widetilde{f}^i(p^i, p_0^{-i}) \right) \\ p_0^i &\in \arg\max_{p^i \in \Delta^i} \left(\lambda \widetilde{g}^i(p^i, p_0^{-i}) + (1-\lambda) q(s^i, p_0^{-i}) w_i + \widetilde{f}^i(p^i, p_0^{-i}) \right) \end{aligned}$$

Proof. A consequence of Fink [11]. □

Corollary 4.6. *$p_0 \in \Delta$ is a stationary equilibrium of the absorbing game if and only if*

$$p_0 \in \arg\min_{p \in \Delta} \max_{i, s^i} \left\{ \frac{\lambda \widetilde{g}^i(s^i, p^{-i}) + \widetilde{f}^i(s^i, p^{-i})}{\lambda q(s^i, p^{-i}) + (1 - q(s^i, p^{-i}))} - \frac{\lambda \widetilde{g}^i(p) + \widetilde{f}^i(p)}{\lambda q(p) + (1 - q(p))} \right\} \tag{4.1}$$

Or equivalently, iff p_0 is a Nash equilibrium of the Loomis game with payoff functions $p \to \frac{\lambda \widetilde{g}^i(p) + \widetilde{f}^i(p)}{\lambda q(p) + (1 - q(p))}$, $i = 1, ..., N$.

Proof. A simple computation shows that $p_0 \in \Delta$ is a stationary equilibrium with payoff $w = \left(w^1, ..., w^N\right) \in \mathbb{R}^N$ if for every $i \in N$:

$$\begin{aligned} w_i &= \max_{s^i \in S^i} \frac{\lambda \widetilde{g}^i(s^i, p_0^{-i}) + \widetilde{f}^i(s^i, p_0^{-i})}{\lambda q(s^i, p_0^{-i}) + (1 - q(s^i, p_0^{-i}))} \\ &= \max_{p^i \in \Delta^i} \frac{\lambda \widetilde{g}^i(p^i, p_0^{-i}) + \widetilde{f}^i(p^i, p_0^{-i})}{\lambda q(p^i, p_0^{-i}) + (1 - q(p^i, p_0^{-i}))} \end{aligned}$$

and

$$p_0^i \in \arg\max_{p^i \in \Delta^i} \frac{\lambda \widetilde{g}^i(p^i, p_0^{-i}) + \widetilde{f}^i(p^i, p_0^{-i})}{\lambda q(p^i, p_0^{-i}) + (1 - q(p^i, p_0^{-i}))}$$

A calculus as in Loomis games shows the equivalence with the statement of the lemma. □

Similarly, the min-max of a discounted absorbing game may be shown to satisfy the following formula:

$$\underline{v}_i = \min_{p^{-i} \in \mathbf{\Delta}^{-i}} \max_{s^i \in S^i} \frac{\lambda \widetilde{g}^i(s^i, p^{-i}) + \widetilde{f}^i(s^i, p^{-i})}{\lambda q(s^i, p^{-i}) + (1 - q(s^i, p^{-i}))}$$

Hence from (4.1) in Corollary 4.6, solving a finite absorbing game reduces to solving a problem $\mathbf{P}$ as defined in (3.2), which again can be solved via the hierarchy of semidefinite relaxations (3.10).

5. Zero-sum polynomial games

Let $\mathbf{K}_1, \mathbf{K}_2 \subset \mathbb{R}^n$ be two basic and closed semi-algebraic sets

$$\mathbf{K}_1 := \{x \in \mathbb{R}^n : g_j(x) \geq 0, \quad j = 1, \ldots, m_1\} \tag{5.1}$$

$$\mathbf{K}_2 := \{x \in \mathbb{R}^n : h_k(x) \geq 0, \quad k = 1, \ldots, m_2\} \tag{5.2}$$

for some polynomials $(g_j h_k) \subset \mathbb{R}[x]$.

Let $P(\mathbf{K}_i)$ be the set of Borel probability measures on $\mathbf{K}_i$, $i = 1, 2$, and consider the following min-max problem:

$$\mathbf{P}: \quad J^* = \inf_{\mu \in P(\mathbf{K}_1)} \sup_{\nu \in P(\mathbf{K}_2)} \int\int p(x,z)\, d\mu(x)\, d\nu(z) \tag{5.3}$$

for some polynomial $p \in \mathbb{R}[x, z]$.

If $\mathbf{K}_1$ and $\mathbf{K}_2$ are compact, it is well-known that

$$\begin{aligned} J^* &= \min_{\mu \in P(\mathbf{K}_1)} \max_{\nu \in P(\mathbf{K}_2)} \int\int p(x,z)\, d\mu(x)\, d\nu(z) \\ &= \max_{\nu \in P(\mathbf{K}_2)} \min_{\mu \in P(\mathbf{K}_1)} \int\int p(x,z)\, d\mu(x)\, d\nu(z), \end{aligned} \tag{5.4}$$

that is, there exist $\mu^* \in P(\mathbf{K}_1)$ and $\nu^* \in P(\mathbf{K}_2)$ such that:

$$J^* = \int\int p(x,z)\, d\mu^*(x)\, d\nu^*(z). \tag{5.5}$$

The probability measures μ^* and ν^* are the optimal strategies of players 1 and 2 respectively.

Semidefinite relaxations for P. With $p \in \mathbb{R}[x, z]$ as in (3.2), write

$$\begin{aligned} p(x,z) &= \sum_{\alpha \in \mathbb{N}^{n_2}} p_\alpha(x)\, z^\alpha \qquad \text{with} \\ p_\alpha(x) &= \sum_{\beta \in \mathbb{N}^{n_1}} p_{\alpha\beta}\, x^\beta, \qquad |\alpha| \le d_z \end{aligned} \tag{5.6}$$

where d_z is the total degree of p when seen as polynomial in $\mathbb{R}[z]$. So, let $p_{\alpha\beta} := 0$ for every $\beta \in \mathbb{N}^{n_1}$ whenever $|\alpha| > d_z$.

Let $r_j := \lceil \deg g_j/2 \rceil$, for every $j = 1, \dots, m_1$, and consider the following semidefinite program:

$$\left\{ \begin{array}{ll} \displaystyle\min_{\mathbf{y}, \lambda, Z^k} & \lambda \\ \text{s.t.} & \displaystyle\lambda\, I_{\alpha=0} - \sum_{\beta \in \mathbb{N}^{n_1}} p_{\alpha\beta}\, y_\beta = \langle Z^0, B_\alpha \rangle + \sum_{k=1}^{m_2} \langle Z^k, B_\alpha^{h_k} \rangle, \quad |\alpha| \le 2d \\ & \\ & M_d(\mathbf{y}) \succeq 0 \\ & M_{d-r_j}(g_j\, \mathbf{y}) \succeq 0, \quad j = 1, \dots, m_1 \\ & y_0 = 1 \\ & \\ & Z^k \succeq 0, \quad k = 0, 1, \dots m_2 \end{array} \right. \tag{5.7}$$

where $\mathbf{y}$ is a finite sequence indexed in the canonical basis (x^α) of $\mathbb{R}[x]_{2d}$. Denote by λ_d^* its optimal value. In fact, with $h_0 \equiv 1$ and $p(\mathbf{y}, \cdot) \in \mathbb{R}[z]$ defined by:

$$z \mapsto p(\mathbf{y}, z) := \sum_{\alpha \in \mathbb{N}^{n_2}} \left(\sum_{\beta \in \mathbb{N}^{n_1}} p_{\alpha\beta}\, y_\beta \right) z^\alpha, \tag{5.8}$$

the semidefinite program (5.7) has the equivalent formulation:

$$\left\{\begin{array}{ll} \min\limits_{\mathbf{y},\lambda,\sigma_k} & \lambda \\ \text{s.t.} & \lambda - p(\mathbf{y},\cdot) = \displaystyle\sum_{k=0}^{m_2} \sigma_k\, h_k \\ & \\ & M_d(\mathbf{y}) \succeq 0 \\ & M_{d-r_j}(g_j\,\mathbf{y}) \succeq 0, \quad j=1,\ldots,m_1 \\ & y_0 = 1 \\ & \\ & \sigma_k \in \Sigma[z];: \deg\sigma_k + \deg h_k \leq 2d, \quad k=0,1,\ldots,m_2. \end{array}\right. \tag{5.9}$$

Observe that for any admissible solution $(\mathbf{y},\lambda)$ and $p(\mathbf{y},\cdot)$ as in (5.8),

$$\lambda \geq \sup_z \{p(\mathbf{y},z) : z \in \mathbf{K}_2\}. \tag{5.10}$$

Similarly, with p as in (3.2), write

$$\begin{aligned} p(x,z) &= \sum_{\alpha\in\mathbb{N}^{n_1}} \hat{p}_\alpha(z)\, x^\alpha \qquad \text{with} \\ \hat{p}_\alpha(z) &= \sum_{\beta\in\mathbb{N}^{n_2}} \hat{p}_{\alpha\beta}\, z^\beta, \qquad |\alpha| \leq d_x \end{aligned} \tag{5.11}$$

where d_x is the total degree of p when seen as polynomial in $\mathbb{R}[x]$. So, let $\hat{p}_{\alpha\beta} := 0$ for every $\beta \in \mathbb{N}^{n_2}$ whenever $|\alpha| > d_x$.

Let $l_k := \lceil \deg h_k/2 \rceil$, for every $k = 1,\ldots,m_2$, and with

$$x \mapsto \hat{p}(x,\mathbf{y}) := \sum_{\alpha\in\mathbb{N}^{n_1}} \left(\sum_{\beta\in\mathbb{N}^{n_2}} \hat{p}_{\alpha\beta}\, y_\beta \right) x^\alpha, \tag{5.12}$$

consider the following semidefinite program (with $g_0 \equiv 1$):

$$\left\{\begin{array}{ll} \max\limits_{\mathbf{y},\gamma,\sigma_j} & \gamma \\ \text{s.t.} & \hat{p}(\cdot,\mathbf{y}) - \gamma = \displaystyle\sum_{j=0}^{m_1} \sigma_j\, g_j \\ & \\ & M_d(\mathbf{y}) \succeq 0 \\ & M_{d-l_k}(h_k\,\mathbf{y}) \succeq 0, \quad k=1,\ldots,m_2 \\ & y_0 = 1 \\ & \\ & \sigma_j \in \Sigma[x];\ \deg\sigma_j + \deg g_j \leq 2d, \quad j=0,1,\ldots,m_1. \end{array}\right. \tag{5.13}$$

where $\mathbf{y}$ is a finite sequence indexed in the canonical basis (z^α) of $\mathbb{R}[z]_{2d}$. Denote by γ_d^* its optimal value. In fact, (5.13) is the dual of the semidefinite program (5.7).

Observe that for any admissible solution $(\mathbf{y},\gamma)$ and $\hat{p}(\cdot,\mathbf{y})$ as in (5.12),

$$\gamma \leq \inf_x \{\hat{p}(x,\mathbf{y}) : x \in \mathbf{K}_1\}. \tag{5.14}$$

Assumption 5.1. $\mathbf{K}_1$ *and* $\mathbf{K}_2$ *are compact and both satisfy Putinar's property:*

Theorem 5.2. *Let* $\mathbf{P}$ *be the min-max problem defined in(3.2) and let Assumption 5.1 hold. Let* λ_d^* *and* γ_d^* *be the optimal values of the semidefinite program (5.9) and (5.13) respectively. Then* $\lambda_d^* \to J^*$ *and* $\gamma_d^* \to J^*$ *as* $d \to \infty$.

We also have a test to detect whether finite convergence has occurred.

Theorem 5.3. *Let* $\mathbf{P}$ *be the min-max problem defined in (3.2) and let Assumption 3.2 hold.*

(a) Let λ_d^* *be the optimal value of the semidefinite program (5.9), and suppose that with* $r := \max_{j=1,\dots,m_1} r_j$*, the condition*

$$\operatorname{rank} M_{d-r}(\mathbf{y}) \;=\; \operatorname{rank} M_d(\mathbf{y}) \quad (=: s_1) \tag{5.15}$$

holds at an optimal solution $(\mathbf{y}, \lambda, \sigma_k)$ *of (5.7).*

Then $\lambda_d^* = J^*$ *and an optimal strategy for player 1 is a probability measure* μ *supported on* s_1 *points of* $\mathbf{K}_1$.

(b) Let γ_d^* *be the optimal value of the semidefinite program (5.13), and suppose that with* $r := \max_{k=1,\dots,m_2} l_k$*, the condition*

$$\operatorname{rank} M_{d-r}(\mathbf{y}) \;=\; \operatorname{rank} M_d(\mathbf{y}) \quad (=: s_2) \tag{5.16}$$

holds at an optimal solution $(\mathbf{y}, \gamma, \sigma_j)$ *of (5.13).*

Then $\gamma_d^* = J^*$ *and an optimal strategy for player 2 is a probability measure* ν *supported on* s_2 *points of* $\mathbf{K}_2$.

For a proof the reader is referred to §7.2.

Remark 5.4. In the univariate case, when $\mathbf{K}_1, \mathbf{K}_2$ are (not necessarily bounded) intervals of the real line, the optimal value $J^* = \lambda_d^*$ (resp. $J^* = \gamma_d^*$) is obtained by solving the single semidefinite program (5.9) (resp. (5.13)) with $d = d_0$. Theorem 5.3 in the univariate case was proved in Parrilo [32].

6. Zero-sum polynomial absorbing games

As in the previous section, consider two compact basic semi-algebraic sets $\mathbf{K}_1 \subset \mathbb{R}^{n_1}$, $\mathbf{K}_2 \subset \mathbb{R}^{n_2}$ and polynomials g, f and $q : \mathbf{K}_1 \times \mathbf{K}_2 \to [0,1]$. Recall that $P(\mathbf{K}_1)$ (resp. $P(\mathbf{K}_2)$) denotes the set of probability measures on $\mathbf{K}_1$ (resp. $\mathbf{K}_2$). The absorbing game is played in discrete time as follows. At stage $t = 1, 2, ...$ player 1 chooses at random $x_t \in \mathbf{K}_1$ (using some mixed action $\mu_t \in P(\mathbf{K}_1)$) and, simultaneously, Player 2 chooses at random $y_t \in \mathbf{K}_2$ (using some mixed action $\nu_t \in P(\mathbf{K}_2)$).

(i) with probability $1 - q(x_t, y_t)$ the game is absorbed and player 1 receives $f(x_t, y_t)$ from that stage and forever (player 2 receives $-f(x_t, y_t)$),

and

(ii) with probability $q(x_t, y_t)$ player 2 receives at that stage $g(x_t, y_t)$ (player 2 receives $-g(x_t, y_t)$) and the interaction continues one step further (the situation is repeated at step $t+1$).

If the stream of payoffs is $r(t)$, $t = 1, 2, ...$, the λ-discounted-payoff of the game is $\sum_{t=1}^{\infty} \lambda(1-\lambda)^{t-1} r(t)$.

Let $\widetilde{g} = g \times q$ and $\widetilde{f} = f \times (1-q)$ and extend $\widetilde{g}$, $\widetilde{f}$ and q multilinearly to $P(\mathbf{K}_1) \times P(\mathbf{K}_2)$.

Player 1 maximizes the expected discounted-payoff and player 2 minimizes that payoff. Using an extension of the Shapley operator [35] one can deduce that the game has a value $v(\lambda)$ that uniquely satisfies,

$$\begin{aligned} v(\lambda) &= \max_{\mu\in P(\mathbf{K}_1)} \min_{\nu\in P(\mathbf{K}_2)} \int_\Theta \left(\lambda\widetilde{g} + (1-\lambda)v(\lambda)p + \widetilde{f}\right)\, d\mu\otimes\nu \\ &= \min_{\nu\in P(\mathbf{K}_2)} \max_{\mu\in P(\mathbf{K}_1)} \int_\Theta \left(\lambda\widetilde{g} + (1-\lambda)v(\lambda)p + \widetilde{f}\right)\, d\mu\otimes\nu \end{aligned}$$

with $\Theta := \mathbf{K}_1 \times \mathbf{K}_2$. A simple computation yields

$$v(\lambda) = \max_{\mu\in P(\mathbf{K}_1)} \min_{\nu\in P(\mathbf{K}_2)} \frac{\int_\Theta P\, d\mu\otimes\nu}{\int_\Theta Q\, d\mu\otimes\nu} = \min_{\nu\in P(\mathbf{K}_2)} \max_{\mu\in P(\mathbf{K}_1)} \frac{\int_\Theta P\, d\mu\otimes\nu}{\int_\Theta Q\, d\mu\otimes\nu} \tag{6.1}$$

where

$$\begin{aligned} (x,y)\mapsto P(x,y) &:= \lambda\widetilde{g}(x,y) + \widetilde{f}(x,y) \\ (x,y)\mapsto Q(x,y) &:= \lambda q(x,y) + 1 - q(x,y) \end{aligned}$$

Or equivalently, $v(\lambda)$ is the unique real t such that

$$\begin{aligned} 0 &= \max_{\mu\in P(\mathbf{K}_1)} \min_{\nu\in P(\mathbf{K}_2)} \left[\int_\Theta (P(x,y) - t\,Q(x,y))\, d\mu(x)d\nu(y)\right] \\ &= \min_{\nu\in P(\mathbf{K}_2)} \min_{\nu\in P(\mathbf{K}_1)} \left[\int_\Theta (P(x,y) - t\,Q(x,y))\, d\mu(x)d\nu(y)\right]. \end{aligned}$$

Actually, the function $s : \mathbb{R} \to \mathbb{R}$ defined by:

$$t \to s(t) := \max_{\mu\in P(\mathbf{K}_1)} \min_{\nu\in P(\mathbf{K}_2)} \left[\int_\Theta (P(x,y) - tQ(x,y))\, d\mu(x)d\nu(y)\right]$$

is continuous, strictly decreasing and goes from $+\infty$ to $-\infty$ as t increases from $-\infty$ to $+\infty$.

In the univariate case, if $\mathbf{K}_1$ and $\mathbf{K}_2$ are both real intervals (not necessarily compact), then evaluating $s(t)$ for some fixed t can be done by solving a *single* semidefinite program; see Remark 5.4. Therefore, in this case, one may approximate the optimal value s^* $(= s(t^*))$ of the game by a dichotomy on t and so, the problem can be solved in a polynomial time. This extends Shah and Parrilo [34].

7. Appendix

7.1. **Proof of Theorem 3.5.** We already know that $\inf \mathbf{Q}_r \leq \rho$ for all $r \geq r_0$. Next, we need to prove that $\inf \mathbf{Q}_r > -\infty$ for sufficiently large r. Let $m' := m+p+2$. Recall that the quadratic module $Q(h) \subset \mathbb{R}[x,z]$ generated by the polynomials $\{h_j\} \subset \mathbb{R}[x,z]$ that define $\widehat{\mathbf{K}}$ is the set

$$Q(h) := \{\, \sigma \in \mathbb{R}[x,z] \quad | \quad \sigma = \sum_{j=0}^{m'} \sigma_j\, h_j \quad \text{with } \{\sigma_j\}_{j=0}^{m'} \subset \Sigma[x,z]\}.$$

In addition, let $Q_t(h) \subset Q(h)$ be the set of elements σ of $Q(h)$ which have a representation $\sigma_0 + \sum_{j=0}^{m'} \sigma_j\, h_j$ for some s.o.s. family $\{\sigma_j\} \subset \Sigma[x,z]$ with $\deg \sigma_0 \leq 2t$ and $\deg \sigma_j h_j \leq 2t$ for all $j = 1, \ldots, m'$.

Let $r \in \mathbb{N}$ be fixed. As $q > 0$ on $\widehat{\mathbf{K}}$, then $q > \delta$ on $\widehat{\mathbf{K}}$ for some scalar $\delta > 0$. Therefore, by Theorem 2.2, $q - \delta \in Q(h)$. Similarly, there exists N such that $N \pm (x,z)^\alpha > 0$ on $\widehat{\mathbf{K}}$, for all $\alpha \in \mathbb{N}^{n+1}$ with $|\alpha| \leq 2r$. Therefore by Theorem 2.2 the polynomial $(x,z) \mapsto N \pm (x,z)^\alpha$ belongs to $Q(h)$. But there is even some $l(r)$

such that $q-\delta \in Q_{l(r)}(h)$ and $(x,z) \mapsto N \pm (x,z)^\alpha \in Q_{l(r)}(h)$ for every $|\alpha| \leq 2r$. Of course we also have $q-\delta \in Q_l(h)$ and $(x,z) \mapsto N \pm (x,z)^\alpha \in Q_l(h)$ for every $|\alpha| \leq 2r$, whenever $l \geq l(r)$. Therefore, let us take $l(r) \geq r_0$, with $r_0 \geq \max_{j=0,\ldots,m'} r_j$.

As $q-\delta \in Q_{l(r)}(h)$, $q-\delta = \sigma_0 + \sum_{j=1}^{m'} \sigma_j h_j$, for some $(\sigma_j) \subset \Sigma[x,z]$ with $\deg \sigma_0 \leq 2l(r)$ and $\deg \sigma_j + \deg h_j \leq 2l(r)$, for all $j = 1,\ldots,m'$. Hence, for every feasible solution $\mathbf{y}$ of $\mathbf{Q}_{l(r)}$ (and of $\mathbf{Q}_l$ with $l \geq l(r)$),

$$1 - \delta y_0 \,=\, L_{\mathbf{y}}(q-\delta) \,=\, L_{\mathbf{y}}(\sigma_0) + L_y(\sum_{j=1}^{m'} \sigma_j h_j) \geq 0,$$

where the last inequality follows from $M_{l(r)}(\mathbf{y}) \succeq 0$ and $M_{l(r)-r_j}(\mathbf{y}\, h_j) \succeq 0$, $j = 1,\ldots,m'$. Therefore, $y_0 \leq \delta^{-1}$.

Similarly, $N \pm (x,z)^\alpha = \sigma_0 + \sum_{j=1}^{m'} \sigma_j\, h_j$ for some $(\sigma_j) \subset \Sigma[x,z]$ with $\deg \sigma_0 \leq 2l(r)$ and $\deg \sigma_j + \deg h_j \leq 2l(r)$, for all $j = 1,\ldots,m'$. Hence, for same reasons as above,

$$N y_0 \pm y_\alpha \,=\, L_{\mathbf{y}}(N \pm (x,z)^\alpha) \,=\, L_{\mathbf{y}}(\sigma_0) + \sum_{j=1}^{m'} L_{\mathbf{y}}(\sigma_j\, h_j) \geq 0,$$

which implies $|y_\alpha| = |\, L_{\mathbf{y}}((x,z)^\alpha)\,| \leq N y_0 \leq N\delta^{-1}$, for all $|\alpha| \leq 2r$.

In particular, $L_{\mathbf{y}}(h_0) \,\geq\, -N\delta^{-1} \sum_\alpha |(h_0)_\alpha|$, which proves that $\inf \mathbf{Q}_{l(r)} > -\infty$, and so $\inf \mathbf{Q}_r > -\infty$ for sufficiently large r.

Next, from what precedes, and with $k \in \mathbb{N}$ arbitrary, let $l(k) \geq k$ be such that $q - \delta \in Q_{l(k)}(h)$ and

$$N_k \pm (x,z)^\alpha \in Q_{l(k)}(h) \qquad \forall\, \alpha \in \mathbb{N}^{n+1} \text{ with } |\alpha| \leq 2k, \tag{7.1}$$

for some N_k. Let $r \geq l(r_0)$, and let $\mathbf{y}^r$ be a nearly optimal solution of $\mathbf{Q}_r$ with value

$$\inf \mathbf{Q}_r \leq L_{\mathbf{y}^r}(h_0) \leq \inf \mathbf{Q}_r + \frac{1}{r} \qquad \left(\leq \rho + \frac{1}{r}\right). \tag{7.2}$$

Fix $k \in \mathbb{N}$. Notice that from (7.1), one has

$$|\, L_{\mathbf{y}^r}((x,z)^\alpha)\,| \;\leq\; N_k y_0 \;\leq\; N_k \delta^{-1}, \quad \forall\, \alpha \in \mathbb{N}^{n+1} \text{ with } |\alpha| \;\leq\; 2k, \quad \forall\, r \geq l(k).$$

Therefore,

$$|y^r_\alpha| \;=\; |\, L_{\mathbf{y}^r}((x,z)^\alpha)\,| \;\leq\; N'_k, \quad \forall\, \alpha \in \mathbb{N}^{n+1} \text{ with } |\alpha| \;\leq\; 2k, \quad \forall\, r \geq r_0. \tag{7.3}$$

where $N'_k = \max[N_k\delta^{-1}, V_k]$, with

$$V_k := \max_{\alpha,r} \{\, |y^r_\alpha| \;:\quad |\alpha| \leq 2k\,;\quad r_0 \leq r \leq l(k)\,\}.$$

Complete each vector $\mathbf{y}^r$ with zeros to make it an infinite bounded sequence in l_∞, indexed in the canonical basis in $u_\infty(x,z)$ of $\mathbb{R}[x,z]$. In view of (7.3), one has $y^r_0 \leq \delta^{-1}$ and

$$|y^r_\alpha| \;\leq N'_k \qquad \forall\, \alpha \in \mathbb{N}^n \text{ with } \quad 2k-1 \leq |\alpha| \leq 2k, \tag{7.4}$$

and for all $k = 1,2,\ldots$.

Hence let $\widehat{\mathbf{y}}^r \in l_\infty$ be a new sequence defined by $\widehat{y}^r_0 = \delta y^r_0$ and

$$\widehat{y}^r_\alpha \;:=\; \frac{y^r_\alpha}{N'_k}, \quad \forall\, \alpha \in \mathbb{N}^{n+1} \text{ with } \quad 2k-1 \leq |\alpha| \leq 2k, \quad \forall\, k = 1,2,\ldots,$$

and in l_∞, consider the sequence $\{\widehat{\mathbf{y}}^r\}_r$, as $r \to \infty$.

Obviously, the sequence $\{\widehat{\mathbf{y}}^r\}_r$ is in the unit ball B_1 of l_∞, and so, by the Banach-Alaoglu theorem (see e.g. Ash [1]), there exists $\widehat{\mathbf{y}} \in B_1$, and a subsequence $\{r_i\}$, such that $\widehat{\mathbf{y}}^{r_i} \to \widehat{\mathbf{y}}$ as $i \to \infty$, for the weak $\star$ topology $\sigma(l_\infty, l_1)$ of l_∞. In particular, pointwise convergence holds, that is,

$$\lim_{i\to\infty} \widehat{y}_\alpha^{r_i} \to \widehat{y}_\alpha \qquad \forall\, \alpha \in \mathbb{N}^{n+1}.$$

Next, define $y_0 = \delta^{-1}\widehat{y}_0$ and

$$y_\alpha := \widehat{y}_\alpha \times N'_k \quad \forall\, \alpha \in \mathbb{N}^{n+1} \text{ with } \quad 2k-1 \le |\alpha| \le 2k, \quad \forall\, k = 1, 2, \ldots$$

Clearly, the pointwise convergence $\widehat{\mathbf{y}}^{r_i} \to \widehat{\mathbf{y}}$ implies $\mathbf{y}^{r_i} \to \mathbf{y}$, i.e.,

$$\lim_{i\to\infty} y_\alpha^{r_i} \to y_\alpha \qquad \forall\, \alpha \in \mathbb{N}^{n+1}. \tag{7.5}$$

Next, let $r \in \mathbb{N}$ be fixed. From the pointwise convergence (7.5) we deduce that

$$\lim_{i\to\infty} M_r(h_j\, \mathbf{y}^{r_i}) = M_r(h_j\, \mathbf{y}) \succeq 0, \quad j = 0, 1, \ldots, m'.$$

As r was arbitrary we obtain

$$M_r(h_j\, \mathbf{y}) \succeq 0, \quad j = 0, 1, \ldots, m'; \quad r = 1, 2, \ldots \tag{7.6}$$

By Theorem 2.2(b), (7.6) implies that $\mathbf{y}$ is the sequence of moments of some finite measure μ with support contained in $\widehat{\mathbf{K}}$.

Next, from the pointwise convergence (7.5) and the constraints of $\mathbf{Q}_r$, one has

$$1 = \lim_{i\to\infty} L_{\mathbf{y}^{r_i}}(q) = L_{\mathbf{y}}(q) = \int q\, d\mu,$$

that is, μ is a feasible solution of $\mathcal{P}$ in (3.5). Finally, the pointwise convergence (7.5) implies $L_{\mathbf{y}^{r_i}}(h_0) \to L_{\mathbf{y}}(h_0) = \int h_0\, d\mu$ ($\le \rho$ by (7.2)), we deduce that $\inf \mathbf{Q}_{r_i} \to \rho = \int h_0\, d\mu$, and in fact the desired result $\inf \mathbf{Q}_r \uparrow \rho$, because the sequence $\{\inf \mathbf{Q}_r\}$ is monotone nondecreasing. $\square$

7.2. **Proof of Theorem 5.2.** We first need the following partial result.

Lemma 7.1. *Let $(\mathbf{y}^d)_d$ be a sequence of admissible solutions of the semidefinite program (5.7). Then there exists $\hat{\mathbf{y}} \in \mathbb{R}^\infty$ and a subsequence (d_i) such that $\mathbf{y}^{d_i} \to \hat{\mathbf{y}}$ pointwise as $i \to \infty$, that is,*

$$\lim_{i\to\infty} y_\alpha^{d_i} = \hat{y}_\alpha, \qquad \forall \alpha \in \mathbb{N}^n. \tag{7.7}$$

The proof is omitted because it is exactly along the same lines as the proof of Theorem 3.5 as among the constraints of the feasible set, one has

$$y_0^d = 1, \quad M_d(\mathbf{y}^d) \succeq 0, \quad M_d(g_j\, \mathbf{y}^d) \succeq 0,\ j = 1, \ldots, m_1.$$

Proof of Theorem 5.2. Let $\mu^* \in P(\mathbf{K}_1), \nu^* \in P(\mathbf{K}_2)$ be optimal strategies of player 1 and player 2 respectively, and let $\mathbf{y}^* = (y^*_\alpha)$ be the sequence of moments of μ^* (well-defined because $\mathbf{K}_1$ is compact). Then

$$\begin{aligned}
J^* &= \sup_{\nu\in P(\mathbf{K}_2)} \int \left(\int p(x,z)d\mu^*(x)\right) d\nu(z)\\
&= \sup_{\nu\in P(\mathbf{K}_2)} \int \sum_{\alpha\in\mathbb{N}^n} \left(\sum_{\beta\in\mathbb{N}^n} p_{\alpha\beta}\int x^\beta\, d\mu^*(x)\right) z^\alpha d\nu(z)\\
&= \sup_{\nu\in P(\mathbf{K}_2)} \int \sum_{\alpha\in\mathbb{N}^n} \left(\sum_{\beta\in\mathbb{N}^n} p_{\alpha\beta} y^*_{\alpha\beta}\right) z^\alpha d\nu(z)\\
&= \sup_{\nu\in P(\mathbf{K}_2)} \int p(\mathbf{y}^*, z)\, d\nu(z)\\
&= \sup_z \{p(\mathbf{y}^*, z) : z\in\mathbf{K}_2\}\\
&= \inf_{\lambda,\sigma_k} \{\lambda : \lambda - p(\mathbf{y}^*,\cdot) \,=\, \sigma_0 + \sum_{k=1}^{m_2}\sigma_k\, h_k; \quad (\sigma_j)_{j=0}^{m_2}\subset \Sigma[z]\}
\end{aligned}$$

with $z\mapsto\ p(\mathbf{y}^*,z)$ defined in (5.8). Therefore, with $\epsilon>0$ fixed arbitrary,

$$J^* - p(\mathbf{y}^*,\cdot) + \epsilon \,=\, \sigma_0^\epsilon + \sum_{k=1}^{m_2} \sigma_k^\epsilon\, h_k, \tag{7.8}$$

for some polynomials $(\sigma_k^\epsilon)\subset\Sigma[z]$ of degree at most $2d_\epsilon^1$. So $(y^*, J^*+\epsilon, \sigma_k^\epsilon)$ is an admissible solution for the semidefinite program (5.9) whenever $d\geq \max_j r_j$ and $d\geq d^1_\epsilon + \max_k l_k$, because

$$2d\geq \deg\sigma_0^\epsilon; \quad 2d\geq\deg\sigma_k^\epsilon + \deg h_k, \quad k=1,\ldots,m_2. \tag{7.9}$$

Therefore,

$$\lambda_d^* \,\leq\, J^*+\epsilon, \qquad \forall d\geq\ \tilde{d}_\epsilon^1 := \max\left[\max_j r_j,\, d_\epsilon^1 + \max_k l_k\right]. \tag{7.10}$$

Now, let (y^d,λ_d) be an admissible solution of the semidefinite program (5.9) with value $\lambda_d\leq\lambda_d^* + 1/d$. By Lemma 7.1, there exists $\hat{\mathbf{y}}\in\mathbb{R}^\infty$ and a subsequence (d_i) such that $\mathbf{y}^{d_i}\to\hat{\mathbf{y}}$ pointwise, that is, (7.7) holds. But then, invoking (7.7) yields

$$M_d(\hat{\mathbf{y}})\succeq 0 \quad\text{and}\quad M_d(g_j\,\hat{\mathbf{y}})\succeq 0, \quad \forall j=1,\ldots,m_1; \quad d=0,1,\ldots$$

By Theorem 2.2, there exists $\hat{\mu}\in P(\mathbf{K}_1)$ such that

$$\hat{y}_\alpha \,=\, \int x^\alpha\, d\hat{\mu}, \qquad \forall\alpha\in\mathbb{N}^n.$$

On the other hand,

$$\begin{aligned}
J^* &\leq \sup_{\nu\in P(\mathbf{K}_2)} \int \left(\int p(x,z)d\hat{\mu}(x)\right) d\nu(z)\\
&= \sup_z \{p(\hat{\mathbf{y}}, z) : z\in\mathbf{K}_2\}\\
&= \inf\{\lambda : \lambda - p(\hat{\mathbf{y}},\cdot) \,=\, \sigma_0 + \sum_{k=1}^{m_2}\sigma_k\, h_k; \quad (\sigma_j)_{j=0}^{m_2}\subset\Sigma[z]\}
\end{aligned}$$

with

$$z \mapsto p(\hat{\mathbf{y}}, z) := \sum_{\alpha \in \mathbb{N}^n} \left(\sum_{\beta \in \mathbb{N}^n} p_{\alpha\beta}\, \hat{y}_\beta \right) z^\alpha.$$

Next, let $\rho := \sup_{z \in \mathbf{K_2}} p(\hat{\mathbf{y}}, z)$ (hence $\rho \geq J^*$), and consider the polynomial

$$z \mapsto p(\mathbf{y}^d, z) := \sum_{\alpha \in \mathbb{N}^n} \left(\sum_{\beta \in \mathbb{N}^n} p_{\alpha\beta}\, y^d_\beta \right) z^\alpha.$$

It has same degree as $p(\hat{\mathbf{y}}, \cdot)$, and by (7.7), $\|p(\hat{\mathbf{y}}, \cdot) - p(\mathbf{y}^{d_i}, \cdot)\| \to 0$ as $i \to \infty$.

Hence, $\sup_{z \in \mathbf{K_2}} p(\mathbf{y}^{d_i}, z) \to \rho$ as $i \to \infty$, and by construction of the semidefinite program (5.9), $\lambda^*_{d_i} \geq \sup_{z \in \mathbf{K_2}} p(\mathbf{y}^{d_i}, z)$.

Therefore, $\lambda^*_{d_i} \geq \rho - \epsilon$ for all sufficiently large i (say $d_i \geq d^2_\epsilon$) and so, $\lambda^*_{d_i} \geq J^* - \epsilon$ for all $d_i \geq d^2_\epsilon$. This combined with $\lambda^*_{d_i} \leq J^* + \epsilon$ for all $d_i \geq \tilde{d}^1_\epsilon$, yields the desired result that $\lim_{i \to \infty} \lambda^*_{d_i} = J^*$ because $\epsilon > 0$ fixed was arbitrary;

Finally, as the converging subsequence (r_i) was arbitrary, we get that the entire sequence (λ^*_d) converges to J^*. $\square$

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