

LEGENDRIAN VARIETIES

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ABSTRACT. We investigate the geometry of Legendrian complex projective manifolds $X \subset \mathbb{P}V$. By definition, this means V is a complex vector space of dimension $2n + 2$, endowed with a symplectic form, and the affine tangent space to X at each point is a maximal isotropic subspace. We establish basic facts about their geometry and exhibit examples of inhomogeneous smooth Legendrian varieties, the first examples of such in dimension greater than one.

1. INTRODUCTION

The initial motivation for this project stems from the study of the holonomy groups of Riemannian manifolds, where the only open case for existence of compact non-homogeneous examples is the quaternion-Kähler case. Thanks to work of Salamon, LeBrun and others (see, e.g., [27, 24]), the question is essentially equivalent to the existence of inhomogeneous contact Fano manifolds (so far none are known). Several people [30, 29] observed that the set of tangent directions to minimal degree rational curves through a general point of a contact Fano manifold is a Legendrian subvariety in its projective span. S. Kebekus [18, 19] then showed:

Theorem 1. *Let Y be a smooth contact Fano manifold with Picard number one, not a projective space. Let y a general point of Y , and denote by $\mathcal{H}_y \subset \mathbb{P}T_y Y$ the set of tangent directions to contact lines on Y passing through y . Then $X = \mathcal{H}_y$ is a smooth Legendrian variety in its linear span.*

Moreover, if at all points of Y the corresponding Legendrian variety is homogenous and equivariantly embedded, Hong [14] proved that Y itself must be homogeneous.

The homogeneous examples of contact Fano manifolds are as follows: let \mathfrak{g} denote a complex simple Lie algebra and G its adjoint group. Then G has a unique closed orbit X_G^{ad} inside $\mathbb{P}\mathfrak{g}$, the projectivization of its Lie algebra – we call this variety the *adjoint variety* of G . It is also the projectivization of the minimal nilpotent orbit in \mathfrak{g} . The adjoint varieties are contact Fano manifolds, and the conjecture of Lebrun and Salamon is that there exists no other.

The set of lines passing through a given point of an adjoint variety is a smooth homogeneous Legendrian variety. We call these varieties, in these particular embeddings, the *subadjoint varieties*. Classical examples are the twisted cubic in \mathbb{P}^3 (coming from the adjoint variety of the exceptional group

G_2), and the products $\mathbb{P}^1 \times \mathbb{Q}^n$ of a projective line with a smooth quadric of dimension $n \geq 1$ (coming from the adjoint varieties of the orthogonal groups). Note that the symplectic groups give empty subadjoint varieties. The adjoint varieties of the other exceptional groups give rise to a remarkable series of homogeneous varieties, which we called the *subexceptional series*: they constitute the third line of the geometric version of Freudenthal's magic square, and were extensively studied in [23, 7, 17]; see also [6] since they are nice examples of varieties with one apparent double point.

Another motivation for studying Legendrian varieties is the question of special projective embeddings. It is well known and easy to prove that any smooth n -dimensional projective variety can be embedded in \mathbb{P}^{2n+1} . Moreover, various obstructions exist to the existence of an embedding inside \mathbb{P}^{2n} . It is thus a natural question to ask for some preferred embeddings in \mathbb{P}^{2n+1} , and Legendrian embeddings are natural candidates. Indeed, any smooth projective curve has a Legendrian embedding in \mathbb{P}^3 , see [2] or §4.1. But this remarkable fact is somewhat misleading, and the case of higher dimensional varieties seems dramatically different.

In §2 we establish a series of Chern class identities for Legendrian varieties. They imply that a smooth Legendrian variety which is a product, must be a $\mathbb{P}^1 \times \mathbb{Q}^n$ (Corollary 6), that the only Legendrian embedding of \mathbb{P}^n is linear (Corollary 5), that a smooth Legendrian variety cannot be \mathbb{P}^k -ruled when $k > 1$ (Corollary 7) or an abelian variety (Corollary 4) and that a homogeneous Legendrian variety with Picard number one, not necessarily equivariantly embedded a priori, must be a projective space or a subadjoint variety (Theorem 11). But the identities do not exclude, for example, that a smooth Legendrian variety has general type, which we do not expect to be possible. In fact most of these results could be seen, somewhat optimistically, as we will see, as evidences that smooth Legendrian varieties of dimension greater than one should be homogeneous. Note, in the same spirit, a very recent result of Buczinsky, that every smooth Legendrian variety whose ideal is generated by quadrics, is indeed homogeneous [4].

In §3 we establish basic properties about the local differential geometry of a Legendrian variety Z . In particular we show that any line having contact to order two with a general point of Z is contained in Z (and thus Z is uniruled).

In §4 we use Bryant's method to construct examples of smooth Legendrian surfaces, which we now explain in more detail.

By Pfaff's theorem, all (holomorphic) contact structures are locally equivalent to the space of one-jets of functions and their Legendrian submanifolds are all locally given by the one-jets of functions. In the algebraic category, the model space for the space of one-jets is $\mathbb{P}(T^*\mathbb{P}^{n+1})$ and the Legendrian varieties are just the lifts $Z^\# := \mathbb{P}N_Z^*$ (Nash blow-ups) of subvarieties $Z \subset \mathbb{P}^{n+1}$. Now $\mathbb{P}(T^*\mathbb{P}^{n+1})$ is birational to \mathbb{P}^{2n+1} and one can take a birational map $\varphi : \mathbb{P}(T^*\mathbb{P}^{n+1}) \dashrightarrow \mathbb{P}^{2n+1}$ that is a linear contactomorphism

on a \mathbb{C}^{2n+1} (a “big cell” in each space). (The inverse rational map and its cousins are studied in detail in [22].)

Hence the idea: choose any variety $Z \subset \mathbb{P}^{n+1}$, then $\tilde{Z} := \varphi(Z^\#)$ will be Legendrian in \mathbb{P}^{2n+1} . The problem is that, except for curves, this has very little chance to produce a *smooth* variety: for example, in dimension two, bitangent planes on a surface $Z \subset \mathbb{P}^3$ tend to produce double points on $\tilde{Z} \subset \mathbb{P}^5$. We analyze the conditions under which \tilde{Z} can be smooth. It turns out that when Z is a *Kummer quartic surface* – a quartic surface with sixteen double points as singularities, the resulting surface is smooth. To give a precise statement, note that we have a diagram

$$\begin{array}{ccc} & Z^\# & \\ \swarrow & & \searrow \\ \mathbb{P}^3 \supset Z & & Z^* \subset \check{\mathbb{P}}^3, \end{array}$$

where $Z^* \subset \check{\mathbb{P}}^3$ is the dual variety of Z , which is again a Kummer quartic surface projectively equivalent to Z . The surface $Z^\#$ is a K3 surface, and its natural maps to Z and Z^* resolve their singularities. Let C and D denote general hyperplane sections of Z and Z^* , pulled back to $Z^\#$. Then C and D meet transversely in 12 points.

Theorem 2. *The blow up of the K3 surface $Z^\#$ in these twelve points can be embedded in \mathbb{P}^5 as a smooth Legendrian surface of degree 20.*

Apart from curves, this is the first example of a non-homogeneous smooth Legendrian variety. In particular, it provides a counter-example to the naïve guess that smooth Legendrian varieties of dimension greater than one, should be rational.

2. CHERN CLASSES OF LEGENDRIAN VARIETIES

In this section we establish Chern class identities for smooth Legendrian varieties. They involve not only the Chern classes of the variety, but also the hyperplane class. We determine a number of consequences, including obstructions to the existence of a Legendrian embedding of a given variety. For example, an abelian variety of dimension at least two has no Legendrian embedding.

2.1. An exact sequence. Let $X \subset \mathbb{P}V$ be a smooth variety, let $x \in X$, let $\hat{T}_x X \subset V$ denote the affine tangent space to X at x , let $\tilde{T}_x X = \mathbb{P}(\hat{T}_x X) \subset \mathbb{P}V$ denote the embedded tangent space and let $T_x X$ denote the Zariski tangent space. We have a natural identification

$$T_x X = \text{Hom}(x, \hat{T}_x X/x) \subset T_x \mathbb{P}V = \text{Hom}(x, V/x).$$

By hypothesis, $\hat{T}_x X$ is a Lagrangian subspace of V , so that the symplectic form induces a canonical identification of $V/\hat{T}_x X$ with the dual of $\hat{T}_x X$.

Consider the commutative diagram of vector bundles, where N denotes the normal bundle to X :

$$\begin{array}{ccccccc}
& & & 0 & & 0 & \\
& & & \uparrow & & \uparrow & \\
& & & (\hat{T}X)^* & = & N \otimes \mathcal{O}_X(-1) & \\
& & & \uparrow & & \uparrow & \\
0 & \rightarrow & \mathcal{O}_X(-1) & \rightarrow & V \otimes \mathcal{O}_X & \rightarrow & T\mathbb{P}V \otimes \mathcal{O}_X(-1) \rightarrow 0 \\
& & \parallel & & \uparrow & & \uparrow \\
0 & \rightarrow & \mathcal{O}_X(-1) & \rightarrow & \hat{T}X & \rightarrow & TX \otimes \mathcal{O}_X(-1) \rightarrow 0 \\
& & & & \uparrow & & \uparrow \\
& & & & 0 & & 0
\end{array}$$

We deduce an exact sequence

$$0 \rightarrow \mathcal{O}_X(-1) \rightarrow N^*(1) \rightarrow TX(-1) \rightarrow 0.$$

2.2. Chern class identities. From the previous exact sequence, we get an identity between Chern characters:

$$\begin{aligned}
ch(TX(-1)) + ch(\mathcal{O}_X(-1)) &= ch(N^*(1)) \\
&= ch(\Omega_{\mathbb{P}V}^1(1)|_X) - ch(\Omega_X^1(1)) \\
&= 2n + 2 - ch(\mathcal{O}_X(1)) - ch(\Omega_X^1(1)).
\end{aligned}$$

Let h denote the hyperplane class on X . We can rewrite this identity as

$$e^{-h}ch(TX) + e^hch(\Omega_X^1) + e^h + e^{-h} = 2n + 2.$$

Extracting the homogeneous components, we obtain:

Proposition 3. *Let $X^n \subset \mathbb{P}^{2n+1} = \mathbb{P}V$ be a smooth Legendrian variety, and $h \in H^2(X, \mathbb{Z})$ the hyperplane class. Then for all $m > 0$, the characteristic class*

$$\sigma_{2m}(X, h) := \sum_{i=0}^{2m} (-1)^i \binom{2m}{i} ch_{2m-i}(TX) h^i + h^{2m}$$

is zero.

In particular, for $m = 1$ we get the identity

$$(1) \quad 2ch_2(TX) = c_1^2 - 2c_2 = 2hc_1 - (n+1)h^2.$$

This already has striking consequences:

Corollary 4. *An abelian variety, more generally a parallelizable variety, has no Legendrian embedding.*

Corollary 5. *The unique Legendrian embedding of the projective space \mathbb{P}^n , with $n > 1$, is the linear embedding $\mathbb{P}^n \subset \mathbb{P}^{2n+1}$.*

Corollary 6. *Suppose that $X = Y \times Z$ has a Legendrian embedding. Then $X = \mathbb{P}^1 \times \mathbb{Q}^{n-1}$, where \mathbb{Q}^{n-1} denotes a smooth $(n-1)$ -dimensional quadric, and the unique Legendrian embedding is the Segre embedding.*

Proof. Note that the class $2ch_2 = c_1^2 - 2c_2$ is additive, $ch_2(E \oplus F) = ch_2(E) + ch_2(F)$. By the Künneth formula, we can decompose our very ample class h as $\ell + \theta + m$ over the rational numbers, where $\ell \in H^2(Y, \mathbb{Q})$ and $m \in H^2(Z, \mathbb{Q})$ are very ample (being the classes of the restriction of the hyperplane divisor to fibers of the projection of X to Z and Y , respectively), and $\theta \in H^1(Y, \mathbb{Q}) \otimes H^1(Z, \mathbb{Q})$. Our Chern classes identity decomposes into the following conditions, where n_Y and n_Z respectively denote the dimensions of Y and Z :

$$\begin{aligned} c_2(Y) &= 2\ell c_1(Y) - (n_Y + n_Z + 1)\ell^2, \\ 0 &= c_1(Y)\theta - (n_Y + n_Z + 1)\ell\theta, \\ 0 &= c_1(Y)m + \ell c_1(Z) - (n_Y + n_Z + 1)\ell m, \\ 0 &= \theta c_1(Z) - (n_Y + n_Z + 1)\theta m, \\ c_2(Z) &= 2m c_1(Z) - (n_Y + n_Z + 1)m^2. \end{aligned}$$

Note that the class $\ell' = -c_1(Y) + (n_Y + n_Z + 1)\ell$ is very ample (ℓ being very ample, this special case of the Fujita conjecture can easily be proved by induction on the dimension). By the hard Lefschetz theorem, the second identity $\ell'\theta = 0$ thus implies that $\theta = 0$.

Now the third condition implies that $c_1(Y)$ (resp. $c_1(Z)$) and ℓ (resp. m) are numerically proportional. Let us write $c_1(Y) = l\ell$ and $c_1(Z) = \mu m$ for some rational numbers l and μ . Then $l + \mu = n_Y + n_Z + 1$. Therefore, we cannot have both $l \leq n_Y$ and $\mu \leq n_Z$, so we may suppose that $l > n_Y$. Then by the Kobayashi-Ochiai theorem, $Y \simeq \mathbb{P}^{n_Y}$ is a projective space, ℓ is the hyperplane class and $l = n_Y + 1$. Hence $\mu = n_Z$, which implies that $Z \simeq \mathbb{Q}^{n_Z}$ is a quadric and m is the hyperplane class. But then the first identity cannot be fulfilled, except if $n_Y = 1$, in which case everything vanishes. \square

Corollary 7. *Let $X = \mathbb{P}E$ be the total space of a \mathbb{P}^p -bundle over a variety Y . Then if $p > 1$, X does not admit a Legendrian embedding.*

Proof. Let $\pi : \mathbb{P}E \rightarrow Y$ denote the projection. We have exact sequences

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_E(1) \otimes \pi^*E^* \longrightarrow T^v X \longrightarrow 0, \\ 0 \longrightarrow T^v X \longrightarrow TX \longrightarrow \pi^*TY \longrightarrow 0, \end{aligned}$$

where $T^v X$ denotes the relative tangent space with respect to π . Let ℓ denote the first Chern class of the relative hyperplane bundle $\mathcal{O}_E(1)$, and let $r = p + 1$, then $c_1(X) = r\ell + (\text{basic})$ and $c_2(Z) = \binom{r}{2}\ell^2 - \ell \cdot (\text{basic}) + (\text{basic})$, where (basic) denotes any class that is the pullback of a class on Y . Suppose that X has a Legendrian embedding, given by a very ample line bundle $h = k\ell + \pi^*L$. Let $q = \dim Y$. Identity (1) implies

$$r - 2rk + (q + r)k^2 = 0.$$

Considering this as a quadratic equation for k , its discriminant is $-4rq$. \square

2.3. The case of surfaces.

Proposition 8. *Let $Z \subset \mathbb{P}^5$ be a ruled Legendrian surface. Then $Z = \mathbb{P}^1 \times \mathbb{P}^1$, embedded by the complete linear system $|H + 2H'|$.*

Proof. Suppose that $Z = \mathbb{P}E$ for some rank 2 vector bundle E on a curve C of genus g , and call π the projection to C . We have exact sequences as above.

Let ℓ denote the first Chern class of the relative hyperplane bundle $\mathcal{O}_E(1)$, then $c_1(Z) = 2\ell - \pi^*(c_1(E) + K_C)$ and $c_2(Z) = \ell^2 - \ell \cdot \pi^*(c_1(E) + 2K_C)$. Hence $c_1^2(Z) = 2c_2(Z)$. Note that $\ell^2 = \deg(E)$, and $\ell \cdot \pi^*L = \deg(L)$ for any line bundle L on C .

We suppose that Z has a Legendrian embedding, given by a very ample line bundle $h = k\ell + \pi^*L$. We must have $2c_1(Z)h = 3h^2$, that is,

$$4k(1 - g) = (3k - 2)(k\deg(E) + 2\deg(L)).$$

But $k > 0$, and $h^2 = k(k\deg(E) + 2\deg(L)) > 0$. This implies $g = 0$, and then either $k = 1$ and $\deg(E) + \deg(L) = 1$, or $k = 2$ and $\deg(E) + 2\deg(L) = 4$. Since C is rational, E is split and we can normalize it as $E = \mathcal{O} \oplus \mathcal{O}(-e)$ with $e \geq 0$. Then the very ampleness of h is equivalent to the condition that $\deg(L) > ke$ ([13], V, Corollary 2.18). We easily deduce that $e = 0$, so that $Z = \mathbb{P}^1 \times \mathbb{P}^1$, and then we already proved that it must be embedded by the $h = H + 2H'$. \square

Proposition 9. *Let $Z \subset \mathbb{P}^5$ be a minimal surface of Kodaira dimension zero, in some Legendrian embedding. Then $\deg(Z) = 8\chi(\mathcal{O}_Z)$, and moreover $\chi(\mathcal{O}_Z) > 1$.*

In particular Z can be neither an abelian nor an Enriques surface. If it is a K3 surface, then its genus must be equal to 9.

Proof. The first Chern class of Z is numerically trivial, so that the formula $c_1^2 - 2c_2 = 2c_1h - 3h^2$ gives $3\deg(Z) = 3h^2 = 2c_2 = 24\chi(\mathcal{O}_Z)$, hence the first claim. Since h is very ample, Riemann-Roch gives

$$h^0(Z, h) = \chi(h) = \frac{h^2}{2} + \chi(\mathcal{O}_Z) = 5\chi(\mathcal{O}_Z).$$

Since Z is nondegenerate in \mathbb{P}^5 , we have $h^0(Z, h) \geq 6$, hence $\chi(\mathcal{O}_Z) > 1$.

As we already know, this excludes abelian surfaces, for which $\chi(\mathcal{O}_Z) = 0$, and also Enriques surfaces, for which $\chi(\mathcal{O}_Z) = 1$. For a K3 surface, $\chi(\mathcal{O}_Z) = 2$, hence $h^2 = 16$. Since the genus of the polarized K3 surface (Z, h) is defined by the identity $h^2 = 2g - 2$, we get $g = 9$. \square

Following Mukai [26], a general K3 surface of genus 9 is a linear section of the symplectic Grassmannian $G_\sigma(3, 6)$, which we already met as an example of a subadjoint (hence Legendrian) variety. Note also that a surface in \mathbb{P}^5 is Legendrian, if and only if the image of its Gauss map is contained in a copy of $G_\sigma(3, 6) \subset G(3, 6)$. This could hint that K3 surfaces of genus 9 do admit Legendrian embeddings, but we have not been able to prove or disprove this.

Proposition 10. *Let $Z \subset \mathbb{P}^5$ be a minimal surface of general type, in some Legendrian embedding. Then $c_1(Z)^2 < 2c_2(Z)$.*

Note that this is stronger than Miyaoka's bound $c_1(Z)^2 \leq 3c_2(Z)$. Since Miyaoka's bound is sharp, we get infinite families of surfaces of general type admitting no Legendrian embedding.

Questions: Does there exist any example of a surface of general type admitting a Legendrian embedding? More generally, what can be the Kodaira dimension of a smooth Legendrian variety? Can it be positive?

2.4. The case of homogeneous spaces. Let $X = G/P$ be homogeneous, i.e. the quotient of a semi-simple algebraic group G by some parabolic subgroup P . We may suppose that G is simple, otherwise X is a product and has already been characterized. We fix a maximal torus and a Borel subgroup of G inside P , hence a set Δ of simple roots. Let $I \subset \Delta$ denote the simple roots which are not roots of P , and Φ_P^+ the positive roots which are not roots of P . The cohomology algebra of X is

$$H^*(G/P, \mathbb{Z}) = \mathbb{Z}[\mathcal{P}]^{W_P} / \mathbb{Z}[\mathcal{P}]_+^W \mathbb{Z}[\mathcal{P}]^{W_P},$$

where \mathcal{P} denotes the root lattice, W the Weyl group of G , $W_P \subset W$ the Weyl group of P , $\mathbb{Z}[\mathcal{P}]^{W_P}$ the algebra of W_P -invariant polynomials, and $\mathbb{Z}[\mathcal{P}]_+^W \mathbb{Z}[\mathcal{P}]^{W_P}$ the ideal generated by homogeneous W -invariants of positive degree [16]. Since we can interpret Φ_P^+ as the set of Chern roots of TX , we have

$$c_1(X) = \sum_{\alpha \in \Phi_P^+} \alpha, \quad 2ch_2(X) = \sum_{\alpha \in \Phi_P^+} \alpha^2.$$

Since there is no invariant of W in degree one, the identity $2ch_2 = 2c_1l - (n+1)l^2$ means that the corresponding quadratic element of $\mathbb{Z}[\mathcal{P}]^{W_P}$ is W -invariant. But W is generated by W_P and the simple reflections s_i for $i \in I$, so we just need to check the invariance under the action of these simple reflections.

Recall that a positive root α is in Φ_P^+ if and only $(\alpha, \phi_j) \geq 0$ for all $j \in I$, and there is at least one $k \in I$ such that $(\alpha, \phi_k) > 0$. For each $i \in I$, define

$$\Phi_P^+(i) = \{\alpha \in \Phi_P^+ \mid (\alpha, \phi_j) = 0 \ \forall j \in I \setminus i, (\alpha, \phi_i) > 0 = (s_i \alpha, \phi_i)\}.$$

We have

$$2ch_2 - 2s_i(ch_2) = \alpha_i \sum_{\alpha \in \Phi_P^+(i)} \alpha(H_i)(\alpha + s_i \alpha).$$

Theorem 11. *Let $X = G/P \neq \mathbb{P}^1$ be a homogeneous space with Picard number one. Suppose that X admits a Legendrian embedding, not necessarily equivariant a priori. Then X is subadjoint. In particular, the Legendrian embedding is the equivariant subadjoint embedding.*

Proof. Since X has Picard number one, there is a unique simple root α_i which is not a root of P . Then the corresponding weight ϕ_i generates $\text{Pic}(X)$, and we can let $c_1 = \gamma\phi_i$ and $\ell = l\phi_i$ for some positive integers γ and l .

Consider the identity $2ch_2 = 2c_1\ell - (n+1)\ell^2$. Applying s_i and dividing by α_i we get the relation

$$\sum_{\alpha \in \Phi_P^+(i)} \alpha(H_i)(\alpha + s_i\alpha) = l(2\gamma - (n+1)l)(2\phi_i - \alpha_i).$$

In particular, the scalar product with ϕ_i gives

$$\sum_{\alpha \in \Phi_P^+(i)} \alpha(H_i)^2 \frac{(\alpha_i, \alpha_i)}{2} = l(2\gamma - (n+1)l)(2\phi_i - \alpha_i, \phi_i) > 0.$$

But $2\phi_i - \alpha_i$ is a linear combination, with positive coefficients, of the fundamental weights corresponding to the nodes of the Dynkin diagram that are connected to i . In particular, $(2\phi_i - \alpha_i, \phi_i) > 0$ and we deduce that $2\gamma - (n+1)l > 0$. This implies that $l = 1$ (otherwise the index of X would be greater than $n+1$, which is impossible), and that X has index $\gamma > \frac{n+1}{2}$.

The index of the rational homogeneous spaces with Picard number one have been computed by Snow [28], and we easily get the following lemma from his results.

Lemma 12. *Let X be a homogeneous space with Picard number one.*

- (1) *If X is adjoint, its index is $\gamma = \frac{n+1}{2}$.*
- (2) *If $\gamma > \frac{n+1}{2}$, then X is either a projective space, $G(2, n)$, $G(3, 7)$, $G_\phi(2, n)$, a quadric, a spinor variety SS_m with $m \leq 7$, the Cayley plane, or a subadjoint variety.*

Then we check that our Chern class identity only holds in the subadjoint case. Moreover, $l = 1$ implies that the embedding is given by the subadjoint embedding, which is Legendrian, possibly followed by a projection, which is excluded by the fact that the dimension of the ambient space would become too small. \square

2.5. More Chern class identities. If the dimension of X is large enough, we can eliminate the hyperplane class from the identities $\sigma_{2m}(X, h) = 0$ to obtain identities between the Chern classes of X . This illustrates the principle that, the greater the dimension, the more difficult it should be to find inhomogeneous smooth Legendrian varieties – if any exist.

Formally, we just interpret our identities as polynomial equations for h , with coefficients in the Chow ring of X . Let $R_{l,m}$ be the resultant of the polynomials σ_{2l} and σ_{2m} , a polynomial of degree $2l + 2m + 2$ in the Chern classes.

Proposition 13. *Let X be a smooth variety of dimension $n \geq 8$, admitting a Legendrian embedding. Then $R_{l,m}(X) = 0$ for $1 \leq l < m$.*

The first identity $R_{1,2}(X) = 0$ comes out in degree 8 and can be written

$$C_{8,4}(n+1)^4 + C_{8,3}(n+1)^3 + C_{8,2}(n+1)^2 + C_{8,1}(n+1) + C_{8,0} = 0,$$

where the classes $C_{8,i}$ are expressed as follows in terms of the Chern classes:

$$\begin{aligned} C_{8,4} &= ch_4^2, \\ C_{8,3} &= 16ch_2ch_3^3 - 8ch_1ch_3ch_4 - 20ch_2^2ch_4, \\ C_{8,2} &= 32ch_1^2ch_2ch_4 - 16ch_1ch_2^2ch_4 + 100ch_2^4, \\ C_{8,1} &= 32ch_1^3ch_2ch_3 - 176ch_1^2ch_2^3 - 16ch_1^4ch_4, \\ C_{8,0} &= 468ch_1^4ch_2^2. \end{aligned}$$

3. LOCAL DIFFERENTIAL GEOMETRY OF LEGENDRIAN VARIETIES

When X is a subexceptional variety (a homogeneous Legendrian variety coming from the adjoint variety of an exceptional Lie algebra), the base locus of the second fundamental form is a *Severi variety* [31]. This suggests that the Lebrun-Salamon conjecture could be derived from Zak's classification of Severi varieties. For this we would need a very precise understanding of the local geometry of Legendrian varieties. In this section we obtain basic results in that direction.

Let $X^n \subset \mathbb{P}V$ be a Legendrian variety and let $z \in X$ be a general point. By Pfaff's theorem (see, e.g., [3] p38) there exist local coordinates x^1, \dots, x^{2n+1} about z such that locally X may be written as a graph of the form

$$\begin{aligned} (2) \quad & x^{2n+1} = f(x^1, \dots, x^n) \\ (3) \quad & x^{n+j} = \frac{\partial f}{\partial x^j}(x^1, \dots, x^n) \quad 1 \leq j \leq n \end{aligned}$$

where f and all its first derivatives vanish at $z = (0, \dots, 0)$. Thus $|III_z|$, the third fundamental form, consists of a single cubic (given by the third order partials of f at z) and the quadrics of the second fundamental form $|II_z|$ consist of the n partial derivatives of $|III_z|$. In particular, letting $N_{2,z} = II_z(S^2T_zX)$ denote the image of the second fundamental form, we may view $II_z \in S^2T_z^* \otimes N_{2,z}$ as having an additional symmetry, using the symplectic form ϕ to identify T_z with $N_{2,z}$ we obtain $II_z \in S^3T_z^*$.

A coordinate free way to see this symmetry is as follows: Let $\gamma : X \rightarrow G(n+1, 2(n+1))$ denote the Gauss map of X . Then X is Legendrian iff the image of γ is contained in the ϕ -isotropic Grassmanian $G_\phi(n+1, 2(n+1))$. Recall that $T_E G_\phi(n+1, 2(n+1)) \simeq S^2E^*$ (see, e.g., [21]) so $II_z \simeq d\gamma_z \in T_z^*X \otimes T_{T_z^*X} G_\phi(n+1, 2(n+1)) = T_z^*X \otimes S^2T_z^*X$. On the other hand, the differential of the Gauss map of any variety is an element of $S^2T_z^*X \otimes T_z^*X$ and the intersection of these two spaces is exactly $S^3T_z^*X$.

This property propagates to higher order differential invariants. Let $F_{k,z} \in S^kT_z^*X \otimes N_{2,z}X$ denote the relative differential invariant of order

k (defined in coordinates in the proof below, see [15] section 3.5 for a coordinate free definition):

Proposition 14. *Let $X \subset \mathbb{P}V$ be Legendrian, let $z \in X$ be a smooth point. Then using the identification $N_{2,z}X \simeq T_z^*X$, we have $F_{k,z} \in S^{k+1}T_z^*X$.*

Proof. The coefficients of $F_{k,z}$ are simply the higher order terms in the Taylor series for the x^{n+j} , which correspond to the k -th derivatives of $\frac{\partial f}{\partial x^j}$. \square

If P denotes the cubic for the third fundamental form and $v \in T_zX$, let $Q_v = \frac{\partial P}{\partial v}$ denote the corresponding quadric in II_z . Let $\text{Base}|II_z| := \{v \in T_zX \mid II(v, v) = 0\}$, the *base locus* of the second fundamental form, the set of tangent directions to lines having contact with X at z to order two.

Corollary 15. $\text{Base}|II_z| = \{v \in T_z \mid v \in (Q_v)_{\text{sing}}\}$

Let $C_z \subset T_zX$ denote the tangent directions to lines on X passing through z . Note that one always has $C_z \subseteq \text{Base}|II_z|$.

Theorem 16. *Let $X \subset \mathbb{P}V$ be Legendrian and let $z \in X$ be a general point. Then $C_z = \text{Base}|II_z|$. In particular, $|III_z|$ is singular if and only if X is uniruled by lines.*

Proof. Let $v \in \text{Base}|II_z|$, so $v \in (Q_v)_{\text{sing}}$. Now [20] (3.1.2) implies $v \in (F_{3,v})_{\text{sing}}$ (here we mean the cubic in the normal direction corresponding to the tangent vector v) but now the symmetry implies $v \in \text{Base}F_3$. But now [20] (3.1.3) implies $v \in (F_{4,v})_{\text{sing}}$ and the symmetry again implies $v \in \text{Base}F_4$. Continuing, $v \in (F_{j,v})_{\text{sing}}$ and $v \in \text{Base}(F_i)$ for all $i \leq j$ implies $v \in (F_{j+1,v})_{\text{sing}}$ and then the symmetry implies in turn $v \in \text{Base}(F_{j+1})$, and these two facts imply $v \in (F_{j+2,v})_{\text{sing}}$ etc... and one obtains $v \in \text{Base}(F_l)$ for all l , i.e., that there is a line having infinite order contact to X at x in the direction of v . \square

We summarize some other known properties of Legendrian varieties:

Proposition 17. *Let $X \subset \mathbb{P}^{2n+1} = \mathbb{P}V$ be a Legendrian variety.*

- (1) *If X is linearly degenerate, then X is a linear subspace or a cone.*
- (2) *If X is not linearly degenerate, then the tangent variety $\tau(X) \subset \mathbb{P}V$ and the dual variety $X^* \subset \mathbb{P}V^*$ are projectively isomorphic hypersurfaces.*

Proof. If X is degenerate, any tangent space to X is contained in some hyperplane. By duality, we get that every tangent space to X passes through some fixed point. In characteristic zero, this implies that X is a cone.

A hyperplane H is tangent to X at the point x if and only if the point $h = H^\perp$ belongs to the embedded tangent space T_xX . This gives a linear identification between X^* and $\tau(X)$. The fact that they are hypersurfaces follows e.g., from the fact that III_z is nonzero (because X is not linearly degenerate) hence, by the infinitesimal calculation of the dimension of the secant variety $\sigma(X)$ in [12], we have $\sigma(X) = \mathbb{P}V$ and thus by the Fulton-Hansen connectedness theorem $\tau(X)$ is a hypersurface. \square

The subexceptional varieties have the remarkable property that their dual varieties are quartic hypersurfaces, whose complements are homogeneous.

In the case of surfaces, the degree of the dual variety, which is sometimes called the *codegree*, does not depend of the embedding but only of the Chern numbers. This seems to be specific to dimension two. Note the curious relation with Miyaoka’s inequality.

Proposition 18. *If $X \subset \mathbb{P}^5$ is a Legendrian surface, its codegree is equal to $3c_2 - c_1^2$.*

Proof. By Katz’s formula ([10], Chapter 2, Theorem 3.4), the degree of X^* is $c_2 - 2hc_1 + 3h^3$. Since for a Legendrian surface we know that $c_1^2 - 2c_2 = 2hc_1 - 3h^3$, the result follows. \square

4. BRYANT’S METHOD

Now that we have various numerical conditions on smooth Legendrian varieties, we expect that it should be a delicate problem to construct explicit examples, especially in higher dimensions. The only method of construction that we are aware of was suggested by Bryant. As explained in the introduction, it is based on the observation that the corresponding problem in $\mathbb{P}(T^*\mathbb{P}^{n+1})$ is easily solved. One then tries to transport the solutions to \mathbb{P}^{2n+1} through an explicit birational map. Unfortunately, there are strong obstructions for this simple and elegant idea to produce smooth varieties and we are only able to make it work in a very special situation.

4.1. The birational map. For convenience we switch n to $n - 1$ and we consider $\mathbb{P}(T^*\mathbb{P}^n)$ as the flag variety $\mathbb{F}_{1,n}(\mathbb{C}^{n+1})$ of pairs of incident lines and hyperplanes in \mathbb{P}^n . It has two projections p and p' on \mathbb{P}^n and its dual $\check{\mathbb{P}}^n$. If we choose homogeneous coordinates x_0, \dots, x_n on \mathbb{P}^n , and dual coordinates y^0, \dots, y^n on $\check{\mathbb{P}}^n$, it is the subvariety of $\mathbb{P}^n \times \check{\mathbb{P}}^n$ defined by the equation $\sum_{i=0}^n x_i y^i = 0$.

Let $[w_1, \dots, w_n, z_1, \dots, z_n]$ be homogeneous coordinates on \mathbb{P}^{2n-1} . We consider the rational map

$$\begin{aligned} \varphi : \mathbb{F}_{1,n}(\mathbb{C}^{n+1}) &\dashrightarrow \mathbb{P}^{2n-1} \\ ([x], [y]) &\mapsto [x_0 y^1, \dots, x_0 y^{n-1}, x_0 y^0 - x_n y^n, x_1 y^n, \dots, x_{n-1} y^n, x_0 y^n]. \end{aligned}$$

This is a birational map, whose inverse is given by

$$\begin{aligned} \varphi^{-1} : \mathbb{P}^{2n-1} &\dashrightarrow \mathbb{F}_{1,n}(\mathbb{C}^{n+1}) \\ [w, z] &\mapsto \left([z_n, z_1, \dots, z_{n-1}, -\frac{1}{2}(w_n + \frac{(w,z)}{z_n})], \right. \\ &\quad \left. [\frac{1}{2}(w_n - \frac{(w,z)}{z_n}), w_1, \dots, w_{n-1}, z_n] \right), \end{aligned}$$

where $(w, z) := \sum_{i=1}^{n-1} w_i z_i$.

Consider the contact structures given by the line-bundle valued one-forms $\theta' = xdy = -ydx$ on $\mathbb{F}_{1,n}(\mathbb{C}^{n+1})$, and $\theta = zdw - wdz$ on \mathbb{P}^{2n-1} .

Lemma 19. *The birational map φ is compatible with the contact structures determined by θ' on $\mathbb{F}_{1,n}(\mathbb{C}^{n+1})$ and θ on \mathbb{P}^{2n-1} .*

Proof. A simple computation shows that $\varphi^*\theta = x_0y^n\theta'$. □

Now let Z be any subvariety of \mathbb{P}^n . Denote by $Z^\# \subset \mathbb{F}_{1,n}(\mathbb{C}^{n+1})$ its *conormal variety*, defined as the Zariski closure of the projectivized conormal bundle $\mathbb{P}N_Z^* \subset \mathbb{F}_{1,n}(\mathbb{C}^{n+1})$, taken over the smooth locus of Z . As recalled in the introduction, it is well known that $Z^\#$ is Legendrian with respect to the contact structure θ . Since by the Lemma the birational map φ preserves the contact structure, we immediately get:

Corollary 20. *Let $Z \subset \mathbb{P}^n$ be a subvariety, and $Z^\# \subset \mathbb{F}_{1,n}(\mathbb{C}^{n+1})$ its conormal variety. Then $\tilde{Z} = \varphi(Z^\#)$ is a Legendrian subvariety of \mathbb{P}^{2n-1} .*

The conormal variety is the closure of the incidence variety of pairs (z, H) where $z \in Z$ is a smooth point and $H \in \mathbb{P}^{n*}$ is such that $\tilde{T}_z Z \subset H$. Its projection to the dual projective space $\check{\mathbb{P}}^n$ is, by definition, the dual variety Z^* of Z . That the image variety \tilde{Z} is Legendrian means that at every smooth point, the affine tangent space is maximal isotropic with respect to θ . Unfortunately, the fact that φ is not an isomorphism will tend to produce singularities on \tilde{Z} . We now analyze φ in some detail to determine conditions under which \tilde{Z} can be smooth.

Fact 0. The exceptional locus $Exc(\varphi)$ of φ is the hyperplane section $x_0y^n = 0$ of $\mathbb{F} = \mathbb{F}_{1,n}(\mathbb{C}^{n+1})$, union of the two irreducible divisors $E_1 := \{([x], [y]) \mid x_0 = 0\}$ and $E_2 := \{([x], [y]) \mid y^n = 0\}$. The indeterminacy locus is $Ind(\varphi) = E_1 \cap E_2$.

Outside the exceptional locus, φ restricts to an isomorphism onto the affine space defined as the complement of the hyperplane $P = (z_n = 0)$.

Let H_0 be the hyperplane $\{x_0 = 0\}$ in \mathbb{P}^n , and $p_0 \in H_0$ the point dual to the hyperplane $y^n = 0$ in $\check{\mathbb{P}}^n$. Geometrically, $E_2 = \{(p, H) \in \mathbb{F} \mid p_0 \in H\}$ and $E_1 = \{(p, H) \in \mathbb{F} \mid p \in H_0\}$.

Fact 1. Two points (p, H) and (p', H') of \mathbb{F} outside the indeterminacy locus are in the same fiber of φ if and only if

$$p = p' \in H_0 \quad \text{or} \quad p_0 \in H = H'.$$

Proof. On $Exc(\varphi) - Ind(\varphi)$, φ is given by the following formulas:

$$\begin{aligned} \varphi([0, x_1, \dots, x_n], [y]) &= [0, \dots, 0, x_n, \dots, x_1, 0], \\ \varphi([x], [y^0, \dots, y^{n-1}, 0]) &= [y^1, \dots, y^{n-1}, -y_0, 0, \dots, 0]. \end{aligned}$$

This implies the claim. □

Let now $Z \subset \mathbb{P}^n$ be some irreducible, possibly singular hypersurface. This case is of primary interest since when Z has codimension bigger than one, \tilde{Z}

is birationally ruled. Note also that when Z is a hypersurface, $Z^\#$ is nothing else than the closure of its Gauss map.

Fact 2. If \tilde{Z} is smooth, then $(Z - Z \cap H_0)^\#$ is smooth outside E_2 .

Proof. This follows from Fact 0, since

$$Z^\# - Exc(\varphi) = (Z - Z \cap H_0)^\# - E_2.$$

□

We say that Z is *quasi-smooth* outside H_0 if $(Z - Z \cap H_0)^\#$ is smooth. For a curve, this means that outside H_0 , Z has only nodes or simple cusps. In general, it seems to be a difficult problem to understand quasi-smooth singularities. A surface with a double curve will be quasi-smooth at smooth points of that curve. An isolated quadratic singularity is also quasi-smooth, but apparently no other simple surface singularity. Arnold and its school have classified what they call (real) stable Legendrian singularities ([1], section 21). This gives examples of quasi-smooth singularities, for example the swallow-tail. But note that Z and its dual have the same conormal variety, so that the dual variety of a smooth variety (which is in general very singular), is always quasi-smooth.

Fact 3. Let $z \in Z \cap H_0$ be a smooth point such that $p_0 \notin T_z Z$. Then $Z^\#$ meets the fiber of φ containing $(z, \tilde{T}_z Z)$ transversely at $(z, \hat{T}_z Z)$.

Proof. By Fact 1, the fiber of φ at $(z, \tilde{T}_z Z)$ is $\{(z, H) \mid z \in H\}$, or, identifying a linear space on \mathbb{F} with a vector subspace of the ambient $V \otimes V^*$, it is the set of elements $\{u \otimes h' \mid [u] = z, \langle u, h' \rangle = 0\}$, which also corresponds to the kernel of φ_* at $(z, \tilde{T}_z Z)$. On the other hand, an easy moving frames calculation shows

$$\hat{T}_{z, \tilde{T}_z Z} Z^\# = \{v \otimes h - x \otimes Q(v, \cdot) \mid [h] = \tilde{T}_z Z, v \in \hat{T}_z Z\} \subset V \otimes V^*$$

where $Q \in S^2 T^*$ is the quadric corresponding to h as an element of the second fundamental form of Z at z . (Here we slightly abuse notation to consider $Q(v, \cdot)$ as an element of V^* which requires a choice of splitting, but this identification is harmless. Recall that $\hat{T}Z^\#$ denotes the affine tangent space to $Z^\# \subset \mathbb{F}$, considered as a subvariety of $\mathbb{P}(V \otimes V^*)$.)

The unique intersection of these two linear spaces is the line $\{u \otimes h \mid [u] = z, [h] = \tilde{T}_z Z\}$, which corresponds to the zero vector in $T_{z, \tilde{T}_z Z} Z^\#$. □

Fact 3'. Let $z \in Z - Z \cap H_0$ be a smooth point such that $p_0 \in T_z Z$. Then $Z^\#$ meets the fiber of φ containing $(z, T_z Z)$ transversely at $(z, \hat{T}_z Z)$, if and only if the Gauss map of Z is immersive at z , that is, if and only if z is not a flex point of Z .

Proof. Here, by Fact 1 again, the fiber of φ containing $(z, \tilde{T}_z Z)$ is the projective space $\{(p, \tilde{T}_z Z) \mid p \in \tilde{T}_z Z\}$ which we identify with the linear subspace $\{v \otimes h \mid [h] = \tilde{T}_z Z, \langle v, h \rangle = 0\}$ of $V \otimes V^*$, which also corresponds to the kernel of φ_* at $(z, \tilde{T}_z Z)$. Note in particular that this space consists of rank one elements. On the other hand, as above

$$\hat{T}_{z, T_z Z} Z^\# = \{v \otimes h - x \otimes Q(v, \cdot) \mid [h] = \tilde{T}_z Z, v \in \hat{T}_z Z\}$$

which, except for the point $v \otimes h$ when $[v] = z$ consists of rank at least two elements, as long as v is not a singular point of Q . If v is a singular point of Q then the two spaces coincide. \square

Fact 3''. More generally, still assuming \tilde{Z} is smooth, let $z \in Z - Z \cap H_0$, not necessarily a smooth point, and z^* a tangent hyperplane at z , containing p_0 . Suppose that $Z^\#$ is smooth at (z, z^*) .

Then $Z^\#$ meets the fiber of φ transversely at (z, z^*) , if and only if the projection $Z^\# \rightarrow Z^*$ is immersive at (z, z^*) .

Proof. Same proof as for Fact 3'. \square

Note that from these facts one can reproduce Bryant's proof that any smooth projective curve can be embedded as a smooth Legendrian curve ([2], Theorem G). Simply project the curve to \mathbb{P}^2 so it has only nodal singularities and make sure it is in sufficiently general position with respect to (p_0, H_0) . See [2] for details.

If Z has dimension greater than one, $Z \cap H_0$ is a positive dimensional hypersurface in H_0 . Its tangent hyperplanes will cover H_0 (as long as $Z \cap H_0$ is not set theoretically a linear space), and in particular some of them will contain p_0 , so that $Z^\#$ will meet $\text{Ind}(\varphi)$. Before resolving the indeterminacies of φ , we make two simple observations.

Fact 4. Assume \tilde{Z} is smooth. Let p, p' be smooth points of $Z - Z \cap H_0$ with the same tangent hyperplane H , passing through p_0 . Then H must be tangent to Z along a curve.

Proof. Otherwise $(Z - Z \cap H_0)^\#$ would meet a fiber of φ along a disconnected subset, and its image would therefore be multibranch at the corresponding point of \mathbb{P}^{2n-1} , contradicting the smoothness assumption. \square

Fact 4'. Assume \tilde{Z} is smooth. Let $p \in H_0$ be a singular point of Z . Then Z has at most one branch at p tangent to a hyperplane not containing p_0 .

Proof. Otherwise $Z^\#$ would contain two points (p, H) and (p, H') in the same fiber of φ (or if $H = H'$, $Z^\#$ would not be unibranch at (p, H)), and its image under φ would not be unibranch, thus would be singular. \square

To go further in our analysis, we need to resolve the indeterminacies of φ . A simple thing to do would be to blow-up the indeterminacy locus, but:

Fact 5. The indeterminacy locus $Ind(\varphi)$ has a quadratic singularity at (p_0, H_0) , and is smooth outside that point.

Proof. We choose local coordinates on $\mathbb{F} := \mathbb{F}_{1,n}(\mathbb{C}^{n+1})$ at (p_0, H_0) as follows: for a pair (p, H) , $p = [x_0, x_1, \dots, x_{n-1}, 1]$ and H is the hyperplane generated by p and $n - 1$ other vectors $e_{n-1} - z_{n-1}e_0, \dots, e_1 - z_1e_0$, so that $H = [1, z_1, \dots, z_{n-1}, -x_0 - (x, z)]$, where $(x, z) = \sum_{1 \leq i \leq n-1} x_i z_i$. Then the condition that $p \in H_0$ is equivalent to $x_0 = 0$, and the condition that $p_0 \in H$ is equivalent to $x_0 + (x, z) = 0$. We thus see that E_1 and E_2 are smooth hypersurfaces meeting nontransversely at (p_0, H_0) along the codimension two subvariety $x_0 = (x, z) = 0$, a quadratic cone in a coordinate hyperplane. \square

Outside this singularity there is no serious problem: blowing-up the indeterminacy locus is enough to resolve the indeterminacies.

Fact 6. Let $\sigma : \tilde{\mathbb{F}} \rightarrow \mathbb{F} - \{p_0 \in H_0\}$ be the blow-up of $Ind(\varphi) - \{p_0, H_0\}$. Then $\psi := \varphi \circ \sigma$ is a morphism.

Proof. We check this in local coordinates. Let (p_1, H_1) be a point of $Ind(\varphi)$. We may suppose that $p_1 = [0, 1, 0, \dots, 0]$ and $H_1 = [0, \dots, 0, 1, 0]$. If $[x_0, 1, x_2, \dots, x_n]$ are affine coordinates around p_1 in \mathbb{P}^n , and $[y^0, \dots, y^{n-2}, 1, y^n]$ are affine coordinates around H_1 in the dual projective space, we can choose $x_0, x_2, \dots, x_n, y^0, y^2, \dots, y^{n-2}, y^n$ as affine coordinates on \mathbb{F} , with the missing coordinate y^1 given by the relation

$$x_0 y^0 + y^1 + \sum_{i=2}^{n-2} x_i y^i + x_{n-1} + x_n y^n = 0.$$

Above the corresponding open set $U \subset \mathbb{F}$, the blow-up $\tilde{\mathbb{F}}$ is the set of points $(x, y, [s, t]) \in U \times \mathbb{P}^1$ such that $s x_0 = t y^n$.

The roles of s and t being symmetric, we may suppose that $s \neq 0$. Then we let $s = 1$ and choose $x_2, \dots, x_n, y^0, y^2, \dots, y^{n-2}, y^n, t$ as local coordinates on $\tilde{\mathbb{F}}$. In these coordinates, ψ is given by

$$\begin{aligned} \psi(x, y, [s, t]) &= \varphi([t y^n, 1, x_2, \dots, x_n], [y^0, y^1, \dots, y^{n-2}, 1, y^n]) \\ &= [t y^n y^1, \dots, t y^n y^{n-1}, x_n y^n - t y^n y^0, \\ &\quad x_{n-1} y^n, \dots, x_2 y^n, y^n, t (y^n)^2] \\ &= [t y^1, \dots, t y^{n-1}, x_n - t y^0, x_{n-1}, \dots, x_2, 1, t y^n]. \end{aligned}$$

This is always defined, and our claim follows. \square

Although we do not use it in what follows, we briefly describe what happens near the singular point. We leave the details to the reader.

We begin by blowing-up (p_0, H_0) in \mathbb{F} , giving a map $\sigma : \mathbb{F}' \rightarrow \mathbb{F}$. We let $\varphi' = \varphi \circ \sigma : \mathbb{F}' \dashrightarrow \mathbb{P}^{2n-1}$, and I' be the strict transform of $Ind(\varphi)$.

Fact 7. The indeterminacy locus $Ind(\varphi')$ has two irreducible components: I' and a smooth hyperplane H inside the exceptional divisor $E \simeq \mathbb{P}^{2n-2}$. These two components meet transversely along the smooth subvariety $I' \cap H$.

Then we blow-up I' by $\sigma' : \mathbb{F}'' \rightarrow \mathbb{F}'$, and we let $\varphi'' = \varphi' \circ \sigma'$.

Fact 8. The indeterminacy locus $Ind(\varphi'')$ is the strict transform H' of H , a smooth subvariety of \mathbb{F}'' of codimension two.

Finally we blow-up H' by $\sigma'' : \mathbb{F}''' \rightarrow \mathbb{F}''$, and we let $\varphi''' = \varphi'' \circ \sigma''$.

Fact 9. $\varphi''' : \mathbb{F}''' \rightarrow \mathbb{P}^{2n-1}$ is a morphism, and its exceptional locus is the union of four irreducible divisors.

4.2. Surfaces. Now suppose that Z is a surface in \mathbb{P}^3 , such that $Z^\#$ does not contain (p_0, H_0) – for example, we can just ask that $p_0 \notin Z$. Then by Fact 6, we just need to blow-up the smooth part of $Ind(\varphi)$ to resolve the indeterminacies of φ on $Z^\#$.

Let $Z^{\#\#}$ denote the strict transform of $Z^\#$ by the blow-up σ . Also we let $E_1^\#$ and $E_2^\#$ be the strict transforms of E_1 and E_2 , and $E_0^\#$ be the exceptional divisor of σ . The morphism ψ restricts to an isomorphism between the complement of $E_0^\# \cup E_1^\# \cup E_2^\#$ and the complement of the hyperplane H in \mathbb{P}^5 . Moreover, ψ maps $E_0^\#$ to a quadratic cone Q inside the hyperplane $P = \{z_3 = 0\}$, and $E_1^\#, E_2^\#$ to planes U_1 and U_2 inside Q , meeting at the vertex of Q .

Proposition 21. *Suppose that:*

- (1) $Z \cap H_0$ is a smooth curve,
- (2) Z has no bitangent plane containing p_0 ,
- (3) $Z \cap H_0$ has no bitangent line containing p_0 .
- (4) $Z^\#$ meets $Ind(\varphi)$ transversely.

Then ψ is injective on $Z^{\#\#}$.

Actually, we can replace condition (3) by the weaker condition (3'): on a line in H_0 containing p_0 , there is at most one point of Z whose tangent hyperplane contains p_0 .

Proof. We just need to check that the fiber of a point of the quadratic cone Q contains at most one point of $Z^{\#\#}$.

Let $q = [u^1, u^2, w, v_2, v_1, 0]$ be a point of $Q - U_1 \cup U_2$. Let $p \in \pi \circ \sigma(\psi^{-1}(q)) \subset Ind(\varphi)$. By the formulas above for the morphism ψ we see that p belongs to the line in H_0 generated by $v = [0, v_1, v_2, 0]$ and p_0 . By (1), Z is smooth at p , and by (3) or (3'), p is uniquely determined. Now, (4) guarantees that the restriction of σ to $Z^\#$ is just the blow-up of the finite set of points on the smooth curve $C = Z \cup H_0$, where the tangent lines to C hits p_0 . Our point p is one of these, and it follows that $Z^{\#\#}$ is smooth over

p and contains the whole fiber of ψ over $(p, T_p Z)$. This fiber is a projective line. But it follows from its expression that ψ restricts to an isomorphism between that line and a line in \mathbb{P}^5 . We conclude that ψ restricted to $Z^{\#\#}$ is injective over $Q - U_1 \cup U_2$.

Because of Fact 1, (2) and (3) ensure that it is also injective over $U_1 \cup U_2$, and our claim follows. \square

Proposition 22. *Suppose moreover that:*

- (1) Z is quasi-smooth,
- (2) the projection $Z^{\#} \rightarrow Z^*$ is immersive over any tangent plane containing p_0 ,
- (3) at any point $p \in Z \cap H_0$ such that $T_p Z$ contains p_0 , the base locus of the second fundamental form of Z does not contain the line $\overline{pp_0}$.

Then ψ defines an embedding of $Z^{\#\#}$ in \mathbb{P}^5 , and its image \tilde{Z} is a smooth Legendrian surface in \mathbb{P}^5 .

Proof. Since $Z^{\#}$ is smooth and meets $Ind(\varphi)$ transversely, $Z^{\#\#}$ is smooth (note that this transversality hypothesis is equivalent to the hypothesis we made on the second fundamental form, as the computation below will show).

By Fact 3 and Fact 3'' the restriction of ψ to $Z^{\#\#}$ is immersive outside $E_0^{\#}$, and what we need to check is that this remains true on this divisor. The intersection $Z^{\#\#} \cap E_0^{\#}$ is a bunch of projective lines – the pre-images under the blow-up σ of the finite number of points inside $Z^{\#} \cap Ind(\varphi)$.

Let (p_1, H_1) be one of these points. We use the notations of the proof of Fact 6, with $n = 3$. Let $F = 0$ be the equation of the surface Z , and $f(x_0, x_2, x_3) = 0$ the equation we get by letting $x_1 = 1$. Our hypothesis on $H_1 = T_{p_1} Z$ means that $\partial f / \partial x_0(0) = \partial f / \partial x_3(0) = 0$, and we may suppose that $\partial f / \partial x_2(0) = 1$. The local equations of $Z^{\#}$ are

$$f(x_0, x_2, x_3) = 0, \quad (\partial f / \partial x_2)y^0 = \partial f / \partial x_0, \quad (\partial f / \partial x_2)y^3 = \partial f / \partial x_3.$$

If we write $f(x_0, x_2, x_3) = x_2 + q(x_0, x_2, x_3) + \text{higher order terms}$, where q is quadratic, we deduce from these equations that the tangent space of $Z^{\#}$ at (p_1, H_1) is given by:

$$\begin{aligned} dx_2 &= 0, \\ dy^0 &= q_{00}dx_0 + q_{03}dx_3, \\ dy^3 &= q_{03}dx_0 + q_{33}dx_3. \end{aligned}$$

In particular, the intersection with $Ind(\varphi)$ is transverse if and only if $q_{33} \neq 0$. But in the system of local coordinates on Z given by x_0 and x_3 , the local analytic equation of that hypersurface is, up to higher order terms, $x_2 + q_{00}x_0^2 + 2q_{03}x_0x_3 + q_{33}x_3^2 = 0$, so that the condition $q_{33} \neq 0$ precisely means that the second fundamental form $II : (x_0, x_3) \mapsto q_{00}x_0^2 + 2q_{03}x_0x_3 + q_{33}x_3^2$ does not vanish identically on the line $x_0 = 0$. This is precisely condition (3) at p_1 .

Now consider the strict transform of $Z^{\#}$, which means that we let $x_0 = ty^3$ and replace x_0 by t in our system of local equations. The equations of the

tangent space T of $Z^{\#\#}$ at the point $(0, 0, [1, t])$ over (p_1, H_1) become:

$$\begin{aligned} dx_2 &= 0, \\ dy^0 &= tq_{00}dy^3 + q_{03}dx_3, \\ dy^3 &= tq_{03}dy^3 + q_{33}dx_3. \end{aligned}$$

But $\psi_*(dt, dx_2, dx_3, dy^0, dy^3) = (-tdx_2, dt, dx_3 - tdy^0, dx_2, tdy^3)$. If $t \neq 0$, the kernel of ψ_* is the line $dt = dx_2 = dy^3 = dx_3 - tdy^0 = 0$, which is not contained in T . If $t = 0$, this kernel is the plane $dt = dx_2 = dx_3 = 0$, which again meets T only at the origin. This implies our claim. \square

The series of conditions given by Propositions 21 and 22 look very restrictive. In particular, a “generic” smooth surface will have a curve (possibly reducible) of bitangent planes, covering the whole of \mathbb{P}^3 . This will not be compatible with condition (2) of Proposition 21, which is really a necessary condition for \tilde{Z} to be smooth.

Thus we look for singular surfaces, but not too singular since they must remain quasi-smooth, at least outside the hyperplane H_0 . Let \check{p}_0 denote the set of hyperplanes passing through p_0 .

Theorem 23. *Let Z be a Kummer quartic surface in \mathbb{P}^3 , in general position with respect to p_0 and H_0 . Let $C = Z \cap H_0$ and $D = Z^* \cap \check{p}_0$, two general hyperplane sections. The pull-backs of these curves to the K3 surface $Z^{\#}$ meet transversely in twelve points. The surface \tilde{Z} is isomorphic to $Z^{\#}$ blown up at these twelve points and is a smooth Legendrian surface in \mathbb{P}^5 .*

Recall (see for example [11]) that a Kummer surface Z in \mathbb{P}^3 is a singular quartic surface in \mathbb{P}^3 with exactly 16 ordinary double points as singularities. In particular, it is quasi-smooth. But the property that will be most useful to us is that the dual of a Kummer surface also only has ordinary double points as singularities. In fact the Kummer surface is projectively isomorphic to its dual surface Z^* . The double points of the dual surface define 16 planes in Z called its *tropes*. These planes are (doubly) tangent to Z along smooth conics, each of which contains exactly 6 double points (this is the famous 16_6 configuration). The map $Z^{\#} \rightarrow Z$ blows-up the 16 double points of Z , and the dual map $Z^{\#} \rightarrow Z^*$ contracts the pull-back of the tropes to the 16 double points of Z^* .

Proof. We verify that the conditions of the previous two Propositions hold for general p_0 and H_0 . The only bitangent planes to Z are the 16 tropes, so the conditions of Proposition 21 clearly hold true in general. Since Z has a finite number of singular points, which are ordinary double points, it is certainly quasi-smooth, which was condition (1) of Proposition 22. If the hyperplane \check{p}_0 of \mathbb{P}^3 does not contain any of the sixteen singular points of Z^* , then the projection $Z^{\#} \rightarrow Z^*$ is an isomorphism above $Z^* \cap \check{p}_0$, and condition (2) is also verified. Finally, (3) is again a general position

condition and will hold in general. We conclude that $Z^{\#\#}$ is smooth and isomorphic with \tilde{Z} .

Now the hyperplane section $Z \cap H_0$, supposed to be general, is a smooth quartic plane curve C whose dual, by the Plücker formulas, is a curve of degree twelve. We conclude that the indeterminacy locus of φ restricted to Z is given by the twelve points on C whose tangent line to C hits p_0 . The surface $Z^{\#\#}$ is therefore the $K3$ -surface $Z^\#$, blown-up at the twelve corresponding points. \square

We can give a precise meaning to the condition that Z be in general position with respect to p_0 and H_0 . Namely, we need that:

- (1) p_0 is not contained in Z nor in any of its tropes,
- (2) H_0 is not tangent to Z and contains none of its double points,
- (3) none of the 28 bitangents of the quartic curve $C = Z \cap H_0$ pass through p_0 ,
- (4) if $p \in C$ is such that the tangent line $T_p C$ contains p_0 , this line is not a bitangent, and is not contained in the kernel of the second fundamental form of Z at p .

These are all non-empty and open conditions.

Note that since $c_2(Z^\#) = 24$, we get $c_2(\tilde{Z}) = 36$, and since $c_1(Z^\#) = 0$, $c_1(\tilde{Z})$ is minus the sum of the 12 exceptional divisors E_1, \dots, E_{12} . In particular, $c_1^2 = -12$ and $2ch_2 = -84$.

On the other hand, let L (resp. L') denote the pull-back to $Z^\#$ of the hyperplane divisor of Z (resp. Z^*). By construction, the hyperplane class on \tilde{Z} is $h = L + L' - E_1 - \dots - E_{12}$. We know that $L^2 = (L')^2 = 4$, while $L.L' = 12$, hence $h^2 = 4 + 2 \times 12 + 4 - 12 = 20$ and $c_1.h = -12$. Finally, $2c_1h - 3h^2 = -2 \times 12 - 3 \times 20 = -84$, in agreement with the identity $2ch_2 = 2c_1h - 3h^2$ which we proved to hold for Legendrian surfaces.

4.3. Relations with homaloïdal polynomials. In this final section we explore the relation of Bryant's method with the subadjoint varieties.

Let P be a homogeneous polynomial of degree d in n variables. Denote by $Z_P \subset \mathbb{P}^{n+1}$ the hypersurface of equation $x_0^{d-1}x_{n+1} = P(x_1, \dots, x_n)$. After applying Bryant's birational map φ to its tranform $Z_P^\# \in \mathbb{F}_{1,n}(\mathbb{C}^{n+1})$, we get a Legendrian, possibly singular variety \tilde{Z}_P . For $d \neq 2$, this variety can also be described as the image of the birational map

$$\psi : [x_0, x] \in \mathbb{P}^n \mapsto [x_0^d, x_0^{d-1}x, x_0\partial P, P] \in \mathbb{P}^{2n+1}.$$

Again the problem is: when is \tilde{Z}_P a smooth Legendrian variety ?

Example 1. Let $P = x_1^3$, so that Z_P is a cuspidal rational plane cubic. Its dual Z_P^* is again a cuspidal cubic, but $Z_P^\#$ is smooth, and \tilde{Z}_P is a normal rational cubic curve in \mathbb{P}^3 .

Example 2. Let $P = x_1 x_2 \dots x_n$. One can check that \tilde{Z}_P is singular for $n \geq 4$. For $n = 3$, $\tilde{Z}_P = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is smooth.

Example 3. Let q be a quadratic form on \mathbb{C}^{n-1} , and let $P = q(x_1, \dots, x_{n-1})x_n$. Then $\tilde{Z}_P \simeq \mathbb{P}^1 \times \mathbb{Q}(q)$, where $\mathbb{Q}(q) \subset \mathbb{P}^n$ denotes the quadric hypersurface of equation $x_0 x_n = q(x_1, \dots, x_{n-1})$. In particular, \tilde{Z}_P is smooth if and only if q is nondegenerate.

We first note that the self-duality of Z is a rather general phenomenon. This generalizes the well-known self-duality of smooth quadrics, which is the case where $d = 2$. We refer to [8] for the terminology in the next statement.

Proposition 24. *Let P denote the unique (up to constant) relative invariant of minimal degree of an irreducible regular prehomogeneous space under some reductive Lie group. Then the hypersurface Z is self dual.*

Proof. As noticed in [8], the polynomial $P(\partial_1 P, \dots, \partial_n P)$ is again a relative invariant, nonzero because of the regularity hypothesis. It must therefore be a nonzero multiple of P^{d-1} . In particular the homogeneous coordinates $[y_0 = (d-1)x_0^{d-2}x_{n+1}, y_1 = \partial_1 P, \dots, y_n = \partial_n P, y_{n+1} = x_0^d]$ are related by an identity

$$P(y_1, \dots, y_n) = cP(x)^{d-1} = c(x_0^d x_{n+1})^{d-1} = \left(\frac{y_0}{d-1}\right)^{d-1} y_{n+1}.$$

This proves the claim. \square

Let us try to understand when \tilde{Z}_P can be smooth at the point $q_0 = [0, \dots, 0, 1]$. We first note that its (reduced) tangent cone contains, for each x such that $P(x) \neq 0$, the point $[0, 0, \partial P(x), 0]$. We can suppose that $x \mapsto \partial P(x)$ is linearly non-degenerate: otherwise, after a change of coordinates we can suppose that $\partial P / \partial x_n = 0$, hence \tilde{Z}_P is linearly degenerate, hence a linear space. Then the reduced tangent cone must contain the n -dimensional linear space $L = [0, 0, *, *]$.

Supposing \tilde{Z}_P to be smooth, L must coincide with its tangent space at q . Then the projection of Z on L with respect to the supplementary space $[*, *, 0, 0]$ must be a local isomorphism in the complex topology. That is, the map $[x_0^d, x_0^{d-1}x, x_0 \partial P, P] \mapsto [x_0 \partial P, P]$ must be a local isomorphism at q – in particular it must be injective. But suppose that we can find two noncolinear vector x_1 and x_2 with $P(x_1), P(x_2) \neq 0$, such that the vectors $\partial P(x_1)$ and $\partial P(x_2)$ are colinear. After multiplying them by a suitable constant, we may suppose that $P(x_1)^{-1} \partial P(x_1) = P(x_2)^{-1} \partial P(x_2)$. Then for x_0 small enough, $[x_0^d, x_0^{d-1}x_1, x_0 \partial P(x_1), P(x_1)]$ and $[x_0^d, x_0^{d-1}x_2, x_0 \partial P(x_2), P(x_2)]$ are two distinct points in \tilde{Z} , arbitrarily close to q , but $[x_0 \partial P(x_1), P(x_1)]$ and $[x_0 \partial P(x_2), P(x_2)]$ coincides – a contradiction with the smoothness of \tilde{Z}_P at q . We conclude that the rational map $[x] \in \mathbb{P}^{n-1} \mapsto [\partial P(x)] \in \mathbb{P}^{n-1}$ must be injective on the open subset $P(x) \neq 0$. In particular, we have proved:

Proposition 25. *Let $Z_P \subset \mathbb{P}^{n+1}$ be the hypersurface $x_0^{d-1}x_{n+1} = P$, and suppose that \tilde{Z}_P is a smooth Legendrian variety. Then P must be homaloïdal.*

Corollary 26. *If $d = 3$ and \tilde{Z}_P is a smooth Legendrian variety, then P must be the determinant of a semisimple Jordan algebra of rank three, and \tilde{Z}_P is a subadjoint variety.*

Proof. That a homaloïdal polynomial of degree three must be the determinant of a semisimple Jordan algebra of rank three is due to Etingof, Kazhdan and Polishchuk [9], see also [5]. The fact that the resulting varieties \tilde{Z}_P are exactly the subadjoint varieties follows from [22], where they were constructed precisely that way. \square

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