Impact of correlation crises in risk theory: asymptotics of finite-time ruin probabilities for heavy-tailed claim amounts when some independence and stationarity assumptions are relaxed[1]

ROMAIN BIARD romain.biard@univ-lyon1.fr
Université de Lyon, Université Claude Bernard Lyon 1, Institut de Science Financière et d'Assurances, 50 Avenue Tony Garnier, F-69007 Lyon, France

CLAUDE LEFÈVRE clefevre@ulb.ac.be
Département de Mathématique, Université Libre de Bruxelles, Campus de la Plaine C.P. 210, B-1050 Bruxelles, Belgium

STÉPHANE LOISEL stephane.loisel@univ-lyon1.fr
Université de Lyon, Université Claude Bernard Lyon 1, Institut de Science Financière et d'Assurances, 50 Avenue Tony Garnier, F-69007 Lyon, France

Abstract. In the renewal risk model, several strong hypotheses may be found too restrictive to model accurately the complex evolution of the reserves of an insurance company. In the case where claim sizes are heavy-tailed, we relax independence and stationarity assumptions and extend some asymptotic results on finite-time ruin probabilities, to take into account possible correlation crises like the one recently bred by the sub-prime crisis: claim amounts, in general assumed to be independent, may suddenly become strongly positively dependent. The impact of dependence and non-stationarity is analyzed and several concrete examples are given.

Keywords: Finite-time ruin probabilities; ruin theory; correlation crisis; Sub-prime effect; processes with dependent increments; asymptotic behavior; non-stationarity; heavy-tailed claim size distribution.

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1 Introduction

In the Solvency II framework, computing the Solvency Capital Requirement (SCR) and the risk margin often involves approximation of finite-time ruin probabilities in internal models. For several kinds of insurance risks, heavy-tailed distributions may be used to model individual claim amounts. Pareto distributions, or more generally regular variation distributions are often preferred to log-normal distributions to fit empirical data. A natural way to tackle this kind of problem could be to use the classical compound renewal risk model with heavy-tailed claim size distribution. In the classical Sparre Andersen risk model, the classical risk process $(R_t)_{t\geq 0}$ is defined as follows: for $t \geq 0$,

$$R(t) = u + ct - S(t),$$

where u is the non-negative amount of initial reserves, $c > 0$ is the premium income rate. The cumulated claim amount up to time t is described by the compound renewal process

$$S(t) = \sum_{i=1}^{N(t)} X_i,$$

where amounts of claims X_i, $i = 1, 2, ...$ are non-negative independent, identically distributed random variables, distributed as X. As usual $S_t = 0$ if $N(t) = 0$. The number of claims $N(t)$ until $t \geq 0$ is modeled by a renewal process $(N(t))_{t\geq 0}$ defined from the inter-occurrence times $(T_k)_{k\geq 1}$ by $N_t = \sum_{k\geq 1} 1_{\{T_k \leq t\}}$. Claim amounts and inter-occurrence times times are assumed to be mutually independent.

What we want to compute is the probability of ruin before time t with initial reserve u denoted by $\psi(u, t)$:

$$\psi(u, t) = P(\exists s \in [0, t], \quad R(s) < 0 \mid R(0) = u), \quad u \geq 0, t > 0.$$

Note that as we consider finite-time ruin probabilities, no profit condition has to be satisfied from a theoretical point of view. In this framework it is possible to adapt directly properties of sums of independent random variables with regular variation with index $-\alpha > 0$ to derive the asymptotics of $\psi(u, t)$ as u tends to infinity (see Section 2 for definitions and details):

$$\psi(u, t) \sim \frac{1}{E(T_1)} u^{-\alpha} \quad \text{as } u \to +\infty. \tag{1.1}$$

The problem is that in real world, the mutual independence of $X_1, \ldots, X_n, \ldots, T_1, \ldots, T_n, \ldots$ is not realistic for a certain number of reasons. First, the claim amounts X_k, $k \geq 1$ are not independent in practice, and may present complex forms of positive dependence: some factors may have an impact on those amounts; some claims of a certain type may have identical (in the sense of comonotonicity) severities depending on the outcomes of trials at the court. Second, weather or economic conditions can create as well strong positive dependence on claim amounts, which can be weakly dependent and independent in the usual regime, and suddenly become strongly positively dependent if a so-called correlation crisis breaks out. The marginal distribution of the claim amount may be modified as well, or remain identical. The most remarkable recent example of such a crisis is certainly the sub-prime crisis. Not only did the number of losses increase, but correlation was also raised.

Dependence between claim amounts has been investigated by Ignatov et al. (2001) and by Lefèvre and Loisel (2008) who provide recursive formulas that involve Appell-type properties for the finite-time ruin probability with dependent claim amounts. Sums of dependent random variables have been studied by many authors, in particular by Barbe et al. (2006) and by

Kortschak and Albrecher (2008). Dependence and non-stationarity have been partially taken into account for infinite-time ruin probabilities by Boudreault et al. (2006), and Albrecher and Boxma (2004) who assume that the sequence (T_k, X_k), $k \geq 1$ is i.i.d. or that the (X_k, T_{k_1}), $k \geq 1$ are i.i.d., and by Asmussen (1989) and many others who assume that the risk process is modulated by a Markovian environment process.
The natural question that arises is the following: what is the impact of this dependence on the level of SCR in internal models of Solvency II, that is can we derive expressions equivalent to (1.1) when those independence and stationarity assumptions are relaxed? What is the impact of potential correlation crises on the asymptotic behavior of finite-time ruin probabilities for heavy-tailed claim amounts? This is the question we address in this paper, in the case where the X_k, $k \geq 1$ have distributions with regular variation.
Our paper is organized as follows: in Section 2, we study a first example to show that the effect of positive dependence between claim amounts may vary a lot. In Section 3, we study a basic model with dependent claim amounts. We also consider stochastic correlation between claim amounts and use stochastic orderings to study the impact of stochastic correlation. In Section 4, we consider more complex dependence structures between the claim amounts. In Section 5, we use the results of the first Sections to analyze our main model, with possible outbreaks of correlation crises. We derive the corresponding of the finite-time ruin probability as the initial reserve tends to infinity. We illustrate our method on two examples in which the stationarity and independence properties mentioned above are relaxed.
Throughout the paper, we assume that claim amount distributions belong to the regular variation class:

Definition 1.1 *(Regular variation). A distribution function F is regularly varying of index $-\alpha$ with $\alpha \geq 0$ (written $F \in \mathcal{R}_{-\alpha}$) if*

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha}, \quad \textit{for } y > 0.$$

Before studying the correlation crisis model, let us first discuss preliminary models with static or stochastic dependence between claim amounts.

2 Varying effects of positive dependence

It is often believed that positive dependence between risks increases the probability of ruin over any given time horizon. This seems to be natural, for example, if the different claims are subjected to some exterior environment. Conclusions in that direction are indeed pointed out, e.g., in Cossette and Marceau (2000), Frostig (2003) and Picard et al. (2003).

In this section, we show through a simple illustration that ruin probabilities can not only increase, but also decrease owing to the presence of positive dependence between claim amounts. Such a decreasing effect is possible in a different model where each claim size depends on the previous claim interval (as, e.g., in Albrecher and Boxma (2004) and Boudreault et al. (2006). In that case, positive dependence corresponds to a kind of mutualisation that plays a protective role. For the present example, the decreasing effect obtained comes rather from the claim size distribution itself as it is a consequence of the max-sum-equivalence property for heavy-tailed distributions.

Specifically, let us consider two particular risk models in which the successive claim amounts $(X_n)_{n\geq 1}$ have the same distributions but are dependent in a comonotonic way. For both models, we will compare the ruin probability $\psi(u, t)$ in the independent case, i.e. when the X_n are independent identically distributed, and in a comonotonic case when all the $X_n = X_1$ almost surely, i.e. under an extremal positive dependence.

(i) Let us assume that the successive claim amounts have a common biatomic distribution given by

$$P(X_1 = 1) = 0.99 \quad \text{and} \quad P(X_1 = 1000) = 0.01. \tag{2.1}$$

Note that this law may be considered as heavy-tailed. Let us take $\lambda = c = 1$, an horizon of length $t = 10$ and $u = 990$ as initial surplus.

Intuitively, as the average number of claims up to t is equal to $\lambda t = 10$, ruin will occur when $u = 990$ if there arises (at least) one large claim (of size 1000) before time t, or if there arise sufficiently many small claims (of size 1), this event being however of small probability. In addition, the probability of getting at least one large claim is clearly smaller in the comonotonic case than in the independent case. Thus, one expects that the ruin probability before time $t = 10$ will be also smaller in the comonotonic case.

Let us show this rigourously. By definition,

$$\psi(990, 10) = P[S(\tau) \leq 990 + \tau \text{ for some } \tau \leq 10].$$

>From (2.1) and since $P[N(10) \geq 991] < 10^{-500}$ is negligible, we can approximate $\psi(990, 10)$ by a quantity $\psi_a(990, 10)$ given by

$$\psi_a(990, 10) = \sum_{j=1}^{990} P[N(10) = j \text{ and at least one these j claims is of size 1000}]. \tag{2.2}$$

In the comonotonic case, (2.2) yields the approximation $\psi_a^{com}(990, 10)$ given by

$$\psi_a^{com}(990, 10) = P(X_1 = 1000)\ P[1 \leq N(10) \leq 990],$$

while in the independent case, the corresponding approximation $\psi_a^{\perp}(990, 10)$ is

$$\psi_a^{\perp}(990, 10) = \sum_{j=1}^{990} \left[1 - P(X_1 = 1)^j\right]\ P[N(10) = j].$$

We so see that

$$\begin{aligned}
[\psi_a^{\perp} - \psi_a^{com}](990, 10) &> P[2 \leq N(10) \leq 990]\ \left\{\left[1 - P(X_1 = 1)^2\right] - P(X_1 = 1000)\right\} \\
&\simeq 0.00227 \ >> \ 10^{-500} > P[N(10) \geq 991].
\end{aligned}$$

Thus, as for the exact ruin probabilities $\psi^{\perp}$ and ψ^{com}, we get the inequality $\psi^{\perp}(990, 10) > \psi^{com}(990, 10)$.

(ii) Let us consider another situation where the common claim amount distribution is still a biatomic law but now given by

$$P(X_1 = 1) = 0.99 \quad \text{and} \quad P(X_1 = 10) = 0.01. \tag{2.3}$$

In comparison with (2.1), this law may be viewed as light-tailed. Let us take $\lambda = c = 1$, a n horizon of length $t = 10$ and $u = 100$ as initial surplus.

This time, large claims (of size 10) will cause ruin before time $t = 10$ only if they are also relatively numerous, which is more probable in the comonotonic case. So, one expects intuitively that the comonotonic case could provide a higher ruin probability than the independent case.

Let us establish this result. First, we observe that ruin is sure when there arise 11 large claims before $t = 10$. Thus, the ruin probability for the comonotonic case, $\psi^{com}(100, 10)$, satisfies

$$\psi^{com}(100, 10) > P[N(10) \geq 11]\; P(X_1 = 10) \simeq 0.00417. \tag{2.4}$$

On the other hand, occurrence of ruin before $t = 10$ implies necessarily that the the total claim amount at t is larger than $u = 100$. Using (2.3), we then have

$$\psi(100, 10) \leq P[S(10) > 100] = 1 - \sum_{j=0}^{100} P[N(10) = j,\ S(10) \leq 100].$$

For $0 \leq j \leq 10$, the event $[N(10) = j, S(10) \leq 100]$ is equivalent to $[N(10) = j]$. For $11 \leq j \leq 100$, the event $[N(10) = j, S(10) \leq 100]$ means that the number of large claims, k say, satisfies the relation $10k + (j - k) \leq 100$; so, in the independent case,

$$P[N(10) = j,\ S(10) \leq 100] = P[N(10) = j] \sum_{k=0}^{\lfloor (100-j)/9 \rfloor} \binom{j}{k} [P(X_1 = 10)]^k\ [P(X_1 = 1)]^{j-k}.$$

Thus, the ruin probability for the independent model, $\psi^{\perp}(100, 10)$, satisfies

$$\begin{aligned}\psi^{\perp}(100, 10) \leq 1 - &\sum_{j=0}^{100} P[N(10) = j] \\ &\left(\sum_{k=0}^{\lfloor (100-j)/9 \rfloor} \binom{j}{k} [P(X_1 = 10)]^k\ P(X_1 = 1)]^{j-k} \right) \simeq\ 10^{-14}.\end{aligned} \tag{2.5}$$

Comparing (2.4) with (2.5) then gives the inequality $\psi^{\perp}(100, 10) < \psi^{com}(100, 10)$.

Clearly, in general positive dependence will not affect ruin probabilities in a monotone way. Nevertheless, the two examples above show that asymptotically as $u \to \infty$, such a property could be true for certain classes of dependent claim amounts with heavy-tailed distributions, as in example (i), or with light-tailed distributions, as in example (ii).

3 A basic situation with heavy-tailed claims

Following the previous illustration, we are going to establish that for certain heavy-tailed claim amount laws, positive dependence affects ruin probabilities in a monotone way, increasing or decreasing, when the initial surplus is large enough. For related questions on the asymptotic tail behaviour of sums of dependent risks, the reader is referred to, e.g., Alink et al. (2004), Albrecher et al. (2006), Kortschak and Albrecher (2008) and Barbe et al. (2006).

In the model under study, the different claim amounts have the same law but they are either independent or with a common level. Clearly, this assumption makes them exchangeable and positively correlated. A notation $\sim$ will mean that the ratio tends to 1 as $u \to \infty$.

Proposition 3.1 *Suppose that premiums arrive at a constant rate c and claims occur according to some point process $N(t)_{t \geq 0}$. Moreover, independently of this arrival claim process, the successive claim amounts $(X_n)_{n \geq 1}$ are described by*

$$X_n = I_n W_0 + (1 - I_n) W_n, \quad n \geq 1, \tag{3.1}$$

where $(W_n)_{n\geq 0}$ *is a sequence of i.i.d positive random variables of distribution function* F_W *with*

$$F_W \in \mathcal{R}_{-\alpha}, \ \alpha \geq 0, \tag{3.2}$$

and $(I_n)_{n\geq 1}$ *is a sequence of i.i.d Bernoulli random variables with*

$$P(I_1 = 1) = p \ \in [0,1], \tag{3.3}$$

these two sequences being mutually independent. Let u *be the initial reserve* u *and denote by* $\psi_p(u,t)$ *the corresponding ruin probability over any fixed finite-time horizon* $(0,t)$*. Then, asymptotically for* u *large enough,*

$$\psi_p(u,t) \ \sim \ \left\{(1-p)\ E[N(t)] + E\left[Z_p(t)\right]^{\alpha}\right\} \ \overline{F_W}(u+ct), \tag{3.4}$$

where $Z_p(t)$ *denotes a mixed binomial random variable* $Bin[N(t),p]$.

Proof. Let

$$S_p(t) = \sum_{n=1}^{N(t)} X_n$$

be the aggregate claim amount. To establish (3.4), we first calculate $P[S_p(t) > x]$ for large x, and we then approximate $\psi_p(u,t)$ by $P[S_p(t) > u]$.

Step 1. A key point is the convolution closure property and the max-sum-equivalence property of the regular variation class (see, e.g., Cai and Tang (2004)). Specifically, when F_1 and F_2 belong to $\mathcal{R}_{-\alpha}$, $\alpha \geq 0$, the convolution closure states that

$$F_1 * F_2 \ \text{ belongs to } \mathcal{R}_{-\alpha},$$

and the max-sum-equivalence means that

$$\overline{F_1 * F_2}(x) \sim \overline{F_1}(x) + \overline{F_2}(x) \ \text{ for large } x.$$

Since $F_W \in \mathcal{R}_{-\alpha}$ by assumption, these properties allow us to write that for any $k \geq 1$ and any pairwise distinct $n_1, ..., n_{k-j} \geq 1$ with $0 \leq j \leq k-1$,

$$\begin{aligned} P\left(W_{n_1} + ... + W_{n_{k-j}} + jW_0 > x\right) \ &\sim \ (k-j)\overline{F_W}(x) + \overline{F_W}\,(x/j) \\ &\sim \ \left(k - j + \frac{\overline{F_W}\,(x/j)}{\overline{F_W}(x)}\right) \overline{F_W}(x) \\ &\sim \ (k-j+j^{\alpha})\ \overline{F_W}(x). \end{aligned} \tag{3.5}$$

Thus, (3.5) yields, for any $k \geq 1$ and $0 \leq j \leq k-1$,

$$P\left[S_p(t) > x | N(t) = k, \sum_{i=1}^{k} I_i = j\right] \ \sim \ (k-j+j^{\alpha})\ \overline{F_W}(x). \tag{3.6}$$

We also have, for $k \geq 1$ and $j = k$,

$$P\left[S_p(t) > x | N(t) = k, \sum_{i=1}^{k} I_i = k\right] = P(kW_0 > x) = \frac{\overline{F_W}\,(x/k)}{\overline{F_W}(x)}\ \overline{F_W(x)} \ \sim \ k^{\alpha}\ \overline{F_W}(x), \tag{3.7}$$

and for $k = j = 0$,

$$P[S_p(t) > x | N(t) = 0] = 0. \tag{3.8}$$

>From (3.6), (3.7) and (3.8), and since $N(t)_{t\geq 0}$, $(W_n)_{n\geq 0}$, $(I_n)_{n\geq 0}$ are mutually independent, we then get

$$P[S_p(t) > x] \sim \left\{ \sum_{k=1}^{\infty} P[N(t) = k] \sum_{j=0}^{k} \binom{k}{j} p^j (1-p)^{k-j} (k - j + j^{\alpha}) \right\} \overline{F_W}(x). \tag{3.9}$$

Obviously, (3.9) can be rewritten as

$$P[S_p(t) > x] \sim \left\{ (1-p)E[N(t)] + E\left[Z_p(t)\right]^{\alpha} \right\} \overline{F_W}(x), \tag{3.10}$$

Step 2. Let us show that for any $c, t > 0$,

$$\psi_p(u, t) \sim P[S_p(t) > u + ct], \tag{3.11}$$

as $u \to \infty$. Indeed, we observe that

$$\begin{aligned} 0 &\leq \frac{\psi_p(u,t) - P[S_p(t) > u + ct]}{\psi_p(u,t)} \\ &\leq \frac{\psi_p(u,t) - P[S_p(t) > u + ct]}{P[S_p(t) > u + ct]} \\ &\leq \frac{P[S_p(t) > u] - P[S_p(t) > u + ct]}{P[S_p(t) > u + ct]} \\ &\sim \frac{\overline{F_W}(u)}{\overline{F_W}(u + ct)} - 1, \end{aligned} \tag{3.12}$$

using (3.10). For any $x \in \mathbb{R}$, one knows that $\overline{F_W}(u)/\overline{F_W}(u+x) \to 1$ as $u \to \infty$ (see Lemma 1.3.5 in Embrechts et al. (1997)). Therefore, the approximation (3.11) follows from (3.12). Finally, combining (3.10) and (3.11) yields the formula (3.4). $\diamond$

By the approximation (3.4), $\psi_p(u,t)$ is simply given as a product of two distinct factors, the former in terms of $N(t)$, p and α, and the latter in terms of F_W, u and c. Note that the claim amount distribution plays a role in both factors. For example, choose $\overline{F_W}(x) \sim l(x)\, x^{-\alpha}$, $x > 0$, where $\alpha > 1$ and $l(x)$ is slowly varying. This covers the Pareto law and the loggamma law, inter alia. From (3.4), if α increases, $\overline{F_W}(u + ct)$ decreases while $E[Z_p(t)]^{\alpha}$ increases.

Let us observe that in particular, (3.4) gives, if $p = 0$ (i.e. when $X_n = W_n$ are i.i.d.),

$$\psi_p(u,t) \sim E[N(t)]\, \overline{F_W}(u + ct),$$

while if $p = 1$ (i.e. when $X_n = W_0$ for all n),

$$\psi_p(u,t) \sim E[N(t)]^{\alpha}\, \overline{F_W}(u + ct).$$

Let us also indicate that by (3.1), any pair of claim amounts (X_n, X_m), $n \neq m$, has a correlation $corr(X_n, X_m) = p^2\, var(W)$, which is positive and increasing in p as expected.

Proposition 3.1 can be extended to the case where the type of claim amount, either W_n or W_0, is influenced by a random environment. More precisely, suppose that the indicators $(I_n)_{n\geq 1}$

have now a common random parameter P, with some distribution on $[0,1]$. Let $\psi_P(u,t)$ be the ruin probability over $(0,t)$. Then, as $u \to \infty$,

$$\psi_P(u,t) \ \sim \ \{[1 - E(P)]\ E[N(t)] + E\,[Z_P(t)]^\alpha\}\ \overline{F_W}(u+ct), \tag{3.13}$$

$Z_P(t)$ being of mixed law $MBin[N(t), P]$. One easily checks that now, $corr(X_n, X_m) = E(P^2)\ var(W)$, $n \neq m$, which is again positive but increasing in $E(P^2)$ (i.e. in $var(P)$ if the mean $E(P)$ is fixed).

Let us examine how the law of P can affect $\psi_p(u,t)$. For that, we use the concept of $s-$convex stochastic ordering (see Lefèvre and Utev (1996) and Denuit et al. (1998)). By definition, given two random variables Y and Z, then for any $s = 1, 2, \ldots$,

$$X \leq^{\mathcal{D}}_{s-cx} Y \ \text{ if } \ E[\phi(Y)] \leq E[\phi(Z)] \ \text{ for all } s\text{-convex function } \phi : \mathcal{D} \to \mathbb{R}, \tag{3.14}$$

i.e. in short, for any function ϕ on $\mathcal{D}$ whose s-th derivative exists and satisfies $\phi^{(s)} \geq 0$. Note that the first $s-1$ moments of Y and Z are then necessarily equal. The order $\leq^{\mathcal{D}}_{1-cx}$ is just the stochastic order, $\leq^{\mathcal{D}}_{2-cx}$ is the usual convex order (which implies $var(Y) \leq var(Z)$) and $\leq^{\mathcal{D}}_{3-cx}$ is also very popular (it means that Y has smaller right-side risk than Z). Put $\alpha_{[s]} = \alpha(\alpha-1)\ldots(\alpha-s+1)$, and $\delta_{s,1} = 1$ (0) if $s = 1$ ($\neq 1$).

Property 3.2 *Asymptotically for u large enough, given any $s = 1, 2, \ldots$,*

$$\textit{if } \ \alpha_{[s]} \leq \ (\textit{resp.} \geq)\ \delta_{s,1}, \ \ \textit{then } P \leq^{[0,1]}_s Q \ \textit{ implies } \ \psi_P(u,t) \geq \ (\textit{resp.} \leq)\ \psi_Q(u,t). \tag{3.15}$$

Proof. From (3.4), we have

$$\psi_P(u,t) - \psi_Q(u,t) \ \sim \ \left(E\left\{[Z_P(t)]^\alpha - Z_P(t)\right\} - E\left\{\left[Z_Q(t)\right]^\alpha - Z_Q(t)\right\}\right)\ \overline{F_W}(u+ct). \tag{3.16}$$

A binomial law $Bin(n,p)$ is stochastically $s-$convex in the parameter p (see Dnuit and Lefèvre (2001)). Thus, if $P \leq^{[0,1]}_s Q$, then $MBin(n,P) \leq^{\{0,\ldots,n\}}_s MBin(n,Q)$, so that $Z_P(t) \leq^{\mathbb{N}}_s Z_Q(t)$. Now, consider the function $f(x) \equiv x^\alpha - x$, $x \in \{0,1,\ldots\}$. We see that $f(x)$ (resp. $-f(x)$) is $s-$convex when $\alpha_{[s]} \geq$ (resp. $\leq$) $\delta_{s,1}$. Therefore, from (3.16), we deduce the announced implication (3.15). ⋄

So, Property 3.2 states that if $P \leq^{[0,1]}_1 Q$, then

$$\psi_P(u,t) \geq \ (\leq)\ \psi_Q(u,t) \ \text{ for } \ \alpha \leq \ (\geq)\ 1.$$

In particular, this is true if P and Q reduce to two constants p and q such that $p \leq q$.

An identical conclusion holds under the condition $P \leq^{[0,1]}_2 Q$, but remember that $E(P) = E(Q)$ here. For instance, one has $P_2^{min} \leq^{[0,1]}_2 P \leq^{[0,1]}_2 P_2^{max}$, where $P_2^{min} = E(P)$ and P_2^{max} is a variable with two atoms, 0 and 1, such that $P(P_2^{max} = 1) = E(P)$. This yields

$$\psi_{P_2^{min}}(u,t) \geq \ (\leq)\ \psi_P(u,t) \geq \ (\leq)\ \psi_{P_2^{max}}(u,t) \ \text{ for } \ \alpha \leq \ (\geq)\ 1,$$

where

$$\psi_{P_2^{max}}(u,t) \ \sim \ [1 - E(P)]\ \{E[N(t)] + E[N(t)]^\alpha\}\ \overline{F_W}(u+ct).$$

If $P \leq^{[0,1]}_3 Q$, then

$$\psi_P(u,t) \geq \ (\leq)\ \psi_Q(u,t) \ \text{ for } \ \alpha \leq 1 \text{ or } \alpha \geq 2 \ (1 \leq \alpha \leq 2).$$

Let us recall that this time, $E(P) = E(Q)$ and $E(P^2) = E(Q^2)$. For instance, one might now use the inequality $P_3^{min} \leq^{[0,1]}_3 P \leq^{[0,1]}_3 P_3^{max}$, where P_3^{min} is a variable with two atoms, 0 and $E(P^2)/E(P)$, such that $P(P_3^{min} = 0) = var(P)/E(P^2)$, and P_3^{max} is a variable with two atoms, $[E(P) - E(P^2)]/[1 - E(P)]$ and 1, such that $P(P_3^{max} = 1) = var(P)/\{[(1 - E(P)]^2 + var(P)\}$.

4 More complex dependent cases

In this part, the framework of Section 3 is retained except that the claim amounts $(X_i)_{i\geq 1}$ are no longer of the form (3.1), but with a dependence structure described by a specific copula. Denote by $C^{(n)}$ the copula of the vector $(X_1, X_2, ..., X_n)$. To satisfy the condition of independence, the following relation must be hold

$$C^{(n)}(u_1, ..., u_{n-1}, 1) = C^{(n-1)}(u_1, ..., u_{n-1}) \,, \;\; n \geq 2. \tag{4.1}$$

4.1 A non-standard class of copulas

In section 3 we studied the asymptotic behavior of the finite-time ruin probability $\psi(u, t)$ from initial reserve u within time t for regular variation claim amounts X_i, $i \geq 0$ such that each X_i is equal to W_0 with probability $p \in [0, 1]$ or equal to W_i with probability $1 - p$, where the W_k, $k \geq 0$ are i.i.d. regular variation random variables. We extend here the result to the more general model case where for $n \geq 2$,

$$(X_1, \ldots, X_n)$$

has again identical marginals but with copula $C^{(n)}$ defined below. Let $\mathfrak{P}\left([1, n]\right)$ denotes the set of partitions of $[1, n]$. For $A^{(n)} \in \mathfrak{P}\left([1, n]\right)$, consider all the subsets a in $A^{(n)}$ and define the associated copula

$$C_{A^{(n)}}(u_1, \ldots, u_n) = \prod_{a \in A^{(n)}} \min_{k \in a}(u_k). \tag{4.2}$$

Then, the copula $C^{(n)}$ is represented as a weighted average of these copulas over all possible partitions $A^{(n)}$, i.e.

$$C^{(n)} = \sum_{A^{(n)} \in \mathfrak{P}([1,n])} \lambda_{A^{(n)}} C_{A^{(n)}}, \tag{4.3}$$

where the nonnegative weights must satisfies, for all $n \geq 1$ and $A^{(n)} \in \mathfrak{P}[1, n]$:

$$\begin{cases} \displaystyle\sum_{A^{(n)} \in \mathfrak{P}([1,n])} \lambda_{A^{(n)}} = 1, \\ \displaystyle\sum_{\substack{A^{(n+1)} \in \mathfrak{P}[1,n+1] \\ A^{(n+1)} \setminus \{n+1\} = A^{(n)}}} \lambda_{A^{(n+1)}} = \lambda_{A^{(n)}}, \end{cases} \tag{4.4}$$

$A^{(n+1)} \setminus \{n + 1\}$ meaning that $\{n + 1\}$ is removed from all subsets of $A^{(n+1)}$ that contain $\{n + 1\}$.

Proposition 4.1 *For all $n \geq 2$, $C^{(n)}$ defined by (4.3) satisfies the condition (4.1).*

Proof. By (4.2) and (4.3), we get

$$\begin{aligned}
C^{(n)}(u_1,...,u_{n-1},1) &= \sum_{A^{(n)}\in\mathfrak{P}([1,n])} \lambda_{A^{(n)}} C_{A^{(n)}}(u_1,...,u_{n-1},1) \\
&= \sum_{A^{(n)}\in\mathfrak{P}([1,n])} \lambda_{A^{(n)}} \prod_{a\in A^{(n)}} \min_{k\in a\setminus\{n\}}(u_k) \\
&= \sum_{A^{(n-1)}\in\mathfrak{P}([1,n-1])} \sum_{\substack{A^{(n)}\in\mathfrak{P}[1,n] \\ A^{(n)}\setminus\{n\}=A^{(n-1)}}} \lambda_{A^{(n)}} \prod_{a\in A^{(n)}} \min_{k\in a\setminus\{n\}}(u_k) \\
&= \sum_{A^{(n-1)}\in\mathfrak{P}([1,n-1])} \sum_{\substack{B\in\mathfrak{P}[1,n] \\ B\setminus\{n\}=A^{(n-1)}}} \lambda_{A^{(n)}} \prod_{a\in A^{(n-1)}} \min_{k\in a}(u_k) \\
&= \sum_{A^{(n-1)}\in\mathfrak{P}([1,n-1])} \lambda_{A^{(n-1)}} \prod_{a\in A^{(n-1)}} \min_{k\in a}(u_k) \\
&= C_{n-1}(u_1,...,u_{n-1}),
\end{aligned}$$

hence (4.1). $\diamond$

Proposition 4.2 *For claim amounts $(X_i)_{i\geq 1}$ that have the same marginal distribution F with $F\in\mathcal{R}_{-\alpha}$, $\alpha>0$, and such that $(X_1,...,X_n)$, $n\geq 2$ has a copula $C^{(n)}$ given by (4.3), then for large u and any $t>0$, we have*

$$\psi(u,t) \sim \left\{\sum_{k=1}^{\infty} P[N(t)=k] \sum_{A^{(k)}\in\mathfrak{P}([1,k])} \lambda_{A^{(k)}} \left(\sum_{a\in A^{(k)}} Card(a)^{\alpha}\right)\right\} \overline{F}(u+ct). \tag{4.5}$$

Proof. As in the proof of Proposition 3.1, we start by computing $P(S_t>x)$. For all $n\geq 1$ and $A^{(n)}\in\mathfrak{P}[1,n]$ denote $(X_1^{(A^{(n)})},...,X_n^{(A^{(n)})})$ a copy of the vector $(X_1,\ldots,X_n)$ with a dependence structure described by the copula $C_{A^{(n)}}$ defined in (4.2).
For all $k\geq 1$ and $x>0$, we have

$$P(X_1+...+X_k>x) = \sum_{A^{(k)}\in\mathfrak{P}[1,k]} \lambda_{A^{(k)}} P\left(X_1^{(A^{(k)})}+...+X_k^{(A^{(k)})}>x\right)$$

For all subsets $a\in A^{(n)}$, denote $X^{(A^{(n)},a)}$ the common variable the variables in $(X_1^{(A^{(n)})},...,X_n^{(A^{(n)})})$ whose index belongs to a are equal to ; then,

$$P(X_1+...+X_k>x) = \sum_{A^{(k)}\in\mathfrak{P}[1,k]} \lambda_{A^{(k)}} P\left(\sum_{a\in A} Card(a) X^{(A^{(k)},a)}>x\right).$$

Arguing as in Step 1 of Proposition 3.1, we obtain that

$$\begin{aligned}
P(X_1+...+X_k>x) &\sim \sum_{A^{(k)}\in\mathfrak{P}[1,k]} \lambda_{A^{(k)}} \sum_{a\in A^{(k)}} P\left(Card(a)X^{(A^{(k)},a)}>x\right) \\
&\sim \left[\sum_{A^{(k)}\in\mathfrak{P}[1,k]} \lambda_{A^{(k)}} \left(\sum_{a\in A^{(k)}} Card(a)^{\alpha}\right)\right] \overline{F}(x).
\end{aligned}$$

Thus, we may conclude in the same way as in Step 2 of Proposition 3.1. $\diamond$

We note that the sequence $(\lambda_{A^{(k)}})_{k\geq 1}$ can be built using a tree and appropriate transition probabilities. For all $k \geq 1$, $A^{(k)} \in \mathfrak{P}([1,k])$ and $A^{(k+1)} \in \mathfrak{P}([1,k+1])$, let $p_{A^{(k)}\to A^{(k+1)}}$ be the probability that $(X_1,...,X_{k+1})$ has its dependence structure described by $C_{A^{(k+1)}}$ knowing that $(X_1,...,X_k)$ has its dependence structure described by $C_{A^{(k)}}$. This is illustrated in Figure 1. Then, for all $k \geq 1$, $A^{(k)} \in \mathfrak{P}([1,k])$, we have

$$\lambda_{A^{(k)}} = \prod_{i=1}^{k-1} p_{A^{(i)}\to A^{(i+1)}}.$$

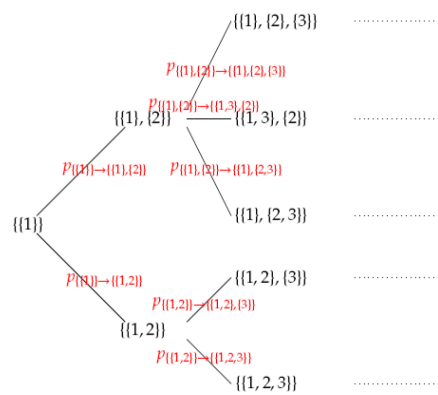

Figure 1: Step-by-step construction of the sequence $(\lambda_{A^{(k)}})_{k\geq 1}$ with transition probabilities $p_{A^{(k)}\to A^{(k+1)}}$.

4.2 Classical copulas

4.2.1 A general result

Definition 4.3 *(Multivariate regular variation). A random vector $\mathbf{X} = (X_1,...,X_n)$ belongs to $\mathcal{MR}_{-\alpha}$, $\alpha \geq 0$ if there exists a $\theta \in \mathbb{S}^{n-1}$, where $\mathbb{S}^{n-1}$ is the unit sphere with respect to a norm $|\cdot|$, such that*

$$\frac{P\left(|\mathbf{X}| > tu, \mathbf{X}/\left|\mathbf{X}\right| \in \cdot\right)}{P\left(|\mathbf{X}| > u\right)} \xrightarrow{v} t^{-\alpha} P_{\mathbb{S}^{n-1}}\left(\theta \in \cdot\right),$$

where $\xrightarrow{v}$ denotes vague convergence on $\mathbb{S}^{n-1}$.

Definition 4.4 *(Extreme value copula). A copula such that*

$$C(u_1^t,...,u_k^t) = C^t(u_1,...,u_k)\ , \quad \forall t > 0,$$

is called an extreme value copula.

Definition 4.5 *(Domain of attraction of copula). Let C be a copula and let C^* be an extreme value copula. The copula C belongs to the domain of attraction of C^*, written $C \in CDA(C^*)$, if for all $\mathbf{u}$*

$$\lim_{m\to\infty} C^m(\mathbf{u}^{1/m}) = C^*(\mathbf{u}).$$

Proposition 4.6 *For claim amounts $(X_i)_{i\geq 1}$, such that for all $n \geq 1$, $(X_1, ..., X_n) \in \mathcal{MR}_{-\alpha}$ with the same marginal distribution F and their dependence structure is described by a copula $C^{(n)}$ which belongs to the domain of attraction of $C^{(n)^*}$, then for large u and any $t > 0$,*

$$\psi(u,t) \sim \left\{ \sum_{k=1}^{\infty} P\left[N(t) = k\right] \; q_{k,\alpha} \right\} \overline{F}(u + ct), \tag{4.6}$$

where

$$q_{k,\alpha} = \int_{\mathbb{T}_k} \left(p_1^{1/\alpha} + ... + p_k^{1/\alpha}\right)^{\alpha} dU_k(\mathbf{p}),$$

where $\mathbb{T}_k$ denotes the k-dimensional unit simplex and U_k is the measure such that :

$$C^{(k)^*}(x_1, .., x_k) = exp\left(- \int_{\mathbb{T}_k} \max_{1\leq j\leq k} \left\{ -p_j log(x_j) \right\} dU_k(\mathbf{p}) \right).$$

Proof. This result follows directly from a theorem of Barbe et al. (2006). Indeed under the assumptions made above, these authors show that

$$P\left(\sum_{p=1}^{k} X_p > x \right) \sim q_{k,\alpha} \overline{F}(x).$$

With the same method as in Proposition 3.1 (step 1 and step 2), we then obtain the approximation 4.6. $\diamond$

Remark. Another relation links C^* with $q_{k,\alpha}$. From Resnick (2004) and Barbe et al. (2006), there exists a Radon measure ν on the punctured space $\mathbb{E} = [0, \infty]^k \backslash \{0\}$ such that

$$q_{k,\alpha} = \nu(\Omega), \tag{4.7}$$

where $\Omega = \left\{ (p_1, ..., p_k) \in [0, \infty)^k : p_1^{1/\alpha} + ... + p_k^{1/\alpha} > 1 \right\}$ and

$$\nu([\mathbf{0}, \mathbf{x}]^c) = -log\left\{ C^*(e^{-\mathbf{x}}) \right\}, \;\; \forall \mathbf{x} \in [0, \infty)^k. \tag{4.8}$$

Property 4.7 *Under the assumptions of Proposition 4.6, comparing the case where $(X_1, \ldots, X_n) \in \mathcal{MR}_{-\alpha}$, with the case where $(X_1, \ldots, X_n) \in \mathcal{MR}_{-\beta}$ yield for large u*

$$\alpha \leq \; (resp. \geq) \; \beta \Rightarrow \psi_\alpha(u,t) \leq \; (resp. \geq) \; \psi_\beta(u,t).$$

Proof. Directly by checking that for fixed $\mathbf{p} \in \mathbb{T}_k$, $\left(p_1^{1/\alpha} + ... + p_k^{1/\alpha}\right)^{\alpha}$ is an increasing function of α. $\diamond$

4.2.2 Independent copula

Definition 4.8 *The independent copula is*

$$C_{ind}(u_1, ..., u_k) = \prod_{i=1}^{k} u_i. \tag{4.9}$$

Remark. The independent copula is an extreme copula since

$$C_{ind}^t(u_1, ..., u_k) = \left(\prod_{i=1}^{d} u_i \right)^t = \prod_{i=1}^{d} u_i^t = C_{ind}(u_1^t, ..., u_k^t).$$

It is easily seen that (4.6) simplifies here as follow.

Proposition 4.9 *Under the assumptions of Proposition 4.6, with in addition $C_{ind}^{(n)}$ as copula, then for large u*

$$\psi(u, t) \sim \lambda t \overline{F}(u + ct).$$

4.2.3 Fréchet upper bound

Definition 4.10 *The comonotonic copula is*

$$C_{com}(u_1, ..., u_k) = min(u_1, ..., u_k). \tag{4.10}$$

Remark. The comonotonic copula is an extreme copula since

$$C_{com}^t(u_1, ..., u_k) = \min(u_1, ..., u_k)^t = \min(u_1^t, ..., u_k^t) = C_{com}(u_1^t, ..., u_k^t)$$

Proposition 4.11 *Under the assumptions of Proposition 4.6, with in addition $C_{com}^{(n)}$ as copula, then for large u*

$$\psi(u, t) \sim E[N(t)]^{\alpha} \overline{F}(u + ct).$$

4.2.4 Gaussian copula

Definition 4.12 *The Gaussian or normal copula is*

$$C_{Ga,\Sigma}(u_1, ..., u_k) = \Phi_{\Sigma}^k \left(\Phi^{-1}(u_1), ..., \Phi^{-1}(u_k) \right), \tag{4.11}$$

where Φ_{Σ}^k denotes the joint distribution function of the $k-$variate standard normal distribution with correlation matrix Σ and Φ^{-1} denotes the inverse of the distribution function of the univariate standard normal distribution.

Remark. The Gaussian copula satisfies the condition (4.1) if the correlation matrix is adapted, that is if

$$C_{Ga,\Sigma_k}^{(k)}(u_1, ..., u_{k-1}, u_k = 1) = C_{Ga,\Sigma_{k-1}}^{(k-1)}(u_1, ..., u_{k-1}), \tag{4.12}$$

with Σ_{k-1} is formed with the first $k-1$ rows and columns of Σ_k.

Lemma 4.13 *(Demarta (2002)) The Gaussian copula belongs to the domain of attraction of the independent copula.*

Proposition 4.14 *Under the assumptions of Proposition 4.6, with in addition $C_{Ga,\Sigma_n}^{(n)}$ as copula, such 4.12 is satisfied, then for large u*

$$\psi(u, t) \sim \lambda t \overline{F}(u + ct).$$

4.2.5 Archimedean copulas

Definition 4.15 *Let $\phi : [0,1] \to [0,\infty]$ be continuous and strictly decreasing with $\phi(0) \le \infty$ and $\phi(1) = 0$. A pseudo inverse of ϕ is defined as*

$$\phi^{[-1]}(t) = \begin{cases} \phi^{-1}(t) & 0 \le t \le \phi(0) \\ 0 & \phi(0) \le t \le \infty \end{cases},$$

where ϕ^{-1} is the classical inverse of ϕ. Functions ϕ are called generators of Archimedean copulas. If $\phi(0) = \infty$, then ϕ is called a strict generator.

Definition 4.16 *A decreasing (resp. increasing) function $f : \mathbb{R} \to \mathbb{R}$ is completely monotonic on an interval I if it is continuous on I and satisfies*

$$(-1)^k \frac{d^k}{dt^k} f(t) \ge 0, \quad \left(resp.\ (-1)^{k-1} \frac{d^k}{dt^k} f(t) \ge 0 \right),$$

for all t in the interior of I and any $k \ge 1$.

As a consequence, if f is completely monotonic on $[0,\infty]$ and $f(c) = 0$ for some $c > 0$, then f must be identically zero on $[0,\infty]$. So if the pseudo-inverse $\phi^{[-1]}$ of an Archimedean generator ϕ is completely monotonic, it must be positive on $[0,\infty]$, i.e. ϕ is a strict generator and $\phi^{[-1]} = \phi^{-1}$.

Lemma 4.17 *(see Nelsen (2006)) Let ϕ be a continuous strictly decreasing function from $[0,1]$ to $[0,\infty]$ such that $\phi(0) = \infty$ and $\phi(1) = 0$. If C is the function from $[0,1]^d$ to $[0,1]$ given by*

$$C_\phi(u_1, ..., u_k) = \phi^{[-1]} \left(\phi(u_1) + ... + \phi(u_k) \right), \tag{4.13}$$

then C is a $k-$copula for all $k \ge 2$ if and only if $\phi^{[-1]}$ is completely monotonic on $[0,\infty]$.

C_ϕ given by (4.13) is named an Archimedean copula with generator ϕ. Since $\phi(1) = 0$, the condition (4.1) is well satisfied.

Proposition 4.18 *Under the assumptions of Proposition 4.6, with in addition $C_\phi^{(n)}$ as copula, such that $\phi(1 - 1/t) \in \mathcal{R}_{-\beta}$ for some $\beta > 1$, then for large u*

$$\psi(u,t) \sim \left\{ \sum_{k=1}^{\infty} P\left[N(t) = k\right] \int_{\mathbb{T}_k} \left(\sum_{i=1}^{k} p_i^{1/\alpha} \right)^{\alpha} u_{k,\beta}(\mathbf{p}) d\mathbf{p} \right\} \overline{F}(u + ct),$$

where $u_{k,\beta}(\mathbf{p}) = \left\{ \prod_{i=1}^{k-1} (i\beta - 1) \right\} \left(\prod_{i=1}^{k} p_i \right)^{-\beta-1} \left(\sum_{i=1}^{k} p_i^{-\beta} \right)^{1/\beta - k}$.

Proof. This follows from Barbe et al. (2006) who proved that

$$P\left(X_1 + ... + X_k > x\right) \sim \left\{ \sum_{k=1}^{\infty} P\left[N(t) = k\right] \int_{\mathbb{T}_k} \left(\sum_{i=1}^{k} p_i^{1/\alpha} \right)^{\alpha} u_{k,\beta}(\mathbf{p}) d\mathbf{p} \right\} \overline{F}(x),$$

in the above notation. ⋄

4.3 Mixture of copulas

Proposition 4.19 *Under the assumptions of Proposition 4.6, with in addition $\tilde{C} = \sum_{i=1}^{n} \gamma_i C_i$ as copula with $\gamma_i \in \mathbb{R}_+$ and $\sum_{i=1}^{n} \gamma_i = 1$ and if we assume that $C_i \in CDA(C_i^*)$ for $i = 1, ..., n$, and C_i^* is linked with a $q_{k,\alpha}^{(i)}$ like in (4.7) and (4.8). Then, for large u,*

$$\psi(u,t) \sim \left\{ \sum_{k=1}^{\infty} P\left[N(t) = k\right] \sum_{i=1}^{n} \gamma_i q_{k,\alpha}^{(i)} \right\} \overline{F}(u+ct). \tag{4.14}$$

Proof. This is immediate since if $(X_1^{(i)}, ..., X_k^{(i)})$ is a copy of $(X_1, ..., X_k)$ with dependence structure described by C_i, then, for all $k \geq 1$ and large x,

$$\begin{aligned} P\left(X_1 + ... + X_k > x\right) &= \sum_{i=1}^{n} \gamma_i P(X_1^{(i)} + ... + X_k^{(i)} > x) \\ &\sim \left(\sum_{i=1}^{n} \gamma_i q_{k,\alpha}^{(i)} \right) \overline{F}(x). \end{aligned}$$

⋄

Example. Let us consider a sequence of r.v. $(X_i)_{i\geq 1}$ such that for $i \geq 1$,

- $P(X_i = Y_i) = p$,
- $P(X_i = Z_i) = 1 - p$,

where $p \in [0,1]$ and for all $k \geq 1$, $(Z_1, ..., Z_k)$ and $(Y_1, ..., Y_k)$ belongs to $\mathcal{MR}_{-\alpha}$ with common cdf F. The dependence structure of $(Y_1, ..., Y_k)$ is described by a Gaussian copula which satisfies (4.1) and the dependence structure of $(Z_1, ..., Z_k)$ is described by the copula (4.3).

Now, consider the risk process $R(t) = u + ct - \sum_{i=1}^{N(t)} X_i$. By (4.5) and (4.14) we get large u that

$$\psi(u,t) \sim \left\{ p\lambda t + (1-p) \sum_{k=1}^{\infty} P\left[N(t) = k\right] \sum_{A^{(k)} \in \mathfrak{P}([1,k])} \lambda_{A^{(k)}} \left(\sum_{a \in A^{(k)}} Card(a)^{\alpha} \right) \right\} \overline{F}(u+ct).$$

In particular we deduce that

$$\alpha > 1 \text{ resp. } \alpha < 1 \;\Rightarrow \psi(u,t) \text{increases (resp. decreases) with } p,$$

and if $\alpha = 1$, $\psi(u,t) = \lambda t \overline{F}(u+ct)$. Indeed,

$$\frac{\partial \psi}{\partial p} = \left\{ \lambda t - \sum_{k=1}^{\infty} P\left[N(t) = k\right] \sum_{A^{(k)} \in \mathfrak{P}([1,k])} \lambda_{A^{(k)}} \left(\sum_{a \in A^{(k)}} Card(a)^{\alpha} \right) \right\} \overline{F}(u+ct),$$

and we see that for all $k \geq 1$ and $A \in \mathfrak{P}([1,k])$, $\sum_{a \in A^{(k)}} Card(a)^{\alpha} \left\{ \begin{array}{c} > \\ < \\ = \end{array} \right. k$ if $\left\{ \begin{array}{l} \alpha > 1 \\ \alpha < 1 \\ \alpha = 1 \end{array} \right.$, and $\sum_{k=1}^{\infty} P\left[N(t) = k\right] \sum_{A^{(k)} \in \mathfrak{P}([1,k])} \lambda_{A^{(k)}} k = \lambda t$.

5 Dependence through an environment process

In this Section, we aim at taking into account the fact that one or several correlation crises may occur: claim amounts, in general independent or weakly dependent, may suddenly become comonotone or strongly positively dependent. The claim size distribution may become more dangerous as well. To this end, the dependence between claim amounts and the claim size distribution and intensity are modulated by a Markovian environment process. More precisely,

- there exists a Markovian environment process $(J(t))_{t\geq 0}$ with states $i = 1, \dots, J \geq 2$
  - with initial distribution π_0,
  - and transition rate matrix Q.
- for $i = 1, ..., J$ the claim amounts $(X_n^i)_{n\geq 1}$ are J independent sequences defined as in Proposition 3.1, i.e.

$$X_n^i = I_n^i W_0^i + (1 - I_n^i) W_n^i \ , \ \ n \geq 1,$$

  - where the $(W_n^i)_{n\geq 0}$ are i.i.d. r.v.'s with cdf $F_W^i \in \mathcal{R}_{-\alpha^i}$,
  - the $(I_n^i)_{n\geq 1}$ are i.i.d. Bernoulli r.v.'s with parameter $p^i \in [0, 1]$,
  - the $(W_n^i)_{n\geq 0}$, are independent from the $(I_k^i)_{k\geq 1}$,
  - and the $(W_n^i)_{n\geq 0}$ and $(I_k^i)_{k\geq 1}$ are independent from a Poisson process $N^i(t)$ with parameter λ^i.

Let us define the J independent processes

$$Y^i(t) = c^i t - \sum_{m^i=1}^{N^i(t)} X_{m^i}^i \ , \ \ i = 1, ..., J.$$

Let T_p be the instant of the p^{th} jump of the process $(J(t))_{t\geq 0}$, and define $(R(t))_{t\geq 0}$ by

$$\begin{aligned} R(t) \ \ &= \ \ u + \sum_{p\geq 1} \sum_{1\leq i\leq n} \left[Y^i(T_p) - Y^i(T_{p-1}) \right] 1_{\{J_{T_{p-1}}=i, T_p \leq t\}} \\ &\quad + \sum_{p\geq 1} \sum_{1\leq i\leq n} \left[Y^i(t) - Y^i(T_{p-1}) \right] 1_{\{J_{T_{p-1}}=i, T_{p-1} \leq t < T_p\}}. \end{aligned}$$

Thus, we have built a modulated risk process. For an illustration see figure 2.

We now discuss our model with two situations of special interest. In the first one, the crisis causes the claim amounts to be more dangerous. In the second one, one pure correlation crisis is considered : the dependence between claim amounts increases, but the claim size distribution remains unchanged.

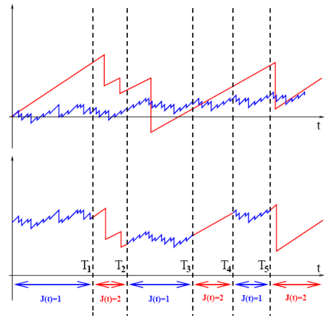

Figure 2: A typical modulated risk process with two states (red and blue).

5.1 Correlation and severity crisis: case where one state dominates

In this Subsection, we suppose that the state 1 is more hazardous then the others states, that is to, for all $i \geq 2$, we have $\alpha^1 < \alpha^i$.

Proposition 5.1 *As $u \to +\infty$, we have for any $t > 0$*

$$\psi(u,t) \sim \left(\sum_{i=1}^{J} \pi_0(i) E\left(M_i^{\perp} + \left[\left(\left[M_i^{com}\right]^{\alpha^1}\right)\right]\right)\right) \bar{F}^1(u), \tag{5.1}$$

- *where $W_i^1(t)$ is the time spent by the environment process in state 1 during $[0,t]$ given that $J(0) = i$,*
- *$M_i^{\perp}$ follows a mixed Poisson distribution with random parameter $\lambda^1(1-p^1)W_i^1(t)$,*
- *and M_i^{com} follows a mixed Poisson distribution with random parameter $\lambda^1 p^1 W_i^1(t)$.*

Proof. First, rewrite $R(t)$ as follow,

$$R(t) = u + C(t) - S(t),$$

where

$$C(t) = \sum_{p\geq 1} \sum_{1\leq i\leq J} \left(c^i(T_p - T_{p-1})\right) \mathbb{1}_{\{J_{T_{p-1}}=i, T_p \leq t\}} + \sum_{p\geq 1} \sum_{1\leq i\leq J} \left(c^i(t - T_{p-1})\right) \mathbb{1}_{\{J_{T_{p-1}}=i, T_{p-1}\leq t\leq T_p\}},$$

and where

$$S(t) = \sum_{p\geq 1} \sum_{1\leq i\leq J} \left(\sum_{m^i=1}^{N^i(T_p)} X^i_{m^i} - \sum_{m^i=1}^{N^i(T_{p-1})} X^i_{m^i} \right) \mathbb{1}_{\{J_{T_{p-1}}=i, T_p\leq t\}} + \sum_{p\geq 1} \sum_{1\leq i\leq J} \left(\sum_{m^i=1}^{N^i(t)} X^i_{m^i} - \sum_{m^i=1}^{N^i(T_{p-1})} X^i_{m^i} \right) \mathbb{1}_{\{J_{T_{p-1}}=i, T_{p-1}\leq t\leq T_p\}}.$$

Then, notice that, for all $t > 0$, $S(t)$ has the same distribution as

$$\tilde{S}(t) = \sum_{j=1}^{J} \sum_{m^j=1}^{N^j(W^j(t))} X^j_{m^j},$$

where for $j = 1, ..., J$ and $t > 0$, $W^j(t)$ is the time spent by the environment process in state j during $[0, t]$. Thus,

$$
\begin{aligned}
P(R(t) < 0) &= \sum_{j=1}^{J} \pi_0(i) P(S(t) > u + C(t) \,|\, J(0) = i) \\
&= \sum_{j=1}^{J} \pi_0(i) P(\tilde{S}(t) > u + C(t) \,|\, J(0) = i).
\end{aligned}
$$

Let us define for $i, j = 1, ..., J$ and $t > 0$, $W_i^j(t)$ as the time spent by the environment process in state j during $[0, t]$ knowing $J(0) = i$. We have, for all $j = 1, ..., J$, $t > 0$ and large u,

$$
\begin{aligned}
P\left(\sum_{m^j=1}^{N^j(W^j(t))} X^j_{m^j} > u \,|\, J(0) = i \right) &= P\left(\sum_{m^j=1}^{N^j(W_i^j(t))} X^j_{m^j} > u \right) \\
&= P\left(\sum_{m_j=1}^{N_j(W_i^j(t))} (I^j_{m^j} W_0^j + (1 - I^j_{m^j}) W^j_{m_j}) > u \right) \\
&= P\left(\sum_{m_j=1}^{N_j^{com}(W_i^j(t))} W_0^j + \sum_{m_j=1}^{N_j^{\perp}(W_i^j(t))} W^j_{m_j} > u \right),
\end{aligned}
$$

using the thinning property of the Poisson process and where $N_j^{com}(W_i^j(t))$ is a Poisson process with parameter $\lambda^j p^j W_i^j(t)$ and $N_j^{\perp}(W_i^j(t))$ is a Poisson process with parameter $\lambda^j p^j W_i^j(t)$. For $i, j = 1, ...J$, $t > 0$ and large u, we have

$$
\begin{aligned}
P\left(\sum_{m_j=1}^{N_j^{com}(W_i^j(t))} W_0^j > u \right) &= P\left(N_j^{com}(W_i^j(t)) W_0^j > u \right) \\
&= \sum_{n=0}^{\infty} P\left(N_j^{com}(W_i^j(t)) = n \right) P\left(W_0^j > \frac{u}{n} \right) \\
&\sim \sum_{n=0}^{\infty} P\left(N_j^{com}(W_i^j(t)) = n \right) n^{\alpha^j} \overline{F}(u) \\
&\sim E(N_j^{com}(W_i^j(t)))^{\alpha^j} \overline{F^j}(u) \in \mathcal{R}_{-\alpha^j},
\end{aligned}
$$

and

$$P\left(\sum_{m_j=1}^{N_j^{\perp}(W_i^j(t))} W_{m_j}^j > u\right) = \sum_{n=0}^{\infty} P\left(N_j^{\perp}(W_i^j(t)) = n\right) P\left(\sum_{m_j=1}^{n} W_{m_j}^j > u\right) \tag{5.2}$$

$$\sim \sum_{n=0}^{\infty} P\left(N_j^{\perp}(W_i^j(t)) = n\right) n\overline{F}(u)$$

$$\sim E(N_j^{\perp}(W_i^j(t)))\overline{F^j}(u) \in \mathcal{R}_{\alpha^j}. \tag{5.3}$$

Since for all $i, j = 1, ...J$ $\sum_{m_j=1}^{N_j^{com}(W_i^j(t))} W_0^j$ and $\sum_{m_j=1}^{N_j^{\perp}(W_i^j(t))} W_{m_j}^j$ are independent and belong to $\mathcal{R}_{-\alpha^j}$, we have for $t > 0$ and large u,

$$P\left(\sum_{m^j=1}^{N^j(W^j(t))} X_{m^j}^j > u\,|J(0) = i\right) \sim \left(E(N_j^{\perp}(W_i^j(t))) + E(N_j^{com}(W_i^j(t)))^{\alpha^j}\right)\overline{F^j(u)} \in \mathcal{R}_{-\alpha^j}$$

Since for all $j = 2, ..., J$, $\alpha^1 < \alpha^j$, and for all $i, j = 1, ..., J$, $i \neq j$ $(X_{m_i}^i)_{m_i \geq 1}$ and $(X_{m_j}^j)_{m_j \geq 1}$ we have for $t > 0$ and large u,

$$\begin{aligned}
P\left(S(t) > u\right) &= \sum_{j=1}^{J} \pi_0(i) P(\tilde{S}(t) > u\,|J(0) = i) \\
&= \sum_{j=1}^{J} \pi_0(i) P(\sum_{j=1}^{J} \sum_{m^j=1}^{N^j(W^j(t))} X_{m^j}^j > u\,|J(0) = i) \\
&\sim \sum_{j=1}^{J} \pi_0(i) P(\sum_{m^1=1}^{N^1(W^1(t))} X_{m^1}^1 > u\,|J(0) = i) \\
&\sim \sum_{j=1}^{J} \pi_0(i) \left(E(N_1^{\perp}(W_i^1(t))) + E(N_1^{com}(W_i^1(t)))^{\alpha^1}\right)\overline{F^1(u)} \\
&\sim \left(\sum_{i=1}^{J} \pi_0(i) E\left(M_i^{\perp} + \left[\left(\left[M_i^{com}\right]^{\alpha^1}\right)\right]\right)\right)\bar{F}^1(u).
\end{aligned}$$

and since for all $s \in \mathbb{R}$, we have for large u $\overline{F^1}(u+s) \sim \overline{F^1(u)}$,

$$P\left(S(t) > u + \sup_{j=1,\ldots,J} c^j t\right) \sim \left(\sum_{i=1}^{J} \pi_0(i) E\left(M_i^{\perp} + \left[\left(\left[M_i^{com}\right]^{\alpha^1}\right)\right]\right)\right)\bar{F}^1(u).$$

We conclude with the inequality :

$$P\left(S(t) > u + \sup_{j=1,\ldots,J} c^j t\right) \leq P(R(t) < 0) \leq P\left(S(t) > u\right).$$

$\diamond$

The next Theorem gives a way to compute the moments of $N^{com}(t)$ when $\alpha^1 \in \mathbb{N}$ using a result in Castella et al. (2007)

Proposition 5.2 *If besides $\alpha^1 \in \mathbb{N}$, we have*

$$E\left(M_i^{\perp}\right) = \lambda^1(1-p^1)E\left[W_i^1(t)\right] = \lambda^1(1-p^1)D_i^1(1,t),$$

and

$$\begin{aligned} E\left(\left[M_i^{com}\right]^{\alpha^1}\right) &= E\left[\sum_{k=0}^{\alpha^1} S(\alpha^1,k)(\lambda^1 p^1)^k \left(W_i^1(t)\right)^k\right] \\ &= \sum_{k=0}^{\alpha^1} S(\alpha^1,k)(\lambda^1 p^1)^k D_i^1(k,t), \end{aligned}$$

where $S(\alpha^1,k)$ is the (α^1,k) Stirling number of the second kind, and where

$$\text{for } m \geq 1, \quad D_i^1(m,t) = E\left[\left(W_i^1(t)\right)^m \mid J(0) = i\right]$$

is the i^{th} component of vector $D^1(m,t)$ defined by $D^1(0,t) = 1$ and for $m \geq 1$,

$$D^1(m,t) = r\int_0^t e^{Q(t-u)}A_{11}D^1(m-1,u)du \text{ where } A_{11} \text{ is } J \times J \text{ with coeff. } \delta_{i1}\delta_{j1}.$$

Proof. Castella et al. (2007) gives the way to compute $D_i^1(m,t)$ for $m \geq 1$ and $i = 1, ..., J$. Moreover we know that the moments of a Poisson distribution X with parameter λ are given by, for $m \geq 1$,

$$EX^m = \sum_{k=1}^{m} S(m,k)\lambda^k.$$

$\diamond$

5.2 Pure correlation crisis: a typical case

In this Subsection, let us assume that there exist three states such that

- in state 1, the X_n^1, $n \geq 1$ are i.i.d.,
- in state 2, there is a light correlation: the X_n^2, $n \geq 1$ have Gaussian copulas,
- and in state 3, the X_n^3, $n \geq 1$ are given by the basic dependence model with parameter p_3.

Moreover claim amounts are identically distributed with common c.d.f. F such that $\overline{F} \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$ and for all $n \geq 1$ $(X_1^2, ..., X_n^2)$ belongs to $\mathcal{MR}_{-\alpha}$.

Proposition 5.3 *We have for any $t > 0$, as $u \to +\infty$,*

$$\psi(u,t) \sim \left(\sum_{i=1}^{3} \pi_0(i)[\lambda^1 D_i^1(1,t) + \lambda^2 D_i^2(1,t) + \lambda^3 \left(1 - p^3\right) D_i^3(1,t) + \sum_{k=0}^{\alpha} S(\alpha,k)(\lambda^3 p^3)^k D_i^3(k,t)] \right) \bar{F}(u).$$

Proof. Since $(X_n^1)_{n\geq 1}$, $(X_n^2)_{n\geq 1}$ and $(X_n^3)_{n\geq 1}$ are independent, we have for $t > 0$ and large u,

$$\begin{aligned} P(S(t) > u) &= P\left(\sum_{j=1}^{3} \sum_{m^j=1}^{N^j(W^j(t))} X_{m^j}^j > u \right) \qquad \text{see the proof of Proposition 5.1,} \\ &\sim \sum_{m^j=1}^{3} P\left(\sum_{m^j=1}^{N^j(W^j(t))} X_{m^j}^j > u \right). \end{aligned}$$

We have for $t > 0$ and large u, we have

$$P\left(\sum_{m^1=1}^{N^1(W^1(t))} X_{m^1}^1 > u \right) \sim \lambda^1 E(W^1(t)) \overline{F}(u),$$

from Proposition 4.14, we have

$$P\left(\sum_{m^2=1}^{N^2(W^2(t))} X_{m^2}^2 > u \right) \sim \lambda^2 E(W^2(t)) \overline{F}(u),$$

and from (5.3), we have

$$P\left(\sum_{m^3=1}^{N^3(W^3(t))} X_{m^3}^3 > u \right) \sim \lambda^3 \left(E(W^3(t)) + E\left[(N^3(W^3(t)))^\alpha \right] \right) \overline{F}(u),$$

We conclude with Proposition 5.2. $\diamond$

Similar results may be obtained with classical copulas, and dependence between different state processes.

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