

TRANSITIVITY OF CODIMENSION ONE ANOSOV ACTIONS OF \mathbb{R}^k ON CLOSED MANIFOLDS

THIERRY BARBOT AND CARLOS MAQUERA

ABSTRACT. In this paper, we consider Anosov actions of \mathbb{R}^k , $k \geq 2$, on a closed connected orientable manifold M , of codimension one, i.e. such that the unstable foliation associated to some element of \mathbb{R}^k has dimension one. We prove that if the ambient manifold has dimension greater than $k + 2$, then the action is topologically transitive. This generalizes a result of Verjovsky for codimension one Anosov flows.

1. INTRODUCTION

It is nowadays a common sense that the Anosov systems lie in the central heart of the theory of dynamical systems, as the most perfect kind of global hyperbolic behavior. It has strong connections with algebra, natural examples arising from number field theory or Lie groups theory (see also [14] for an example illustrating the deep interplay between Anosov systems and representation theory), and also with topology, the dynamics of an Anosov system usually reflecting the ambient manifold topology.

The notion has been introduced by V.V. Anosov in the 60's in [1], but one should also consider previous works by precursors, including Hadamard, Morse, etc...

An Anosov system is (*topologically*) *transitive* if it admits a dense orbit. There is a quite extensive literature devoted to transitivity for certain classes of Anosov systems. In particular, by a celebrated result of Newhouse [18] and Franks [8], every codimension one Anosov diffeomorphism on a compact manifold is topologically mixing (more than transitive). As a corollary from this theorem and [8, Corollary (6.4)], up to finite coverings, Anosov diffeomorphisms of codimension one (i.e. such that the unstable subbundle has dimension one) on closed manifolds of dimension ≥ 3 are topologically conjugate to hyperbolic toral automorphisms. For flows, in the three-dimensional case, Franks–Williams [9] construct an Anosov flow that is not topologically transitive. In the higher dimensional

Date: July 29, 2008.

2000 Mathematics Subject Classification. Primary: 37C85.

Key words and phrases. Anosov action, compact orbit, closing lemma, robust transitivity, Anosov flow. The second author would like to thank CNPq of Brazil for financial support Grant 200464/2007-8.

case, Verjovsky [35] proved that codimension one Anosov flows on manifolds of dimension greater than three are transitive.

A natural question arises: “what about transitivity for actions of higher dimensional groups (particularly \mathbb{R}^k , $k \geq 2$)?”

The development which concerns us here deals with the case of Anosov actions of the group \mathbb{R}^k (some element $r \in \mathbb{R}^k$ acts normally hyperbolically with respect to the orbit foliation). This concept was originally introduced by Pugh–Shub [24] in the early seventies, and more recently received a strong impetus under the contribution of A. Katok and R.J. Spatzier. The rigidity aspects of these actions receives nowadays a lot of attention, in the framework of Zimmer program.

In this paper we undertake the study of transitivity of codimension one Anosov actions of \mathbb{R}^k , $k > 1$. An action \mathbb{R}^k is called *topologically transitive* if it admits a dense orbit. Our main result is the following theorem.

Theorem 1. *Every codimension one Anosov action of \mathbb{R}^k on a closed manifold of dimension greater than $k + 2$ is topologically transitive.*

The denomination, coming from the usual one for Anosov diffeomorphisms or flows, may be confusing: here, codimension one does not mean that the orbits of \mathbb{R}^k have codimension one, but that the unstable foliation of some element of \mathbb{R}^k has dimension one. See § 2.2.

Note that if a closed n -manifold support a codimension one Anosov action of \mathbb{R}^k and $m < k + 3$, then $m = k + 2$. In this case, the Theorem does not hold: take the product (cf. Example 4) of the by Franks–Williams example ([9]) by a flat torus is a non transitive codimension one Anosov action of \mathbb{R}^k on a $(k + 2)$ -manifold of the form $N^3 \times \mathbb{T}^{k-1}$, where N^3 is closed three manifold.

Actually, we will prove slightly more. The theorem above states that under the hypothesis there is a dense \mathbb{R}^k -orbit, but we can wonder if there is a one parameter subgroup of \mathbb{R}^k whose orbit on M is dense. Actually, this stronger statement does not hold in general: just consider once more as above the product of an Anosov flow, transitive or not, by a flat torus. However, it is nearly true, in a weak sense, as explained just below.

An element of \mathbb{R}^k is said Anosov if it acts normally hyperbolically with respect to the orbit foliation. Every connected component of the set of Anosov elements is an open convex cone in \mathbb{R}^k , called a chamber. More generally, a *regular subcone* \mathcal{C} is an open convex cone in \mathbb{R}^k containing only Anosov elements. One should consider \mathcal{C} as a semi-group in \mathbb{R}^k : the sum of two elements in the cone still lies in the cone. The \mathcal{C} -orbit of a point x in M is the subset comprising the iterates $\phi^a(x)$ for a describing \mathcal{C} .

Theorem 2. *Let ϕ be a codimension one Anosov action of \mathbb{R}^k on a closed manifold M of dimension greater than $k + 2$. Then any regular subcone \mathcal{C} admits a dense orbit in M .*

Theorem 1 is obviously a direct corollary of Theorem 2. On the other hand, given a in \mathbb{R}^k , we can apply Theorem 2 to every small regular cone containing a : hence we can loosely have in mind that, up to arbitrarily small errors, ϕ^a admits a dense orbit.

The cornerstone of the proof is the study of the codimension foliation \mathcal{F}^s tangent to the stable subbundle of Anosov elements in \mathcal{C} . The unstable foliation \mathcal{F}^{uu} for \mathcal{C} has dimension one, and the first step is to prove that every leaf of \mathcal{F}^{uu} admits an affine structure, preserved by the action of \mathbb{R}^k (cf. Theorem 6). The existence of this affine structure provides a very good information about the transverse holonomy of \mathcal{F}^s , giving *in fine*, through the classical theory of codimension one foliations, many information about the topology of the various foliations involved. In particular, the orbit space of the lifting of ϕ to the universal covering \widetilde{M} is a Hausdorff manifold, homeomorphic to \mathbb{R}^{n-k} (cf. Theorem 8).

On the other hand, one can produce a generalization for \mathbb{R}^k -actions of the classical Spectral decomposition Theorem (Theorem 5). It allows to reduce the proof of the transitivity to the proof that stable leaves are dense (Lemma 7). Now, if some leaf of \mathcal{F}^s is not dense, then there must be some *non bi-homoclinic orbit* of \mathbb{R}^k (cf. Proposition 4). One then get the final result by using some clever argument, involving Jordan-Schönflies Theorem, and already used in Verjovsky proof as rewritten in [3] or [15].

Actually, all the strategy above mostly follows the guideline used in Verjovsky proof, but is more than a simple transposition. New phenomena arise, even enlightening the case of Anosov flows.

- *Convex cones:* The case $k = 1$ is somewhat greatly simplified by the fact that regular subcones in \mathbb{R}^k are simply half-lines, and that the only non Anosov element of \mathbb{R}^k is the origin 0. One can compare the classical Closing Lemma with the general version, more technical in its statement: Theorem 4.

- *Reducibility:* Given an Anosov \mathbb{R}^k action on some manifold M , one can always take the product $M \times \mathbb{T}^l$ by some torus $\mathbb{T}^l = \mathbb{R}^l / \mathbb{Z}^l$ and consider the locally free action of \mathbb{R}^{k+l} on this product manifold. Then this action is still Anosov. This construction can be generalized to twisted products through a representation $\rho : \pi_1(M) \rightarrow \mathbb{R}^l$ (cf § 5). Of course, this construction gives examples with $k > 1$, hence doesn't appear in the case of Anosov flows. Therefore, an important step is to put aside these examples. In Theorem 7, we prove that every codimension one action of \mathbb{R}^k splits uniquely as a principal torus bundle over some manifold \bar{M} such that the \mathbb{R}^k actions permutes the fibers, and thus induces an action on \bar{M} . Moreover, the fibers are precisely the orbits of some subgroup $H_0 \subset \mathbb{R}^k$, and the induced

action is Anosov. Finally, this splitting is maximal, i.e. \bar{M} cannot be decomposed further: it is *irreducible*. Many properties, among them transitivity, is obviously satisfied by the Anosov action on M if and only if it is satisfied by the induced action on \bar{M} . Therefore, the proof of Theorem 2 reduces to the irreducible case.

The irreducibility of an Anosov \mathbb{R}^k action can be equivalently defined as requiring that the codimension one stable foliation \mathcal{F}^s has trivial holonomy cover; in a less pedantic way, it means that homotopically non-trivial loops in leaves have non-trivial holonomy (cf. Remark 5). Irreducible Anosov actions enjoy many nice topological properties. Among them (cf. Proposition 3):

Let ϕ be an irreducible codimension one action of \mathbb{R}^k on a manifold M . Then the isotropy subgroup of every element of M is either trivial, or a lattice in \mathbb{R}^k .

We can observe, as a corollary, that if a codimension one Anosov action of \mathbb{R}^k admits an orbit homeomorphic to $\mathbb{T}^{k-1} \times \mathbb{R}$, then it is a twisted product by flat tori \mathbb{T}^{k-1} over an Anosov flow.

The paper is organized as follows: in the preliminary section 2, we give definitions, and present first results, as the generalized Closing Lemma for actions of \mathbb{R}^k , and the spectral decomposition of the non-wandering set as a finite union of basic blocks. In § 3 we present the known examples of Anosov actions of codimension one. In § 4 we establish the reduction Theorem 7 (it includes the proof of the \mathbb{R}^k -invariant affine structures along unstable leaves). In § 5 we prove the Main Theorem 2. In the last section 6 we give additional comments, and present forecoming works in progress.

Acknowledgments. This paper was written while the second author stayed at Unité de Mathématiques Pures et Appliquées, École Normale Supérieure de Lyon. He thanks the members of UMPA, especially Professor Etienne Ghys for his hospitality.

2. PRELIMINARIES

2.1. Definitions and notations. Now, we outline some basic results about actions of \mathbb{R}^k which will be used in the proof of the main theorem. Recall that, for any action $\phi : \mathbb{R}^k \times M \rightarrow M$ of \mathbb{R}^k on a manifold M , $\mathcal{O}_p := \{\phi(\omega, p), \omega \in \mathbb{R}^k\}$ is the orbit of $p \in M$ and $\Gamma_p := \{\omega \in \mathbb{R}^k : \phi(\omega, p) = p\}$ is called the isotropy group of p . The action ϕ is said to be *locally free* if the isotropy group of every point is discrete. In this case the orbits are diffeomorphic to $\mathbb{R}^\ell \times \mathbb{T}^{k-\ell}$, where $0 \leq \ell \leq k$.

Let \mathcal{F} be a continuous foliation on a manifold M . We denote the leaf that contains $p \in M$ by $\mathcal{F}(p)$. For an open subset U of M , let $\mathcal{F}|_U$ be the foliation on U such that $(\mathcal{F}|_U)(p)$ is the connected component of $\mathcal{F}(p) \cap U$ containing $p \in M$. A coordinate $\varphi = (x_1, \dots, x_n)$

on U is called a *foliation coordinate* of \mathcal{F} if x_{m+1}, \dots, x_n are constant functions on each leaf of $\mathcal{F}|_U$, where m is the dimension of \mathcal{F} . A foliation is of class C^{r+} if it is covered by C^{r+} foliation coordinates. We denote the tangent bundle of M by TM . If \mathcal{F} is a C^1 foliation, then we denote the tangent bundle of \mathcal{F} by $T\mathcal{F}$.

We fix a Riemannian metric $\|\cdot\|$, and denote by d the associated distance map on M .

2.2. Anosov \mathbb{R}^k -actions. Let us recall the definitions and basic properties of Anosov actions.

Definition 1. Let M be a C^∞ manifold and ϕ a locally free C^{1+} action of \mathbb{R}^k on M . By $T\phi$, we denote the k -dimensional subbundle of TM that is tangent to the orbits of ϕ .

- (1) We say that $a \in \mathbb{R}^k$ is an *Anosov element* for ϕ if $g = \phi(a, \cdot)$ acts normally hyperbolically with respect to the orbit foliation. That is, there exist real numbers $\lambda > 0$, $C > 0$ and a continuous Dg -invariant splitting of the tangent bundle

$$TM = E_a^{ss} \oplus T\phi \oplus E_a^{uu}$$

such that

$$\begin{aligned} \|Dg^n|_{E_a^{ss}}\| &\leq Ce^{-\lambda n} \quad \forall n > 0 \\ \|Dg^n|_{E_a^{uu}}\| &\leq Ce^{\lambda n} \quad \forall n < 0 \end{aligned}$$

- (2) Call ϕ an *Anosov action* if some $a \in \mathbb{R}^k$ is an Anosov element for ϕ .

Hirsch, Pugh and Shub developed the basic theory of normally hyperbolic transformations in [11]. As consequence of this we obtain that the splitting is Hölder continuous and the subbundles E_a^{ss} , E_a^{uu} , $T\phi \oplus E_a^{ss}$, $T\phi \oplus E_a^{uu}$ are integrable. The corresponding foliations, \mathcal{F}_a^{ss} , \mathcal{F}_a^{uu} , \mathcal{F}_a^s , \mathcal{F}_a^u , are called *the strong stable foliation*, *the strong unstable foliation*, *the weak stable foliation*, and *the weak unstable foliation*, respectively.

From now ϕ is an Anosov action of \mathbb{R}^k on M , and a an Anosov element fixed once for all. For simplicity, the foliations corresponding to a will be denoted by \mathcal{F}^{ss} , \mathcal{F}^{uu} , \mathcal{F}^s and \mathcal{F}^u . For all $\delta > 0$, $\mathcal{F}_\delta^i(x)$ denote the open ball in $\mathcal{F}^i(x)$ under the induced metric which centering at x with radius δ , where $i = ss, uu, s, u$.

Theorem 3 (of product neighborhoods). *Let $\phi : \mathbb{R}^k \times M \rightarrow M$ be an Anosov action. There exists a $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ and for all $x \in M$, the applications*

$$[\cdot, \cdot]^u : \mathcal{F}^s(x) \times \mathcal{F}^{uu}(x) \rightarrow M; \quad [y, z]^u = \mathcal{F}_{2\delta}^s(z) \cap \mathcal{F}_{2\delta}^{uu}(y)$$

$$[\cdot, \cdot]^s : \mathcal{F}^{ss}(x) \times \mathcal{F}^u(x) \rightarrow M; \quad [y, z]^s = \mathcal{F}_{2\delta}^{ss}(z) \cap \mathcal{F}_{2\delta}^u(y)$$

are homeomorphisms on their images.

Remark 1. Every foliation \mathcal{F}_a^{ss} , \mathcal{F}_a^{uu} , \mathcal{F}_a^s or \mathcal{F}_a^u is preserved by every diffeomorphism commuting with a . In particular, it is \mathbb{R}^k -invariant. Another standard observation is that, since every compact domain in a leaf of \mathcal{F}_a^{ss} (respectively of \mathcal{F}_a^{uu}) shrinks to a point under positive (respectively negative) iteration by ϕ^a , every leaf of \mathcal{F}_a^{ss} or \mathcal{F}_a^{uu} is a *plane*, i.e. diffeomorphic to \mathbb{R}^ℓ for some ℓ .

Let F be a weak leaf, let say a weak stable leaf. For every strong stable leaf L in F , let Γ_L be the subgroup of \mathbb{R}^k comprising elements a such that $\phi^a(L) = L$, and let \mathcal{O}_L be the saturation of L under ϕ . Thanks to Theorem 3 we have:

- \mathcal{O}_L is open in F ,
- Γ_L is discrete.

Since F is connected, the first item implies $F = \mathcal{O}_L$: the ϕ -saturation of a strong leaf is an entire weak leaf. Therefore, Γ_L does not depend on L , only on F . The second item implies that the quotient $P = \Gamma_L \backslash \mathbb{R}^k$ is a manifold, more precisely, a flat cylinder, diffeomorphic to $\mathbb{R}^p \times \mathbb{T}^q$ for some p, q . For every x in F , define $p_F(x)$ as the equivalence class $a + \Gamma_L$ such that x belongs to $\phi^a(L)$. The map $p_F : F \rightarrow P$ is a locally trivial fibration and the restriction of p_F to any ϕ -orbit in F is a covering map. Since the fibers are contractible (they are leaves of \mathcal{F}^{ss} , hence planes), the fundamental group of F is the fundamental group of P , i.e. Γ_L for any strong stable leaf L inside F .

Observe that if \mathcal{F}^{ss} is oriented, then the fibration p_F is trivial: in particular, F is diffeomorphic to $P \times \mathbb{R}^p$, where p is the dimension of \mathcal{F}^{ss} .

Of course, analogous statements hold for the strong and weak unstable leaves.

We say ϕ is a *codimension-one* Anosov action if E_a^{uu} is one-dimensional for some a in \mathbb{R}^k . In this case, we will always assume that the fixed Anosov element has one dimensional strong stable foliation.

Remark 2. Let $\mathcal{A} = \mathcal{A}(\phi)$ be the set of Anosov elements of ϕ .

- (1) \mathcal{A} is always an open subset of \mathbb{R}^k . In fact, by the structural stability theorem for normally hyperbolic transformations by Hirsch, Pugh and Shub a map C^1 -close to a normally hyperbolic transformation is again normally hyperbolic for a suitable foliation [11]. For an element in \mathbb{R}^k close to an Anosov element, this suitable foliation is forced to be the orbit foliation of the action.
- (2) Every connected component of \mathcal{A} is an open convex cone in \mathbb{R}^k . Let a be an Anosov element. Every element near a must share the same stable and unstable bundles, therefore, all the Anosov elements in the same connected component than a admits the same stable/unstable splitting. The contracting or expanding property along

a given bundle is stable by composition and by multiplication of the generating vector field by a positive constant factor; it follows that the connected component is a convex cone, as claimed.

We call such a connected component a *chamber*, by analogy with the case of Cartan actions. More generally, a *regular subcone* is an open convex cone contained in a chamber.

- (3) If \mathcal{A}_a is the chamber containing a , then $\mathcal{F}_a^{ss} = \mathcal{F}_b^{ss}$, $\mathcal{F}_a^{uu} = \mathcal{F}_b^{uu}$, $\mathcal{F}_a^s = \mathcal{F}_b^s$ and $\mathcal{F}_a^u = \mathcal{F}_b^u$ for all $b \in \mathcal{A}_a$.
- (4) Any ϕ -orbit whose isotropy subgroup contains an Anosov element v is compact. Indeed, let y be a point in the closure of the orbit. It is clearly fixed by ϕ^v , and there is a local cross-section Σ to ϕ containing x such that for every z in Σ near y the image $\phi^v(z)$ lies in Σ . Then, y is a fixed point of ϕ^v of saddle type, in particular, it is an isolated ϕ^v -fixed point. Hence y lies in the ϕ -orbit of x .

Another standard fact about Anosov \mathbb{R}^k -actions is an Anosov-type closing lemma which is a straightforward generalization of a similar statement for Anosov flows (cf. [12, Theorem 2.4]).

Theorem 4 (Closing Lemma). *Let $a \in \mathbb{R}^k$ be an Anosov element of an Anosov \mathbb{R}^k -action ϕ on a closed manifold M . There exist positive constants ε_0 , C and λ depending continuously on ϕ in the C^1 -topology and a such that: if for some $x \in M$ and $t \in \mathbb{R}$*

$$d(\phi(ta, x), x) < \varepsilon_0,$$

then there exists a point $y \in M$, a differentiable map $\gamma : [0, t] \rightarrow \mathbb{R}^k$ such that for all $s \in [0, t]$ we have

- (1) $d(\phi(sa, x), \phi(\gamma(s), y)) < Ce^{-\lambda(\min\{s, t-s\})}d(\phi(ta, x), x)$;
- (2) $\phi(\gamma(t), y) = \phi(\delta, y)$ where $\|\delta\| < Cd(\phi(ta, x), x)$;
- (3) $\|\gamma' - a\| < Cd(\phi(ta, x), x)$.

Remark 3. Let \mathcal{C} be a regular subcone containing a (for example, a chamber). Once a is fixed, item (3) in the Theorem above implies that if $d(\phi(ta, x), x)$ is sufficiently small, the velocity γ' lies in \mathcal{C} , therefore, that the image of γ is contained in \mathcal{C} . Moreover, once more if $d(\phi(ta, x), x)$ is sufficiently small, item (2) implies that $\gamma(t) - \delta$ belongs to \mathcal{C} . According to Remark 2 the orbit of y is compact.

Definition 2 (The nonwandering set). A point $x \in M$ is *nonwandering* with respect to a regular subcone \mathcal{C} if for any open set U containing x there is a $v \in \mathcal{C}$, $\|v\| > 1$, such that

$\phi^v(U) \cap U \neq \emptyset$, where $\phi^v = \phi(v, \cdot)$. The set of all nonwandering points, with respect to \mathcal{C} , is denoted by $\Omega(\mathcal{C})$.

By using the Closing Lemma for Anosov \mathbb{R}^k -actions we obtain:

Proposition 1. *For any regular subcone \mathcal{C} , the union of compact orbits of \mathbb{R}^n is dense in $\Omega(\mathcal{C})$.*

Proof. For $x \in \Omega(\mathcal{C})$ and $\varepsilon > 0$ denote by U_ε the $\varepsilon/(2C + 1)$ -neighborhood of x in M , where C is as in the Closing Lemma. Then there exists $v \in \mathcal{C}$ such that $\phi^v(U_\varepsilon) \cap U_\varepsilon \neq \emptyset$. For $y \in \phi^{-v}(U_\varepsilon) \cap U_\varepsilon \neq \emptyset$ we have $d(\phi^v(y), y) < 2\varepsilon/(2C + 1)$ and hence by the Closing Lemma and Remark 3 there is a point z such that $\phi^v(z) = z$, \mathcal{O}_z is compact and $d(y, z) < Cd(\phi^v(y), y)$, consequently $d(z, x) \leq d(y, x) + d(y, z) < \varepsilon$. It prove that $\text{Comp}(\phi)$ is dense in $\Omega(\mathcal{C})$ and finishes the proof. \square

Remark 4. *Let a be any non-trivial element of \mathbb{R}^k . The nonwandering set $\Omega(\phi^{ta})$ of the (semi-)flow generated by a is clearly contained in $\Omega(\phi)$. On the other hand, the nonwandering set of any linear flow on a torus is the entire torus. Hence, compact orbits of \mathbb{R}^k are contained in $\Omega(\phi^{ta})$. Hence, it follows from the proposition above that the nonwandering sets $\Omega(\phi^{ta})$ and $\Omega(\phi)$ coincide. In particular, the nonwandering set $\Omega(\mathcal{C})$ is independent from the regular subcone \mathcal{C} .*

Lemma 1. *The isotropy subgroup of any compact orbit contains an element in \mathcal{C} .*

Proof. Let $R > 0$ such that every Euclidean ball of radius R in \mathbb{R}^k intersects every orbit of the isotropy subgroup Γ . Let B be a closed Euclidean ball of radius r in the open convex cone \mathcal{C} . Then, for $t > R/r$, the ball tB is contained in \mathcal{C} and has radius $> R$, and thus intersects the Γ -orbit of 0. The lemma follows. \square

The Riemannian metric induces an area form on every ϕ -orbit.

Lemma 2. *For every $C > 0$, there is only a finite number of compact ϕ -orbits of area $\leq C$.*

Proof. Assume by contradiction the existence of an infinite sequence of distinct compact orbits \mathcal{O}_n of area $\leq C$. For each of them, let Γ_n be the isotropy group of \mathcal{O}_n : it is an element of $\mathcal{R} = \text{GL}(k, \mathbb{R})/\text{SL}(k, \mathbb{Z})$, the space of lattices in \mathbb{R}^k . Since ϕ is locally free, the length of elements of Γ_n is uniformly bounded from below, independently from n . By the Mahler's criterion ([25]), it ensures that, up to a subsequence, the Γ_n converges to some lattice Γ_∞ . In particular, for every v_∞ in Γ_∞ , there is a sequence of elements v_n of Γ_n converging in \mathbb{R}^k to v_∞ . Furthermore, according to (the proof of) Lemma 1, we can select v_∞ in \mathcal{C} . Up

to a subsequence, we can also pick up a sequence of elements x_n in each \mathcal{O}_n converging to some x_∞ in M . Then, since $\phi^{v_n}(x_n) = x_n$, at the limit we have $\phi^{v_\infty}(x_\infty) = x_\infty$. Since v_∞ is Anosov, the ϕ -orbit \mathcal{O}_∞ of x_∞ is compact. Consider a local section Σ to ϕ containing x_∞ : the first return map on Σ along the orbit of ϕ^{v_∞} is hyperbolic, admitting x_∞ as an isolated fixed point. On the other hand, by pushing slightly along ϕ , we can assume without loss of generality that every x_n belongs to Σ . Since the v_n converges to v_∞ , the ϕ^{v_n} -orbit of x_n approximates the ϕ^{v_∞} -orbit of x_∞ , showing that the x_n are also fixed points of the first return map. It is a contradiction, since they accumulate to the isolated fixed point x_∞ . \square

Theorem 5 (Spectral decomposition). *Let M be a closed smooth manifold and let ϕ be an Anosov action on M . The nonwandering set of ϕ can be partitioned into a finite number of ϕ -invariant closed subsets, called basic blocks:*

$$\Omega = \bigcup_{i=1}^{\ell} \Lambda_i$$

such that for every regular subcone \mathcal{C} , every Λ_i is \mathcal{C} -transitive, i.e. contains a dense \mathcal{C} -orbit.

Proof. Let $\text{Comp}(\phi)$ be the set of compact orbits of ϕ . By Proposition 1 we have $\overline{\text{Comp}(\phi)} = \Omega(\phi)$. We define a relation on $\text{Comp}(\phi)$ by: $x \sim y$ if and only if $\mathcal{F}^u(x) \cap \mathcal{F}^s(y) \neq \emptyset$ and $\mathcal{F}^s(x) \cap \mathcal{F}^u(y) \neq \emptyset$ with both intersections transverse in at least one point. We want to show that this is an equivalence relation and obtain each Λ_i as the closure of an equivalence class.

Note that \sim is trivially reflexive and symmetric. In order to check the transitivity suppose that $x, y, z \in \text{Comp}(\phi)$ and $p \in \mathcal{F}^u(x) \cap \mathcal{F}^s(y)$, $q \in \mathcal{F}^u(y) \cap \mathcal{F}^s(z)$ are transverse intersection points. There exists $v \in \mathcal{A}_a$ such that $\phi^v(x) = x$. Since the images of a ball around p in $\mathcal{F}^u(p) = \mathcal{F}^u(x) = \phi^v(\mathcal{F}^u(x))$ accumulate on $\mathcal{F}^u(y)$, we obtain that $\mathcal{F}^u(x)$ and $\mathcal{F}^s(z)$ have a transverse intersection. Analogously, we obtain that $\mathcal{F}^s(x)$ and $\mathcal{F}^u(z)$ have a transverse intersection.

By Theorem 3 any two sufficiently near points are equivalent, so by compactness we have finitely many equivalence classes whose (pairwise disjoint) closures we denote by $\Lambda_1, \Lambda_2, \dots, \Lambda_\ell$.

It remains to show that every Λ_i is \mathcal{C} -transitive for every regular subcone \mathcal{C} . Notice first that if $p \in \Lambda_i \cap \text{Comp}(\phi)$ and $p \sim q$ with $q \in \text{Comp}(\phi)$, then there is $z \in \mathcal{F}^u(p) \cap \mathcal{F}^s(q)$. Let $v \in \mathcal{A}_a$ such that $\phi^v(p) = p$. As the iterates under ϕ^v of a ball around z in $\mathcal{F}^u(p) = \mathcal{F}^u(z) = \phi^v(\mathcal{F}^u(z))$ accumulate on $\mathcal{F}^u(q)$, and since z belongs to $\mathcal{F}^s(q)$, we obtain that $\mathcal{F}^u(p)$ is dense in $\Lambda_i \cap \text{Comp}(\phi)$, hence in Λ_i .

Now, for the transitivity, we need to show that for any two open sets U and V in Λ_i there exists $v \in \mathcal{C}$ such that $\phi^v(U) \cap V \neq \emptyset$. The density of compact orbits in Λ_i implies the existence of $p \in U$ and $v \in \mathcal{C}$ such that $\phi^v(p) = p$ (cf. Lemma 1). Let $\mathcal{F}_\delta^{uu}(p)$ be a neighborhood of p in $\mathcal{F}^{uu}(p)$ that is contained in U . Since the \mathbb{R}^k -orbit of p is compact, there is a compact domain K in \mathbb{R}^k so that the leaf $\mathcal{F}^u(p)$ is equal to $K \cdot \cup_{j=0}^\infty \phi^{jv}(\mathcal{F}_\delta^{uu}(p))$. Since this leaf is dense in Λ_i , there exists for every $m \in \mathbb{N}$ sufficiently big an element $g_m \in K$ such that $V \cap [g_m \cdot \cup_{j=0}^m \phi^{jv}(\mathcal{F}_\delta^{uu}(p))] \neq \emptyset$, hence, $V \cap [\cup_{j=0}^m \phi^{jv+g_m}(\mathcal{F}_\delta^{uu}(p))] \neq \emptyset$. Thus $V \cap \phi^{mv+g_m}(\mathcal{F}_\delta^{uu}(p)) \neq \emptyset$, consequently, $V \cap \phi^{mv+g_m}(U) \neq \emptyset$. The Theorem follows, since for m sufficiently large, $mv + g_m = m(v + g_m/m)$ lies in \mathcal{C} . \square

3. EXAMPLES

Let us give some examples of Anosov actions of \mathbb{R}^k . We will especially focus on codimension one examples.

Example 1. Let G be a real semi-simple Lie group, with Lie algebra \mathcal{G} , Γ a torsion-free uniform lattice in G , and A a split Cartan subgroup of G . The centralizer of A in G is a product AK , where A commutes with K . Then the action at the right induces a \mathbb{R}^k -action on the compact quotient $M = \Gamma \backslash G/K$. An essential starting point in the theory of root systems has a strong dynamical system flavor: this action is Anosov! More precisely, the classical first step is to prove that the adjoint action of A on \mathcal{G} preserves a splitting :

$$\mathcal{G} = \mathcal{K} + \mathcal{A} + \sum_{\alpha \in \Sigma} \mathcal{G}^\alpha$$

where \mathcal{K} , \mathcal{A} are the Lie algebras of K , A , and where every α (the *roots*) are linear forms describing the restriction of the adjoint action of a on \mathcal{G}^α : it is simply the multiplication by $\alpha(a)$. The classical way is then to prove that the elements a of A for which $\alpha(a) \neq 0$ is a Zariski open subset, and these elements, called *regular*, are precisely the ones which are Anosov in our terminology for \mathbb{R}^k -action. They form an union of open convex cones, called Weyl chambers, of the form $\{\alpha > 0; \forall \alpha \in \Sigma^*\}$ where Σ^* is a subsystem of a certain kind, called *reduced root system*.

This family of examples, called *Weyl chamber flows* in [12], is certainly one the the most interesting, but is never of codimension one, except in the case $G = \mathrm{SL}(2, \mathbb{R})$. Indeed, the root system is always equal to its own opposite. Hence if the associated \mathbb{R}^k Anosov action has codimension one, then Σ contains exactly two elements, and our assertion follows. In this very special case, the examples we obtain are Anosov flows (i.e. $k = 1$), and

more precisely, up to finite coverings, geodesic flows of compact Riemannian surfaces with constant curvature -1 .

This dynamical feature in algebra is very useful. For example, the (simple) fact that Anosov actions admit compact orbits implies that every uniform lattice Γ in G admits a conjugate $g\Gamma g^{-1}$ which is a lattice in A . In particular, Γ contains a free abelian subgroup of the same rank than G .

Example 2. Consider an action of \mathbb{Z}^k on a closed manifold S . The *suspension* of this action is the quotient M of $S \times \mathbb{R}^k$ by the relation identifying each (x, u) with $(k.x, u + k)$ for every k in \mathbb{Z}^k . The translation on the second factor \mathbb{R}^k induces a \mathbb{R}^k -action on M . It is easy to prove that this action is Anosov if and only some element of \mathbb{Z}^k induces an Anosov diffeomorphism on S . Observe also that this action has codimension one if and only if one of the Anosov element of \mathbb{Z}^k has codimension one. Hence, by Franks-Newhouse Theorem reported in the introduction, if the suspension has codimension one, then codimension one Anosov elements of \mathbb{Z}^k are (up to finite coverings and topological conjugation) hyperbolic toral automorphism on some torus \mathbb{T}^n . Every homeomorphism of the torus commuting with a hyperbolic toral automorphism is also an automorphism (i.e. linear). Hence the only possible examples of codimension one suspensions are the ones described below, arising from number field theory, maybe after restriction to a subgroup of $\mathbb{Z}^k \subset \text{Aut}(\mathbb{T}^n)$.

The suspension process can be generalized to a version including Weyl chamber flows (see twisted Weyl chamber flows in [12, 13]), but this new family of examples are never of codimension one.

Example 3. Generically, the centralizer of an Anosov diffeomorphism f reduces to the iterates f^k ($k \in \mathbb{Z}$) (see [20]). Hence the construction of Anosov actions of \mathbb{Z}^k for $k \geq 2$ requires special features.

Let $K = \mathbb{Q}[\alpha]$ be a field extension of the field \mathbb{Q} of finite degree n , \mathcal{O}_K the ring of algebraic integers of K , and \mathcal{O}_K^* the group of units of \mathcal{O}_K . Then, the quotient of $K \otimes \mathbb{R}$ by the additive action of \mathcal{O}_K is a compact torus of dimension n , on which \mathcal{O}_K^* acts by multiplication. According to Dirichlet unit Theorem, the torsion-free part of \mathcal{O}_K^* is isomorphic to \mathbb{Z}^k , with $k = r_1 + r_2 - 1$ where r_1 is the number of real embeddings and r_2 the number of conjugate pairs of complex embeddings of K . Hence every finite extension of \mathbb{Q} naturally provides an action of \mathbb{Z}^k on a torus. More precisely, the real and complex embeddings provide altogether a realization of $K \otimes \mathbb{R}$ as a vector subspace of $\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$, preserved by the multiplicative action of \mathcal{O}_K^* , which is diagonalizable, the eigenvalues being the various conjugates. Hence, this action is Anosov if and only if some unit has no conjugate of norm 1.

A concrete way to produce such examples is to take some algebraic number α admitting no conjugate of norm 1, and to consider the extension $K = \mathbb{Q}[\alpha]$. Of course, one can forget part of the unit group and just consider some subgroup. It is actually what we do when defining linear Anosov diffeomorphisms.

In order to get codimension one \mathbb{Z}^k -actions, it is sufficient to select as algebraic integer α any *Pisot number*, which is, by definition precisely a real algebraic integer α exceeding 1, and such that its conjugate elements are all less than 1 in absolute value. Concretely, examples of Pisot numbers are roots of $x^3 - x - 1$, $x^4 - x^3 - 1$, etc...

Example 4. Let N be a n -dimensional manifold supporting a codimension one Anosov flow (clearly, $n \geq 3$). We construct a codimension one action of \mathbb{R}^k , $k \geq 2$, on $M = N \times \mathbb{T}^{k-1}$. Consider the coordinate system (x, θ) in M , $x \in N$, $\theta \in \mathbb{T}^{k-1}$. In what follows, for a real function $a(x, \theta)$, by $a(x, \theta) \frac{\partial}{\partial x}$ we mean $a_1 \frac{\partial}{\partial x_1} + \dots + a_n \frac{\partial}{\partial x_n}$ where x_1, \dots, x_n are coordinates in N .

Let $\phi \in A^1(\mathbb{R}^k, M)$ be defined by X_1 and Y_1, \dots, Y_{k-1} , such that $X_1 = a(x) \frac{\partial}{\partial x}$ is a codimension one Anosov flow in N and $X_j := \frac{\partial}{\partial \theta_j}$, where $\theta_1, \dots, \theta_{k-1}$ are coordinates in \mathbb{T}^{k-1} . Then

- ϕ is a codimension one Anosov action of \mathbb{R}^k on M .
- for $n > 3$, by Verjovsky Theorem, ϕ is transitive.
- if $n = 3$ and X_1 is the Anosov flow defined by Franks-Williams in [9], then ϕ is a codimension one Anosov action of \mathbb{R}^k on the $(k + 2)$ -manifold M which is not transitive.

Example 5. More generally, let ϕ be an Anosov \mathbb{R}^k -action on a closed n -dimensional manifold M , and let $p : \widehat{M} \rightarrow M$ be a principal flat \mathbb{T}^ℓ -bundle over M . By flat, we mean that it is equipped with a flat \mathbb{T}^ℓ -invariant connection, i.e. a n -dimensional foliation \mathcal{H} transverse to the fibers of p and preserved by \mathbb{T}^ℓ (there is a 1-1 correspondence between principal flat \mathbb{T}^k -bundles and group homomorphisms $\rho : \pi_1(M) \rightarrow \mathbb{T}^k$). Then, the \mathbb{R}^ℓ -action on M lifts uniquely as a \mathbb{R}^ℓ action tangent to \mathcal{H} . Moreover, this action commutes with the right action of \mathbb{T}^ℓ tangent to the fibers. Hence, both action combine to a $\mathbb{R}^{k+\ell}$ -action on \widehat{M} , which is clearly Anosov.

4. REDUCING CODIMENSION ONE ANOSOV ACTIONS

In this section we to show that any codimension one Anosov action, up to a reduction through a principal torus bundle, has the several topological properties, including:

- The universal covering of the ambient manifold is diffeomorphic to \mathbb{R}^n ;

- The fundamental group of every compact orbit injects into the fundamental group of the ambient manifold;
- The holonomy of the codimension one foliation along a homotopically non-trivial loop in a leaf is non-trivial.

The crucial ingredient is the construction along each strong leaf of dimension one of an affine structure, preserved by the action of ϕ .

4.1. Affine structures over the leaves strong unstables. We begin by remembering that an affine structure of class C^2 on \mathbb{R} is equivalent to given a differential 1-form on \mathbb{R} . If f is a real-valued C^2 map defined on an interval of \mathbb{R} on which the derivative vanishes nowhere, we may define the following differential 1-form:

$$\eta(f) = \frac{f''}{f'} dt$$

It follows from the previous definition that

$$\eta(f \circ g) = g^* \eta(f) + \eta(g)$$

where $g^* \eta(f)$ is the pull-back of $\eta(f)$ by g :

$$g^* \eta(f) = \frac{f'' \circ g}{f' \circ g} g' dt$$

On the other hand, the maps f satisfying $\eta(f) = 0$ are characterized as the restrictions of affine maps (that is, of the form $t \rightarrow \lambda t + b$). Consequently, if g is affine, we have

$$\eta(g \circ f) = \eta(f).$$

Hence, there is a correspondence between a differential 1-form on an interval of \mathbb{R} and an affine structure on this interval. In fact, if $\omega(t)dt$ is a differential 1-form on an interval I , then the differential equation $\eta(f) = \omega dt$ has local solutions which are local diffeomorphisms between I and an open set of \mathbb{R} . Moreover, two of these diffeomorphisms differ by right composition by an affine diffeomorphism. Then, the family of this local solutions is a system of affine charts on I .

Conversely, if $(U_i, f_i)_i$ is a system of charts that defines an affine structure of class C^2 , then the differential 1-form defined by $\omega(t)dt = \eta(f_i)$ if $t \in U_i$ is independent of the choice of $U_i \ni t$.

We consider a C^∞ Anosov action of \mathbb{R}^k on M whose stable foliation \mathcal{F}^s is of codimension one. Then, each leaf of \mathcal{F}^{uu} is C^∞ diffeomorphic to \mathbb{R} . We may assume that \mathcal{F}^{uu} is orientable, otherwise we consider the double covering of M . Consequently, it is possible

to parametrize \mathcal{F}^{uu} by $u : \mathbb{R} \times M \rightarrow M$, an application such that the signed distance of $u(t, x) \in \mathcal{F}^{uu}(x)$ to x is t . Here we consider the induced metric on \mathcal{F}^{uu} .

Lemma 3. *The application u is continuous. For x fixed, the application $u_x : \mathbb{R} \rightarrow M$ defined by $u_x(t) = u(t, x)$ is C^∞ . Furthermore, the derivatives $\frac{\partial}{\partial t^\ell} u(0, x)$, $\ell \in \mathbb{N}$, depend continuously on x .*

Proof. Since u is a flow, it is sufficient to establish the lemma for small values of t .

Let $C^\infty(\mathbb{R}, M)$ be the space of C^∞ immersions of \mathbb{R} in M provided of the C^∞ uniform topology. It follows from the theory developed in [11] that for all $x \in M$, there exist an open neighborhood U of x and a continuous application $\mathcal{I} : U \rightarrow C^\infty(\mathbb{R}, M)$, such that, for every $y \in U$, the immersion $\mathcal{I}_y = \mathcal{I}(y)$ is a diffeomorphism between \mathbb{R} and a neighborhood of y in \mathcal{F}^{uu} . There exist $t_0 > 0$ such that, for all $(t, y) \in (-t_0, t_0) \times U$ we have that $u(t, y) \in \mathcal{I}_y(\mathbb{R})$. Hence $u(t, y) = \mathcal{I}_y(s(t, y))$ where $s(t, y) \in \mathbb{R}$ is defined by equation

$$\int_0^{s(t,y)} \|\mathcal{I}'_y(\alpha)\| d\alpha = t$$

As $\|\mathcal{I}'_y(\alpha)\|$ is continuous with respect to y and smooth with respect to α , it follows that, $s(t, y)$ is continuous with respect to y and smooth with respect to t . This proves that u is continuous and u_x is of class C^∞ . The last statement of the lemma is trivial. \square

For a continuous application $\omega : M \rightarrow \mathbb{R}$, the parametrization u of \mathcal{F}^{uu} permits us to associate affine structures on the leaves $\mathcal{F}^{uu}(x)$ which are defined by the differential 1-form $\omega(u(t, x))$. This structure will be called of *affine structure along the leaves of \mathcal{F}^{uu}* defined by ω .

We say that an affine structure along the leaves of \mathcal{F}^{uu} is *invariant by the action ϕ* if, for each $v \in \mathbb{R}^k$, the application $\phi^v|_{\mathcal{F}^{uu}(x)} : \mathcal{F}^{uu}(x) \rightarrow \mathcal{F}^{uu}(\phi^v(x))$, $x \in M$, is an affine diffeomorphism.

Theorem 6. *Let ϕ be a codimension one Anosov action on M and suppose that \mathcal{F}^{uu} is one dimensional. There exists an unique affine structure along the leaves of \mathcal{F}^{uu} depending continuously on the points and invariant by the action ϕ .*

Proof. For each $(v, x) \in \mathbb{R}^k \times M$, let $\tau_x^v : \mathbb{R} \rightarrow \mathbb{R}$ be the application defined by:

$$\phi^v(u(t, x)) = u(\tau_x^v(t), \phi^v(x))$$

We claim that a continuous application $\omega : M \rightarrow \mathbb{R}$ defines an invariant affine structure along the leaves of \mathcal{F}^{uu} if and only if

$$\omega = n^v + \delta^v \omega \circ \phi^v = \mathcal{A}^v(\omega), \text{ for all } v \in \mathbb{R}^k,$$

where $\delta^v \omega(x) = (\tau_x^v)'(0)$ and $n^v(x) = (\tau_x^v)''(0)/(\tau_x^v)'(0)$. In fact, f_y , $y = \phi^v(x)$ is an affine chart on $\mathcal{F}^{uu}(\phi^v(x))$ if only $f_y \circ \tau_x^v$ is an affine chart on $\mathcal{F}^{uu}(x)$. Equivalently:

$$\omega(u(\tau_x^v(t), y)) = \frac{f_y''(\tau_x^v(t))}{f_y'(\tau_x^v(t))} = \omega \circ \phi^v(u(t, x)) \Leftrightarrow \omega(u(t, x)) = \frac{(f_y \circ \tau_x^v)''(t)}{(f_y \circ \tau_x^v)'(t)}.$$

Hence, as $\tau_{u(t,x)}^v(s) = \tau_x^v(t + s)$, we obtain:

$$\begin{aligned} \omega(u(t, x)) &= \frac{(f_y \circ \tau_x^v)''(t)}{(f_y \circ \tau_x^v)'(t)} \\ &= \frac{(\tau_x^v)''(t)}{(\tau_x^v)'(t)} + (\tau_x^v)'(t) \frac{f_y''(\tau_x^v(t))}{f_y'(\tau_x^v(t))} \\ &= \frac{(\tau_{u(t,x)}^v)''(0)}{(\tau_{u(t,x)}^v)'(0)} + (\tau_{u(t,x)}^v)'(0) \omega \circ \phi^v(u(t, x)) \\ &= n^v(u(t, x)) + \delta^v(u(t, x)) \omega \circ \phi^v(u(t, x)) \\ &= \mathcal{A}^v(\omega)(u(t, x)) \end{aligned}$$

This proves our claim.

The applications \mathcal{A}^v acting on the Banach space of the continuous applications of M on \mathbb{R} provided of the uniform norm. By definition of Anosov action, if v is an element of the Anosov chamber, we have that δ^{sv} , $s < 0$ has uniform norm less than one. This implies that \mathcal{A}^{sv} , $s < 0$ is a contraction, hence \mathcal{A}^{sv} , $s < 0$ admit a unique fixed point. Finally, since $\mathcal{A}^v \circ \mathcal{A}^w = \mathcal{A}^w \circ \mathcal{A}^v$ for all $v, w \in \mathbb{R}^k$, there exists a unique fixed point ω for all \mathcal{A}^v . This finishes the proof. \square

Real affine structures on the real line are well-known: they are all affinely isomorphic to the segment $(0, 1)$, the half-line $(0, +\infty)$, or the complete affine line $(-\infty, +\infty)$. In the latter case, the affine structure is said *complete*.

Lemma 4. *Every leaf of \mathcal{F}^{uu} , endowed with the affine structure provided by Theorem 6, is complete.*

Proof. For every x in M , there is a unique affine map $f_x : \mathcal{F}^{uu}(x) \rightarrow \mathbb{R}$ mapping x on 0 and the point $u(1, x)$ at distance 1 on 1. The image of f_x is an interval $(\alpha(x), \beta(x))$. We aim to prove that $\alpha(x) = -\infty$ and $\beta(x) = +\infty$.

For every v in \mathbb{R}^k and x in M , the restriction of ϕ^v on $\mathcal{F}^{uu}(x)$ induces an affine transformation, even linear, of the affine line, of the form $z \rightarrow \lambda(v, x)z$:

$$f_{\phi^v(x)} \circ \phi^v = \lambda(v, x) f_x$$

Hence:

$$\beta(\phi^v(x)) = \lambda(v, x)\beta(x)$$

Recall that:

$$\phi^v(u(t, x)) = u(\tau_x^v(t), \phi^v(x))$$

Since by definition $f_x(u(1, x)) = 1$, we get:

$$\lambda(v, x) = f_{\phi^v(x)}(u(\tau_x^v(t), \phi^v(x)))$$

Hence, for $v = ta$, where a is the codimension Anosov element, $\lambda(-ta, x)$ is arbitrarily small if $t > 0$ is sufficiently big. Therefore, if $\beta(x)$ is not $+\infty$, $\beta(\phi^{-ta}(x))$ takes arbitrarily small value. This is a contradiction since obviously $\beta > 1$ everywhere. Therefore, β is infinite everywhere.

The proof of $\alpha = -\infty$ is similar. □

4.2. Irreducible codimension one Anosov actions. A codimension one Anosov action ϕ of \mathbb{R}^k on M is said to be *irreducible* if for any $v \in \mathbb{R}^k - \{0\}$ and $x \in M$ with $\phi^v(x) = x$ we have that Hol_γ , the holonomy along of $\gamma = \{\phi^{sv}(x); s \in [0, 1]\}$ of $\mathcal{F}^s(x)$, is a topological contraction or a topological expansion.

Remark 5. It follows from Theorem 6 that the holonomy along γ is differentially linearizable. Therefore, an equivalent definition of irreducibility is to require that the holonomy along γ is non-trivial.

Remark 6. When $k = 1$, the case that the action is a flow, all the codimension one Anosov actions are irreducibles.

Theorem 7. Let $\phi : \mathbb{R}^k \times M \rightarrow M$ be a codimension one Anosov action. Then, there exists a free abelian subgroup $H_0 \approx \mathbb{R}^\ell$ of \mathbb{R}^k , a lattice $\Gamma_0 \subset H_0$, a smooth $(n - \ell)$ -manifold \bar{M} , and $p : M \rightarrow \bar{M}$ a smooth \mathbb{T}^ℓ -principal bundle such that:

- (1) Γ_0 is the kernel of ϕ ;
- (2) every orbit of $\phi_0 = \phi|_{H_0 \times M}$ is a fiber of $p : M \rightarrow \bar{M}$. In particular, \bar{M} is the orbit space of ϕ_0 ;
- (3) ϕ induces an irreducible codimension one Anosov action $\bar{\phi} : \bar{H} \times \bar{M} \rightarrow \bar{M}$ where $\bar{H} = \mathbb{R}^k / H_0$.

The proof of Theorem 7 essentially relies on the following lemma:

Lemma 5. *Let v be an element of \mathbb{R}^k and x an element of M such that $\phi^v(x) = x$. Then, either x is a repelling of attracting (and therefore, unique) fixed point of the restriction of ϕ^v to $\mathcal{F}^{uu}(x)$, or the action of ϕ^v on the entire manifold M is trivial.*

Proof. For every x in M and every w in \mathbb{R}^k such that $\mathcal{F}^{ss}(x) = \phi^w(\mathcal{F}^{ss}(x))$ we consider any loop in $\mathcal{F}^s(x)$ which is the composition of $t \in [0, 1] \rightarrow \phi^{tw}(y)$ with any path in $\mathcal{F}^{ss}(x)$ joining $\phi^w(x)$ to x . Since $\mathcal{F}^{ss}(x)$ is a plane (Remark 1) all these loops are homotopic one to the other in $\mathcal{F}^s(x)$; in particular, the holonomy of \mathcal{F}^s along any of them is well-defined and does not depend on the loop. We denote it by h_x^w .

Assume that x is the one appearing in the statement of the lemma. Then according to Theorem 6, the restriction of ϕ^v to $\mathcal{F}^{uu}(x)$ is conjugated to an affine transformation of the real affine line. Therefore, in order to prove the theorem, we just have to consider the case where h_x^v is trivial.

We define Ω_w as the set comprising the points x in M such that $\phi^w(x) \in \mathcal{F}^{ss}(x)$ and for which the holonomy h_x^w is trivial. This set is obviously ϕ -invariant. By the discussion above, we can assume that Ω_v is non empty. Moreover, for every x in Ω_w , and every y in $\mathcal{F}^{ss}(x)$, the loops considered above associated to respectively x, y , are freely homotopic one to the other in $\mathcal{F}^s(x)$. Hence $h_y^w = h_x^w$. It follows that Ω_w is saturated by \mathcal{F}^s .

Finally, for every subset U of \mathbb{R}^k , let Ω_U be the union of the Ω_w for w in U . For every x in Ω_v , since the holonomy h_x^v is trivial, for every y in $\mathcal{F}^{uu}(x)$ near x the point $\phi^v(y)$ lies in the local stable leaf of y . It follows that y lies in Ω_w for some w close to v in \mathbb{R}^k . Since M is compact, for every neighborhood U of v in \mathbb{R}^k , there exists $\delta > 0$ such that every y in M , lying on a local unstable leaf $\mathcal{F}_\delta^{uu}(x)$ with x in Ω_v , belongs to Ω_U .

Now, at the one hand we know that Ω_U is ϕ -invariant. On the other hand, since the \mathcal{F}^{uu} -saturation of any \mathcal{F}^s -invariant subset is the entire M , for every y in M the point $\phi^{ta}(y)$ lies in $\mathcal{F}_\delta^{uu}(x)$ for some $t < 0$, where x is an element of Ω_v . It follows that Ω_U is the entire M . Since U is arbitrary, we get the equality $M = \Omega_v$.

Consider now a compact ϕ -orbit \mathcal{O} . For some $\delta > 0$ and every x in \mathcal{O} , the intersection $\mathcal{O} \cap \mathcal{F}_\delta^{ss}(x)$ is reduced to x . For every y in $\mathcal{F}^{ss}(x)$, and for every $t > 0$ sufficiently big, $\phi^{ta}(y)$ belongs to $\mathcal{F}_\delta^{ss}(\phi^{ta}(x))$. Hence $\mathcal{F}^{ss}(x) \cap \mathcal{O} = \{x\}$. It follows that every point in $\text{Comp}(\phi) \cap \Omega_v = \text{Comp}(\phi)$ is fixed by ϕ_v . Hence, the restriction of ϕ^v to the closure $\Omega(\phi)$ of $\text{Comp}(\phi)$ is trivial.

Finally, assume that x is an arbitrary element of $M = \Omega_v$. Let $(t_n)_{(n \in \mathbb{N})}$ be a sequence of positive real number diverging to $+\infty$ and such that $x_n = \phi^{-t_n a}(x)$ converges to some element x_∞ of $\Omega(\phi) \subset \text{Fix}(\phi)$. Then, due to the proximity to x_∞ , for every $\epsilon > 0$, and for n is sufficiently big, there is a path c_n in $\mathcal{F}^{ss}(x_n)$ of length $\leq \epsilon$ joining x_n to $\phi^v(x_n)$. Then,

$\phi^{tn^a}(c_n)$ is a path of length $\leq \epsilon$ joining x to $\phi^v(x)$. Since ϵ is arbitrary, we get $\phi^v(x) = x$. This achieves the proof of the lemma. \square

Proof of Theorem 7. Let Γ_0 be the kernel of ϕ . Since the action is locally free, Γ_0 is discrete, isomorphic to \mathbb{Z}^ℓ for some integer $\ell \geq 0$. Let H_0 be the subspace of \mathbb{R}^k generated by Γ_0 . Observe that for every x in M , and every v in Γ_0 , the holonomy h_x^v is well-defined, and trivial (cf. the notations introduced in the proof of Lemma 5).

The torus $\mathbb{T}^\ell = \Gamma_0 \backslash H_0$ acts on M ; since it is compact, this action is proper. Moreover, this action is free: indeed, if some v in H_0 fixes some x , then the holonomy h_x^v is trivial since v is a linear combination of elements of Γ_0 (or, better to say, since some of the iterates nv , for integers n , are arbitrarily approximated by elements of Γ_0). According to Lemma 5, the action of v on M is trivial, i.e. v belongs to Γ_0 .

Therefore, the quotient space \bar{M} is a closed $(n-\ell)$ -dimensional manifold, and the quotient map $p : M \rightarrow \bar{M}$ is a principal \mathbb{T}^ℓ -bundle. The action of \mathbb{R}^k on M induces an action of $\bar{H} = \mathbb{R}^k/H_0$. It is straightforward to check that this action is Anosov, and of codimension one.

Finally, if $\bar{\phi}$ is not irreducible, there is a non-trivial element \bar{v} of \bar{H} fixing a point \bar{x} in \bar{M} and such that $h_{\bar{x}}^{\bar{v}}$ is trivial. There is a representant v of \bar{v} in \mathbb{R}^k fixing a point x in M above \bar{x} , and such that h_x^v is trivial. According to Lemma 5, v belongs to $\Gamma_0 \subset H_0$. Hence \bar{v} is trivial. This contradiction achieves the proof of the theorem. \square

Remark 7. Let $\phi : \mathbb{R}^k \times M \rightarrow M$ be a codimension one Anosov action and $\bar{\phi}$ an action of \bar{H} (that is isomorphic to $\mathbb{R}^{k-\ell}$) on \bar{M} as in Theorem 7. Then:

- (1) $\dim M > k + 2 \iff \dim \bar{M} > \dim \bar{H} + 2$;
- (2) ϕ is transitive $\iff \bar{\phi}$ is transitive;
- (3) ϕ is irreducible $\iff \bar{H} = \mathbb{R}^k$.

4.3. The orbit space of an irreducible codimension one Anosov action. Let $\pi : \widetilde{M} \rightarrow M$ be the universal covering map of M and $\tilde{\phi}$ be the lift of ϕ on \widetilde{M} . The foliations \mathcal{F}^{ss} , \mathcal{F}^{uu} , \mathcal{F}^s and \mathcal{F}^u lift to foliations $\tilde{\mathcal{F}}^{ss}$, $\tilde{\mathcal{F}}^{uu}$, $\tilde{\mathcal{F}}^s$ and $\tilde{\mathcal{F}}^u$ in \widetilde{M} . We denote by Q^ϕ be the orbit space of $\tilde{\phi}$ and $\pi^\phi : \widetilde{M} \rightarrow Q^\phi$ be the canonical projection. This section is devoted to the proof of the following theorem, which is a keystone of the proof of the main theorem.

Theorem 8. *If ϕ is an irreducible codimension one Anosov action of \mathbb{R}^k on M then Q_ϕ , the orbit space of $\tilde{\phi}$, is homeomorphic to \mathbb{R}^{n-k} .*

Let $p = n - 1 - k$. Since by hypothesis \mathcal{F}^{uu} is one dimensional, p is the dimension of \mathcal{F}^{ss} .

Proposition 2. *Every loop in M transverse to \mathcal{F}^s is homotopically non-trivial in M .*

Proof. By a Theorem of Haefliger (see [4, Proposition 7.3.2]), if some transverse loop is homotopically trivial, then there is a leaf F , of \mathcal{F}^s containing a loop $c : [0, 1] \rightarrow F$, homotopically non-trivial in F , and such that the holonomy of \mathcal{F}^s along c is trivial on one side.

According to Remark 1 there is a fibration $p_F : F \rightarrow P$ where P is a flat cylinder, and whose fibers are strong stable leaves. Moreover, the restriction of p_F to the orbit \mathcal{O}_x of $x = c(0)$ is a covering map. Hence we can lift in \mathcal{O}_x the curve $p \circ c$. In other words, there is a continuous path $c' : [0, 1] \rightarrow \mathbb{R}^k$ such that for every t in $[0, 1]$ the image $c'(t)$ lies in the same leaf of \mathcal{F}^{ss} than $\phi^{c(t)}(x)$. Let $v = c'(1)$: since c is homotopic to the loop obtained by composing c' with any path in $\mathcal{F}^{ss}(c(0))$ joining $c'(1)$ to $c(1) = c(0) = x$, we get that the holonomy h_x^v is trivial on one side. Since it is linearizable, h_x^v is trivial. According to Lemma 5, and since ϕ is irreducible, it means that $c'(1) = 0$. Hence c' is homotopically trivial in \mathbb{R}^k . Hence $p_F \circ c'$ is homotopically trivial in P . But since p_F is a trivial fibration with contractible leaves and $p_F \circ c = p_F \circ c'$, we obtain that c is homotopically trivial in F . Contradiction. \square

Corollary 1. *The orbits of ϕ are incompressible: every loop in a ϕ -orbit \mathcal{O} which is homotopically non-trivial in \mathcal{O} is homotopically non-trivial in M .*

Proof. Every loop in \mathcal{O} is homotopic to a trajectory $t \rightarrow c(t) = \phi^{tv}(x)$; $t \in [0, 1], v \in \mathbb{R}^n, \phi^v(x) = x$. Since ϕ is irreducible, the holonomy of \mathcal{F}^s along c is non-trivial. It follows that there is a loop homotopic to c and transverse to \mathcal{F}^s . The corollary follows from Proposition 2. \square

A foliation is said to be *by closed planes* if all the leaves are closed and images of embeddings of \mathbb{R}^n .

Corollary 2. *Let ϕ be an irreducible codimension one Anosov action on M . The foliations $\tilde{\mathcal{F}}^{uu}, \tilde{\mathcal{F}}^{ss}, \tilde{\mathcal{F}}^u, \tilde{\mathcal{F}}^s$ and the foliation defined by $\tilde{\phi}$ are by closed planes. The intersection between a leaf of $\tilde{\mathcal{F}}^u$ and a leaf of $\tilde{\mathcal{F}}^s$ is at most an orbit of $\tilde{\phi}$. Every orbit of $\tilde{\phi}$ meets a leaf of $\tilde{\mathcal{F}}^{uu}$ or $\tilde{\mathcal{F}}^{ss}$ at most once.*

Proof. According to Corollary 1, $\tilde{\phi}$ is a free action. Moreover, if $\tilde{\mathcal{F}}^{ss}(x) = \tilde{\mathcal{F}}^{ss}(\tilde{\phi}^v(x))$ for some non-trivial v , then $h_{\pi(x)}^v$ is non-trivial (since ϕ is irreducible), but is also the holonomy of \mathcal{F}^s along a closed loop in $\mathcal{F}^s(\pi(x))$ homotopically trivial in M . It is in contradiction with Proposition 2.

Therefore, every orbit of $\tilde{\phi}$ intersects every leaf of $\tilde{\mathcal{F}}^{ss}$ at most once. Since every leaf of $\tilde{\mathcal{F}}^s$ is the saturation under $\tilde{\phi}$ of a leaf of $\tilde{\mathcal{F}}^{ss}$, it follows that it is an injective immersion of \mathbb{R}^{p+k} . Similarly, every leaf of $\tilde{\mathcal{F}}^u$ is an injective immersion of \mathbb{R}^{2+k} .

It is easy to show, by a standard argument, that if a leaf $\tilde{\mathcal{F}}^s$ is not closed, then there is a loop in \tilde{M} transverse to $\tilde{\mathcal{F}}^s$, giving a contradiction with Proposition 2. Hence $\tilde{\mathcal{F}}^s$ is a foliation by closed planes. The statement for leaves of $\tilde{\mathcal{F}}^{ss}$ follows.

Since \tilde{M} is simply connected, every foliation in it are oriented and transversely oriented. Being closed hypersurfaces, leaves of $\tilde{\mathcal{F}}^s$ disconnect \tilde{M} . Hence, a leaf of $\tilde{\mathcal{F}}^{uu}$ intersecting a leaf F of $\tilde{\mathcal{F}}^s$ enters in one side of F , and cannot cross F once more afterwards, due to orientation considerations. In other words, every leaf of $\tilde{\mathcal{F}}^{uu}$ intersects every leaf of $\tilde{\mathcal{F}}^s$ at most once. In particular, it cannot accumulate somewhere, i.e. it is closed.

In order to achieve the proof of the corollary, we just have to prove that the $\tilde{\phi}$ -orbits are closed. But this is clear, since each of them is the intersection between a weak stable leaf and a weak unstable leaf, that we have shown to be closed. \square

Consequently, by a theorem of Palmeira [21]:

Corollary 3. *With the same hypotheses of above proposition, the universal covering of M is diffeomorphic to \mathbb{R}^n .* \square

Lemma 6. *If ϕ be a codimension one Anosov action on M , then the orbit space of $\tilde{\phi}$ is Hausdorff.*

Proof. By contradiction, we assume that there exist two different $\tilde{\phi}$ -orbits $\tilde{\mathcal{O}}_{x_1}$ and $\tilde{\mathcal{O}}_{x_2}$ which are non-separable. Then, the saturation by $\tilde{\mathcal{F}}^{uu}$ of $\tilde{\mathcal{F}}^s(x_1)$ and $\tilde{\mathcal{F}}^s(x_2)$ are two non disjoint neighborhoods of $\tilde{\mathcal{O}}_{x_1}$ and $\tilde{\mathcal{O}}_{x_2}$, respectively.

First, we assume that $\tilde{\mathcal{O}}_{x_1}$ and $\tilde{\mathcal{O}}_{x_2}$ are contained in the same leaf of $\tilde{\mathcal{F}}^s$. Hence, we can assume that $\tilde{\mathcal{F}}^{ss}(x_1) = \tilde{\mathcal{F}}^{ss}(x_2) = F_0$. Let U_1 and U_2 be the disjoint neighborhoods in F_0 of x_1 and x_2 , respectively. It follows, from Corollary 2, that the saturation by $\tilde{\mathcal{F}}^u$ of U_1 and U_2 are two disjoint $\tilde{\phi}$ -invariant neighborhoods of $\tilde{\mathcal{O}}_{x_1}$ and $\tilde{\mathcal{O}}_{x_2}$. This contradicts our assumption.

Hence, $\tilde{\mathcal{F}}^s(x_1) \neq \tilde{\mathcal{F}}^s(x_2)$. The saturation by $\tilde{\mathcal{F}}^{uu}$ of $\tilde{\mathcal{F}}^s(x_1)$ and $\tilde{\mathcal{F}}^s(x_2)$ cannot be disjoint since they are neighborhoods of respectively x_1 , x_2 . There exist $y_1 \in \tilde{\mathcal{F}}^s(x_1)$ and $y_2 \in \tilde{\mathcal{F}}^s(x_2)$ such that $\tilde{\mathcal{F}}^{uu}(y_1) = \tilde{\mathcal{F}}^{uu}(y_2)$. Since $y_1 \neq y_2$, there exist disjoint neighborhoods U_1 and U_2 in $\tilde{\mathcal{F}}^{uu}(y_1)$ of y_1 and y_2 , respectively. The saturation by $\tilde{\mathcal{F}}^s$ of U_1 and U_2 are two $\tilde{\phi}$ -invariant neighborhoods of $\tilde{\mathcal{O}}_{x_1}$ and $\tilde{\mathcal{O}}_{x_2}$ which, by our assumption, are non disjoint. In this case, a leaf of $\tilde{\mathcal{F}}^s$ passing by a point in the intersection of these neighborhoods meet $\tilde{\mathcal{F}}^{uu}(y_1)$ in two points, this contradicts the Corollary 2 and finishes the proof. \square

Proof of Theorem 8. Let $x \in \widetilde{M}$ and U be a neighborhood of x which is a lift of a product neighborhood in the sense of Theorem 3. Let $\Sigma \subset U$ be a smooth $(n - k)$ -submanifold which is transverse to ϕ . First we are going to show that every orbit of ϕ meets Σ in at most one point. By contradiction, we assume that there exists an orbit \mathcal{O} that meets Σ in two points. As \mathcal{O} meets a leaf of $\widetilde{\mathcal{F}}^{ss}$ in at most one point (Corollary 2), then \mathcal{O} meets U along two different leaves of $\widetilde{\mathcal{F}}^s|_U$. Hence, since U is a product neighborhood, we have that there are leaves of $\widetilde{\mathcal{F}}^{uu}$ that meets a leaf of $\widetilde{\mathcal{F}}^{ss}$ at two points. But this is impossible by Corollary 2.

Let $(\Sigma_i)_{i \in I}$ be a family of transverses as above whose union meets all the orbits of ϕ . Then $\{(\Sigma_i, \pi^\phi|_{\Sigma_i}); i \in I\}$ defines a differentiable structure on Q^ϕ whose class of differentiability is the same that of the action ϕ . Moreover, π^ϕ is a locally trivial bundle. Thus Q^ϕ is a manifold of dimension $n - k$, Hausdorff and, as shown by the exact sequence of homotopy groups for the bundle $\pi^\phi : \widetilde{M} \rightarrow Q^\phi$, simply connected. Since $\widetilde{\mathcal{F}}^s$ induces on Q^ϕ a codimension one foliation by planes, we conclude, once more by Palmeira's Theorem, that Q^ϕ is diffeomorphic to \mathbb{R}^{n-k} . \square

Proposition 3. *All the non compact orbits of an irreducible codimension one Anosov action are planes.*

Proof. Let \mathcal{O} be an orbit of an irreducible codimension one Anosov action $\phi : \mathbb{R}^k \times M \rightarrow M$. Suppose that \mathcal{O} is not a plane, i.e. that $\phi^v(x) = x$ on \mathcal{O} for some $v \in \mathbb{R}^k - \{0\}$. Let $y \in \overline{\mathcal{O}}$. Then there exist a sequence $\{x_n\}$ of elements of \mathcal{O} such that $x_n \rightarrow y$. Thus, $\phi^v(y) = y$. Since the action is irreducible, the holonomy h_y^v is non trivial. It follows that all the x_n , for n sufficiently big, lie in the same local stable leaf $\mathcal{F}_\delta^s(y)$.

Hence the closure $\overline{\mathcal{O}}$ in M , which is compact, is contained in the weak stable leaf $F = \mathcal{F}^s(y) = \mathcal{F}^s(x)$. It follows that the space $P = p_F(\mathcal{O}) = p_F(\overline{\mathcal{O}})$ of strong stable leaves in F is compact (cf. the notations in Remark 1), hence Γ_L is a lattice in \mathbb{R}^k . According to Lemma 1, Γ_L contains an Anosov element b contained in the chamber \mathcal{A}_a . The restriction of ϕ^a to the strong stable leaf L is a contraction, hence contains a fixed point z . By Remark 2, the ϕ -orbit of z is compact, and Γ_L is the isotropy group of z . Now, since $\overline{\mathcal{O}}$ is compact, the same is true for the intersection $L \cap \overline{\mathcal{O}}$. On the other hand, the negative iterates $\phi^{-na}(z')$ of a point z' in L different from z escape from any compact subset of L . Therefore, $\overline{\mathcal{O}}$ is the compact orbit of z . \square

As an immediate corollary of Theorem 7 and Proposition 3 we get:

Corollary 4. *Let $\phi : \mathbb{R}^k \times M \rightarrow M$ be a codimension one Anosov action, not necessarily irreducible. Every non-compact ϕ -orbit is diffeomorphic to $\mathbb{T}^\ell \times \mathbb{R}^{k-\ell}$.* \square

5. CODIMENSION ONE ANOSOV \mathbb{R}^k -ACTIONS ARE TRANSITIVE

In this section we prove the main Theorem:

Theorem 2. *Let ϕ be a codimension one Anosov action of \mathbb{R}^k on a closed manifold M of dimension greater than $k + 2$. Then, any regular subcone \mathcal{C} admits a dense orbit in M .*

In what follows, we consider a codimension one Anosov action $\phi : \mathbb{R}^k \times M \rightarrow M$, with $\dim M > k + 2$ and a regular subcone \mathcal{C} . Furthermore, we will consider the foliations \mathcal{F}^{ss} , \mathcal{F}^{uu} , \mathcal{F}^s and \mathcal{F}^u which corresponds to the chamber \mathcal{A}_a containing \mathcal{C} .

5.1. Bi-homoclinic points and transitivity. Let u be the parametrization of \mathcal{F}^{uu} which was studied in Lemma 3. We define

$$\begin{aligned}\mathcal{H}^+ &= \{x \in M; (x, +\infty) \cap \mathcal{F}^s(x) = \emptyset\} \\ \mathcal{H}^- &= \{x \in M; (-\infty, x) \cap \mathcal{F}^s(x) = \emptyset\}\end{aligned}$$

where $(x, +\infty) = \{u(\tau, x); \tau > 0\}$ and $(-\infty, x) = \{u(\tau, x); \tau < 0\}$.

Definition 3. A point $x \in M$ is said to be *bi-homoclinic* if $x \notin \mathcal{H}^+ \cup \mathcal{H}^-$.

The following results establish a criterion for transitivity.

Proposition 4. *If every point in M is bi-homoclinic, then every leaf of \mathcal{F}^s is dense.*

Proof. For all $\tau_0 \in \mathbb{R}$ we consider the set $\{x \in M; (x, u(\tau_0, x)) \cap \mathcal{F}^s(x) \neq \emptyset\}$. It follows, from Theorem 3, that this set is open. Hence, since M is compact, there is $\tau_0 \in \mathbb{R}$ such that

$$\text{for all } x \in M, \text{ there exists } \tau \in (0, \tau_0) \text{ such that } u(\tau, x) \in \mathcal{F}^s(x)$$

Similarly, increasing τ_0 , if necessary, we have

$$\text{for all } x \in M, \text{ there exists } \tau \in (-\tau_0, 0) \text{ such that } u(\tau, x) \in \mathcal{F}^s(x).$$

Thus, there exists $\ell > 0$ such that if I is an interval contained in a leaf of \mathcal{F}^{uu} whose arc length is greater than ℓ , then each leaf of \mathcal{F}^s intercepting I contains at least three points of I .

On the other hand, any interval contained in a leaf of \mathcal{F}^{uu} admits an iterate by ϕ^a (a Anosov element) whose arc length is greater than ℓ . This implies that any interval of $\mathcal{F}^{uu}(x)$ meets $\mathcal{F}^s(x)$, and hence the closure of $\mathcal{F}^s(x)$ is an open set. Therefore $\mathcal{F}^s(x)$ is dense in M . \square

Lemma 7. *If every leaf of \mathcal{F}^s is dense in M , then \mathcal{C} admits a dense orbit.*

Proof. Let $a \in \mathcal{C}$ be an Anosov element and ϕ^{ta} the corresponding flow. According to Remark 4:

$$\Omega(\phi) = \Omega(\phi^{ta}).$$

By a result of Conley [6], there exists $L : M \rightarrow \mathbb{R}$ a complete Lyapunov function for the flow ϕ^{ta} , meaning that for $t > 0$ we have $L(\phi^{ta}(x)) \leq L(x)$, and the equality holds only if x lies in $\Omega(\phi^{ta})$. On the other hand, Theorem 5 and the hypotheses imply that the $\Omega(\phi)$ admits only one basic block, in particular, that there \mathcal{C} admits a dense orbit in $\Omega(\phi)$. Moreover, for every x, y in the basic block $\Omega(\phi)$, the unstable leaf of x intersects the stable leaf of y , and the stable leaf of x intersects the unstable leaf of y . From the former we get $L(x) \geq L(y)$, and from the latter, $L(y) \geq L(x)$. Hence, the restriction of L to $\Omega(\phi)$ is constant, say, vanishes. Then, for every x in M , the inequalities $L(x) \geq 0$ and $L(x) \leq 0$ hold (the former because the α -limit set of the ϕ^{ta} -orbit of x is non-empty, the latter because the ω -limit set is non-empty). It follows that L vanishes everywhere, and $\Omega(\phi) = M$. Therefore \mathcal{C} is transitive. \square

5.2. Proof of Main Theorem.

Lemma 8. *Assume that ϕ is an irreducible codimension one Anosov action. The sets \mathcal{H}^- and \mathcal{H}^+ are unions of compact orbits.*

Proof. Note that \mathcal{H}^- and \mathcal{H}^+ are closed invariant sets. We will show the lemma for \mathcal{H}^+ , the case of \mathcal{H}^- is analogous.

Cover M by a finite collection $(U_i)_{1 \leq i \leq N}$ of product neighborhoods as in Theorem 3. We claim that the intersection of any orbit in \mathcal{H}^+ with every U_i is connected. This will show the lemma. Indeed, since the U_i are in finite number, the orbit under consideration is compact. Moreover, since the area of local orbits contained in every U_i is uniformly bounded from above, it also implies that there is a uniform bound on the area of compact orbits in \mathcal{H}^+ . According to Lemma 2, \mathcal{H}^+ is the union of a finite number of compact orbits.

We are going to show our claim above. Let $y_0 \in \mathcal{H}^+$, and let F be its weak stable leaf. By the very definition of \mathcal{H}^+ , for every i , the intersection between the orbit \mathcal{O}_0 of y_0 and the product neighborhood U_i must be contained in a single plaque F_i . The union of the closure of the F_i is compact; it follows that \mathcal{O}_0 is relatively compact, not only in the manifold M , but also in the leaf F equipped with its own leaf topology.

Recall that there is there is a bundle map $p_F : F \rightarrow P$, whose restriction to \mathcal{O}_0 is a covering map (remark 1). Since \mathcal{O}_0 is relatively compact in F , the base manifold $P = \mathbb{R}^k / \Gamma_F$ is compact. By Lemma 1, the lattice Γ_L must contain an element v of \mathcal{A}_a . The restriction of ϕ^v to $\mathcal{F}^{ss}(y_0)$ is then a contracting map, hence admitting a fixed point y_1 .

Moreover, since \mathcal{O}_0 is relatively compact, the leaf distance between the iterates $\phi^{-tv}(y_0)$ and $\phi^{-tv}(y_1)$ is bounded. It is possible if and only if $y_0 = y_1$, in particular, y_0 is fixed by the Anosov element ϕ^v .

According to item (4) of Remark 2, the orbit \mathcal{O}_0 is compact. The claim and the lemma then follow, since F contains at most one compact orbit, and that this compact orbit intersects every strong stable leaf at most once. \square

Proof of Main Theorem. As by Remark 7 we can assume that ϕ is irreducible, then by Proposition 4 and Lemma 7, it is sufficient to show that the sets \mathcal{H}^+ and \mathcal{H}^- are empty. We will show by contradiction that $\mathcal{H}^+ = \emptyset$, the case $\mathcal{H}^- = \emptyset$ is analogous.

Fix a point $x_0 \in \mathcal{H}^+$ which we consider as the basepoint of M and put $\Gamma = \pi_1(M, x_0)$. We consider the action of Γ on Q^ϕ which is induced by the action of Γ on \widetilde{M} by covering automorphisms. Let \mathcal{G}^s and \mathcal{G}^u be the foliations on Q^ϕ which are induced by $\widetilde{\mathcal{F}}^s$ and $\widetilde{\mathcal{F}}^u$, respectively. They are both preserved by the Γ -action.

Let θ_0 be the $\tilde{\phi}$ -orbit of a lift of x_0 in \widetilde{M} . Since the ϕ -orbit of x_0 is diffeomorphic to \mathbb{T}^k and incompressible, the isotropy group Γ_0 of θ_0 is isomorphic to \mathbb{Z}^k . Let F_0 be the leaf through θ_0 of \mathcal{G}^s and put $F'_0 = F_0 - \{\theta_0\}$.

As the foliation \mathcal{G}^u is orientable and one dimensional, their leaves admit a natural order. For all $x \in Q^\phi$ the subset of $\mathcal{G}^u(x)$ comprising elements above x is denoted by $(x, +\infty)$. The fact that $x_0 \in \mathcal{H}^+$ means:

$$(\theta_0, +\infty) \cap \Gamma \cdot F_0 = \emptyset.$$

On the other hand, since θ_0 is only point of F_0 which is the lift of a compact orbit, all the points of F'_0 are not lifts of compact orbits. This means that each point $x \in F'_0$ is not a lift of an orbit contained in \mathcal{H}^+ , equivalently, $(x, +\infty) \cap \Gamma \cdot F_0 \neq \emptyset$. Let $h(x)$ be the infimum of $(x, +\infty) \cap \Gamma \cdot F_0$. We observe that h is, by definition, injective.

Claim 1. *Every $x \in F'_0$ is strictly inferior to $h(x)$.* Indeed, there exists $[x, \beta_x)$ a neighborhood of x in $[x, +\infty)$ such that all the leaves of \mathcal{G}^s which meets $[x, \beta_x)$ also meets $[\theta_0, +\infty)$. Hence, since none of these leaves is of the form $\gamma \cdot F_0$, $\gamma \in \Gamma$, we obtain that $\beta_x \leq h(x)$. Consequently $h(x) > x$.

Claim 2. *The image of F'_0 by h is contained in a leaf F_1 of \mathcal{G}^s .* Given x , consider a small product neighborhood around $h(x)$. Then it is clear that $h(y)$ and $h(x)$ lies in the same leaf of \mathcal{G}^s . Hence, for every weak stable leaf F' , the subset $\Omega(F')$ of F'_0 comprising elements whose image by h belongs to F' is open. F'_0 is the disjoint union of all the $\Omega(F')$,

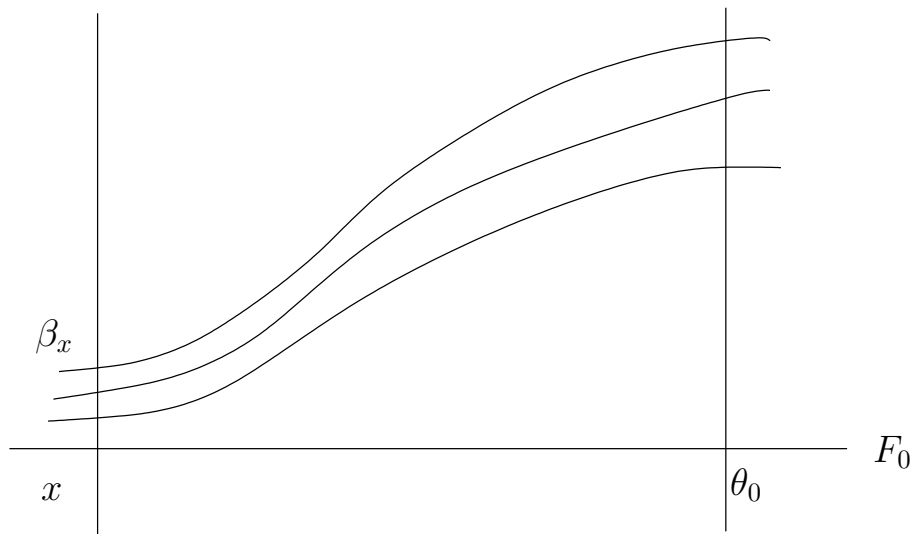


FIGURE 1.

and is connected (here we use the hypothesis $n > k + 2$), therefore, all the $\Omega(F')$ are empty, except one, $\Omega(F_1)$.

Claim 3. *The leaf F_1 is Γ_0 invariant, and the map $h : F'_0 \rightarrow F_1$ is a Γ_0 -equivariant injective local homeomorphism onto its image.* It is clear that $h \circ \gamma = \gamma \circ h$ for every γ in Γ_0 . The claim follows.

According to Claim 1, F_1 and F_0 are disjoint. Consider the stable leaf $\bar{F}_1 = \pi(F_1)$: it is a bundle with contractible fibers over a flat cylinder \mathbb{R}^k/Γ_1 . Its fundamental group is Γ_0 , hence Γ_1 is isomorphic to \mathbb{Z}^k , and thus, a lattice in \mathbb{R}^k . According to Lemma 1, it contains an Anosov element, which acts as a contraction in every strong stable leaf. Therefore, \bar{F}_1 contains a unique compact orbit. This compact orbit lifts as an element θ_1 , which is the unique Γ_0 -fixed point in F_1 . Let $F'_1 = F_1 - \{\theta_1\}$. Observe that since h is injective, that θ_1 is a Γ_0 -fixed point, and that Γ_0 admits no fixed point in F'_0 , the image of h is contained in F'_1 .

Claim 4. *The map $h : F'_0 \rightarrow F_1$ is a homeomorphism.* The only remaining point to show is the fact that $h(F'_0) = F'_1$. According to item (4) of Remark 2, Γ_0 contains an element γ_0 such that, for some Anosov element v of \mathbb{R}^k we have $\phi^v(x) = \gamma_0 x$ for every x in θ_0 . It follows that the action of γ_0 on F_0 is contracting, admitting θ_0 as its unique fixed point. Therefore, the action of γ_0 on $F'_0 \approx \mathbb{R}^p - \{0\}$ is free, properly discontinuous, and the quotient space is diffeomorphic to $\mathbb{S}^{p-1} \times \mathbb{S}^1$. If we knew that the action of γ_0 on F_1 is also a contraction, then the claim would follow immediately from the fact that h induces a continuous map between the quotient spaces $F'_0 / \langle \gamma_0 \rangle$ and $F'_1 / \langle \gamma_0 \rangle$, and from

the compactness of these quotient spaces. Unfortunately, there is no warranty that γ_0 acts properly on F'_2 , hence we need a slightly more intricate argument.

Since γ_0 is a contraction, there is an embedded codimension 1 sphere S_0 in F'_0 , boundary of a closed ball B_0 containing θ_0 , such that $\gamma_0(S_0)$ is another embedded sphere, contained inside B_0 , and disjoint from S_0 . The union $S_0 \cup \gamma_0(S_0)$ is the boundary of a subdomain $W_0 \approx \mathbb{S}^{p-1} \times [0, 1]$ of B_0 . Now, the union of all the iterates $\gamma_0^n W_0$ covers the entire F'_0 .

Since h is injective, its image is a Γ_0 -invariant domain W_∞ of $F'_1 \approx \mathbb{S}^{p-1} \times \mathbb{R}$ diffeomorphic to $\mathbb{S}^{p-1} \times \mathbb{R}$, containing the embedded sphere $S_1 = h(S_0)$. Now observe that even if γ_0 might not be contracting in F_1 , the same argument as the one used in F'_0 ensures that some element γ_1 of Γ_0 is contracting. Then, for N sufficiently big, $\gamma_1^N S_1$ is disjoint from S_1 . By construction, S_1 does not bound a ball inside W_∞ , hence the same is true for $\gamma_1^N S_1$: there are both incompressible spheres inside $W_\infty \approx \mathbb{S}^{p-1} \times \mathbb{R}$. It follows that their union is the boundary of a compact domain $W_1 \subset W_\infty$. Considering W_1 as a compact domain in F'_1 , we get that F'_1 is the union of the iterates under γ_1^N of W_1 . Therefore, W_∞ is the entire F'_1 .

Conclusion. Consider the sphere S_0 introduced in the previous step, and its image S_1 by h . By construction, S_1 bounds a ball B_0 in F_0 , containing θ_0 . According to Jordan-Schönflies Theorem, S_1 is also the boundary of a closed ball B_1 in F_1 . If B_1 does not contain θ_1 , then it would be contained in F'_1 , and $h^{-1}(B_1)$ would be a closed ball in F'_0 bounded by S_0 : contradiction.

Let \mathcal{C} be the union of all unstable segments $[x, h(x)]$ for x describing S_0 . The union \mathcal{S} of \mathcal{C} with B_0 and B_1 is then a submanifold of Q^ϕ , homeomorphic to a sphere of codimension one. Since Q^ϕ is homeomorphic to \mathbb{R}^{n-k} , \mathcal{S} is the boundary of a closed topological ball \mathcal{B} .

We not get the concluding final contradiction as follows: the \mathcal{G}^u -leaf ℓ_1 through θ_1 is a closed line in Q^ϕ , crossing \mathcal{S} at θ_1 . Since \mathcal{B} is compact, \mathcal{G}^u must escape from it, and thus, cross \mathcal{S} at another point. This intersection cannot occur in \mathcal{C} , since \mathcal{C} is tangent to \mathcal{G}^u . It cannot occur in B_1 , since, as a leaf of \mathcal{G}^u , it intersects every \mathcal{G}^s -leaf in at most one point. Therefore, ℓ_1 must intersect B_0 , and this intersection is reduced to one point. Finally, since F_0 and ℓ_1 are Γ_0 -invariant, this intersection point must be fixed by Γ_0 : hence, it is θ_0 .

Therefore, ℓ_1 contains two Γ_0 -fixed points: θ_0 and θ_1 . This is a contradiction with the fact that unstable leaves contain at most one compact orbit (see figure 2)

□

6. CONCLUSION

As we already mentioned in the introduction, codimension one Anosov flows has been extensively studied, from the 60's until nowadays. It is reasonable to expect that all these

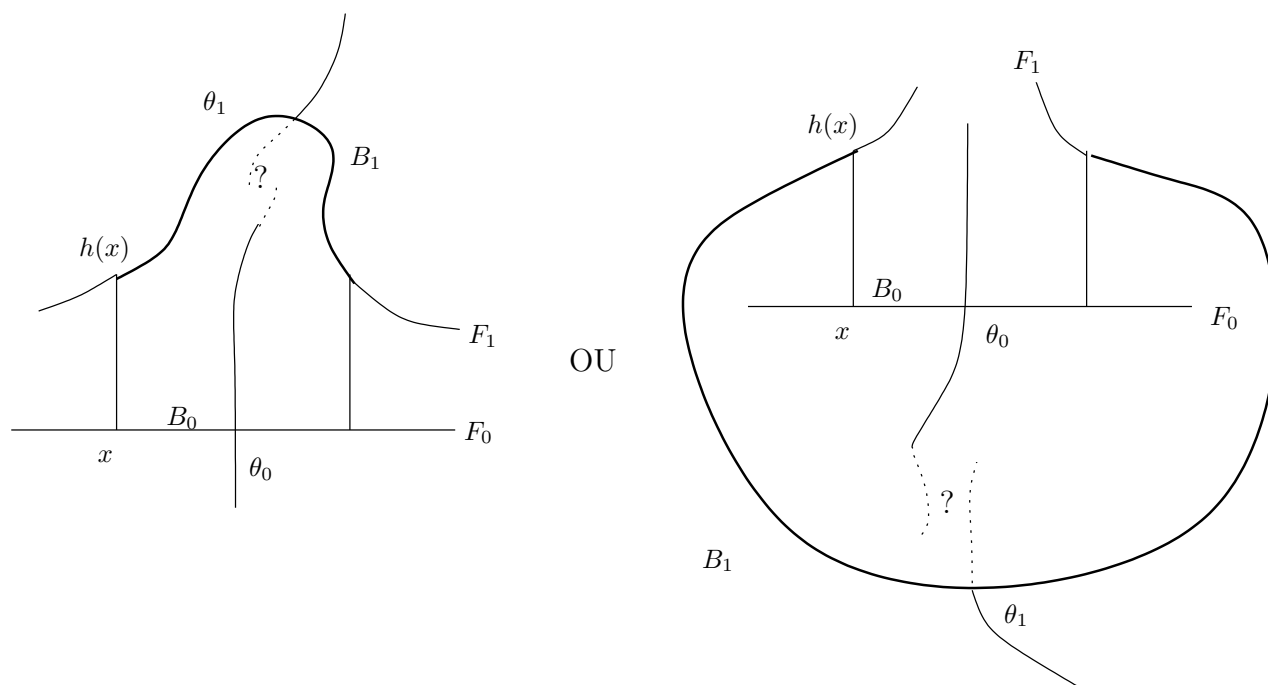


FIGURE 2.

results admit natural extensions to (irreducible) Anosov actions of \mathbb{R}^k , but most work still has to be done.

A *symmetric flow* is the flow defined by a one-parameter subgroup g^t of a Lie group G by right translations on a quotient manifold $\Gamma \backslash G/K$, where Γ is a lattice of G and K a compact subgroup commuting with g^t . In [32] P. Tomter classified Anosov symmetric flows up to finite coverings and conjugacy when G is semisimple or solvable. He proved that in the former case, the symmetric flow is (commensurable to) the geodesic flow of a rank 1 symmetric space, and in the former case, the flow is (commensurable to) the suspension of hyperbolic automorphisms of a compact infranilmanifold. He further pursued his study to the more general case ([33]).

This definition of symmetric flows extends naturally to the notion of symmetric actions of \mathbb{R}^k . It is natural to ask about the classification of these actions for $k > 1$, at least in the case of irreducible actions. But the case $k = 1$ is already quite intricate. In a forecoming paper, we will classify irreducible symmetric actions of \mathbb{R}^k of codimension one: either they are Anosov symmetric flows, or suspensions of hyperbolic automorphisms of tori (cf. examples 2, 3).

In [10], E. Ghys proved that Anosov flows of codimension one on a manifold of dimension ≥ 4 , preserving a volume form and for which the sum of the stable and the unstable bundles is C^1 , is topologically equivalent to the suspension of an Anosov diffeomorphism (and hence a hyperbolic automorphism of the torus). In a forecoming paper, we also extend this result to the $k \geq 2$ case.

Actually, this last statement, for Anosov flows, has been recently highly improved: S. Simic proved that the same conclusion holds if the sum of the stable and unstable bundles is only Lipschitz regular (but this is still a restrictive hypothesis) ([30]), and Asoako furthermore proved that the volume form preserving form hypothesis can be removed ([2]). All these impressive results are important steps towards the Verjovsky conjecture: *every codimension one Anosov flow on a manifold of dimension ≥ 4 is topologically equivalent to a suspension (of a hyperbolic toral automorphism)*. Moreover, S. Simic announced a complete solution of Verjovsky's conjecture ([31]).

Therefore, it seems reasonable to conjecture:

Conjecture. *Every irreducible codimension one Anosov action of \mathbb{R}^k on a manifold of dimension $\geq k + 3$ is topologically conjugate to the suspension of an Anosov action of \mathbb{Z}^k on a closed manifold.*

REFERENCES

- [1] D.V. Anosov. Roughness of geodesic flows on compact manifolds of negative curvature (in Russian). *Dokl. Akad. Nauk SSSR*, 145, 707–709, 1962.
- [2] M. Asaoka. On Invariant volumes of codimension-one Anosov flows and the Verjovsky conjecture. ArXiv math.DS/0703334.
- [3] T. Barbot. Géométrie transverse des flots d'Anosov. Ph D Thesis, ENS Lyon, 1992.
- [4] A. Candel, L. Conlon. Foliations I. Graduate Studies in Mat., Vol. 23, 2000.
- [5] G. Chatelet, H. Rosenberg, and D. Weil. A classification of the topological types of \mathbf{R}^2 -actions on closed orientable 3-manifolds. *Inst. Hautes Études Sci. Publ. Math.*, 43:261–272, 1974.
- [6] C. Conley. Isolated Invariant Sets and the Morse Index. *CBMS Regional Conf. Ser. in Math.*, 38 , Amer. Math. Soc., Providence, R.I. 1978.
- [7] L. J. Díaz, E. Pujals, and R. Ures. Partial hyperbolicity and robust transitivity. *Acta Math.*, 183:1–43, 1999.
- [8] J. Franks. Anosov diffeomorphisms. *Global Analysis (Proc. Symp. Pure Math.)*, 14, Amer. Math. Soc., 61–93, 1970.
- [9] J. Franks and B. Williams. Anomalous Anosov flows. *Lecture Notes in Math.*, 819, Springer, Berlin, 158–174, 1980.

- [10] E. Ghys. Codimension one Anosov flows and suspensions. *Lecture Notes in Math.*, 1331, Springer, Berlin, 59–72, 1988.
- [11] M. Hirsch, C. Pugh and M. Shub. Invariant manifolds. *Lecture Notes in Math.*, 583, Springer, Berlin 1977.
- [12] A. Katok, R. Spatzier. First cohomology of Anosov actions of higher rank abelian groups and applications to rigidity. *Inst. Hautes Études Sci. Publ. Math.*, 79:131–156, 1994.
- [13] A. Katok, R. Spatzier. Differential rigidity of Anosov actions of higher rank abelian groups and algebraic lattice actions. *Proc. Steklov Inst. Math.*, no. 1 (216), 287–314, 1997.
- [14] F. Labourie. Anosov flows, surface groups and curves in projective space. *Invent. Math.*, 165:51–114, 2006.
- [15] S. Matsumoto. Codimension one Anosov flows. *Lecture Notes Series, Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul*, 27, 1995.
- [16] Ming Li, Shaobo Gan and Lan Wen. Robustly transitive singular sets via approach of an extended linear Poincaré flow. *Discrete Contin. Dyn. Syst.*, 13(2):239–269, 2005.
- [17] C. A. Morales, M. J. Pacifico, and E. R. Pujals. Robust transitive singular sets for 3-flows are partially hyperbolic attractors or repellers. *Ann. of Math. (2)*, 160(2):375–432, 2004.
- [18] S. E. Newhouse. On codimension one Anosov diffeomorphisms. *Amer. J. Math.*, 92:761–770, 1970.
- [19] M. Oka. Expansive flows and their centralizers. *Nagoya Math. J.*, 64: 1–15, 1976.
- [20] J. Palis and J.C. Yoccoz. Centralizers of Anosov diffeomorphisms on tori. *Ann. Sci. École Norm. Sup.*, (4) 22: 99–108, 1989.
- [21] C.F.B. Palmeira. Open manifolds foliated by planes. *Ann. of Math.*, 107:109–131, 1978.
- [22] M. M. Peixoto. Structural stability on two-dimensional manifolds. *Topology*, 1:101–120, 1962.
- [23] C. Pugh. The closing lemma. *Amer. J. of Math.*, 89:956–1009, 1967.
- [24] C. Pugh and M. Shub. Ergodicity of Anosov actions. *Invent. Math.*, 15:1–23, 1972.
- [25] M.S. Raghunathan. Discrete subgroups of Lie groups. *Springer-Verlag, New York-Heidelberg*, 1972.
- [26] H. Rosenberg and R. Roussarie. Reeb foliations. *Ann. of Math. (2)*, 91:1–24, 1970.
- [27] H. Rosenberg, R. Roussarie, and D. Weil. A classification of closed orientable 3-manifolds of rank two. *Ann. of Math. (2)*, 91:449–464, 1970.
- [28] Robert Roussarie and Daniel Weil. Extension du “closing lemma” aux actions de R^2 sur les variétés de dim = 3. *J. Differential Equations*, 8:202–228, 1970.
- [29] Richard Sacksteder. Foliations and pseudogroups. *Amer. J. Math.*, 87:79–102, 1965.
- [30] S.N. Simic. Codimension one Anosov flows and a conjecture of Verjovsky. *Ergodic Theory Dynam. Systems.*, 17:1211–1231, 1997.
- [31] S.N. Simic. Volume preserving codimension one Anosov flows in dimensions greater than three are suspensions. ArXiv math.DS/0508024.
- [32] P. Tomter. Anosov flows on infra-homogeneous spaces. *Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968)*, 299–327, 1970.
- [33] P. Tomter. On the classification of Anosov flows. *Topology*, 14:179–189, 1975.
- [34] Thérèse Vivier. Flots robustement transitifs sur les variétés compactes. *C. R. Math. Acad. Sci. Paris*, 337(12):791–796, 2003.
- [35] Verjovsky, Alberto. Codimension one Anosov flows. *Bol. Soc. Mat. Mexicana*, 19(2):49–77, 1974.

THIERRY BARBOT, CNRS, UMR 5669, UMPA, ENS LYON 46, ALLÉE D'ITALIE 69364 LYON,

CARLOS MAQUERA, CNRS, UMR 5669, UMPA, ENS LYON 46, ALLÉE D'ITALIE 69364 LYON,
AND, UNIVERSIDADE DE SÃO PAULO - SÃO CARLOS, INSTITUTO DE CIÊNCIAS MATEMÁTICAS E DE
COMPUTAÇÃO, AV. DO TRABALHADOR SÃO-CARLENSE 400, 13560-970 SÃO CARLOS, SP, BRAZIL

E-mail address: `barbot@umpa.ens-lyon.fr`

E-mail address: `camaquer@umpa.ens-lyon.fr`