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Insertion and expansion operations for n-Dimensional Generalized Maps[⋆] [⋆⋆]

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Abstract. Hierarchical representations, such as irregular pyramids, are the bases of several applications in the field of discrete imagery. So, n-dimensional "bottom-up" irregular pyramids can be defined as stacks of successively reduced n-dimensional generalized maps (n-G-maps) [11], each n-G-map being defined from the previous level by using removal and contraction operations defined in [8]. Our goal is to build a theoretical framework for defining and handling n-dimensional "top-down" irregular pyramids. To do so, we propose in this paper to study the definition of both insertion and expansion operations that allow to conceive these kinds of pyramids.

1 Introduction

Hierarchical representations form the bases of several applications in the field of discrete imagery. Our goal is the study of basic problems related to the definition of hierarchical structures. To achieve this goal, removal and contraction operations have been defined in [8]. In this paper, we define two others basic operations: *insertion* and *expansion*, which allow to define "top-down" pyramids.

Many works deal with regular or irregular image pyramids for multi-level analysis and treatments. The first ones are [7, 12, 16, 17]. In irregular pyramids, each level represents a partition of the pixel set into cells, i.e. *connected subsets of pixels*. There are two ways to build an irregular pyramid: "botom-up" and "top-down"[4]. In the first case, the number of cells increases between two contiguous levels of a pyramid, while in the second case this number of cells decreases. To manipulate these models, it is neccessary to handle a (topological) representation and some basic operations, for instance dual graphs [13] and removal and contraction operations [8].

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[4] In the following sections, we use the terms "top-down" and "bottom-up" pyramids to refer to the pyramids built by using "top-down" and "bottom-up" approaches, respectively.

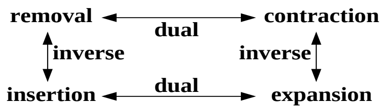


Fig. 1. Links between the basic operations which allow to handle irregular pyramids.

Grasset-Simon and *al* [11] build a theoretical framework for defining and handling n-dimensional "bottom-up" irregular pyramids. To do so, they use the removal and contraction operations, defined in [8], in order to get consistent definitions of data structures for any dimension. Our goal is the same as [11]: build a theoretical framework, but for defining and handling n-dimensional "top-down" irregular pyramids. Therefore, we study the definition of insertion and expansion of i-dimensional cells within n-dimensional objects, in order to define the relations between two consecutive levels of a "top-down" pyramid. We also study the definition of these operations in order to set the links between "bottom-up" and "top-down" irregular pyramids. Indeed, insertion and expansion operations (basic operations used for the definition of "top-down" irregular pyramids) are, respectively, the inverse of removal and contraction operations (Fig. 1).

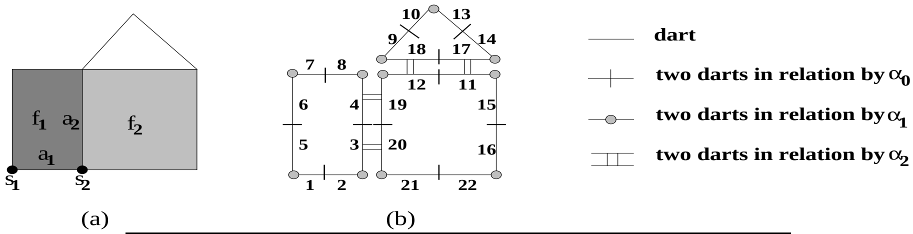
We choose to study the definitions of insertion and expansion operations for n-dimensional generalized maps, since this model enables us to unambiguously represent the topology of quasi-manifolds, which is a well-defined class of subdivisions [15]. Note that several models based on combinatorial maps [9] have been proposed for handling two-dimensional [6, 10] and three-dimensional segmented or multi-level images [1–3]. We prefer to use generalized maps instead of combinatorial maps, since their algebraic definition is homogeneous; therefore, we can provide simpler definitions of data structures and operations with generalized maps, which are also more efficiency for the conception of softwares (note that several kernels of geometric modeling softwares are based upon data structures derived from this notion). Last, we know how to deduce combinatorial maps from generalized maps, so the results presented in this article can be extended to combinatorial maps. Precise relations between generalized, combinatorial maps and other classical data structures are presented in [14].

We recall in Section 2 the notion of generalized maps, removal and contraction operations. Then we define the insertion operation of one i-dimensional cell in Section 3, and expansion operation by duality (in Section 4). In Section 5, we show that it is possible to simultaneously insert and expand several cells of the same dimension. Last, we conclude and give some perspectives in Section 6.

2 Recalls

2.1 Generalized maps

An n-dimensional generalized map is a set of abstract elements, called darts, and applications defined on these darts:



	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
α_0	2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15	18	17	20	19	22	21
α_1	5	3	2	8	1	7	6	4	18	13	15	19	10	17	11	22	14	9	12	21	20	16
α_2	1	2	20	19	5	6	7	8	9	10	17	18	13	14	15	16	11	12	4	3	21	22

Fig. 2. (a) A 2D subdivision. (b) The corresponding 2-G-map (involutions are explicitly given in the array). Darts are represented by numbered black segments.

Definition 1. *(Generalized map) Let $n \geq 0$. A n-dimensional generalized map (or n-G-map) is $G = (B, \alpha_0, \ldots, \alpha_n)$ where:*

1. *B is a finite set of darts;*
2. *$\forall i, 0 \leq i \leq n, \alpha_i$ is an involution[5] on B;*
3. *$\forall i, j, 0 \leq i < i+2 \leq j \leq n, \alpha_i \alpha_j$ is an involution (condition of quasi-manifolds).*

Let G be an n-G-map, and S be the corresponding subdivison. A dart of G corresponds to an (n+1)-tuple of cells $(c_0, ..., c_n)$, where c_i is an i-dimensional cell that belongs to the boundary of c_{i+1} [4]. α_i associates darts corresponding with $(c_0, \ldots, c_n)$ and $(c'_0, \ldots, c'_n)$, where $c_j = c'_j$ for $j \neq i$, and $c_i \neq c'_i$ (α_i swaps the two i-cells that are incident to the same $(i-1)$ and $(i+1)$-cells). When two darts b_1 and b_2 are such that $b_1 \alpha_i = b_2$ $(0 \leq i \leq n)$, b_1 is said i-sewn with b_2. Moreover, if $b_1 = b_2$ then b_1 is said i-free. In Fig. 2, Dart 1 corresponds to (s_1, a_1, f_1), dart $2 = 1\alpha_0$ corresponds to (s_2, a_1, f_1), $3 = 2\alpha_1$ corresponds to (s_2, a_2, f_1), and $20 = 3\alpha_2$ corresponds to (s_2, a_2, f_2). G-maps provide an implicit representation of cells:

Definition 2. *(i-cell) Let G be an n-G-map, b a dart and $i \in N = \{0, ., n\}$. The i-cell incident to b is the orbit[6] $\langle\rangle_{N-\{i\}}(b) = \langle \alpha_0, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n \rangle (b)$.*

An i-cell is the set of all darts which can be reached starting from b, by using any combination of all involutions except α_i. In Fig. 2, the 0-cell (vertex) incident to dart 2 is the orbit $\langle \alpha_1, \alpha_2 \rangle (2) = \{2, 3, 20, 21\}$, the 1-cell (edge) incident to dart 3 is $\langle \alpha_0, \alpha_2 \rangle (3) = \{3, 4, 19, 20\}$, and the 2-cell (face) incident to dart 9 is $\langle \alpha_0, \alpha_1 \rangle (9) = \{9, 10, 13, 14, 17, 18\}$. The set of i-cells is a partition of the darts of

[5] An involution f on S is a one mapping from S onto S such that $f = f^{-1}$.

[6] Let $\{\Pi_0, ..., \Pi_n\}$ be a set of permutations on B. The orbit of an element b relatively to this set of permutations is $\langle \Pi_0, ..., \Pi_n \rangle (b) = \{\Phi(b), \Phi \in \langle \Pi_0, ..., \Pi_n \rangle\}$, where $\langle \Pi_0, \ldots, \Pi_n \rangle$ denotes the group of permutations generated by $\{\Pi_0, \ldots, \Pi_n\}$.

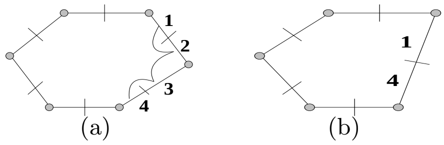


Fig. 3. 0-removal in 1D. (a) Initial 1-G-map. (b) Result. Vertex $C = \langle\alpha_1\rangle(2) = \{2,3\}$ and $C\alpha_0 = \{1,4\} = B^S$. 0-removal consists in setting $1\alpha'_0 = 1\,(\alpha_0\alpha_1)\,\alpha_0 = 4 \in B^S$ and $4\alpha'_0 = 4\,(\alpha_0\alpha_1)\,\alpha_0 = 1 \in B^S$.

the G-map, for each i between 0 and n. Two cells are disjoined if their intersection is empty, i.e. when no dart is shared by the cells. More details about G-maps are provided in [15].

2.2 Removal and contraction operations

In a general way for an n-dimensional space, the removal of an i-cell consists in removing this cell and in merging its two incidents $(i+1)$-cells: so removal can be defined for $0\dots(n-1)$-cells.

Definition 3. *(i-cell removal [8]) Let $G = (B, \alpha_0, \dots, \alpha_n)$ be an n-G-map, $i \in \{0,\dots,n-1\}$ and $C = \langle\rangle_{N-\{i\}}(b)$ be an i-cell, such that: $\forall b' \in C, b'\alpha_{i+1}\alpha_{i+2} = b'\alpha_{i+2}\alpha_{i+1}$[7]. Let $B^S = C\alpha_i - C$, the set of darts i-sewn to C that do not belong to C (Fig. 3). The n-G-map resulting from the removal of C is $G' = (B', \alpha'_0, \dots, \alpha'_n)$ defined by:*

- $B' = B - C$;
- $\forall j \in \{0,\dots,n\} - \{i\}, \alpha'_j = \alpha_j | B'$; [8]
- $\forall b' \in B' - B^S, b'\alpha'_i = b'\alpha_i$;
- $\forall b' \in B^S, b'\alpha'_i = b'\,(\alpha_i\alpha_{i+1})^k\,\alpha_i$ *(where k is the smallest integer such that $b'\,(\alpha_i\alpha_{i+1})^k\,\alpha_i \in B^S$).*

The last expression redefines involution α_i for each dart $b \in B^S$ in G. Indeed, the image of b by α'_i is dart $b' \in B^S$ such that b and b' both are ends of path $(\alpha_i\alpha_{i+1})^k\,\alpha_i$. In Fig. 3a, darts 1 and 4 are the extremities of path $(\alpha_0\alpha_1)\,\alpha_0$ which is represented by a curving line.

Note that G' can contain only one n-cell, and may even be empty if G contains only one i-cell. Note also that contraction operation can be defined directly by duality (see section 4 for explanations on duality in G-map). More details about removal and contraction operations are provided in [8].

[7] This constraint corresponds to the fact that the local degree of i-cell C is 2 (a vertex locally incident to exactly two edges or an edge locally incident to two faces or a face locally incident to two volumes...).

[8] α'_j is equal to α_j restricted to B', i.e. $\forall b \in B', b\alpha'_j = b\alpha_j$.

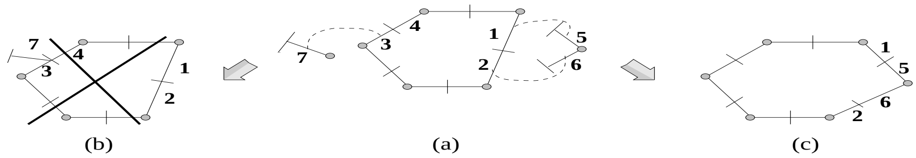


Fig. 4. 0-insertion in 1D. (a) Initial 1-G-map G corresponds to the orbit $\langle \alpha_0, \alpha_1 \rangle (1)$, 0-cell $C_1 = \langle \alpha_1 \rangle (7) = \{7\}$, $E_1 = \{3\}$, $F_1 = \{7\}$, involution γ is represented by dashed lines: γ sews darts $1, 2$ and 3 with darts $5, 6$ and 7, respectively. 0-cell $C_2 = \langle \alpha_1 \rangle (5) = \{5, 6\}$, $E_2 = \{1, 2\}$, $F_2 = \{5, 6\}$. (b) Invalid result after inserting C_1 into G (α_0 is no more an involution since $3\alpha_0 = 7$ and $4\alpha_0 = 3$). (c) Valid result after inserting C_2 into G (the precondition $b\alpha_0 = b\gamma\alpha_1\gamma$ is satisfied for all darts of E_2).

3 Insertion

The insertion of an i-cell C into an n-G-map G consists (conversely to removal operation) in adding this cell to G and splitting an $(i+1)$-cell where C should be inserted: so insertion can be defined for $0 \ldots (n-1)$-cells. In this section, we present some useful cases of 0 and 1-insertions in 1- and 2-dimensional space before giving the general definition in nD.

3.1 Dimension 1: 0-insertion

In dimension 1, only the 0-insertion exists, which consists in adding a vertex and splitting an edge where that vertex should be inserted. Let $G = (B, \alpha_0, \alpha_1)$ be the initial 1-G-map, $C = \langle \alpha_1 \rangle (b)$ be the vertex to insert (belonging to another G-map) and E and F be two subsets of darts such that: $E \subseteq B$ and $F \subseteq C$. These subsets allow to explicit where and how cell C will be inserted in G-map G by using an additional involution[9] γ (Fig. 4). The 1-G-map resulting from 0-insertion, called $G' = (B', \alpha'_0, \alpha'_1)$, is obtained only by redefining α_0 for the darts of E and F as follows: $\forall b \in E \cup F, b\alpha'_0 = b\gamma$, where $B' = B \cup C$ (α'_0 is unchanged for the darts of $B' - (E \cup F)$, $\alpha'_0 = \alpha_0$ and α'_1 is unchanged for the darts of B', $\alpha'_1 = \alpha_1$). Furthermore, 0-insertion can be applied only if the following preconditions are satisfied: (1) $\forall b \in C, b$ is 0-free and (2) $\forall b \in E, b\alpha_0 = b\gamma\alpha_1 (\alpha_0\alpha_1)^k \gamma$ (where k is the smallest integer such that $b\gamma\alpha_1 (\alpha_0\alpha_1)^k \in F$). These constraints ensure the validity of the operation and the fact that 0-insertion is the inverse operation of 0-removal, by avoiding the following cases:

1. $b \in E$ is not 0-free and $b\gamma$ is 1-free. In such a case, we have $C = F = \{b\gamma\}$, $b' = b\alpha_0 \in B - E$ and $b' \neq b\gamma$. Thus, we have $b'\alpha'_0 = b'\alpha_0 = b$ and $b'\alpha'_0\alpha'_0 = b\alpha'_0 = b\gamma \neq b'$. α'_0 is not well-defined since it is not an involution (Fig. 4b);
2. $b \in E$ is not 0-free and it does not exist a path between darts b and $b\alpha_0$ expressed by the following composition of involutions $\gamma\alpha_1 (\alpha_0\alpha_1)^k \gamma$. Indeed, for the same reason that the previous case, α'_0 is not well-defined and is not an involution;

[9] γ is defined on set $E \cup F$ as follows: $b \in E \Leftrightarrow b\gamma \in F$.

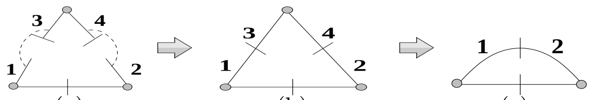


Fig. 5. Example where the precondition of the 0-insertion is not satisfied. (a) Darts of the vertex C to insert and to remove are numbered 3 and 4. Involution γ is marked with dashed line. E and F are two subsets of 0-free darts defined as: $E = \{1, 2\}$ and $F = \{3, 4\}$. (b) Resulting 1-G-map G' after sewing the darts of F with those of E (0-insertion without precondition). (c) Resulting 1-G-map after removing C from G': darts 1 and 2 are now sewed by α_0, whereas they were 0-free before insertion.

3. it exists a path[10] between two darts b and b' of F such that $b \neq b'$ and $b\gamma$ is 0-free or $b\gamma\alpha_0 \neq b'\gamma$. In such a case, we do not obtain the same 1-G-map G after successively inserting then removing a 0-cell C; hence, the 0-insertion is not the inverse operation of 0-removal (Fig. 5).

In Fig. 4c, the 0-insertion of C_2 is obtained by redefining α_0 for the darts of $E_2 = \{1, 2\}$ and $F_2 = \{5, 6\}$ such that: $1\alpha'_0 = 5$ and $2\alpha'_0 = 6$. Note that α_1 is not modified by 0-insertion.

3.2 Dimension 2

0-insertion It consists in adding a 0-cell $C = \langle\alpha_1, \alpha_2\rangle(b)$ into an initial 2-G-map $G = (B, \alpha_0, \alpha_1, \alpha_2)$. There are several ways to carry out this operation. Indeed, several possible combinations enable to link the darts of C with those of B (Figs. 6b and 6c). Then, it is necessary to remove this ambiguity by defining an involution that enables to characterize these links in a single way. Let γ be this involution and E and F be two subsets of darts (γ, E and F are defined as above and respect the same properties). The 2-G-map resulting from 0-insertion, called $G' = (B', \alpha'_0, \alpha'_1, \alpha'_2)$, is obtained by redefining α_0 for the darts of E and F as follows: $\forall b \in E \cup F, b\alpha'_0 = b\gamma$. In Fig. 6c, 0-insertion consists in setting $1\alpha'_0 = 5$, $2\alpha'_0 = 8$, $3\alpha'_0 = 7$ and $4\alpha'_0 = 6$. Note that this redefinition of α_0 is the same as for dimension 1 but concerns different darts, since it is a 0-cell within a 2D object (intuitively, in the general case, this operation consists in applying the 0-insertion defined for dimension 1 twice).

0-insertion can be applied only if the following preconditions are satisfied:

1. $\forall b \in E \cup F$ such that $b\alpha_2 \in E \cup F$ then $b\alpha_2\gamma = b\gamma\alpha_2$. This constraint enables to guarantee the quasi-manifold of the resulting 2-G-map: $\forall b \in E \cup F, \alpha'_0\alpha'_2$ is an involution, by using the following substitutions: $\forall b \in E \cup F, b\gamma = b\alpha'_0$ and $\forall b \in E \cup F, b\alpha_2 = b\alpha'_2$. Then, we obtain $\forall b \in E \cup F, b\alpha_2\gamma = b\gamma\alpha_2 \Leftrightarrow b\alpha'_2\alpha'_0 = b\alpha'_0\alpha'_2$. Thus, $\alpha'_0\alpha'_2$ is an involution for all darts in $E \cup F$;

[10] The path between two darts b and b' of F is expressed by the following composition of involutions $\alpha_1(\alpha_0\alpha_1)^k$, where k is the smallest integer such that $b\alpha_1(\alpha_0\alpha_1)^k = b'$.

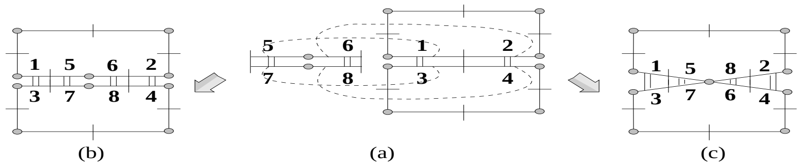


Fig. 6. 0-insertion in 2D. (a) 0-cell $C = \langle\alpha_1, \alpha_2\rangle(5) = \{5, 6, 7, 8\}$, $E = \{1, 2, 3, 4\}$, $F = C$, initial 2-G-map G corresponds to the orbit $\langle\alpha_0, \alpha_1, \alpha_2\rangle(1)$. Involution γ_1 is represented by dashed lines and links darts $1, 2, 3$ and 4 with $5, 6, 7$ and 8, respectively. Involution γ_2 (not shown on the figure) links darts $1, 2, 3$ and 4 with $5, 8, 7$ and 6, respectively. (b) Resulting 2-G-map after inserting C into G by using involution γ_1. (c) Resulting 2-G-map after inserting C into G by using involution γ_2.

2. $\forall b \in C, b\alpha_1\alpha_2 = b\alpha_2\alpha_1$: this constraint corresponds, in the general case, to the fact that the local degree of the vertex is equal to 2. If this constraint is not satisfied, we cannot remove C after inserting it. Indeed, $\forall b \in C, b\alpha_1\alpha_2 = b\alpha_2\alpha_1$ is also a precondition of the removal operation [8]. Thus, in order to define insertion operation as inverse of removal operation, it is necessary to check this precondition;
3. $\forall b \in E, b\alpha_0 = b\gamma\alpha_1 (\alpha_0\alpha_1)^k \gamma$ (where k is the smallest integer such that $b\gamma\alpha_1 (\alpha_0\alpha_1)^k \in F$): this constraint is the same as the one defined in the previous section.

1-insertion It consists in adding a 1-cell $C = \langle\alpha_0, \alpha_2\rangle(b)$ into an initial 2-G-map $G = (B, \alpha_0, \alpha_1, \alpha_2)$. Let E and F be two subsets of darts and γ an involution (E, F and γ are defined as in subsection 3.1). The 2-G-map G', resulting from 1-insertion, is obtained by redefining α_1 for the darts of E and F as follows: $\forall b \in E \cup F, b\alpha'_1 = b\gamma$. Examples of this operation are represented in Figs. 7 and 8. 1-insertion can be applied only if the following precondition is satisfied: $\forall b \in E, b\alpha_1 = b\gamma\alpha_2 (\alpha_1\alpha_2)^k \gamma$ (where k is the smallest integer such that: $b\gamma\alpha_2 (\alpha_1\alpha_2)^k \in F$).

The first two preconditions, defined in the previous subsection, are always satisfied. Indeed, for the first case, G' is a quasi-manifold since only α_1 is redefined by 1-insertion operation, and $\alpha_0\alpha_2$ is an involution in G; for the second case, the local degree of an edge in a 2-dimensional quasi-manifold is always equal to 2.

3.3 Dimension n

The general definition of i-cell insertion for an n-dimensional G-map is a direct extension of the previous cases. Let C be the i-cell to insert, E and F be two subsets of darts. i-insertion of C in G is carried out by sewing darts of F with those of E by an involution γ. Then i-insertion consists in redefining α_i for the darts of E and F in the following way: $b\alpha'_i = b\gamma$. We obtain the general definition of the i-insertion operation:

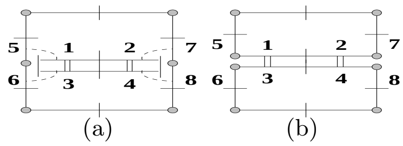


Fig. 7. 1-insertion in 2D in the general case. (a) Darts of the edge to insert are numbered 1, 2, 3 and 4. Involution γ is represented by dashed line. (b) Result. The precondition of the 1-insertion is satisfied: $5\alpha_1 = 5\gamma\alpha_2\gamma = 6$ (here, $k = 0$).

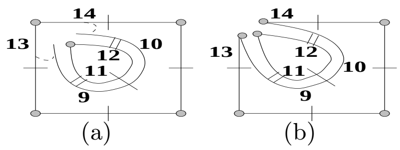


Fig. 8. 1-insertion of a loop. (a) Initial 2-G-map with the loop to insert and involution γ. (b) Result. The precondition of the 1-insertion is satisfied: $13\alpha_1 = 13\gamma\alpha_2 (\alpha_1\alpha_2) \gamma = 14$.

Definition 4. *(i-cell insertion) Let* $G = (B, \alpha_0, \alpha_1, \ldots, \alpha_n)$ *be an n-G-map,* $i \in \{0, \ldots, n-1\}$, $C = \langle \alpha_0, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n \rangle (b)$ *be the i-cell to insert, E and F be two subsets of darts such that:* $E \subseteq B$ *and* $F \subseteq C$ *and* γ *be an involution defined on set* $E \cup F$ *such that:* $b \in E \Leftrightarrow b\gamma \in F$. *This operation can be applied only when:*

- $\forall b \in F, b$ *is i-free;*
- $\forall b \in C, b\alpha_{i+1}\alpha_{i+2} = b\alpha_{i+2}\alpha_{i+1}$[11]*;*
- $\forall b \in E \cup F, \forall j\ 0 \leq j \leq n$ *such that* $|i - j| \geq 2$ *and* $b\alpha_j \in E \cup F$ *then* $b\alpha_j\gamma = b\gamma\alpha_j$*;*
- $\forall b \in E, b\alpha_i = b\gamma\alpha_{i+1} (\alpha_i\alpha_{i+1})^k \gamma$ *(where k is the smallest integer such that* $b\gamma\alpha_{i+1} (\alpha_i\alpha_{i+1})^k \in F$*).*

The n-G-map resulting from the insertion of C is $G' = (B', \alpha'_0, \ldots, \alpha'_n)$ *defined by:*

- $B' = B \cup C$*;*
- $\forall j \in \{0, \ldots, n\} - \{i\}, \forall b \in B' : b\alpha'_j = b\alpha_j$*;*
- $\forall b \in B' - (E \cup F) : b\alpha'_i = b\alpha_i$*;*
- $\forall b \in E \cup F : b\alpha'_i = b\gamma$.

The constraints of definition 4 enable to ensure that: the local degree of C is equal to 2, the resulting n-G-map is a quasi-manifold and insertion operation is the inverse of removal operation.

Note that, in the definition of G', only α_i is redefined for the darts of $E \cup F$. Indeed, i-insertion involves the sewing of darts of F with those of E by α'_i which is equal to γ. For everything else, the initial n-G-map remains unchanged. Now, we prove the validity of the operation by showing that the new structure G' is a n-G-map.

Theorem 1 *G' is an n-G-map.*

[11] Note that this condition does not apply for $i = n - 1$, so we can always insert any $(n-1)$-cell.

Proof. We differentiate three cases. First for $j \neq i$, involutions α_j are not redefined but only restricted to the darts of the final G-map. Then, for $j = i$, we distinguish two cases, depending on if darts belong or not to $E \cup F$:

1. for $b_1 \in B - E$: We show that $b_2 = b_1\alpha_i \in B - E$. Assume that $b_2 \notin B - E$, then $b_2 \in E$. In this case, $b_1 = b_2\alpha_i \in E$[12] so $b_1 \notin B - E$: contradiction. Moreover, $b_1\alpha'_i = b_1\alpha_i \in B - E$ and $b_1\alpha'_i\alpha'_i = b_1$: α'_i is well-defined and is an involution (we use the same method for the darts of $C - F$).
2. for $b_1 \in E \cup F$, We show that $b_2 = b_1\gamma \in E \cup F$. That is true because γ is well-defined and is an involution on set $E \cup F$. Moreover, $b_1\alpha'_i = b_1\gamma \in E \cup F$ and $b_1\alpha'_i\alpha'_i = b_1\gamma\gamma = b_1$: α'_i is well-defined and is an involution.

We have now to prove that $\forall j, 0 \leq j \leq n, \forall k, j+2 \leq k \leq n : \alpha'_j\alpha'_k$ is an involution.

- for $j \neq i$ et $k \neq i$: this is obvious since $\alpha'_j = \alpha_j$ and $\alpha'_k = \alpha_k$. As G is a G-map, $\alpha_j\alpha_k = \alpha'_j\alpha'_k$ is an involution.
- for $j = i$: we show that $\forall b_1 \in B'$, we have $b_1\alpha'_i\alpha'_k = b_1\alpha'_k\alpha'_i$:
  1. for $b_1 \in B - E$: $b_1\alpha'_i = b_1\alpha_i$ and $\alpha'_k = \alpha_k$. Since G is a G-map, $b_1\alpha_i\alpha_k = b_1\alpha_k\alpha_i$; since $b_1\alpha_k = b_1\alpha'_k \in B - E$ (indeed, assume that $b_2 = b_1\alpha_k \in E$. In this case, $b_1 = b_2\alpha_k \in E$: contradiction), $b_1\alpha_k\alpha_i = b_1\alpha'_k\alpha_i$ whence $b_1\alpha_k\alpha_i = b_1\alpha'_k\alpha'_i$; thus $b_1\alpha'_i\alpha'_k = b_1\alpha'_k\alpha'_i$.
  2. for $b_1 \in C - F$: similar to the previous case.
  3. for $b_1 \in E$: $b_1\alpha_k \in E$ and $b_1\alpha_k\gamma = b_1\gamma\alpha_k$ (precondition of the insertion operation). So, $b_1\alpha'_i = b_1\gamma$, $b_1\alpha_k\alpha'_i = b_1\alpha'_i\alpha_k$. Moreover, $\forall b \in B, b\alpha_j = b\alpha'_j$; thus, $b_1\alpha'_j\alpha'_i = b_1\alpha'_i\alpha'_j$ (same method for the darts of F).
- for $k = i$: similar to the previous case. □

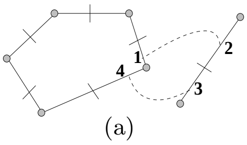
4 Expansion

Informally, i-expansion consists in adding an i-cell "inside" an $(i-1)$-cell. Expansion is the dual of the insertion operation. The dual of a subdivision is a subdivision of the same space, for which an $(n-i)$-cell is associated with each initial i-cell, and incidence relations are kept the same. A nice property of G-maps is the fact that the dual G-map of $G = (B, \alpha_0, \dots, \alpha_n)$ is $G' = (B, \alpha_n, \dots, \alpha_0)$: we only need to reverse the involution order.

We can thus easily deduce the definition of i-expansion from the general definition of i-insertion. We only have to replace '+' with '-' for indices of involutions for preconditions and operations, i.e. $\alpha_{i+1}\alpha_{i+2} \rightarrow \alpha_{i-1}\alpha_{i-2}$ and $\alpha_i\alpha_{i+1} \rightarrow \alpha_i\alpha_{i-1}$ (see two examples of expansion in Figs. 9 and 10).

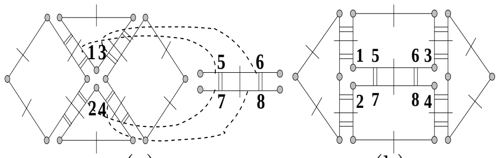
Definition 5. *(i-expansion) Let $G = (B, \alpha_0, \alpha_1, \dots, \alpha_n)$ be an n-G-map, $i \in \{0, \dots, n-1\}$, $C = \langle\rangle_{N-\{i\}}(b)$ be the i-cell to expand, E and F be two subsets of darts such that: $E \subseteq B$ and $F \subseteq C$, γ be an involution defined on set $E \cup F$ such that: $b \in E \Leftrightarrow b\gamma \in F$. This operation can be applied only when:*

[12] According to the previous preconditions, $b_2\alpha_i = b_2\gamma\alpha_{i+1}(\alpha_i\alpha_{i+1})^k\gamma$. Since $b_2\gamma\alpha_{i+1}(\alpha_i\alpha_{i+1})^k \in F$ then $b_2\alpha_i = b_2\gamma\alpha_{i+1}(\alpha_i\alpha_{i+1})^k\gamma \in E$.



(a) (b)

Fig. 9. 1-expansion in 1D. (a) Darts of the edge to expand are numbered 2 and 3. Involution γ are marked with dashed line. (b) Result.



(a) (b)

Fig. 10. 1-expansion in 2D. (a) Initial 2-G-map with the edge to expand and involution γ. (b) Result.

- $\forall b \in F, b$ *is* i*-free;*
- $\forall b \in C, b\alpha_{i-1}\alpha_{i-2} = b\alpha_{i-2}\alpha_{i-1}$;[13]
- $\forall b \in E \cup F, \forall j\ 0 \leq j \leq n$ *such that* $|i-j| \geq 2$ *and* $b\alpha_j \in E \cup F$ *then* $b\alpha_j\gamma = b\gamma\alpha_j$;
- $\forall b \in E$ *et* $b\alpha_i = b\gamma\alpha_{i-1}\left(\alpha_i\alpha_{i-1}\right)^k\gamma$ *(where* k *is the smallest integer such that* $b\gamma\alpha_{i-1}\left(\alpha_i\alpha_{i-1}\right)^k \in F$*).*

The n*-G-map resulting from the expansion of this* i*-cell is* $G' = (B', \alpha'_0, \ldots, \alpha'_n)$ *defined by:*

- $B' = B \cup C$;
- $\forall j \in \{0, \ldots, n\} - \{i\}, \forall b \in B' : b\alpha'_j = b\alpha_j$;
- $\forall b \in B' - (E \cup F) : b\alpha'_i = b\alpha_i$;
- $\forall b \in E \cup F : b\alpha'_i = b\gamma$.

Theorem 2 G' *is an* n*-G-map.*

The proof for the expansion operation is equivalent by duality (exchange $\alpha_{(i+1)}$ and $\alpha_{(i-1)}$) to the proof of theorem 1. □

5 Generalisation

Previous definitions enable us to insert or to expand a single cell. For some applications, it could be more efficient to simultaneously apply several operations. In practice, let G be an n-G-map and G' another n-G-map[14] to insert (resp. to expand). The only difference with the definition of one insertion (resp. expansion) is that G' can contain several i-cells. Thus, the only modification in definition 4 (resp. definition 5) consists in replacing C, the i-cell to insert (resp. to expand) by G', the set of i-cells.

This allows us to apply simultaneously a set of operations and to obtain the same result as if we had successively applied the operations. For this generalisation, there is no additional precondition since i-cells are always disjoined in a n-G-map.

[13] Note that this condition does not apply for $i = 1$, so we can always expand any edge.

[14] By definition, a G-map is always a partition into i -cells, for $0 \leq i \leq n$.

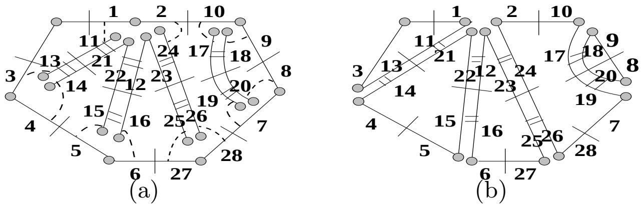


Fig. 11. An example in 2D of simultaneous insertion of 1-cells. (a) 2-G-map before operation with the 1-cells to insert and involution γ (represented by dashed lines). (b) The resulting 2-G-map. The darts belonging to inserted 1-cells are numbered $11, 12, \ldots, 26$. For instance, $1\alpha_1 = 1\gamma\alpha_2\,(\alpha_1\alpha_2)^2\,\gamma = 2$ since three edges $\langle\alpha_0, \alpha_2\rangle\,(11)$, $\langle\alpha_0, \alpha_2\rangle\,(12)$ and $\langle\alpha_0, \alpha_2\rangle\,(23)$ are inserted around the same vertex.

We now show that it is possible to simultaneously perform insertions (resp. expansions) of several i-cells for a given i $(0 \leq i \leq n)$.

Generalisation 1 *We can easily prove that the previous definition of insertion (resp. expansion) stands for the insertion (resp. expansion) of a set of cells with the same dimension i.*

In order to prove this claim, we can follow the same method described in the proof of theorem 1: just consider the sewing between the darts of G and the darts of different i-cells of G'.

In Fig. 11, the 2-G-map G' to insert contains four 1-cells: $\langle\alpha_0, \alpha_2\rangle\,(11)$, $\langle\alpha_0, \alpha_2\rangle$ (12), $\langle\alpha_0, \alpha_2\rangle\,(23)$ and $\langle\alpha_0, \alpha_2\rangle\,(17)$. The first three edges are linked by α_1, and the fourth edge is independent. Furthermore, the simultaneous insertion of these edges consists in sewing the darts of $E = \{1, 2, \ldots, 10, 27, 28\}$ and those of $F = \{11, 12, \ldots, 20, 23, \ldots, 26\}$ (it is the same process as the one given in definition 4).

6 Conclusion and Perspectives

In this paper, we have defined insertion and expansion operations, which can be applied to one i-cell of any n-G-map, whatever their respective dimensions. Moreover, we have studied how to perform the same operations simultaneously. These definitions are homogeneous for any dimension. Since combinatorial maps can be easily deduced from orientable generalized maps, these operations can also be defined on combinatorial maps.

In order to conceive efficient algorithms, an interesting perspective is to perform different operations simultaneously: insertion and expansion of cells of different dimensions. We think that, in this case, we can apply the same preconditions as the ones used for insertion and expansion operations. Next, we will study "top-down" pyramids of n-dimensional generalized maps defined as stacks of n-G-map where each n-G-map is built from the previous level by inserting or expanding cells. This will be made for example within the Fogrimmi project whose goal is to define "top-down" pyramids to analyse biological medical images.

References

1. Bertrand, Y., Damiand, G., Fiorio, C.: Topological encoding of 3d segmented images, in: Discrete Geometry for Computer Imagery, number 1953 in LNCS, Uppsala, Sweden, 2000, pp. 311-324.
2. Braquelaire, J.P., Desbarats, P., Domenger, J.P.: 3d split and merge with 3 maps, in: Workshop on Graph based Representations, Ischia, Italy, May, (2001), pp. 32-43, IAPR-TC15.
3. Braquelaire, J.P., Desbarats, P., Domenger, J.P., Wüthrich, C.A.: A topological structuring for aggregates of 3d discrete objects, in: Workshop on Graph based Representations, Austria, (1999), pp. 193-202, IAPR-TC15.
4. Brisson, E.: Representing geometric structures in d dimensions: topology and order, Discrete and Computational Geometry 9 (1) (1993) 387–426.
5. Brun, L., Kropatsch, W.G.: Dual contraction of combinatorial maps, in: Workshop on Graph based Representations, Austria, (1999) pp. 145-154, IAPR-TC15.
6. Brun, L.: Segmentation d'images couleur à base topologique, PhD Thesis, Université de Bordeaux I, 1996.
7. Burt, P., Hong, T.H., Rosenfeld, A.: Segmentation and estimation of image region properties through cooperative hierarchical computation. IEEE Transactions on Systems, Man and Cybernetics 11 (1981) 802-809.
8. Damiand, G., Dexet-Guiard, M., Lienhardt, P., Andres, E.: Removal and contraction operations to define combinatorial pyramids: application to the design of a spatial modeler, Image and Vision Computing 2 (23) (2005) 259–269.
9. Edmonds, J.: A combinatorial representation for polyhedral surfaces, Notices of the American Mathematical Society 7 (1960).
10. Fiorio, C.: Approche interpixel en analyse d'images: une topologie et des algorithms de segmentation, PhD Thesis, Université Montpellier II, 24 (1995).
11. Grasset-simon, C., Damiand, G., Lienhardt, P.: nD generalized map pyramids: Definition, representations and basic operations, Pattern Recognition 39 (2006) 527-538.
12. Jolion, J., Montanvert, A.: The adaptive pyramid : a framework for 2d image analysis. Computer Vision, Graphics and Image Processing 55 (1992) 339-348.
13. Kropatsch, W.: Building irregular pyramids by dual-graph contraction. Vision, Image and Signal Processing 142 (1995) 366-374.
14. Lienhardt, P.: Topological models for boundary representation: a comparison with n-dimensional generalized maps. Commputer Aided Design 23 (1991) 59-82.
15. Lienhardt, P.: N-dimensional generalized combinatorial maps and cellular quasi-manifolds, International Journal of Computational Geometry and Applications 4 (3) (1994) 275–324.
16. Meer, P.: Stochastic image pyramids. Computer Vision, Graphics and Image Processing 45 (1989) 269-294.
17. Moutanvert, A., Meer, P., Rosenfeld, A.: Hierarchical image analysis using irregular tesselations, IEEE Transactions on Pattern Analysis and Machine Intelligence 13 (4) (1991) 307-316.