SCATTERING NORM ESTIMATE NEAR THE THRESHOLD FOR ENERGY-CRITICAL FOCUSING SEMILINEAR WAVE EQUATION

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ABSTRACT. We consider the energy-critical semilinear focusing wave equation in dimension $N = 3,4,5$. An explicit solution W of this equation is known. By the work of C. Kenig and F. Merle, any solution of initial condition (u_0,u_1) such that $E(u_0,u_1) < E(W,0)$ and $\|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}$ is defined globally and has finite $L_{t,x}^{\frac{2(N+1)}{N-2}}$-norm, which implies that it scatters. In this note, we show that the supremum of the $L_{t,x}^{\frac{2(N+1)}{N-2}}$-norm taken on all scattering solutions at a certain level of energy below $E(W,0)$ blows-up logarithmically as this level approaches the critical value $E(W,0)$. We also give a similar result in the case of the radial energy-critical focusing semilinear Schrödinger equation. The proofs rely on the compactness argument of C. Kenig and F. Merle, on a classification result, due to the authors, at the energy level $E(W,0)$, and on the analysis of the linearized equation around W.

1. Introduction

We consider the focusing energy-critical wave equation on an interval I $(0 \in I)$

$$\begin{cases} \partial_t^2 u - \Delta u - |u|^{\frac{4}{N-2}} u = 0, & (t,x) \in I \times \mathbb{R}^N \\ u_{\restriction t=0} = u_0 \in \dot{H}^1, & \partial_t u_{\restriction t=0} = u_1 \in L^2, \end{cases} \tag{1.1}$$

where u is real-valued, $N \in \{3,4,5\}$, $L^2 := L^2(\mathbb{R}^N)$ and $\dot{H}^1 := \dot{H}^1(\mathbb{R}^N)$. The equation (1.1) is locally well-posed in $\dot{H}^1 \times L^2$ (see [Pec84], [GSV92] and [SS94]): if $(u_0,u_1) \in \dot{H}^1 \times L^2$, there exists an unique solution u, defined on a maximal time of existence $I_{\max}$ and such that for all interval J

$$J \Subset I_{\max} \Longrightarrow \|u\|_{S(J)} < \infty, \text{ where } S(J) := L^{\frac{2(N+1)}{N-2}}\left(J \times \mathbb{R}^N\right).$$

Furthermore, the solution u of (1.1) scatters forward in time in $\dot{H}^1 \times L^2$ if and only

$$[0,+\infty) \subset I_{\max} \text{ and } \|u\|_{S(0,+\infty)} < \infty.$$

Thus the norm $S(\mathbb{R})$ measures the nonlinear effect for a given solution. The energy

$$E(u(t),\partial_t u(t)) = \frac{1}{2}\int |\partial_t u(t,x)|^2 dx + \frac{1}{2}\int |\nabla u(t,x)|^2 dx - \frac{N-2}{2N} |u(t,x)|^{\frac{2N}{N-2}} dx$$

is conserved for solutions of (1.1).

The *defocusing* case (equation (1.1) with sign $+$ instead of $-$ in front of the nonlinearity) has been the object of intensive studies in the last decades (see for example [SS98] and references

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therein). In this case the solutions are known to scatter, which implies, for any solution u, a bound of the norm $S(\mathbb{R})$ by an unspecified function of the defocusing energy

$$E_d = \frac{1}{2}\int |\partial_t u|^2 + \frac{1}{2}\int |\nabla u|^2 + \frac{N-2}{2N}\int |u|^{\frac{2N}{N-2}}.$$

In three spatial dimension, an explicit upper bound was proven by T. Tao [Tao06]: for any solution u of the defocusing equation,

$$\|u\|_{L^4_t L^{12}_x} \leq C(1+E_d)^{CE_d^{105/2}},$$

which gives, by Strichartz and interpolation estimate, a similar bound for $\|u\|_{S(\mathbb{R})}$.

Going back to the *focusing* case, consider the explicit $\dot{H}^1$ stationnary solution of (1.1)

$$W := \frac{1}{\left(1+\frac{|x|^2}{N(N-2)}\right)^{\frac{N-2}{2}}}. \tag{1.2}$$

In [KM06b], C. Kenig and F. Merle have described the dynamics of (1.1) below the energy threshold $E(W,0)$. Namely, if $E(u_0,u_1) < E(W,0)$, then $\int |\nabla u_0|^2 \neq \int |\nabla W|^2$ and the solution u scatters (both forward and backward in time) if and only if $\int |\nabla u_0|^2 < \int |\nabla W|^2$. This implies that for $\varepsilon > 0$ the following supremum is finite:

$$\mathcal{I}_\varepsilon = \sup_{u\in F_\varepsilon} \int_{\mathbb{R}\times\mathbb{R}^N} |u(t,x)|^{\frac{2(N+1)}{N-2}}\, dtdx = \sup_{u\in F_\varepsilon} \|u\|_{S(\mathbb{R})}^{\frac{2(N+1)}{N-2}},$$

where

$$F_\varepsilon := \left\{ u \text{ solution of (1.1) such that } E(u_0,u_1) \leq E(W,0) - \varepsilon^2 \text{ and } \int |\nabla u_0|^2 < \int |\nabla W|^2 \right\}.$$

Furthermore, the existence of the non-scattering solution W at the energy threshold shows that

$$\lim_{\varepsilon\to 0^+} \mathcal{I}_\varepsilon = +\infty.$$

The purpose of this note is to give an equivalent of $\mathcal{I}_\varepsilon$ for small ε. Consider the negative eigenvalue $-\omega^2$ ($\omega > 0$) of the linearized operator associated to (1.1) around W:

$$-\omega^2 = \inf_{\substack{u\in H^1\\ \int u^2=1}} \int_{\mathbb{R}^N} |\nabla u|^2 - \frac{N+2}{N-2}\int_{\mathbb{R}^N} W^{\frac{4}{N-2}}|u|^2.$$

(See §3.1 for details). Then

Theorem 1.

$$\lim_{\varepsilon\to 0^+} \frac{\mathcal{I}_\varepsilon}{|\log\varepsilon|} = \frac{2}{\omega}\int_{\mathbb{R}^N} W^{\frac{2(N+1)}{N-2}}.$$

Remark 1.1. It would be interesting to get an explicit value of the limit $\frac{2}{\omega}\int_{\mathbb{R}^N} W^{\frac{2(N+1)}{N-2}}$. A straightforward computation gives:

$$\int_{\mathbb{R}^N} W^{\frac{2(N+1)}{N-2}} = \frac{(N(N-2))^{\frac{N}{2}}}{2^{2N+1}} \times \frac{N!}{\left(\left(\frac{N}{2}\right)!\right)^2} \times \pi \qquad \text{if } N \text{ is even,}$$

$$\int_{\mathbb{R}^N} W^{\frac{2(N+1)}{N-2}} = \frac{(N(N-2))^{\frac{N}{2}}}{2} \times \frac{\left(\frac{N-1}{2}\right)!}{N!} \qquad \text{if } N \text{ is odd.}$$

However we do not know any explicit expression of ω.

Let us give an outline of the proof of Theorem 1. In Section 2, we show that a sequence of solutions (u_n) such that

$$E(u_n(0), \partial_t u_n(0)) < E(W, 0), \qquad \int |\nabla u_n(0)|^2 < \int |\nabla W|^2 \text{ and } \lim_{n\to+\infty} \|u_n\|_{S(\mathbb{R})} = +\infty$$

must converge to W up to modulation for a well-chosen time sequence. This relies on the compactness argument of [KM06b, Section 4], using the profile decomposition of Bahouri-Gérard [BG99], and on the classification of the solutions of (1.1) at the threshold of energy in our previous work [DM07b]. The second step of the proof is an analysis of the behaviour of solutions whose initial conditions are close to $(W, 0)$, which is carried out in Section 3. We show, as a consequence of the existence of the negative eigenvalue $-\omega^2$, that such solutions go away from the solution W in a time which is of logarithmic order with respect to the distance of the initial condition to $(W, 0)$. In Section 4 we put together the preceding arguments to prove Theorem 1.

Our arguments do not depend strongly on the nature of equation (1.1), and we except that a logarithmic estimate of the scattering norm $S(\mathbb{R})$ near the threshold holds in similar situations, as long as the linearized operator around the ground state admits real nonzero eigenvalues. In Section 5 we give a result and a sketch of proof in the case of the radial, energy-critical focusing nonlinear Schrödinger equation.

2. Convergence to W and W^- near the threshold

In all the article, we will denote by $\|\cdot\|_p$ the L^p norm on $\mathbb{R}^N$.

Equation (1.1) enjoys the following invariances: if u is a solution and $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^N$, $\lambda_0 > 0$, $\delta_0, \delta_1 \in \{-1, +1\}$, then

$$v(t, x) = \frac{\delta_0}{\lambda_0^{(N-2)/2}} u\Big(\frac{t_0 + \delta_1 t}{\lambda_0}, \frac{x + x_0}{\lambda_0}\Big)$$

is also a solution. Note that the energy of u and, if u is globally defined, the norm $\|u\|_{S(\mathbb{R})}$ are not changed by these transformations.

We recall the following classification Theorem, proven in [KM06b], for the case $E(u_0, u_1) < E(W, 0)$, and in [DM07b] for the existence of W^- and the case $E(u_0, u_1) = E(W, 0)$:

Theorem A (Kenig-Merle,Duyckaerts-Merle)**.** *There exists a global solution W^- of* (1.1) *such that*

$$E(W^-(0), \partial_t W^-(0)) = E(W, 0), \quad \|\nabla W_-(0)\|_2 < \|\nabla W\|_2$$
$$\|W^-\|_{S(-\infty,0)} < \infty, \quad \lim_{t\to+\infty} \|\nabla(W^-(t) - W)\|_2 + \|\partial_t(W^-(t) - W)\|_2 = 0.$$

Moreover, if u is a solution of (1.1) *such that $E(u_0, u_1) \le E(W, 0)$ and $\|\nabla u_0\|_2 \le \|\nabla W\|_2$, then u is globally defined. If furthermore $\|u\|_{S(\mathbb{R})} = \infty$, then $u = W^-$ or $u = W$ up to the invariances of the equation.*

We will also need the following simple version of long-time perturbation theory results (see e.g. [KM06b, Theorem 2.20]).

Lemma 2.1. *Let $M > 0$. Then there exist positive constants $\varepsilon(M)$ and $C(M)$ such that for all solutions v and u of* (1.1)*, with initial conditions (v_0, v_1) and (u_0, u_1), if the forward time of* *existence of v is infinite and*

$$\|v\|_{S(0,+\infty)} \leq M \text{ and } \|\nabla(u_0 - v_0)\|_2 + \|u_1 - v_1\|_2 \leq \varepsilon(M),$$

then u is globally defined for positive times and $\|u\|_{S(0,+\infty)} \leq C(M)$. A similar statement holds for negative times.

In this section we show the following:

Proposition 2.2. *Let u_n be a family of solutions of* (1.1)*, such that*

$$E\big(u_n(0), \partial_t u_n(0)\big) < E(W,0), \quad \|\nabla u_n(0)\|_2 < \|\nabla W\|_2. \tag{2.1}$$

and $\lim_{n\to+\infty} \|u_n\|_{S(\mathbb{R})} = +\infty$. Let $(t_n)_n$ be a time sequence.

(a) *Assume*

$$\lim_{n\to+\infty} \|u_n\|_{S(-\infty,t_n)} = \lim_{n\to+\infty} \|u_n\|_{S(t_n,+\infty)} = +\infty.$$

Then, up to the extraction of a subsequence there exist $\delta_0 \in \{-1,+1\}$ and sequences of parameters $x_n \in \mathbb{R}^n$, $\lambda_n > 0$ such that

$$\lim_{n\to+\infty} \left\| \frac{\delta_0}{\lambda_n^{N/2}} \nabla u_n \left(t_n, \frac{\cdot - x_n}{\lambda_n} \right) - \nabla W \right\|_2 + \left\| \frac{\partial u_n}{\partial t}(t_n) \right\|_2 = 0.$$

(b) *Assume that there exists $C_0 \in (0,+\infty)$ such that*

$$\lim_{n\to+\infty} \|u_n\|_{S(-\infty,t_n)} = +\infty \text{ and } \lim_{n\to+\infty} \|u_n\|_{S(t_n,+\infty)} = C_0.$$

Then, up to the extraction of a subsequence there exist $t_0 \in \mathbb{R}$, $\delta_0, \delta_1 \in \{-1,+1\}$, and sequences of parameters $x_n \in \mathbb{R}^n$, $\lambda_n > 0$ such that

$$\lim_{n\to+\infty} \left\| \frac{\delta_0}{\lambda_n^{N/2}} \nabla u_n \left(t_n, \frac{\cdot - x_n}{\lambda_n} \right) - \nabla W^-(t_0) \right\|_2 + \left\| \frac{\delta_0}{\lambda_n^{N/2}} \frac{\partial u_n}{\partial t} \left(t_n, \frac{\cdot - x_n}{\lambda_n} \right) - \frac{\partial W^-}{\partial t}(t_0) \right\|_2 = 0.$$

Remark 2.3. Case (b) will not be used in the proof of Theorem 1, and is stated only for its own interest.

Sketch of Proof. We will sketch the proof (a), the proof of (b) is similar and left to the reader. Translating in time all the u_n, we may assume that $t_n = 0$ for all n, and thus

$$\lim_{n\to+\infty} \|u_n\|_{S(-\infty,0)} = \lim_{n\to+\infty} \|u_n\|_{S(0,+\infty)} = +\infty. \tag{2.2}$$

In view of (2.1) and (2.2), one can show, using the profile decomposition of [BG99] as in [KM06b, Proposition 4.2], that there exist (up to the extraction of a subsequence) parameters $\lambda_n > 0$, $x_n \in \mathbb{R}^N$, and functions $(v_0, v_1) \in \dot{H}^1 \times L^2$ such that

$$\lim_{n\to+\infty} \left\| \frac{1}{\lambda_n^{N/2}} \nabla u_n \left(0, \frac{\cdot - x_n}{\lambda_n} \right) - \nabla v_0 \right\|_2 + \left\| \frac{1}{\lambda_n^{N/2}} \frac{\partial u_n}{\partial t} \left(0, \frac{\cdot - x_n}{\lambda_n} \right) - v_1 \right\|_2 = 0.$$

We refer to [KM06a, Section 4] and also [DM07a, Lemma 2.5] for proofs in the case of nonlinear Schrödinger equations that readily apply to our case. Note that

$$\|\nabla v_0\|_2 \leq \|\nabla W\|_2, \quad E(v_0, v_1) \leq E(W,0). \tag{2.3}$$

Let v be the solution of (1.1) with initial conditions (v_0, v_1). Theorem A and (2.3) imply that v is globally defined. By Lemma 2.1 and by (2.2),

$$\|v\|_{S(-\infty,0)} = +\infty, \quad \|v\|_{S(0,+\infty)} = +\infty.$$

This shows, again by Theorem A, that $v = W$, up to the invariances of equation (1.1) concluding the proof. □

3. Estimates near the threshold

3.1. **Preliminaries on the linearized equation.** In this subsection, we recall results on the linearized equation near W. We refer to [DM07b] for the details. Let u be a solution of (1.1) which is close to W. Write $u = W + h$. Then h is solution to the equation

$$(\partial_t^2 + L)h = R(h). \tag{3.1}$$

$$L := -\Delta - \frac{N+2}{N-2}W^{\frac{4}{N-2}}, \quad R(h) := |W+h|^{\frac{4}{N-2}}(W+h) - W^{\frac{N+2}{N-2}} - \frac{N+2}{N-2}W^{\frac{4}{N-2}}h.$$

Let

$$W_0 = a\left(\frac{N-2}{2}W + x\cdot\nabla W\right), \quad W_j = b\partial_{x_j}W,\ j = 1\dots N. \tag{3.2}$$

where the constants a and b are chosen so that $\|\nabla W_j\|_2 = 1$ for $j = 0, 1, \dots, N$. By the invariances of equation (1.1), $L(W_j) = 0$ for $j = 0, \dots, N$. As a consequence the functions W_j, $j = 0, \dots, N$ are in the kernel of the quadratic form

$$Q(h) = \frac{1}{2}\int Lh\,h = \frac{1}{2}\int |\nabla h|^2 - \frac{N+2}{2(N-2)}\int W^{\frac{4}{N-2}}h^2.$$

Observe that if $(h, \partial_t h)$ is small in $\dot{H}^1 \times L^2$,

$$E(W+h, \partial_t h) = E(W, 0) + Q(h) + \frac{1}{2}\int |\partial_t h|^2 + O\left(\|h\|^3_{\frac{2N}{N-2}}\right). \tag{3.3}$$

Furthermore, one can check that the infimum of $Q(h)$ for $h \in H^1$, $\|h\|_{L^2} = 1$, is negative, and thus that L admits a positive, radial eigenfunction $\mathcal{Y}$ with eigenvalue $-\omega^2 < 0$. We normalize $\mathcal{Y}$ such that $\int \mathcal{Y}^2 = 1$. The self-adjointness of L implies

$$\int \mathcal{Y}W_j = 0, \quad j = 0\dots N. \tag{3.4}$$

Consider

$$G_\perp := \left\{f \in \dot{H}^1,\ \int \mathcal{Y}f = \int \nabla W_0\cdot\nabla f = \dots = \int \nabla W_N \cdot \nabla f = 0\right\}.$$

The following result (see Proposition 5.5 of [DM07b]) shows in particular that $-\omega^2$ is the only negative eigenvalue of L:

Claim 3.1. *There exists a constant $c_Q > 0$ such that*

$$\forall h \in G_\perp, \quad Q(h) \geq c_Q\|\nabla h\|^2_{L^2}.$$

As a consequence of the Strichartz estimates for the linear wave equation (see [GV95] and [LS95]), we easily get the following Strichartz-type estimate for equation (3.1).

Claim 3.2. *There exist constants $\tilde{c}, C > 0$ such that if h is a solution of* (3.1) *on an interval* $[t_0, t_1]$ *such that*

$$\|\nabla h\|_2 + \|\partial_t h\|_2 + |t_0 - t_1| \leq \tilde{c}.$$

Then

$$\|h\|_{S(t_0,t_1)} \leq C\left(\|\nabla h(t_0)\|_2 + \|\partial_t h(t_0)\|_2\right).$$

3.2. **Estimate on the exit time.** In this subsection, we consider a sequence u_n of solutions of (1.1) such that

$$\lim_{n\to+\infty} \|\nabla(u_n(0) - W)\|_2 + \|\partial_t u_n(0)\|_2 = 0 \tag{3.5}$$

$$E(W,0) - E(u_n, \partial_t u_n) = \varepsilon_n^2 \underset{n\to+\infty}{\longrightarrow} 0 \text{ and } \forall n > 0,\ \|\nabla u_n(0)\|_2 < \|\nabla W\|_2. \tag{3.6}$$

Let $h_n = u_n - W$ and decompose h_n as

$$h_n(t) = \beta_n(t)\mathcal{Y} + \sum_{j=0}^{N} \gamma_{j,n}(t) W_j + g_n(t), \quad g_n(t) \in G_\perp. \tag{3.7}$$

For this, observe that the condition $g_n(t) \in G_\perp$ is equivalent to

$$\beta_n(t) = \int h_n(t)\mathcal{Y}, \tag{3.8}$$

$$\gamma_{0,n}(t) = \int \nabla\left(h_n(t) - \beta_n(t)\mathcal{Y}\right)\cdot\nabla W_0, \quad \gamma_{j,n}(t) = \int \nabla h_n(t)\cdot\nabla W_j,\ j = 1,\ldots,N. \tag{3.9}$$

(we used that $\mathcal{Y}$ and W being radial, $\int \nabla\mathcal{Y}\cdot\nabla W_j = b\int\nabla\mathcal{Y}\cdot\partial_{x_j}\nabla W = 0$ if $j \in \{1,\ldots,N\}$ and by a similar argument, $\int \nabla W_j \cdot \nabla W_k = 0$ if $j \neq k$). We have:

Claim 3.3. *Assume* (3.5)*. Then there exists sequences $\lambda_n \in (0,+\infty)$, $x_n \in \mathbb{R}^N$ such that*

$$\lim_{n\to+\infty} \lambda_n = 1, \quad \lim_{n\to+\infty} x_n = 0,$$

and for all n, noting $c_0 = \int \nabla\mathcal{Y}\cdot\nabla W_0$,

$$\frac{1}{\lambda_n^{\frac{N}{2}}}\int \nabla u_n\left(\frac{x - x_n}{\lambda_n}\right)\cdot\nabla W_0(x)dx - \frac{c_0}{\lambda_n^{\frac{N-2}{2}}}\int\left(u_n\left(\frac{x-x_n}{\lambda_n}\right) - W(x)\right)\mathcal{Y}(x)dx = 0 \tag{3.10}$$

$$\forall j \in \{1,\ldots,N\}, \quad \frac{1}{\lambda_n^{\frac{N}{2}}}\int \nabla u_n\left(\frac{x-x_n}{\lambda_n}\right)\cdot\nabla W_j(x) = 0. \tag{3.11}$$

Sketch of proof. Consider the mapping $J : (\lambda, X, u) \mapsto (J_0, J_1, \ldots, J_N)$ where

$$J_0 = \frac{1}{\lambda^{\frac{N}{2}}}\int \nabla u\left(\frac{x-X}{\lambda}\right)\cdot\nabla W_0(x)dx - \frac{c_0}{\lambda^{\frac{N-2}{2}}}\int\left(u\left(\frac{x-X}{\lambda}\right) - W(x)\right)\mathcal{Y}(x)dx$$

$$J_k = \frac{1}{\lambda^{\frac{N}{2}}}\int \nabla u\left(\frac{x-X}{\lambda}\right)\cdot\nabla W_k(x), \quad k \in \{1,\ldots,N\}.$$

A straightforward computation shows that $J = 0$ and $\left(\frac{\partial J}{\partial\lambda}, \frac{\partial J}{\partial X_1}, \ldots, \frac{\partial J}{\partial X_N}\right)$ is diagonal and invertible at the point $(1, 0, \ldots, 0, W)$. The Claim then follows from (3.5) and the implicit function theorem. □

Observe that the conditions (3.10) and (3.11) are equivalent, by (3.8) and (3.9), to the condition that the parameters γ_{jn} corresponding to the modulated solution $\frac{1}{\lambda_n^{\frac{N-2}{2}}} u_n\left(\frac{t}{\lambda_n}, \frac{x}{\lambda_n}\right)$ vanish at $t = 0$. By Claim 3.3 we can assume, up to translation and scaling, that

$$\forall n,\ \forall j \in \{0, 1, \ldots, N\}, \quad \gamma_{j,n}(0) = 0. \tag{3.12}$$

The main result of this subsection is the following:

Proposition 3.4. *There exist a constant η_0, such that for all $\eta \in (0, \eta_0)$, for all sequence (u_n) satisfying (3.5), (3.6), (3.12) and such that $\beta_n(0)\beta_n'(0) \geq 0$, if*

$$T_n(\eta) = \inf\left\{t \geq 0 \ :\ |\beta_n(t)| \geq \eta\right\},$$

then for large n, $\beta_n(0) \neq 0$, $T_n(\eta) \in (0, +\infty)$ and

$$\lim_{n\to+\infty} \frac{T_n(\eta)}{|\log|\beta_n(0)||} = \frac{1}{\omega}. \tag{3.13}$$

Furthermore,

$$\liminf_{n\to+\infty} |\beta_n'(T_n(\eta))| \geq \omega\eta. \tag{3.14}$$

Remark 3.5. If $\beta_n(0)\beta_n'(0) < 0$, we may achieve the condition $\beta_n(0)\beta_n'(0) \geq 0$ by considering the solution $u_n(-t, x)$ instead of $u_n(t, x)$.

The remainder of this subsection is devoted to the proof of Proposition 3.4. We first give, as a consequence of the orthogonality conditions (3.12), a purely variational lower bound on $|\beta_n(0)|$ (Claim 3.6). We then give (Lemma 3.7) precise estimates on $\beta_n(t)$ and $\|\partial_t h_n(t)\|_2 + \|\nabla h_n(t)\|_2$, on an interval $(0, t_n)$ where a priori bounds are assumed. These estimates will give the desired bounds on the exit time $T_n(\eta)$. We will write:

$$\|\partial_{t,x} h_n(t)\|_2 = \|\nabla h_n(t)\|_2 + \|\partial_t h_n(t)\|_2.$$

Claim 3.6. *There exists $M_0 > 0$ such that for all sequence (u_n) of solutions of (1.1) satisfying (3.5), (3.6) and (3.12) we have*

$$\beta_n(0) \neq 0 \ \text{and}\ \limsup_{n\to+\infty} \frac{\|\partial_{t,x} h_n(0)\|_2 + \varepsilon_n}{|\beta_n(0)|} \leq M_0.$$

Proof. Developping the energy as in (3.3), we get

$$E(W,0) - \varepsilon_n^2 = E(W + h_n, \partial_t h_n) = E(W,0) + Q(h_n) + \frac{1}{2}\|\partial_t h_n\|_2^2 + O\left(\|\nabla h_n\|_2^3\right).$$

The expression (3.7) of h at $t = 0$ yields, in view of (3.12)

$$\|\nabla h_n(0)\|_2 \leq C(|\beta_n(0)| + \|\nabla g_n(0)\|_2).$$

Furthermore, taking into account that $Q(\mathcal{Y}) < 0$ and that the functions W_j are in the kernel of Q for $j = 0 \ldots N$, we get

$$Q(h_n(0)) = -\beta_n^2(0)|Q(\mathcal{Y})| + Q(g_n(0)).$$

Combining the preceding estimates, we obtain

$$\beta_n^2(0)|Q(\mathcal{Y})| = \varepsilon_n^2 + Q(g_n(0)) + \frac{1}{2}\|\partial_t h_n(0)\|_2^2 + O\left(\beta_n^3(0) + \|\nabla g_n(0)\|_2^3\right).$$

By Claim 3.1, $Q(g_n(0)) \geq c_Q \|\nabla g_n(0)\|_2^2$. This yields for large n,

$$2\beta_n^2(0)\,|Q(\mathcal{Y})| \geq \varepsilon_n^2 + c_Q\|\nabla g_n(0)\|_2^2 + \frac{1}{2}\|\partial_t h_n(0)\|_2^2 \geq \varepsilon_n^2 + c\|\nabla h_n(0)\|_2^2 + \frac{1}{2}\|\partial_t h_n(0)\|_2^2,$$

which concludes the proof of the claim. □

Our next result is the following Lemma:

Lemma 3.7 (Growth on $[0,T_n]$)**.** *Let us fix ω^+ and ω^-, close to ω, such that $\omega^- < \omega < \omega^+$. There exist positive constants τ_0, K_0 (depending only on the choice of $\omega_\pm$) with the following property. Let $(u_n)_n$ be a sequence of solutions of* (1.1) *satisfying* (3.5), (3.6) *and such that $\beta_n(0){\beta_n}'(0) \geq 0$. Let $M > M_0$ (where M_0 is given by Claim 3.6). Let η such that*

$$0 < \eta < \frac{1}{K_0 M^3}. \tag{3.15}$$

Define

$$t_n = t_n(M,\eta) = \inf\left\{t \geq 0 \;:\; \|\partial_{t,x} h_n(t)\|_2 \geq M|\beta_n(t)| \text{ or } |\beta_n(t)| \geq \eta\right\}. \tag{3.16}$$

Then there exists $\tilde{n} > 0$ such that for $n \geq \tilde{n}$,

$$\forall t \in [\tau_0, t_n)\,,\quad \omega^-|\beta_n(t)| \leq |\beta_n'(t)| \leq \omega^+|\beta_n(t)| \tag{3.17}$$

$$\forall t \in [\tau_0, t_n)\,,\quad \frac{1}{K_0}\,|\beta_n(0)|\,e^{\omega^- t} \leq |\beta_n(t)| \leq K_0\,|\beta_n(0)|\,e^{\omega^+ t} \tag{3.18}$$

$$\forall t \in [0, t_n),\quad \|\partial_{t,x} h_n(t)\|_2 \leq K_0|\beta_n(t)|. \tag{3.19}$$

Before proving the lemma, we will show that it implies Proposition 3.4. For this we take $M = 1 + \max\{M_0, K_0\}$ and apply Lemma 3.7. Then by (3.19), $\|\partial_{t,x} h_n(t)\|_2 < M|\beta_n(t)|$ on $[0, t_n]$ and thus

$$t_n(\eta, M) = \inf\left\{t \geq 0 \;:\; |\beta_n(t)| \geq \eta\right\} = T_n(\eta).$$

This shows by (3.18) that $T_n(\eta) \in (0, +\infty)$ for large n, and by continuity of β_n, that $\beta_n(T_n(\eta)) = \eta$. In particular, $T_n(\eta)$ must tend to infinity; otherwise, as $\beta_n(0)$ tends to 0, the continuity of the flow would imply that $u_n(T_n(\eta))$ tends to W and $\beta_n(T_n(\eta))$ to 0, a contradiction. By (3.18), we get for large n,

$$\frac{1}{K_0}\,|\beta_n(0)|\,e^{\omega^- T_n(\eta)} \leq \eta \leq K_0\,|\beta_n(0)|\,e^{\omega^+ T_n(\eta)}.$$

By the upper bound inequality we get (noticing that $\log|\beta_n(0)|$ is negative for large time)

$$\big|\log|\beta_n(0)|\big| + \log\eta \leq \log(K_0) + \omega^+ T_n(\eta).$$

Hence, using that $\beta_n(0)$ tends to 0, as n goes to infinity,

$$\frac{1}{\omega^+} \leq \liminf_{n\to+\infty} \frac{T_n(\eta)}{\big|\log|\beta_n(0)|\big|}.$$

Letting ω_+ tends to ω we get

$$\frac{1}{\omega} \leq \liminf_{n\to+\infty} \frac{T_n(\eta)}{\big|\log|\beta_n(0)|\big|}.$$

By the same argument, we get

$$\limsup_{n\to+\infty} \frac{T_n(\eta)}{\big|\log|\beta_n(0)|\big|} \leq \frac{1}{\omega},$$

which concludes the proof of (3.13).

To conclude the proof of Proposition 3.4 observe that (3.17) implies, for large n,

$$\omega^-\eta = \omega^-|\beta_n(T_n(\eta))| \leq |\beta_n'(T_n(\eta))|,$$

which yields (3.14).

In the remainder of this subsection we prove Lemma 3.7.

Proof of Lemma 3.7. In view of Claim 3.6, the fact that $\beta_n(0)$ tends to 0 and the continuity of β_n and $\|\partial_{t,x}h_n\|_2$, the time t_n is strictly positive. Furthermore,

$$\forall n,\ \forall t \in (0,t_n), \quad |\beta_n(t)| \leq \eta \tag{3.20}$$

$$\forall n,\ \forall t \in (0,t_n), \quad \|\partial_{t,x}h_n(t)\|_2 \leq M\,|\beta_n(t)|\,. \tag{3.21}$$

Proof of (3.17).

Let

$$m = \frac{1}{2}\min\left\{\omega^2-(\omega^-)^2,(\omega^+)^2-\omega^2\right\}.$$

We first show that if η satisfies (3.15), then

$$\left|\beta_n{''}-\omega^2\beta_n\right| \leq m\,|\beta_n(t)|\,. \tag{3.22}$$

Differentiating twice the equality $\beta_n = \int h_n\mathcal{Y}$, we get, by equation (3.1),

$$\beta_n{''}-\omega^2\beta_n = \int \partial_t^2 h_n\,\mathcal{Y} - \omega^2\int h_n\mathcal{Y} = \int R(h_n)\mathcal{Y} - \int\left(Lh_n+\omega^2h_n\right)\mathcal{Y} = \int R(h_n)\mathcal{Y}.$$

Thus there exists a constant C_1, independent of all parameters, such that

$$\left|\beta_n{''}-\omega^2\beta_n\right| \leq C_1\|\partial_{t,x}h_n\|_2^2. \tag{3.23}$$

By (3.20) and (3.21)

$$\left|\beta_n{''}-\omega^2\beta_n\right| \leq C_1M^2\beta_n^2 \leq C_1M^2\eta|\beta_n|.$$

which yields, if $C_1M^2\eta \leq m$ (which follows from (3.15) if K_0 is large enough), the desired estimate (3.22).

In what follows, we will assume that $\beta_n(0)\geq 0$ and $\beta_n{'}(0)\geq 0$ (otherwise, replace β_n by $-\beta_n$ in the forthcoming argument). We next show that for $t\in(0,t_n)$,

$$\beta_n{''}(t)>0, \quad \beta_n{'}(t)>0, \quad \beta_n(t)>0. \tag{3.24}$$

Indeed by (3.22),

$$(\omega^2-m)\beta_n(t) \leq \beta_n{''}(t) \leq (\omega^2+m)\beta_n(t). \tag{3.25}$$

As $\beta_n(0)>0$ by Claim 3.6, we get that β, β' and β'' are (strictly) positive for small positive t. This shows that (3.24) holds near 0, and by an elementary monotonicity argument, that it holds for all $t\in(0,t_n]$.

We are now ready to show (3.17). For this we write, as a consequence of (3.25)

$$\left(\beta_n'-\omega^-\beta_n\right)' = \beta_n''-\omega^-\beta_n' \geq -\omega^-\left(\beta_n'-\omega^-\beta_n\right)+(\omega^2-(\omega^-)^2-m)\beta_n \geq -\omega^-\left(\beta_n'-\omega^-\beta_n\right)+m\beta_n.$$

Hence (using that β_n increases with time)

$$\frac{d}{dt}\left[e^{\omega^-t}\left(\beta_n'-\omega^-\beta_n\right)\right] \geq me^{\omega^-t}\beta_n(0).$$

Integrating between 0 and t we get

$$e^{\omega^- t}\left(\beta_n'-\omega^-\beta_n\right)\geq m\beta_n(0)\int_0^t e^{\omega^- s}ds+\beta_n'(0)-\omega^-\beta_n(0)\geq\beta_n(0)\left[m\frac{e^{\omega^- t}-1}{\omega^-}-\omega^-\right].$$

Chosing τ_0 large enough we get a positive right hand side for $t\geq\tau_0$, hence the left inequality in (3.17). The right inequality follows similarly by differentiating $\beta_n'-\omega^+\beta_n$ and we omit the details of the proof.

Proof of (3.18). Assume as in the proof of (3.17) that $\beta_n(0)\geq 0$ and $\beta_n'(0)\geq 0$. By (3.25), and using that β_n is positive on $(0,T)$,

$$\forall t\in[0,t_n],\quad \beta_n''(t)-{\omega^+}^2\beta_n(t)\leq\left(\omega^2+m-(\omega^+)^2\right)\beta_n(t)<0.$$

This shows by a standard ODE argument that $\beta_n(t)\leq\tilde{\beta}_n(t)$, where $\tilde{\beta}_n(t)$ is the solution of the differential equation $\tilde{\beta}_n''-{\omega^+}^2\tilde{\beta}_n=0$ with initial conditions $\tilde{\beta}_n(0)=\beta_n(0)$, $\tilde{\beta}_n'(0)=\beta_n'(0)$. Hence

$$\forall t\in[0,t_n],\quad \beta_n(t)\leq\beta_n(0)\cosh(\omega^+t)+\frac{\beta_n'(0)}{\omega^+}\sinh(\omega^+t). \tag{3.26}$$

By (3.7),

$$\partial_t h_n(0)=\beta_n'(0)\mathcal{Y}+\sum_{j=0}^N\gamma_{j,n}'(0)W_j+g_n(0),\quad g_n(0)\in G_\perp.$$

Taking the L^2-scalar product with $\mathcal{Y}$ and recalling that W_j, $j=0\dots N$, and $g_n(0)$ are orthogonal to $\mathcal{Y}$, we get $|\beta_n'(0)|\leq\|\partial_t h_n(0)\|_2$. Thus, in view of Claim 3.6, for large n:

$$\beta_n'(0)\leq(M_0+1)\beta_n(0).$$

By (3.26)

$$\beta_n(\tau_0)\leq\beta_n(0)\cosh(\omega^+\tau_0)+\frac{M_0+1}{\omega^+}\beta_n(0)\sinh(\omega^+\tau_0)\leq K_1\beta_n(0), \tag{3.27}$$

for some constant K_1 depending only on the choice of ω^+. By (3.17),

$$\forall t\geq\tau_0,\quad e^{\omega^-(t-\tau_0)}\beta_n(\tau_0)\leq\beta_n(t)\leq e^{\omega^+(t-\tau_0)}\beta_n(\tau_0).$$

Using (3.27) for the upper bound and the fact that β_n increases for the lower bound , we get

$$\forall t\geq\tau_0,\quad e^{\omega^-(t-\tau_0)}\beta_n(0)\leq\beta_n(t)\leq K_1e^{\omega^+(t-\tau_0)}\beta_n(0),$$

which yields (3.18).

Proof of (3.19).

We divide the proof into two steps.

Step 1. Estimates on the coefficients

We first show that there exist a constant $C_1>0$, independent of the parameters M and η, such that for all $t\in[0,t_n]$

$$\frac{1}{C_1}|\beta_n|-C_1\|\partial_{t,x}h_n\|_2^{3/2}\leq\|\nabla g_n\|_2+\|\partial_t h_n\|_2+\varepsilon_n\leq C_1\beta_n+C_1\|\partial_{t,x}h_n\|_2^{3/2} \tag{3.28}$$

$$\frac{1}{C_1}\|\partial_{t,x}h_n\|_2\leq|\beta_n|+\sum_{j=0}^N|\gamma_{j,n}|\leq C_1\|\partial_{t,x}h_n\|_2. \tag{3.29}$$

We have

$$E\left(W+h_n,\partial_t h_n\right)=E(W,0)-\varepsilon_n^2$$

Thus there exists a constant $C_2>0$ (independent of the parameters) such that

$$\left|Q(h_n)+\int|\partial_t h_n|^2+\varepsilon_n^2\right|\leq C_2\|\partial_{t,x}h_n(t)\|_2^3.$$

Furthermore, by (3.7) (and the fact that the functions W_j, $j=0\dots N$ are in the kernel of Q)

$$Q(h_n)=-{\beta_n}^2|Q(\mathcal{Y})|+Q\left(g_n\right).$$

Which yields

$$\left|-{\beta_n}^2|Q(\mathcal{Y})|+Q\left(g_n\right)+\int|\partial_t h_n|^2+\varepsilon_n^2\right|\leq C_2\|\partial_{t,x}h_n(t)\|_2^3. \tag{3.30}$$

As $g_n\in G_\perp$, we have $Q\left(g_n\right)\approx\|\nabla g_n\|_2^2$, which yields (3.28).

Let us show (3.29). Note that the upper bound follows immediately from the definitions of β_n and $\gamma_{j,n}$ (see (3.8) and (3.9)). It remains to show the lower bound. We have

$$h_n(t)=\beta_n(t)\mathcal{Y}+\sum_{j=0}^N\gamma_{j,n}(t)W_j+g_n(t),$$

and hence, by (3.28)

$$\|\nabla h_n\|_2\leq C\left[|\beta_n|+\sum_{j=0}^N|\gamma_{j,n}|+\|\nabla g_n\|_2\right]\leq C\left[|\beta_n|+\sum_{j=0}^N|\gamma_{j,n}|+\|\partial_{t,x}h_n\|_2^{\frac{3}{2}}\right]$$

$$\|\partial_{t,x}h_n\|_2=\|\nabla h_n\|_2+\|\partial_t h_n\|_2\leq C\left[|\beta_n|+\sum_{j=0}^N|\gamma_{j,n}|+\|\partial_{t,x}h_n\|_2^{\frac{3}{2}}\right].$$

As a consequence of (3.20) and (3.21), we obtain

$$\|\partial_{t,x}h_n\|_2=\leq C\left[|\beta_n|+\sum_{j=0}^N|\gamma_{j,n}|\right]+C\|\partial_{t,x}h_n\|_2M^{1/2}\eta^{1/2}.$$

by (3.15), we get the lower bound in (3.29)

Step 2. Bound on $\gamma_{j,n}$.

We are now ready to show (3.19). According to (3.29), it is sufficient to show that there exists a constant C_3 independent of M and $\eta\leq\frac{1}{K_0M^3}$ such that

$$\forall j\in\{0,\dots,N\},\ \forall t\in[0,t_n],\quad|\gamma_{j,n}(t)|\leq C_3\,|\beta_n(t)|. \tag{3.31}$$

We have, for $j=0\dots N$.

$$\gamma_{j,n}'(t)=\int\nabla\left(\partial_t h_n(t)-{\beta_n}'(t)\mathcal{Y}\right)\nabla W_j.$$

Note that $\int\nabla W_j\nabla\mathcal{Y}=0$ if $j\geq1$, but we won't need this fact in the sequel. The preceding inequality yields

$$\left|\gamma_{j,n}'(t)\right|\leq C\left(\|\partial_t h_n(t)\|_2+\left|{\beta_n}'(t)\right|\right). \tag{3.32}$$

By (3.28) and assumptions (3.20) and (3.21),

$$\|\partial_t h_n\|_2 \leq C_1 \left(|\beta_n| + \|\partial_{t,x} h_n\|_2^{3/2}\right) \leq C_1 |\beta_n| \left(1+\eta^{1/2} M^{3/2}\right).$$

Taking η small enough so that $\eta^{1/2}M^{3/2} \leq 1$, we get

$$\|\partial_t h_n\|_2 \leq 2C_1|\beta_n|. \tag{3.33}$$

By (3.32), taking a larger constant C,

$$\left|\gamma_{j,n}'\right| \leq C\left(|\beta_n| + |\beta_n'|\right). \tag{3.34}$$

Integrating between 0 and $t \leq \tau_0$, and using that $\gamma_{j,n}(0) = 0$, that $|\beta_n|$ increases and that the sign of $\beta_n'(t)$ is independant of $t \in [0, t_n]$ (see (3.24)), we obtain

$$\forall t \in [0, \tau_0], \quad |\gamma_{j,n}(t)| \leq C(t+1)|\beta_n(t)|. \tag{3.35}$$

This yields (3.31) for $t \leq \tau_0$.

Now by (3.17) and (3.34), and using that the signs of β_n and β_n' do not depend on time,

$$\forall t \geq \tau_0, \quad \left|\gamma_{j,n}'(t)\right| \leq C\left|{\beta_n}'(t)\right|.$$

Integrating between τ_0 and $t \in [\tau_0, t_n]$, we get

$$|\gamma_{j,n}(t)| \leq C\left(|\beta_n(t)| + |\gamma_{j,n}(\tau_0)|\right).$$

Using (3.35) at $t = \tau_0$ and the fact that $|\beta_n|$ increases, we get (3.31) for $t \geq \tau_0$. The proof is complete. $\square$

4. Proof of main result

This section is devoted to the proof of Theorem 1. The proof is divided into 3 steps. In Step 1, we show the lower bound, in the next two steps the upper bound.

Step 1. Lower bound.

We must show

$$\liminf_{\varepsilon \to 0^+} \frac{\mathcal{I}_\varepsilon}{|\log \varepsilon|} \geq \frac{2}{\omega} \int_{\mathbb{R}^N} W^{\frac{2(N+1)}{N-2}}. \tag{4.1}$$

For this we first note that

$$\int \nabla W \cdot \nabla \mathcal{Y} = -\int \Delta W \mathcal{Y} = \int W^{\frac{N+2}{N-2}} \mathcal{Y} > 0,$$

as $\mathcal{Y}$ and W are positive. Consider the family of solutions $(u^a)_{a>0}$ of (1.1) with initial conditions

$$u_0^a = W - a\mathcal{Y}, \quad u_1^a = 0.$$

For small $a > 0$,

$$\int |\nabla u_0^a|^2 = \int |\nabla W|^2 - 2a \int \nabla W \cdot \nabla \mathcal{Y} + a^2 \int |\nabla \mathcal{Y}|^2 < \int |\nabla W|^2.$$

We have

$$E\left(u_0^a, u_1^a\right) = E(W,0) + Q\left(-a\mathcal{Y}\right) + O\left(a^3\right) = E(W,0) - a^2\left|Q\left(\mathcal{Y}\right)\right| + O\left(a^3\right). \tag{4.2}$$

We argue by contradiction. If (4.1) does not hold, there exists a sequence ε_n which tends to 0 such that for some $\rho > \omega$

$$\forall n,\quad \frac{2}{\rho}\int_{\mathbb{R}^N} W^{\frac{2(N+1)}{N-2}} \geq \frac{\mathcal{I}_{\varepsilon_n}}{|\log \varepsilon_n|}. \tag{4.3}$$

By (4.2), and using that $E(u_0^a, u_1^a)$ is a continuous function of a, there exists a sequence a_n such that

$$\varepsilon_n^2 = E(W,0) - E\left(u_0^{a_n}, u_1^{a_n}\right),$$

Furthermore,

$$\varepsilon_n \sim a_n\sqrt{|Q(\mathcal{Y})|} \text{ as } n \to +\infty. \tag{4.4}$$

Let $u_n = u^{a_n}$. Observe that

$$\partial_t u_n(0) = 0, \quad \|\nabla(u_n(0) - W)\|_2 = a_n\|\nabla\mathcal{Y}\|_2 \underset{n\to+\infty}{\longrightarrow} 0.$$

Furthermore, $\beta_n(0) = -a_n$, $\beta_n'(0) = 0$, which shows that the assumptions of Proposition 3.4 are satisfied. Consider a small $\eta > 0$. By Proposition 3.4

$$\lim_{n\to+\infty}\frac{T_n(\eta)}{|\log a_n|} = \lim_{n\to+\infty}\frac{T_n(\eta)}{\left|\log|\beta_n(0)|\right|} = \frac{1}{\omega}$$

By (4.4),

$$\lim_{n\to+\infty}\frac{T_n(\eta)}{\left|\log|\varepsilon_n|\right|} = \frac{1}{\omega}. \tag{4.5}$$

Let us give a lower bound for $\|u_n\|_{S(0,+\infty)}$. From now on we will write T_n instead of $T_n(\eta)$ for the sake of simplicity. As $u_n = W + h_n$, we have

$$\|u_n\|_{S(0,T_n)} \geq \|W\|_{S(0,T_n)} - \|h_n\|_{S(0,T_n)}.$$

Furthermore,

$$\|W\|_{S(0,T_n)} = T_n^{\frac{N-2}{2(N+1)}}\|W\|_{\frac{2(N+1)}{N-2}}.$$

Write

$$\|h\|_{S(0,T_n)}^{\frac{2(N+1)}{N-2}} = \sum_{I\in E_{T_n}}\|h\|_{S(I)}^{\frac{2(N+1)}{N-2}},$$

where E_{T_n} is a set of at most $\frac{T_n}{\tilde{c}}+1$ subinterval of $(0,T_n)$, of length at most $\tilde{c}$ (given by Lemma 3.2) such that $(0,T_n) = \bigcup_{I\in E_{T_n}}\overline{I}$. By Lemma 3.2 and the fact that $\|\nabla h_n\|_2 + \|\partial_t h_n\|_2 \leq M\eta$ on $(0,T_n)$, we get for small $\eta > 0$,

$$\|h_n\|_{S(0,T_n)}^{\frac{2(N+1)}{N-2}} \leq C\left(\frac{T_n}{\tilde{c}}+1\right)\eta^{\frac{2(N+1)}{N-2}}.$$

Hence a constant $C > 0$ such that

$$\|h_n\|_{S(0,T_n)} \leq C\eta\, T_n^{\frac{N-2}{2(N+1)}}.$$

Combining the preceding estimates, we obtain

$$\int_0^{T_n}\int_{\mathbb{R}^N}|u_n|^{\frac{2(N+1)}{N-2}} \geq T_n\left[\|W\|_{\frac{2(N+1)}{N-2}} - C\eta\right]^{\frac{2(N+1)}{N-2}}.$$

Hence with (4.5),

$$\liminf_{n\to+\infty} \frac{1}{|\log \varepsilon_n|} \int_0^{+\infty} \int_{\mathbb{R}^N} |u_n|^{\frac{2(N+1)}{N-2}} \geq \frac{1}{\omega} \left[\|W\|_{\frac{2(N+1)}{N-2}} - C\eta \right]^{\frac{2(N+1)}{N-2}}.$$

Letting η tends to 0 we obtain

$$\liminf_{n\to+\infty} \frac{1}{|\log \varepsilon_n|} \int_0^{+\infty} |u_n|^{\frac{2(N+1)}{N-2}} \geq \frac{1}{\omega} \|W\|_{\frac{2(N+1)}{N-2}}^{\frac{2(N+1)}{N-2}}.$$

Next, notice that as $\partial_t u(0) = 0$, the uniqueness in the Cauchy problem (1.1) implies $u(t,x) = u(-t,x)$ and thus

$$\liminf_{n\to+\infty} \frac{1}{|\log \varepsilon_n|} \int_{-\infty}^{0} |u_n|^{\frac{2(N+1)}{N-2}} \geq \frac{1}{\omega} \|W\|_{\frac{2(N+1)}{N-2}}^{\frac{2(N+1)}{N-2}}.$$

Finally,

$$\liminf_{n\to+\infty} \frac{\mathcal{I}_{\varepsilon_n}}{|\log \varepsilon_n|} \geq \liminf_{n\to+\infty} \frac{1}{|\log \varepsilon_n|} \int_{-\infty}^{+\infty} |u_n|^{\frac{2(N+1)}{N-2}} \geq \frac{2}{\omega} \|W\|_{\frac{2(N+1)}{N-2}}^{\frac{2(N+1)}{N-2}},$$

contradicting (4.3). Step 1 is complete.

Step 2. Estimate before the exit time.

We next show the upper bound on $\mathcal{I}_\varepsilon$, i.e that

$$\limsup_{\varepsilon\to 0^+} \frac{\mathcal{I}_\varepsilon}{|\log \varepsilon|} \leq \frac{2}{\omega} \int_{\mathbb{R}^N} W^{\frac{2(N+1)}{N-2}}. \tag{4.6}$$

For this we will show that if $\varepsilon_n > 0$ is a sequence that goes to 0 and u_n a sequence of solutions of (1.1) such that

$$\|\nabla u_n(0)\|_2 < \|\nabla W\|_2, \quad E(W,0) - E(u_n, \partial_t u_n) = \varepsilon_n^2, \tag{4.7}$$

then

$$\limsup_{n\to+\infty} \frac{1}{|\log \varepsilon_n|} \int_{\mathbb{R}\times\mathbb{R}^N} |u_n|^{\frac{2(N+1)}{N-2}} \leq \frac{2}{\omega} \int_{\mathbb{R}^N} W^{\frac{2(N+1)}{N-2}}. \tag{4.8}$$

Possibly time-translating u_n, we may assume

$$\|u_n\|_{S(-\infty,0)} = \|u_n\|_{S(0,+\infty)} \underset{n\to+\infty}{\longrightarrow} +\infty \tag{4.9}$$

By Proposition 2.2, rescaling and space-translating u_n if necessary, we can assume

$$\lim_{n\to+\infty} u_n = W.$$

Consider the functions h_n and g_n, and the parameters β_n and $\gamma_{j,n}$ defined in the beginning of §3.2. Replacing $u_n(x,t)$ by $u_n(x,-t)$ if it is not the case, we may assume

$$\beta_n(0){\beta_n}'(0) \geq 0. \tag{4.10}$$

Furthermore, by Claim 3.3, we may also assume (3.12).

Fix a small $\eta > 0$, and consider $T_n = T_n(\eta)$ defined by Proposition 3.4. In this step, we show that there exists a constant $C > 0$ such that

$$\limsup_{n\to+\infty} \frac{1}{|\log \varepsilon_n|} \int_0^{T_n(\eta)} \int_{\mathbb{R}^N} |u_n|^{\frac{2(N+1)}{N-2}} \leq \frac{1}{\omega} \left[\|W\|_{\frac{2(N+1)}{N-2}} + C\eta \right]^{\frac{2(N+1)}{N-2}}. \tag{4.11}$$

Indeed, by Claim 3.6, for large n,

$$\varepsilon_n \leq M_0|\beta_n(0)|.$$

Hence by Proposition 3.4,

$$\limsup_{n\to+\infty} \frac{T_n}{|\log \varepsilon_n|} \leq \frac{1}{\omega}. \tag{4.12}$$

By the same argument as in Step 1, we get

$$\int_0^{T_n} \int_{\mathbb{R}^N} |u_n|^{\frac{2(N+1)}{N-2}} \leq T_n \left[\|W\|_{\frac{2(N+1)}{N-2}} + C\eta \right]^{\frac{2(N+1)}{N-2}}.$$

Hence

$$\limsup_{n\to+\infty} \frac{1}{T_n} \int_0^{T_n} \int_{\mathbb{R}^N} |u_n|^{\frac{2(N+1)}{N-2}} \leq \left[\|W\|_{\frac{2(N+1)}{N-2}} + C\eta \right]^{\frac{2(N+1)}{N-2}}.$$

Combining with (4.12), we obtain (4.11).

Step 3. Estimate for large time.

To conclude the proof, we will show that if η is small enough, there exists a constant $C(\eta) > 0$ such that for large n

$$\|u_n\|_{S(T_n(\eta),+\infty)} \leq C(\eta). \tag{4.13}$$

Assuming (4.13), we obtain by (4.11),

$$\limsup_{n\to+\infty} \frac{1}{|\log \varepsilon_n|} \|u_n\|_{S(0,+\infty)}^{\frac{2(N+1)}{N-2}} = \limsup_{n\to+\infty} \frac{1}{|\log \varepsilon_n|} \|u_n\|_{S(0,T_n(\eta))}^{\frac{2(N+1)}{N-2}} \leq \frac{1}{\omega} \left[\|W\|_{\frac{2(N+1)}{N-2}} + C\eta \right]^{\frac{2(N+1)}{N-2}}.$$

Letting η tend to 0 we get

$$\limsup_{n\to+\infty} \frac{1}{|\log \varepsilon_n|} \|u_n\|_{S(0,+\infty)}^{\frac{2(N+1)}{N-2}} \leq \frac{1}{\omega} \|W\|_{\frac{2(N+1)}{N-2}}^{\frac{2(N+1)}{N-2}},$$

which shows, in view of (4.9), the desired estimate (4.8).

It remains to show (4.13). We will argue by contradiction. If (4.13) does not hold, there exist a subsequence of (u_n), still denoted by (u_n) such that

$$\|u_n\|_{S(T_n,+\infty)} \xrightarrow[n\to\infty]{} +\infty. \tag{4.14}$$

Furthermore, by (4.9)

$$\|u_n\|_{S(-\infty,T_n)} \geq \|u_n\|_{S(-\infty,0)} = \|u_n\|_{S(0,+\infty)} \xrightarrow[n\to\infty]{} +\infty. \tag{4.15}$$

In view of (4.14) and (4.15), Proposition 2.2 (a) implies that there exists sequences $\lambda_n > 0$, $x_n \in \mathbb{R}^N$, and $\delta_0 \in \{-1, +1\}$ such that

$$\lim_{n\to+\infty} \left\| \frac{\delta_0}{\lambda_n^{N/2}} \nabla u_n \left(T_n, \frac{\cdot - x_n}{\lambda_n} \right) - \nabla W \right\|_2 + \left\| \frac{\partial u_n}{\partial t}(T_n) \right\|_2 = 0. \tag{4.16}$$

By Proposition 3.4,

$$\liminf_{n\to+\infty} |\beta_n'(T_n)| \geq \omega\eta.$$

By the decomposition (3.7) of h_n,

$$\int \partial_t u_n(T_n)\mathcal{Y} = \int \partial_t h_n(T_n)\mathcal{Y} = \beta_n'(T_n).$$

This shows by (4.16) that $\beta_n'(T_n)$ must tend to 0, yielding a contradiction. This concludes the proof of (4.13) and thus of Theorem 1.

5. Estimate of the scattering norm for energy-critical focusing NLS

In this section we briefly adress the case of the radial energy critical focusing semilinear Schrödinger equation

$$i\partial_t u + \Delta u + |u|^{\frac{4}{N-2}} u = 0, \quad u_{\restriction t=0} = u_0 \in \dot{H}^1_r, \tag{5.1}$$

where $N \in \{3,4,5\}$ and $\dot{H}^1_r$ is the subset of $\mathbb{R}^N$ of spherically symmetric functions. The equation (5.1) is locally well-posed (see [CW90]) in the energy space $\dot{H}^1_r$. Furthermore, if $I_{\max} \ni 0$ is the maximal interval of definition then

$$J \Subset I_{\max} \Longrightarrow \|u\|_{\tilde{S}(J)} < \infty, \text{ where } \tilde{S}(J) = L^{\frac{2(N+2)}{N-2}},$$

and globally defined solutions of (5.1) such that $\|u\|_{\tilde{S}(\mathbb{R})}$ is finite scatter (see [Bou99b, Bou99a]).

The energy

$$\mathcal{E}(u(t)) = \frac{1}{2}\int |\nabla u(t)|^2 - \frac{N-2}{2N}\int |u(t)|^{\frac{2N}{N-2}}$$

is conserved.

In the defocusing case, all solutions are known to be globally defined and scatter [Bou99b, Tao05]. Furthermore, in [Tao05], T. Tao gave a bound of $\|u\|_{\tilde{S}(\mathbb{R})}$ in term of an exponential of a power of the conserved defocusing energy $\frac{1}{2}\int |\nabla u_0|^2 + \frac{N-2}{2N}\int |u|^{\frac{2N}{N-2}}$.

In the focusing case, the function W, defined in (1.2) is still a stationnary solution of W. The following theorem shown in [KM06a] for the case $\mathcal{E}(u_0) < \mathcal{E}(W)$ and in [DM07a] for the case $\mathcal{E}(u_0) = \mathcal{E}(W)$, is the analoguous of Theorem A for equation (5.1).

Theorem B (Kenig-Merle,Duyckaerts-Merle)**.** *There exists a global solution $\widetilde{W}_-$ of (5.1) such that*

$$\mathcal{E}\big(\widetilde{W}^-\big) = \mathcal{E}(W), \quad \big\|\nabla \widetilde{W}_-(0)\big\|_2 < \|\nabla W\|_2$$
$$\big\|\widetilde{W}^-\big\|_{\tilde{S}(-\infty,0)} < \infty, \quad \lim_{t\to+\infty} \big\|\nabla\big(\widetilde{W}^-(t) - W\big)\big\|_2 = 0.$$

Moreover, if u is a radial solution of (5.1) such that $\mathcal{E}(u_0) \leq \mathcal{E}(W)$ and $\|\nabla u_0\|_2 \leq \|\nabla W\|_2$, then u is globally defined. If furthermore $\|u\|_{\tilde{S}(\mathbb{R})} = \infty$, then $u = \widetilde{W}^-$ or $u = W$ up to the invariances of the equation.

Defining

$$\tilde{F}_\varepsilon := \left\{ u \text{ radial solution of (5.1) such that } \mathcal{E}(u_0) \leq \mathcal{E}(W) - \varepsilon^2 \text{ and } \int |\nabla u_0|^2 < \int |\nabla W|^2 \right\}.$$

we get in particular that for $\varepsilon > 0$ the supremum

$$\widetilde{\mathcal{I}}_\varepsilon = \sup_{u\in\tilde{F}_\varepsilon} \int_{\mathbb{R}\times\mathbb{R}^N} |u(t,x)|^{\frac{2(N+2)}{N-2}}\, dtdx = \sup_{u\in\tilde{F}_\varepsilon} \|u\|^{\frac{2(N+2)}{N-2}}_{\tilde{S}(\mathbb{R})},$$

is finite, and that

$$\lim_{\varepsilon\to 0^+} \widetilde{\mathcal{I}}_\varepsilon = +\infty.$$

We wish again to estimate of $\widetilde{\mathcal{I}}_\varepsilon$ when ε goes to 0. As in the case of the wave equation, the behavior of $\widetilde{\mathcal{I}}_\varepsilon$ is determined by the linearized operator near W. If $u = W + h$ is a solution of (5.1), then, identifying h with the column vector $(\operatorname{Re} h, \operatorname{Im} h)^T = (h_1, h_2)^T$.

$$\partial_t h + \mathcal{L}(h) + R(h) = 0, \quad \mathcal{L} := \begin{pmatrix} 0 & \Delta + W^{\frac{4}{N-2}} \\ -\Delta - \frac{N+2}{N-2} W^{\frac{4}{N-2}} & 0 \end{pmatrix},$$

where an appropriate norm of $R(h)$ is bounded by $\|\nabla h\|_2^2$ when h is small. It is known (see [DM07a, Section 7.1]) that the essential spectrum of $\mathcal{L}$ is $i\mathbb{R}$ and that $\mathcal{L}$ admits only two nonzero real eigenvalues, $\widetilde{\omega} > 0$ and $-\widetilde{\omega}$, with eigenfunctions $\widetilde{\mathcal{Y}}_\pm$ which are in the space of Schwartz functions. Then:

Theorem 2.

$$\lim_{\varepsilon \to 0^+} \frac{\widetilde{\mathcal{I}}_\varepsilon}{|\log \varepsilon|} = \frac{2}{\widetilde{\omega}} \int_{\mathbb{R}^N} W^{\frac{2(N+2)}{N-2}}.$$

Our result is restricted to the radial case in spatial dimensions $N \in \{3, 4, 5\}$. In view of the recent work [KV08] on non-radial energy-critical focusing NLS in dimension $N \geq 5$, it is natural to expect that the same estimate holds in a more general situation.

The proof of Theorem 2 is very similar to the one of Theorem 1, and we will only sketch it, highlighting the minor differences. In §5.1 we recall a few facts about the operator $\mathcal{L}$ and state without proof the analoguous of Propositions 2.2, Propositions 3.4 and Claim 3.6. In §5.2 we briefly explain how to use these results to show Theorem 2.

5.1. **Convergence to W and estimate on the exit time.** In view of Theorem B, and the use of the profile decomposition method in [KM06a, Section 4] (see also [DM07a, Lemma 2.5]), the proof of Proposition 5.1 adapts easily to show:

Proposition 5.1. *Let u_n be a family of radial solutions of* (5.1)*, such that*

$$\mathcal{E}\big(u_n(0)\big) < \mathcal{E}(W), \quad \|\nabla u_n(0)\|_2 < \|\nabla W\|_2. \tag{5.2}$$

and $\lim_{n\to+\infty} \|u_n\|_{\tilde{S}(\mathbb{R})} = +\infty$*. Let $(t_n)_n$ be a time sequence. Assume*

$$\lim_{n\to+\infty} \|u_n\|_{\tilde{S}(-\infty,t_n)} = \lim_{n\to+\infty} \|u_n\|_{\tilde{S}(t_n,+\infty)} = +\infty.$$

Then, up to the extraction of a subsequence, there exist $\theta_0 \in \mathbb{R}$ and a sequence of parameters $\lambda_n > 0$ such that

$$\lim_{n\to+\infty} \left\| \frac{e^{i\theta_0}}{\lambda_n^{N/2}} \nabla u_n \left(t_n, \frac{\cdot}{\lambda_n}\right) - \nabla W \right\|_2 = 0.$$

We next recall some spectral properties of the operator $\mathcal{L}$. We refer to [DM07a, §5.1] for the details. We will often identify a complex-valued function f with an $\mathbb{R}^2$-valued function $(f_1, f_2)^T$, with $f_1 = \operatorname{Re} f$, $f_2 = \operatorname{Im} f$. Developping the energy around W, we get, for small functions $h \in \dot{H}^1$,

$$\mathcal{E}(W + h) = \mathcal{E}(W) + \widetilde{Q}(h) + O\Big(\|h\|^3_{\frac{2N}{N-2}}\Big),$$

where $\widetilde{Q}$ is the quadratic form $\widetilde{Q}(h) = B(h, h)$ and B is defined by

$$B(g, h) = \frac{1}{2}\int \nabla g_1 \cdot \nabla h_1 - \frac{N+2}{2(N-2)} \int g_1 h_1 W^{\frac{4}{N-2}} + \frac{1}{2}\int \nabla g_2 \cdot \nabla h_2 - \frac{1}{2}\int g_2 h_2 W^{\frac{4}{N-2}}.$$

Denote by $\mathcal{Y}_+$ the eigenfunction of $\mathcal{L}$ for the eigenvalue $\widetilde{\omega}$ and $\mathcal{Y}_- = m\overline{\mathcal{Y}}_+$ the eigenfunction of $\mathcal{L}$ for the eigenvalue $-\widetilde{\omega}$ ($m \neq 0$ is a real normalization constant), and recall the definition of W_0 in (3.2). One may show that W_0 and iW are in the kernel of $\widetilde{Q}$. Furthermore, $\widetilde{Q}(\mathcal{Y}_+) = \widetilde{Q}(\mathcal{Y}_-) = 0$ and we may chose m such that $B(\mathcal{Y}_+, \mathcal{Y}_-) = -1$. Let

$$\tilde{G}_\perp := \left\{ h \in \dot{H}^1 \; : \; \int \nabla W \cdot \nabla h_2 = \int \nabla W_0 \cdot \nabla h_1 = B(\mathcal{Y}_+, h) = B(\mathcal{Y}_-, h) = 0 \right\}.$$

By [DM07a, Lemma 5.2], there exists a constant $c > 0$ such that

$$\forall h \in \tilde{G}_\perp, \quad Q(h) \geq c\|\nabla h\|_2^2.$$

We consider as in §3.2 a sequence u_n of radial solutions of (5.1) such that

$$\mathcal{E}(u_n) \leq \mathcal{E}(W) - \varepsilon_n^2, \quad \|\nabla u_n(0)\|_2 < \|\nabla W\|_2, \tag{5.3}$$

$$\lim_{n\to+\infty} \|\nabla u_n(0) - \nabla W\|_2 = 0, \tag{5.4}$$

and develop $h_n = u_n - W$ as follows

$$h_n(t) = \beta_n^+(t)\mathcal{Y}_+ + \beta_n^-(t)\mathcal{Y}_- + \gamma_n(t)W_0 + \delta_n(t)iW + g_n(t), \quad g_n(t) \in \tilde{G}_\perp. \tag{5.5}$$

Arguing as in Claim 3.3, we may assume

$$\gamma_n(0) = \delta_n(0) = 0 \tag{5.6}$$

Then we have the following analog of Propositions 3.4 and Claim 3.6. We skip the proofs, that are very similar to the previous ones.

Proposition 5.2. *There exist a constant η_0, such that for all $\eta \in (0, \eta_0)$, for all sequence (u_n) satisfying* (5.3)*,* (5.4) *and* (5.6) *if*

$$T_n^+(\eta) = \inf \left\{ t \geq 0 \; : \; |\beta_n^-(t)| \geq \eta \right\}$$
$$T_n^-(\eta) = -\sup \left\{ t \leq 0 \; : \; |\beta_n^+(t)| \geq \eta \right\}.$$

then for large n, $T_n^+(\eta)$ and $T_n^-(\eta)$ are finite and

$$\lim_{n\to+\infty} \frac{T_n^+(\eta)}{\log|\beta_n^-(0)|} = \lim_{n\to+\infty} \frac{T_n^-(\eta)}{\log|\beta_n^+(0)|} = \frac{1}{\widetilde{\omega}}. \tag{5.7}$$

Furthermore,

$$\liminf_{n\to+\infty} |\beta_n'(T_n^+(\eta))| \geq \frac{\eta\widetilde{\omega}}{2}, \quad \liminf_{n\to+\infty} |\beta_n'(T_n^-(\eta))| \geq \frac{\eta\widetilde{\omega}}{2}. \tag{5.8}$$

Observe that in contrast with the wave equation case, there are two eigenfunctions, and that we have distinguished between the coefficient β_n^- of $\mathcal{Y}_-$, which tends to grow for positive times, and the one of $\mathcal{Y}_+$, which plays a similar role for negative times.

Claim 5.3. *There exists $M_0 > 0$ such that for all sequence (u_n) of solutions of* (5.1) *satisfying* (5.3)*,* (5.4) *and* (5.6) *we have*

$$\beta_n^+\beta_n^-(0) \neq 0 \textit{ and } \limsup_{n\to+\infty} \frac{\varepsilon_n^2}{|\beta_n^+(0)\beta_n^-(0)|} \leq M_0.$$

5.2. **Sketch of the proof of Theorem 2.**

Step 1. Lower bound.

We first show

$$\liminf_{\varepsilon\to 0^+} \frac{\widetilde{\mathcal{I}}_\varepsilon}{|\log \varepsilon|} \geq \frac{2}{\widetilde{\omega}} \int_{\mathbb{R}^N} W^{\frac{2(N+2)}{N-2}}. \tag{5.9}$$

Multiplying $\mathcal{Y}_+$ and $\mathcal{Y}_-$ by -1 if necessary, we may assume $\operatorname{Re}\int \nabla W \cdot \nabla \mathcal{Y}_\pm > 0$. Consider the family of solutions $(u^a)_{a>0}$ of (5.1) with initial conditions

$$u_0^a = W - a\mathcal{Y}_+ - a\mathcal{Y}_-.$$

Then for small $a>0$, $\int |\nabla u_0^a|^2 < \int |\nabla W|^2$. Furthermore

$$\mathcal{E}\left(u_0^a\right) = \mathcal{E}(W) - 2a^2 + O\left(a^3\right). \tag{5.10}$$

We argue by contradiction. If (5.9) does not hold, there exists a sequence ε_n which tends to 0 such that for some $\rho > \widetilde{\omega}$

$$\forall n, \quad \frac{2}{\rho}\int_{\mathbb{R}^N} W^{\frac{2(N+2)}{N-2}} \geq \frac{\widetilde{\mathcal{I}}_{\varepsilon_n}}{|\log \varepsilon_n|}. \tag{5.11}$$

We then chose a sequence a_n such that

$$\varepsilon_n^2 = \mathcal{E}(W) - \mathcal{E}\left(u_0^{a_n}\right), \quad \varepsilon_n^2 \sim 2a_n^2 \text{ as } n\to+\infty. \tag{5.12}$$

Let $u_n = u^{a_n}$. Then the assumptions of Proposition 5.2 are satisfied. Consider a small $\eta>0$. By Proposition 5.2, noting that $\beta_n^+(0) = \beta_n^-(0) = -a_n$, we get

$$\lim_{n\to+\infty} \frac{T_n^+(\eta)}{|\log a_n|} = \lim_{n\to+\infty} \frac{T_n^-(\eta)}{|\log a_n|} = \frac{1}{\widetilde{\omega}}$$

By (5.12),

$$\lim_{n\to+\infty} \frac{T_n^-(\eta)}{|\log|\varepsilon_n||} = \lim_{n\to+\infty} \frac{T_n^+(\eta)}{|\log|\varepsilon_n||} = \frac{1}{\widetilde{\omega}}. \tag{5.13}$$

Writing $u_n = W + h_n$, and arguing as in Step 1 of Section 4, we obtain

$$\int_0^{T_n^+(\eta)} \int_{\mathbb{R}^N} |u_n|^{\frac{2(N+2)}{N-2}} \geq T_n^+(\eta)\left[\|W\|_{\frac{2(N+2)}{N-2}} - C\eta\right]^{\frac{2(N+2)}{N-2}}.$$

Hence with (5.13), and letting η tends to 0,

$$\liminf_{n\to+\infty} \frac{1}{|\log \varepsilon_n|} \int_0^{+\infty} |u_n|^{\frac{2(N+2)}{N-2}} \geq \frac{1}{\widetilde{\omega}} \|W\|_{\frac{2(N+2)}{N-2}}^{\frac{2(N+2)}{N-2}}.$$

Arguing similarly for negative time, we obtain

$$\liminf_{n\to+\infty} \frac{\widetilde{\mathcal{I}}_{\varepsilon_n}}{|\log \varepsilon_n|} \geq \liminf_{n\to+\infty} \frac{1}{|\log \varepsilon_n|} \int_{-\infty}^{+\infty} |u_n|^{\frac{2(N+2)}{N-2}} \geq \frac{2}{\widetilde{\omega}} \|W\|_{\frac{2(N+2)}{N-2}}^{\frac{2(N+2)}{N-2}},$$

contradicting (5.11). Step 1 is complete.

Step 2. Upper bound.

To show the upper bound on $\widetilde{\mathcal{I}}_\varepsilon$, we must show that for any sequence $\varepsilon_n > 0$ that goes to 0 any sequence u_n of solutions of (5.1) such that

$$\|\nabla u_n(0)\|_2 < \|\nabla W\|_2, \quad \mathcal{E}(W) - \mathcal{E}(u_n) = \varepsilon_n^2, \tag{5.14}$$

we have

$$\limsup_{n\to+\infty}\frac{1}{|\log\varepsilon_n|}\int_{\mathbb{R}\times\mathbb{R}^N}|u_n|^{\frac{2(N+2)}{N-2}}\leq\frac{2}{\widetilde{\omega}}\int_{\mathbb{R}^N}W^{\frac{2(N+2)}{N-2}}. \tag{5.15}$$

In view of Proposition 5.1 and the analoguous of Claim 3.3, we may assume that u_n satisfy the assumptions of Proposition 5.2.

Fix a small $\eta>0$, and consider $T_n^{\pm}(\eta)$ defined by Proposition 5.2. Then by the same proof than in Step 2 of Section 4, one can show that there exists a constant $C>0$ such that

$$\limsup_{n\to+\infty}\frac{1}{\left|\log|\beta_n^-(0)|\right|}\int_0^{T_n^+(\eta)}\int_{\mathbb{R}^N}|u_n|^{\frac{2(N+2)}{N-2}}\leq\frac{1}{\widetilde{\omega}}\left[\|W\|_{\frac{2(N+2)}{N-2}}+C\eta\right]^{\frac{2(N+2)}{N-2}}$$

$$\limsup_{n\to+\infty}\frac{1}{\left|\log|\beta_n^+(0)|\right|}\int_{T_n^-(\eta)}^{0}\int_{\mathbb{R}^N}|u_n|^{\frac{2(N+2)}{N-2}}\leq\frac{1}{\widetilde{\omega}}\left[\|W\|_{\frac{2(N+2)}{N-2}}+C\eta\right]^{\frac{2(N+2)}{N-2}}.$$

By Claim 5.3, for large n,

$$2\,|\log\varepsilon_n|\geq\left|\log|\beta_n^-(0)|+\log|\beta_n^+(0)|\right|+o_n(1),$$

which yields

$$\limsup_{n\to+\infty}\int_{-T_n^-(\eta)}^{T_n^+(\eta)}\int_{\mathbb{R}^N}|u_n|^{\frac{2(N+2)}{N-2}}\leq\frac{2\,|\log\varepsilon_n|}{\widetilde{\omega}}\left[\|W\|_{\frac{2(N+2)}{N-2}}+C\eta\right]^{\frac{2(N+2)}{N-2}} \tag{5.16}$$

It remains to show, as in Step 3 of Section 4, that if η is small enough, there exists a constant $C(\eta)>0$ such that for large n

$$\|u_n\|_{S(-\infty,-T_n^-(\eta))}+\|u_n\|_{S(T_n^+(\eta),+\infty)}\leq C(\eta). \tag{5.17}$$

Combining (5.16) and (5.17) and letting η tends to 0 we would get (5.15).

To show (5.17), we argue by contradiction. Assume that there exists a subsequence of (u_n), such that (from now on, we will write $T_n^+=T_n^+(\eta)$)

$$\|u_n\|_{S(T_n^+,+\infty)}\underset{n\to\infty}{\longrightarrow}+\infty.$$

Then by Proposition 5.1, there exists $\theta_0\in\mathbb{R}$ and a sequence $\lambda_n>0$, such that

$$\lim_{n\to+\infty}\left\|\frac{e^{i\theta_0}}{\lambda_n^{N/2}}\nabla u_n\left(T_n^+,\frac{\cdot}{\lambda_n}\right)-\nabla W\right\|_2=0. \tag{5.18}$$

As in Step 3 of Section 4, we will get a contradiction by showing that $\frac{d\beta_n^-}{dt}(T_n^+(\eta))$ tends to 0. Unlike the case of the wave equation, the convergence to 0 of the time derivative of u is not given directly by the compactness argument of Proposition 5.1. However, (5.18) and the fact that u_n is a solution of (5.1) which is in $C^0(\mathbb{R},\dot{H}^1)$ shows that

$$\lim_{n\to+\infty}\left\|\partial_t u_n(T_n^+)\right\|_{H^{-1}}=0.$$

As $\mathcal{Y}_+$ is a Schwartz function (see [DM07a, §7.2.2]), we get, at the point $t=T_n^+$,

$$\lim_{n\to+\infty}\frac{d}{dt}B(u_n(t),\mathcal{Y}_+))=0. \tag{5.19}$$

The condition $g_n(t) \in G_\perp$ implies that $\beta_n^-(t) = -B\big(u_n(t) - W, \mathcal{Y}_+\big)$. Thus (5.8) contradicts (5.19). This shows

$$\|u_n\|_{S(T_n^+(\eta),+\infty)} \leq C(\eta).$$

By a similar argument for negative time, we get (5.17). Combining (5.16) and (5.17), we obtain (5.15), which concludes the sketch of the proof of Theorem 2.

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