Modal Intervals Revisited
Part 2: A Generalized Interval Mean-Value Extension

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Abstract

In Modal Intervals Revisited Part 1, new extensions to generalized intervals (intervals whose bounds are not constrained to be ordered), called AE-extensions, have been defined. They provide the same interpretations as the extensions to modal intervals and therefore enhance the interpretations of the classical interval extensions (for example, both inner and outer approximations of function ranges are in the scope of the AE-extensions). The construction of AE-extensions is similar to the one of classical interval extensions. In particular, a natural AE-extension has been defined from the Kaucher arithmetic which simplified some central results of the modal intervals theory.

Starting from this framework, the mean-value AE-extension is now defined. It represents a new way to linearize a real function, which is compatible with both inner and outer approximations of its range. With a quadratic order of convergence for real-valued functions, it allows to overcome some difficulties which were encountered using a preconditioning process together with the natural AE-extensions. Some application examples are finally presented, displaying the application potential of the mean-value AE-extension.

1 Introduction

Classical intervals

One fundamental concept of the classical intervals theory is the extension of real functions to intervals (see [15, 2, 16]). These extensions are constructed so as to provide supersets of the range of real functions over boxes. However, computing the minimal interval extension of a real function, i.e. the interval

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hull of the range of a function over a box, is a NP-hard problem with respect to the number of variables (see [14]). One of the main work of interval researchers has been to construct computable extensions which lead to good approximations of functions ranges. Some intensively used interval extensions are for example the natural extensions and the mean-value extensions. This latter has better properties than the natural extension and have been intensively studied (see [4, 19, 1, 17]). It relies upon a linearization of the function which leads to an interval linear function whose range contains the range of the original non-linear function. This provides the mean-value extension with a good behavior when evaluated over small enough intervals: formally, the mean-value extension has a quadratic order of convergence. Furthermore it allows to apply to non-linear interval systems some algorithms dedicated to linear interval systems.

AE-extensions

The modal intervals theory enhances the classical intervals theory providing richer interpretations (see [23, 24] for a description of the theory and [3, 20, 21, 8, 9] for some promising applications of the enhanced interpretations). In particular, both inner and outer approximations of the function ranges over boxes are in the scope of extensions to modal intervals. The modal intervals theory has been revisited and reformulated in Modal Intervals Revisited Part 1 (see [5]). New extensions to generalized intervals, called AE-extensions, have been defined which provide the same enhanced interpretations as the modal intervals theory. However, the construction of the AE-extensions is similar to the construction of the extensions to classical intervals. In particular, the natural AE-extensions have been defined and the order of convergence of AE-extensions has been introduced. If the natural AE-extension was proved to have a linear order of convergence in Modal Intervals Revisited Part 1, it was also illustrated that such an order of convergence was not sufficient in some situations in particular when some preconditioning process has to be involved.

Contribution

The mean-value AE-extensions are defined. They provide a new way to linearize a nonlinear function which is compatible with the enhanced interpretations of AE-extensions (in particular with both inner and outer approximations of the function range over boxes). Similarly to the classical interval mean-value extensions, the mean-value AE-extensions are proved to have a quadratic order of convergence in the case of real-valued functions $f:\mathbb{R}^n \longrightarrow \mathbb{R}$. The usefulness of the mean-value AE-extension is illustrated. Given a continuously differentiable function $f:\mathbb{R}^n \longrightarrow \mathbb{R}^n$, the mean-value AE-extension is used so as to construct a parallelepiped which is included inside the range of f over a box $\mathbf{x}$.

Outline of the paper

Basic definitions related to generalized intervals and AE-extensions are presented in Section 2. The mean-value AE-extensions for real-valued and vector-valued functions are defined in Section 3. Their order of convergence are investigated in Section 4. The mean-value AE-extension is used together a preconditioning process so as to construct a inner approximation of the range of a function over a box in Section 5.

Notations

When dealing with sets, the usual set union, set intersection and set difference are respectively denoted by $A \cup B$, $A \cap B$ and $A \backslash B$ and defined by

$$x \in A \cup B \iff x \in A \vee x \in B$$

$A \cap B = \{x \in A | x \in B\}$ and $A \backslash B = \{x \in A | x \notin B\}$.

Intervals, interval functions and interval matrices will be denoted by boldface letters, for example $\mathbf{x}$, $\mathbf{f}$ and $\mathbf{A}$. The set of classical intervals is denoted by $\mathbb{IR}$. An interval $\mathbf{x} \in \mathbb{IR}^n$ is equivalently considered as a subset of $\mathbb{R}^n$ or as vector of intervals. The interval hull of a subset $\mathbb{E}$ of $\mathbb{R}^n$ is denoted by $\square\,\mathbb{E}$. The lower and upper bounds of an interval $\mathbf{x}$ are denoted respectively by $\inf \mathbf{x}$ and $\sup \mathbf{x}$. The interval join and meet operations, which are different from the set union and intersection, will be respectively denoted by the symbols $\vee$ and $\wedge$ (the join of two intervals is sometimes called their hull). Given $E \subseteq \mathbb{R}^n$, it will be useful to denote $\{\mathbf{x} \in \mathbb{IR}^n | \mathbf{x} \subseteq E\}$ by $\mathbb{I}E$. The notations of classical intervals will be used for generalized intervals and their related objects, the set of generalized intervals being denoted by $\mathbb{KR}$ (see Section 2.1 for more details). The following notation will be used:

Notation. Sets of indices are denoted by calligraphic letters. Let $\mathcal{I} = \{i_1, \ldots, i_n\}$ be an ordered set of indices with $i_k \leq i_{k+1}$. Then, the vector $(x_{i_1}, \ldots, x_{i_n})^T$ is denoted by $x_{\mathcal{I}}$.

This notation is similar to the one proposed in [13]. It will be used with any kind of objects, for example real vectors, interval vector or function vectors. The involved set of indices will be ordered with the usual lexicographic order. Intervals of integers are denoted by $[n..m] \subseteq \mathbb{N}$ where $n, m \in \mathbb{N}$ with $n \leq m$. The vector $x_{[1..n]} = (x_1, \ldots, x_n)^T$ will be denoted by the usual notation x when no confusion is possible.

The real functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ are respectively called real-valued functions and vector-valued functions when emphasis has to be put on this difference. Their derivatives are defined homogeneously in the following way: $f'(x) \in \mathbb{R}^{m \times n}$ is defined by

$$\big(f'(x)\big)_{ij} = \frac{\partial f_i}{\partial x_j}(x)$$

So, if $m = 1$ then $f'(x) \in \mathbb{R}^{1 \times n}$ is the gradient of f; if $m = n = 1$ then $f'(x) \in \mathbb{R}^{1 \times 1}$ is identified to the usual derivative of f. Finally, vectors of

$\mathbb{R}^n$ (respectively $\mathbb{IR}^n$ or $\mathbb{KR}^n$) are identified to the column matrices of $\mathbb{R}^{n\times 1}$ (respectively $\mathbb{IR}^{n\times 1}$ or $\mathbb{KR}^{n\times 1}$). Therefore, with $x, y \in \mathbb{R}^n$, the matrix product $x^T \times y$ stands for

$$\sum_{i\in[1..n]} x_i y_i$$

2 AE-extensions of real functions

This section presents the generalized intervals and the new formulation of the modal intervals theory that was proposed in Modal Intervals Revisited Part 1 ([5]). Only results that are relevant for the developments proposed in the present paper are summarized in this section.

2.1 Generalized intervals

Generalized intervals are intervals whose bounds are not constrained to be ordered, for example $[-1, 1]$ or $[1, -1]$ are generalized intervals. They have been introduced in [18, 11, 12] so as to improve both the algebraic structure and the order structure of the classical intervals. The set of generalized intervals is denoted by $\mathbb{KR}$ and is decomposed into three subsets:

- The set of proper intervals which bounds are ordered increasingly. These proper intervals are identified with classical intervals. The set of proper intervals is denoted by the same symbol as the one used for classical intervals, i.e. $\mathbb{IR} = \{[a, b] | a \leq b\}$.
- The set of improper intervals which bound are orderer decreasingly. It is denoted by $\overline{\mathbb{IR}} = \{[a, b] | a \geq b\}$.
- The set of degenerated intervals $[a, a]$, where $a \in \mathbb{R}$, which are both proper and improper. A degenerated interval $[a, a]$ will be identified to the corresponding real a.

Therefore, from a set of reals $\{x \in \mathbb{R} | a \leq x \leq b\}$, one can build the two generalized intervals $[a, b]$ and $[b, a]$. It will be useful to change one to the other keeping the underlying set of reals unchanged using the following operators:

- dual operator: $\text{dual}\,[a, b] = [b, a]$;
- proper projection: $\text{pro}\,[a, b] = [\min\{a, b\}, \max\{a, b\}]$;

The operations mid , rad and $|\,.\,|$ are defined as in the case of classical intervals.

- $\text{mid}\,[a, b] = \frac{a+b}{2}$;
- $\text{rad}\,[a, b] = \frac{b-a}{2}$;
- $|[a, b]| = \max\{|a|, |b|\}$.

The width is defined as $\operatorname{wid}\mathbf{x} = 2\operatorname{rad}\mathbf{x}$. Both the radius and the width are positive for proper intervals and negative for improper intervals (and null for degenerated intervals). Given a set of indices E with $\operatorname{card}E = n$ and a generalized interval $\mathbf{x}_{\mathcal{E}} \in \mathbb{KR}^n$, the following functions allow to pick up the indices of the proper and improper components of $\mathbf{x}_{\mathcal{E}}$:

- $\mathcal{P}(\mathbf{x}_{\mathcal{E}}) = \{i \in E | \mathbf{x}_i \in \mathbb{IR}\}$
- $\mathcal{I}(\mathbf{x}_{\mathcal{E}}) = \{i \in E | \mathbf{x}_i \notin \mathbb{IR}\}$

Remark. Degenerated components are counted as proper intervals by convention. The other choice would have been equivalent.

The distance between two generalized intervals $\mathbf{x} \in \mathbb{KR}$ and $\mathbf{y} \in \mathbb{KR}$ is defined in the following way:

$$\operatorname{dist}(\mathbf{x},\mathbf{y}) = \max\{|\underline{\mathbf{x}} - \underline{\mathbf{y}}|, |\overline{\mathbf{x}} - \overline{\mathbf{y}}|\}$$

As shown in [11, 12], $\mathbb{KR}$ then becomes a complete metric space. The generalized intervals are ordered by an inclusion which prolongates the inclusion of classical intervals. Consider two generalized intervals $\mathbf{x} = [\underline{\mathbf{x}}, \overline{\mathbf{x}}]$ and $\mathbf{y} = [\underline{\mathbf{y}}, \overline{\mathbf{y}}]$

$$\mathbf{x} \subseteq \mathbf{y} \iff \underline{\mathbf{y}} \le \underline{\mathbf{x}} \;\wedge\; \overline{\mathbf{x}} \le \overline{\mathbf{y}}$$

The generalized interval join and meet are formally the same as their classical counterparts: consider $E \subseteq \mathbb{KR}$ bounded set of generalized interval then

$$(\vee E) = [\ \inf_{\mathbf{x}\in E}(\inf \mathbf{x})\ ,\ \sup_{\mathbf{x}\in E}(\sup \mathbf{x})\ ]$$
$$\text{and} \quad (\wedge E) = [\ \sup_{\mathbf{x}\in E}(\inf \mathbf{x})\ ,\ \inf_{\mathbf{x}\in E}(\sup \mathbf{x})\ ].$$

Remark. In the context of generalized intervals, it becomes important to use two different signs for the set intersection and for the interval meet. For example, $[0,1] \bigcap [2,3] = \emptyset$ whereas $[0,1] \wedge [2,3] = [2,1]$.

The inclusion is related to the dual operation in the following way.

$$\mathbf{x} \subseteq \mathbf{y} \iff (\operatorname{dual}\mathbf{x}) \supseteq (\operatorname{dual}\mathbf{y})$$

The so-called Kaucher arithmetic extends the classical interval arithmetic. Its definition can be found in [11, 12] or in [22]. When it is not misunderstanding, the Kaucher multiplication will be denoted by $\mathbf{xy}$ instead of $\mathbf{x}\times\mathbf{y}$. The Kaucher arithmetic has better algebraic properties than the classical interval arithmetic: The Kaucher addition is a group. The opposite of an interval $\mathbf{x}$ is $-\operatorname{dual}\mathbf{x}$, i.e.

$$\mathbf{x} + (-\operatorname{dual}\mathbf{x}) = \mathbf{x} - \operatorname{dual}\mathbf{x} = [0,0]$$

The Kaucher multiplication restricted to generalized intervals which proper projection do not contains 0 is also a group. The inverse of such a generalized interval $\mathbf{x}$ is $1/(\operatorname{dual}\mathbf{x})$, i.e.

$$\mathbf{x} \times (1/\operatorname{dual}\mathbf{x}) = \mathbf{x}/(\operatorname{dual}\mathbf{x}) = [1,1]$$

A useful property of the Kaucher arithmetic is its monotonicity with respect to the inclusion: whatever are $\circ \in \{+, \times, -, \div\}$ and $\mathbf{x}, \mathbf{y}, \mathbf{x}', \mathbf{y}' \in \mathbb{KR}$,

$$\mathbf{x} \subseteq \mathbf{x}' \wedge \mathbf{y} \subseteq \mathbf{y}' \implies (\mathbf{x} \circ \mathbf{y}) \subseteq (\mathbf{x}' \circ \mathbf{y}')$$

Furthermore, the Kaucher arithmetic is linked to the distance between generalized intervals in the following way: for any intervals $\mathbf{x}, \mathbf{y}, \mathbf{x}', \mathbf{y}' \in \mathbb{KR}$,

$$\begin{aligned} \operatorname{dist}(\mathbf{x}\mathbf{y}, \mathbf{x}\mathbf{y}') &\leq |\mathbf{x}| \operatorname{dist}(\mathbf{y}, \mathbf{y}') \\ \operatorname{dist}(\mathbf{x}+\mathbf{y}, \mathbf{x}'+\mathbf{y}') &\leq \operatorname{dist}(\mathbf{x}, \mathbf{x}') + \operatorname{dist}(\mathbf{y}, \mathbf{y}') \end{aligned}$$

Generalized interval vectors $\mathbf{x} \in \mathbb{KR}^n$ and generalized interval matrices $\mathbf{A} \in \mathbb{KR}^{n\times n}$ are defined like in the classical interval theory. The operations mid , rad , $|\,.\,|$, dual and pro are performed on vectors and matrices elementwise. The metric is extended to vectors and matrices in the usual way: given $\mathbf{x}, \mathbf{y} \in \mathbb{KR}^n$ and $\mathbf{A}, \mathbf{B} \in \mathbb{KR}^n$,

$$\operatorname{dist}(\mathbf{x}, \mathbf{y}) = \max_{i\in[1..n]} \operatorname{dist}(\mathbf{x}_i, \mathbf{y}_i) \quad \text{and} \quad \operatorname{dist}(\mathbf{A}, \mathbf{B}) = \max_{\substack{i\in[1..n] \\ j\in[1..m]}} \operatorname{dist}(\mathbf{A}_{ij}, \mathbf{B}_{ij})$$

Given $E \subseteq \mathbb{R}^n$, it will be useful to denote the set of generalized interval $\{\mathbf{x} \in \mathbb{KR}^n | \operatorname{pro} \mathbf{x} \subseteq E\}$ by $\mathbb{K}E$. Finally, the distance is used to define a continuity: in the sequel the local Lipschitz continuity will be useful.

Definition 2.1 (Goldsztejn, [5])**.** The interval function $\mathbf{f} : \mathbb{KR}^n \longrightarrow \mathbb{KR}^m$ is locally Lipschitz continuous if and only if for any $\mathbf{x}^{\mathsf{ref}} \in \mathbb{IR}^n$, there exists $\lambda > 0$ which satisfies for all $\mathbf{x}, \mathbf{y} \in \mathbb{K}\mathbf{x}^{\mathsf{ref}}$,

$$\operatorname{dist}(\mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{y})) \leq \lambda \operatorname{dist}(\mathbf{x}, \mathbf{y})$$

Remark. This definition is naturally specialized to functions $\mathbf{f} : \mathbb{IR}^n \longrightarrow \mathbb{IR}^m$ and $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ considering respectively all $\mathbf{x}, \mathbf{y} \in \mathbb{I}\mathbf{x}^{\mathsf{ref}}$ and all $x, y \in \mathbf{x}^{\mathsf{ref}}$. Also, it stands for functions $\mathbf{f} : \mathbb{IR}^n \longrightarrow \mathbb{IR}^{m\times p}$.

Obviously, a function $\mathbf{f}$ is locally Lipschitz continuous if and only if all its components $\mathbf{f}_i$ are locally Lipschitz continuous.

2.2 AE-extensions

The classical interval extensions are defined so as to compute outer approximations of functions ranges over boxes. The condition $\operatorname{range}(f, \mathbf{x}) \subseteq \mathbf{z}$ can be equivalently stated by the following quantified proposition:

$$\big(\forall x \in \mathbf{x}\big)\big(\exists z \in \mathbf{z}\big)\big(z = f(x)\big)$$

The AE-extensions are defined allowing more general quantified propositions. The quantifier corresponding to a variable is determined using the proper/improper quality of the generalized interval corresponding to this variable. The denomination "AE" is related to the specific ordering of the quantifier, the universal quantifiers preceding the existential ones ("A(ll)E(xists)"). The following definition formalizes this idea.

Definition 2.2 (Goldsztejn, [5]). Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $\mathbf{x} \in \mathbb{KR}^n$ and $\mathbf{z} \in \mathbb{KR}^m$. The interval $\mathbf{z}$ is interpretable with respect to f and $\mathbf{x}$ (or shortly $(f, \mathbf{x})$-interpretable) if and only if the following quantified proposition is true:

$$\big(\forall x_{\mathcal{P}} \in \mathbf{x}_{\mathcal{P}}\big)\big(\forall z_{\mathcal{I}'} \in \text{pro } \mathbf{z}_{\mathcal{I}'}\big)\big(\exists z_{\mathcal{P}'} \in \mathbf{z}_{\mathcal{P}'}\big)\big(\exists x_{\mathcal{I}} \in \text{pro } \mathbf{x}_{\mathcal{I}}\big)\big(z = f(x)\big) \tag{1}$$

where $\mathbf{z} = \mathbf{g}(\mathbf{x})$ and $\mathcal{P} = \mathcal{P}(\mathbf{x})$, $\mathcal{I} = \mathcal{I}(\mathbf{x})$, $\mathcal{P}' = \mathcal{P}(\mathbf{z})$ and $\mathcal{I}' = \mathcal{I}(\mathbf{z})$ are the sets of indices corresponding respectively to the proper and improper components of $\mathbf{x}$ and $\mathbf{z}$ (if one of these sets of indices is empty then the corresponding quantifier block it canceled).

When $m = 1$, the quantified proposition (1) can be formulated using a quantifier which depends on the proper/improper quality of the interval $\mathbf{z}$:

$$\big(\forall x_{\mathcal{P}} \in \mathbf{x}_{\mathcal{P}}\big)\big(\mathsf{Q}^{(\mathbf{z})} z \in \mathbf{z}\big)\big(\exists x_{\mathcal{I}} \in \text{pro } \mathbf{x}_{\mathcal{I}}\big)\big(z = f(x)\big)$$

where $\mathsf{Q}^{(\mathbf{z})} = \exists$ if $\mathbf{z} \in \mathbb{IR}$ and $\mathsf{Q}^{(\mathbf{z})} = \forall$ otherwise. Here are some possible interpretations of a $(f, \mathbf{x})$-interpretable interval $\mathbf{z}$ in the special case $m = 1$:

1. When $\mathbf{x}$ is proper, $\mathbf{z}$ has to be proper and a $(f, \mathbf{x})$-interpretable interval $\mathbf{z}$ satisfies
$$\big(\forall x \in \mathbf{x}\big)\big(\exists z \in \mathbf{z}\big)\big(z = f(x)\big),$$
i.e. $\mathbf{z}$ is an outer approximation of range $(f, \mathbf{x})$.

2. When $\mathbf{x}$ is improper, $\mathbf{z}$ can be either proper or improper.

   (a) If $\mathbf{z}$ is improper then it satisfies
   $$\big(\forall z \in \text{pro } \mathbf{z}\big)\big(\exists x \in \text{pro } \mathbf{x}\big)\big(z = f(x)\big),$$
   i.e. pro $\mathbf{z}$ is an inner approximation of range $(f, \mathbf{x})$.

   (b) If $\mathbf{z}$ is proper then it satisfies
   $$\big(\exists z \in \text{pro } \mathbf{z}\big)\big(\exists x \in \text{pro } \mathbf{x}\big)\big(z = f(x)\big),$$
   i.e. pro $\mathbf{z} \cap$ range $(f, \text{pro } \mathbf{x}) \neq \emptyset$.

3. When $\mathbf{x}_1$ is proper and $\mathbf{x}_2$ is improper, $\mathbf{z}$ can be either proper or improper.

   (a) If $\mathbf{z}$ is improper then it satisfies
   $$\big(\forall x_1 \in \mathbf{x}_1\big)\big(\forall z \in \text{pro } \mathbf{z}\big)\big(\exists x_2 \in \text{pro } \mathbf{x}_2\big)\big(z = f(x)\big),$$
   i.e. for any fixed $z_0 \in$ pro $\mathbf{z}$, the interval $\mathbf{x}_1$ is inside the projection of the relation $f(x_1, x_2) = z_0$ on the axis x_1.

   (b) If $\mathbf{z}$ is proper then it satisfies
   $$\big(\forall x_1 \in \mathbf{x}_1\big)\big(\exists z \in \mathbf{z}\big)\big(\exists x_2 \in \mathbf{x}_2\big)\big(z = f(x)\big).$$

When $m > 1$, some additional interpretations are available. In all cases, the more interesting interpretations are (1), (2a) and (3a).

The inclusion between generalized intervals provides a way to compare $(f, \mathbf{x})$-interpretable intervals: as illustrated by the next example, if two $(f, \mathbf{x})$-interpretable intervals $\mathbf{z}$ and $\mathbf{z}'$ are related by $\mathbf{z} \subseteq \mathbf{z}'$ then $\mathbf{z}$ is more accurate than $\mathbf{z}'$ (i.e. $\mathbf{z}$ provides more information than $\mathbf{z}'$).

Example 2.1. Consider the function $f(x) = x^2$ and the proper interval $\mathbf{x} = [-1, 3]$. So $\text{range}\,(f, \mathbf{x}) = [0, 9]$. The proper intervals $[-1, 10]$ and $[-2, 11]$ are both $(f, \mathbf{x})$-interpretable. As $[-1, 10] \subseteq [-2, 11]$, the first is more accurate than the second. Indeed, the first is a more accurate outer approximation of $\text{range}\,(f, \mathbf{x})$ than the second. Now, the improper intervals $[8, 1]$ and $[7, 2]$ are both $(f, \text{dual}\ \mathbf{x})$-interpretable: indeed both $\text{pro}\,[8, 1]$ and $\text{pro}\,[7, 2]$ are inner approximations of $\text{range}\,(f, \mathbf{x}))$. As $[8, 1] \subseteq [7, 2]$, the first is more accurate than the second. Indeed, $\text{pro}\,[8, 1]$ is a more accurate inner approximation of $\text{range}\,(f, \mathbf{x})$ than $\text{pro}\,[7, 2]$.

This leads naturally to the following definition for the minimality of $(f, \mathbf{x})$-interpretable intervals: the $(f, \mathbf{x})$-interpretable interval $\mathbf{z}$ is minimal if and only if for any $(f, \mathbf{x})$-interpretable interval $\mathbf{z}'$,

$$\mathbf{z}' \subseteq \mathbf{z} \Longrightarrow \mathbf{z}' = \mathbf{z}$$

This definition of minimality generalizes its corresponding one in the context of classical interval extensions. Indeed, if $\mathbf{x}$ is proper, then the only minimal $(f, \mathbf{x})$-interpretable interval is $\square\,\text{range}\,(f, \mathbf{x})$ (as in the context of classical interval extensions). However, when $\mathbf{x}$ is not proper, there are in general several minimal $(f, \mathbf{x})$-interpretable intervals. This is illustrated by the next example.

Example 2.2. Let $f(x) = Mx$ with $M = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} [1, -1] \\ [1, -1] \end{pmatrix} \in \overline{\mathbb{IR}}^2$. An improper $(f, \mathbf{x})$-interpretable interval $\mathbf{z} \in \overline{\mathbb{IR}}$ is an inner approximation of $\text{range}\,(f, \text{pro}\ \mathbf{x})$, i.e. it satisfies

$$\big(\forall z \in \text{pro}\ \mathbf{z}\big)\big(\exists x \in \text{pro}\ \mathbf{x}\big)\big(z = f(x)\big).$$

If the inner approximation ($\text{pro}\ \mathbf{z}$) is maximal then $\mathbf{z}$ is an minimal $(f, \mathbf{x})$-interpretable interval (see [5]). The following improper intervals have a proper projection which is a maximal inner approximation of $\text{range}\,(f, \text{pro}\ \mathbf{x})$:

$$\mathbf{z}_\lambda = \frac{1}{2}\begin{pmatrix} [1 - \lambda\ ,\ \lambda - 1] \\ [\lambda + 1\ ,\ -\lambda - 1] \end{pmatrix}$$

where $\lambda \in [-1, 1]$. Therefore, they are minimal $(f, \mathbf{x})$-interpretable and in this case there is a manifold of minimal $(f, \mathbf{x})$-interpretable intervals.

The definition of the AE-extensions is constructed from the usual definition of an extension to classical intervals changing the condition "$\text{range}\,(f, \mathbf{x}) \subseteq \mathbf{z}$" by its generalization to generalized intervals "$\mathbf{z}$ is $(f, \mathbf{x})$-interpretable".

Definition 2.3 (Goldsztejn, [5]). Consider a continuous real function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$. An interval function $\mathbf{g} : \mathbb{KR}^n \longrightarrow \mathbb{KR}^m$ is an AE-extension of f if and only if both following conditions are satisfied:

1. $\big(\forall x \in \mathbb{R}^n\big)\big(\mathbf{g}(x) = f(x)\big)$
2. $\big(\forall \mathbf{x} \in \mathbb{KR}^n\big)\big(\mathbf{g}(\mathbf{x}) \text{ is } (f, \mathbf{x})\text{-interpretable}\big)$

Also, $\mathbf{g}$ is minimal if for all $\mathbf{x} \in \mathbb{KR}^n$ the $(f, \mathbf{x})$-interpretable interval $\mathbf{g}(\mathbf{x})$ is minimal.

Remark. As in [5], the following simplification will be used: all the functions met in the sequel will be defined in $\mathbb{R}^n$. When other functions have to be considered, some attention should be given to the involved definition domains.

This definition is indeed a generalization of the definition of classical interval extensions as when $\mathbf{x}$ is proper, $\mathbf{g}(\mathbf{x})$ has to be proper and we have

$$\big(\forall x \in \mathbf{x}\big)\big(\exists z \in \mathbf{z}\big)\big(z = f(x)\big),$$

that is, range $(f, \mathbf{x}) \subseteq \mathbf{g}(\mathbf{x})$. When dealing with rounded computations, an AE-extension cannot satisfy (1). An interval function which only satisfies (2) is called a weak AE-extension. Some questions which are obvious in the context of classical interval extensions have to be investigated when dealing with AE-extensions. It is proved in [5] that:

- every continuous function has at least one AE-extension.
- for every AE-extension, there exists an minimal AE-extension which is more accurate.

Once the minimality has been defined, the quality of AE-extensions can be measured using the order of convergence. Generalizing its definition in the context of classical interval extensions, the order of convergence of AE-extensions can be defined in the following way:

Definition 2.4 (Goldsztejn, [5]). Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a continuous function and $\mathbf{g} : \mathbb{KR}^n \longrightarrow \mathbb{KR}^m$ be an AE-extension of f. The AE-extension $\mathbf{g}$ has a convergence order $\alpha \in \mathbb{R}$, $\alpha > 0$, if and only if the exists an minimal AE-extension $\mathbf{f}$ of f more accurate that $\mathbf{g}$ such that for any $\mathbf{x}^{\mathsf{ref}} \in \mathbb{IR}^n$, there exists $\gamma > 0$ such that for any $\mathbf{x} \in \mathbb{K}\mathbf{x}^{\mathsf{ref}}$,

$$\operatorname{dist}(\mathbf{g}(\mathbf{x}), \mathbf{f}(\mathbf{x})) \leq \gamma(||\operatorname{wid}\,\mathbf{x}||)^{\alpha}$$

Remark. It is obvious that an AE-extension which has an order of convergence α has also an order of convergence α' for any $0 < \alpha' \leq \alpha$. Also, the usually considered orders of convergence are integers. An order of convergence 1 is called a linear order of convergence, and an order of convergence 2 a quadratic order of convergence.

It is proved in [5] that any locally Lipschitz continuous AE-extension has a linear order of convergence.

The construction of AE-extensions is done in two steps: first, the special case of real-valued functions is investigated leading to the expressions of the minimal AE-extensions of a class of elementary functions. Then, the natural AE-extensions of compound real functions are defined.

2.3 AE-extensions of real-valued functions

Any continuous real-valued function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ has an unique minimal AE-extension which is denoted by $f^* : \mathbb{KR}^n \longrightarrow \mathbb{KR}$. From the definition of the minimality of AE-extensions, f^* is the only interval function which satisfies for all $\mathbf{z} \in \mathbb{KR}^n$,

$$\mathbf{z} \text{ is } (f, \mathbf{x})\text{-interpretable} \quad \Longleftrightarrow \quad f^*(\mathbf{x}) \subseteq \mathbf{z}$$

From this latter characterization, one can get the following expression of f^*:

$$\begin{aligned} f^*(\mathbf{x}) &= \bigvee_{x_\mathcal{P} \in \mathbf{x}_\mathcal{P}} \; \bigwedge_{x_\mathcal{I} \in (\text{pro } \mathbf{x}_\mathcal{I})} f(x) \\ &= [\min_{x_\mathcal{P} \in \mathbf{x}_\mathcal{P}} \; \max_{x_\mathcal{I} \in (\text{pro } \mathbf{x}_\mathcal{I})} f(x) \; , \; \max_{x_\mathcal{P} \in \mathbf{x}_\mathcal{P}} \; \min_{x_\mathcal{I} \in (\text{pro } \mathbf{x}_\mathcal{I})} f(x)] \end{aligned}$$

where $\mathcal{P} = \mathcal{P}(\mathbf{x})$ et $\mathcal{I} = \mathcal{I}(\mathbf{x})$.

Remark. When $\mathcal{P} = \emptyset$ or $\mathcal{I} = \emptyset$, the expressions of f^* are respectively

$$f^*(\mathbf{x}) = [\max_{x \in (\text{pro } \mathbf{x})} f(x), \min_{x \in (\text{pro } \mathbf{x})} f(x)] \;\; \text{and} \;\; f^*(\mathbf{x}) = [\min_{x \in \mathbf{x}} f(x), \max_{x \in \mathbf{x}} f(x)]$$

Computing $f^*(\mathbf{x})$ is NP-hard in general. However, it can be easily computed for simple functions that present good monotonicity properties. In particular, the AE-extensions of elementary functions can be computed from this expression of f^*. The elementary functions here considered are the following, their definition domain being the usual ones:

- two variables functions: $\Omega = \{ \; x + y \; , \; x - y \; , \; x \times y \; , \; x/y \}$
- one variable functions: $\Phi = \{ \; \exp x, \; \ln x, \; \sin x, \; \cos x, \; \tan x, \; \arccos x, \arcsin x, \; \arctan x, \; \text{abs}\, x, \; x^n, \; \sqrt[n]{x} \}$

In the cases of $+$, $-$, $\times$ and $/$, it is proved in [5] that f^* coincides with the Kaucher arithmetic.

Example 2.3. Consider the function $f(x, y) = x + y$, $\mathbf{x} \in \mathbb{KR}$ and $\mathbf{y} \in \mathbb{KR}$. Then $\mathbf{z} = \mathbf{x} + \mathbf{y}$ is the unique minimal $(f, \mathbf{x}, \mathbf{y})$-interpretable interval. For example,

$$[-1, 1] + [12, 8] = [11, 9]$$

means that the reals $[9, 11]$ is the largest interval $\mathbf{z}$ which satisfies

$$\big(\forall z \in \mathbf{z}\big)\big(\forall x \in [-1, 1]\big)\big(\exists y \in [8, 12]\big)\big(z = x + y\big)$$

The minimal AE-extensions of monotonic one variable elementary function are easily computed: for example, $\exp(\mathbf{x}) = [\exp(\inf \mathbf{x}), \exp(\sup \mathbf{x})]$ and, for $\text{pro } \mathbf{x} \subseteq [-1, 1]$, $\arccos(\mathbf{x}) = [\arccos(\sup \mathbf{x}), \arccos(\inf \mathbf{x})]$. In the cases of non monotonic functions, the algorithms dedicated to the computation of the classical interval extensions can be used with only minor modifications regarding rounded computations.

Remark. Not all continuous functions can be considered as elementary functions. See [5] for the condition which has to be satisfied by the elementary functions. For example the two variable function $f(x, y) = 1-(x-y)^2$ cannot be considered as an elementary function.

2.4 The natural AE-extension

The interval evaluation $\mathsf{f}(\mathbf{x})$ of an expression f of a function f using the minimal AE-extensions of the involved elementary functions is $(f, \mathbf{x})$-interpretable provided that each variable has only one occurrence inside the expression.

Example 2.4. Consider the function $f(u, v, w) = u(v + w)$ and the intervals $\mathbf{u} = [1, 2]$, $\mathbf{v} = [-1, 1]$ and $\mathbf{w} = [20, 8]$. The expression of f involving only one occurrence of each variable, the interval $\mathbf{z} = \mathbf{u}(\mathbf{v} + \mathbf{w}) = [19, 18]$ is $(f, \mathbf{u}, \mathbf{v}, \mathbf{w})$-interpretable, that is, the following quantified proposition is true:

$$\big(\forall u \in \mathbf{u}\big)\big(\forall v \in \mathbf{u}\big)\big(\forall z \in \text{pro } \mathbf{z}\big)\big(\exists w \in \text{pro } \mathbf{w}\big)\big(z = f(u, v, w)\big) \tag{2}$$

Such special cases of expressions are sufficient for the coming developments. The construction of the natural AE-extensions of more general functions is described in [5] and needs some modifications of the expressions f (some operations pro have to be inserted before all but one occurrences of each variable). Now, two properties of the interval evaluation of an expression containing only one occurrence of each variable are provided. First, in the special case of bilinear functions $f(x, y) = x^T \times y$, the interval evaluation is minimal, i.e. $f^*(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T \times \mathbf{y}$. Second, Proposition 2.1 will play a key role in the following developments. Skolem functions of quantified propositions like

$$\big(\forall x_{\mathcal{A}} \in \mathbf{x}_{\mathcal{A}}\big)\big(\exists x_{\mathcal{E}} \in \mathbf{x}_{\mathcal{E}}\big)\big(\phi(x_{\mathcal{A} \bigcup \mathcal{E}})\big) \tag{3}$$

where $\mathcal{A}$ and $\mathcal{E}$ are disjoint sets of indices such that $\text{card}\,\mathcal{A} + \text{card}\,\mathcal{E} = n$ and $\mathbf{x}_{\mathcal{A} \bigcup \mathcal{E}} \in \mathbb{IR}^n$ and ϕ is a real relation of $\mathbb{R}^n$, are defined using an analogy with first order logic Skolem functions (see e.g. [25]): a Skolem function of (3) is a function

$$s_{\mathcal{E}} : \mathbf{x}_{\mathcal{A}} \longrightarrow \mathbf{x}_{\mathcal{E}} \quad \text{s.t.} \quad x_{\mathcal{E}} = s_{\mathcal{E}}(x_{\mathcal{A}}) \Longrightarrow \phi(x_{\mathcal{A} \bigcup \mathcal{E}}).$$

Remark. It is implied that the previous implication stands for all $x_{\mathcal{A}} \in \mathbf{x}_{\mathcal{A}}$.

Example 2.5. The quantified proposition (2) is true. Therefore it has a Skolem function, i.e. a function $s : (\mathbf{u}, \mathbf{v}, \text{pro } \mathbf{z})^T \longrightarrow \text{pro } \mathbf{w}$ that satisfies $w = s(u, v, z) \Longrightarrow z = f(u, v, w)$.

Proposition 2.1 (Goldsztejn, [5])**.** *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous function and f an expression of this function involving elementary functions of Ω and Φ where* **each variable has only one occurrence**. *For any $\mathbf{x}_{[1..n]} \in \mathbb{KR}^n$, define $\mathbf{x}_0 = \mathsf{f}(\mathbf{x}_{[1..n]})$ where the evaluation is done using the Kaucher arithmetic. Furthermore define the sets of indices $\mathcal{A} = \mathcal{P}(\mathbf{x}_{[1..n]}) \bigcup \mathcal{I}(\mathbf{x}_{\{0\}})$ and $\mathcal{E} = \mathcal{I}(\mathbf{x}_{[1..n]}) \bigcup \mathcal{A}(\mathbf{x}_{\{0\}})$ (so that $\mathcal{A}$ contains the indices of the universally quantified variable and $\mathcal{E}$ contains the indices of the existentially quantified ones). Then both $\mathcal{A}$ and $\mathcal{E}$ are nonempty and the quantified proposition*

$$\big(\forall x_{\mathcal{A}} \in \text{pro } \mathbf{x}_{\mathcal{A}}\big)\big(\exists x_{\mathcal{E}} \in \text{pro } \mathbf{x}_{\mathcal{E}}\big)\big(f(x_{[1..n]}) = x_0\big)$$

has a **continuous** *Skolem function (and is therefore true).*

Example 2.6. Proposition 2.1 proves that the quantified proposition (2) has a continuous Skolem function.

3 The mean-value AE-extension

The mean-value AE-extension is first defined for continuously differentiable functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ in Subsection 3.2. Then the AE-extensions of continuously differentiable functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ are defined in Subsection 3.3. First of all, an improved mean-value theorem is needed.

3.1 An improved mean-value theorem

Given a continuously differentiable function $f : \mathbb{R} \longrightarrow \mathbb{R}$, $\mathbf{x} \in \mathbb{IR}$, $\tilde{x} \in \mathbf{x}$ and $\Delta \supseteq \text{range}\,(f', \mathbf{x})$, the mean-value theorem (see appendix A) entails the following quantified proposition

$$\big(\forall x \in \mathbf{x}\big)\big(\exists \delta \in \Delta\big)\big(f(x) = f(\tilde{x}) + \delta(x - \tilde{x})\big)$$

The next proposition provides a stronger property: it proves that this quantified proposition has a continuous Skolem function.

Proposition 3.1. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuously differentiable function, $\mathbf{x} \in \mathbb{IR}^n$, $\tilde{x} \in \mathbf{x}$ and $\Delta \in \mathbb{IR}^n$ such that for all $k \in [1..n]$,*

$$\Delta_k \supseteq range\left(\frac{\partial f}{\partial x_k}, \mathbf{x}_1, \ldots, \mathbf{x}_k, \tilde{x}_{k+1}, \ldots, \tilde{x}_n\right)$$

Then, the quantified proposition

$$\big(\forall x \in \mathbf{x}\big)\big(\exists \delta \in \Delta\big)\big(f(x) = f(\tilde{x}) + \delta^T(x - \tilde{x})\big)$$

has a continuous Skolem function (and is therefore true).

Proof. We have to prove that there exists a continuous function $s : \mathbf{x} \longrightarrow \Delta$ which satisfies $f(x) = f(\tilde{x}) + s(x)^T \times (x - \tilde{x})$. To this end, the function $f(x_{[1..n]})$ is written in following way:

$$f(x_{[1..n]}) = f(\tilde{x}_{[1..n]}) + \sum_{k \in [1..n]} g_k(x_{[1..n]}), \tag{4}$$

with

- $g_1(x_{[1..n]}) = f(x_1, \tilde{x}_{[2..n]}) - f(\tilde{x}_{[1..n]})$
- $g_k(x_{[1..n]}) = f(x_{[1..k]}, \tilde{x}_{[k+1..n]}) - f(x_{[1..k-1]}, \tilde{x}_{[k..n]})$ for $k \in [2..n-1]$
- $g_n(x_{[1..n]}) = f(x_{[1..n]}) - f(x_{[1..n-1]}, \tilde{x}_n)$.

Example. For $n = 2$, the previous expression becomes

$$\begin{aligned} f(x_{[1..2]}) &= f(\tilde{x}_{[1..2]}) + g_1(x_{[1..2]}) + g_2(x_{[1..2]}) \\ &= f(\tilde{x}_{[1..2]}) + \big(f(x_1, \tilde{x}_2) - f(\tilde{x}_{[1..2]})\big) + \big(f(x_{[1..2]}) - f(x_1, \tilde{x}_2)\big). \end{aligned}$$

Example. For $n = 3$, the previous expression becomes

$$\begin{aligned} f(x_{[1..3]}) &= f(\tilde{x}_{[1..3]}) + g_1(x_{[1..3]}) + g_2(x_{[1..3]}) + g_2(x_{[1..3]}) \\ &= f(\tilde{x}_{[1..3]}) + \big(f(x_1, \tilde{x}_2, \tilde{x}_3) - f(\tilde{x}_{[1..3]})\big) \\ &\quad + \big(f(x_1, x_2, \tilde{x}_3) - f(x_1, \tilde{x}_2, \tilde{x}_3)\big) \\ &\quad + \big(f(x_{[1..3]}) - f(x_1, x_2, \tilde{x}_3)\big). \end{aligned}$$

Let us define $s_k(x)$ in the following way:

$$s_k(x) = \frac{g_k(x)}{x_k - \tilde{x}_k} \quad \text{if} \quad x_k \neq \tilde{x}_k$$

and

$$s_k(x) = \frac{\partial f}{\partial x_k}(x_{[1..k-1]}, \tilde{x}_{[k..n]}) \quad \text{otherwise.}$$

Three claims has to be proved.

Claim 1: $f(x) = f(\tilde{x}) + s(x)^T \times (x - \tilde{x})$. Thanks to (4) we just have to prove that $g_k(x) = s_k(x)(x_k - \tilde{x}_k)$ for all $x \in \mathbf{x}$. On one hand, if $x_k \neq \tilde{x}_k$ then

$$s_k(x)(x_k - \tilde{x}_k) = \frac{g_k(x)}{x_k - \tilde{x}_k}(x_k - \tilde{x}_k) = g_k(x).$$

On the other hand, if $x_k = \tilde{x}_k$ then $g_k(x) = 0 = s_k(x) \times 0$.

Claim 2: s_k in continuous inside $\mathbf{x}$. Let us consider $x \in \mathbf{x}$ and prove that s_k is continuous at x. On one hand, if $x_k \neq \tilde{x}_k$ then s_k is a composition of continuous functions and is therefore continuous. On the other hand, if $x_k = \tilde{x}_k$

we consider a sequence $x^{(i)}$ which converges to x. Then, we have by definition $s_k(x) = \frac{\partial f}{\partial x_k}(x_{[1..k-1]}, \tilde{x}_{[k..n]})$ and thanks to the mean-value theorem

$$s_k(x^{(i)}) = \frac{f(x^{(i)}_{[1..k]}, \tilde{x}_{[k+1..n]}) - f(x^{(i)}_{[1..k-1]}, \tilde{x}_{[k..n]})}{x^{(i)}_k - \tilde{x}_k} = \frac{\partial f}{\partial x_k}(x_{[1..k-1]}, \xi^{(i)}_k, \tilde{x}_{[k+1,n]})$$

with $\xi^{(i)}_k \in x^{(i)}_k \vee \tilde{x}_k$. As $x^{(i)}_k$ converges to $\tilde{x}_k$, the sequence $\xi^{(i)}_k$ also converges to $\tilde{x}_k$. Therefore, $s_k(x^{(i)})$ converges to $s_k(x)$ because $\frac{\partial f}{\partial x_k}$ is continuous. As a consequence, s_k is eventually continuous at x.

Claim 3: $s_k(x) \in \Delta_k$. On one hand, if $x_k \neq \tilde{x}_k$ then the mean-value theorem proves that for any $x \in \mathbf{x}$,

$$\begin{aligned} \frac{f(x_{[1..k]}, \tilde{x}_{[k+1..n]}) - f(x_{[1..k-1]}, \tilde{x}_{[k..n]})}{x_k - \tilde{x}_k} &\in \text{range}\left(\frac{\partial f}{\partial x_k}, x_{[1..k-1]}, x_k \vee \tilde{x}_k, \tilde{x}_{[k+1..n]}\right) \\ &\subseteq \Delta_k \end{aligned}$$

On the other hand, if $x_k = \tilde{x}_k$ then $s_k(x) \in \Delta_k$ by definition of s_k. □

Remark. The use of the expression

$$\Delta_k \supseteq \text{range}\left(\frac{\partial f}{\partial x_k}, \mathbf{x}_1, \ldots, \mathbf{x}_k, \tilde{x}_{k+1}, \ldots, \tilde{x}_n\right)$$

instead of

$$\Delta_k \supseteq \text{range}\left(\frac{\partial f}{\partial x_k}, \mathbf{x}\right)$$

was initially proposed in [7], in the context of classical intervals extensions. The second expression, which is simpler, will be used in the sequel so as to simplify the proposed statements. It can be replaced by the first expression with no influence on the statements then providing significant improvements in the computations.

Remark. Obviously, the hypothesis $\tilde{x} \in \mathbf{x}$ can be changed to $\tilde{x} \in \mathbb{R}^n$ provided that

$$\Delta_k \supseteq \text{range}\left(\frac{\partial f}{\partial x_k}, \mathbf{x}_1, \ldots, \mathbf{x}_k, \tilde{x}_{k+1}, \ldots, \tilde{x}_n\right)$$

is changed to

$$\Delta_k \supseteq \text{range}\left(\frac{\partial f}{\partial x_k}, \mathbf{x}_1 \vee \tilde{x}_1, \ldots, \mathbf{x}_k \vee \tilde{x}_k, \tilde{x}_{k+1}, \ldots, \tilde{x}_n\right).$$

3.2 The mean-value AE-extension of real-valued functions

The mean-value AE-extension is first illustrated on the special case of a one variable function. Given a continuously differentiable function $f : \mathbb{R} \longrightarrow \mathbb{R}$, $\mathbf{x} \in \overline{\mathbb{IR}}$, $\tilde{x} \in \text{pro } \mathbf{x}$ and $\Delta \supseteq \text{range}\,(f', \text{pro } \mathbf{x})$, define the interval $\mathbf{z}$ by

$$\mathbf{z} = f(\tilde{x}) + \Delta(\mathbf{x} - \tilde{x}) \tag{5}$$

which is improper (multiplication by an improper interval which contains 0 leads to an improper interval which contains 0). Therefore, so as to prove that $\mathbf{z}$ is $(f,\mathbf{x})$-interpretable, the validity of the following quantified proposition has to be proved:

$$\big(\forall z \in \text{pro } \mathbf{z}\big)\big(\exists x \in \text{pro } \mathbf{x}\big)\big(z = f(x)\big) \tag{6}$$

On one hand, the expression (5) corresponds to the natural AE-extension of the function $m(x,\delta) = f(\tilde{x}) + \delta(\mathbf{x} - \tilde{x})$ evaluated at $\mathbf{x} \in \overline{\mathbb{IR}}$ and $\Delta \in \mathbb{IR}$. Therefore the quantified proposition

$$\big(\forall \delta \in \Delta\big)\big(\forall z \in \text{pro } \mathbf{z}\big)\big(\exists x \in \text{pro } \mathbf{x}\big)\big(z = m(x,\delta)\big) \tag{7}$$

has a continuous Skolem function by Proposition 2.1. I.e. there exists a continuous function $s' : (\Delta, \text{pro } \mathbf{z})^T \longrightarrow \text{pro } \mathbf{x}$ that satisfies $x = s'(\delta, z) \Longrightarrow z = m(x,\delta)$. On the other hand, Proposition 3.1 proves that the quantified proposition

$$\big(\forall x \in \text{pro } \mathbf{x}\big)\big(\exists \delta \in \Delta\big)\big(f(x) = m(x,\delta)\big) \tag{8}$$

also has a continuous Skolem function. I.e. there exists a continuous function $s'' : \text{pro } \mathbf{x} \longrightarrow \Delta$ which satisfies $\delta = s''(x) \Longrightarrow f(x) = m(x,\delta)$. In order to prove (6), the continuous function $s : (\text{pro } \mathbf{x}, \text{pro } \mathbf{z})^T \longrightarrow \text{pro } \mathbf{x}$ is constructed in the following way: $s(x,z) = s'(s''(x), z)$. One can easily check that it satisfies

$$\begin{aligned} x = s(x,z) &\Longrightarrow \big(\exists \delta \in \Delta\big)\big(x = s'(\delta,z) \wedge \delta = s''(x)\big) \\ &\Longrightarrow \big(\exists \delta \in \Delta\big)\big(z = m(x,\delta) \wedge f(x) = m(x,\delta)\big) \\ &\Longrightarrow z = f(x) \end{aligned} \tag{9}$$

Now, for each $z \in \text{pro } \mathbf{z}$, the continuous function $s(.,z)$ has pro $\mathbf{x}$ as domain and pro $\mathbf{x}$ as co-domain. So, by the Brouwer fixed point theorem (see appendix A) it has a fixed point $x \in \text{pro } \mathbf{x}$. Therefore, the following quantified proposition is true:

$$\big(\forall z \in \text{pro } \mathbf{z}\big)\big(\exists x \in \text{pro } \mathbf{x}\big)\big(x = s(x,z)\big)$$

Finally, thanks to (9), the previous quantified proposition entails (6) and $\mathbf{z}$ is proved to be $(f,\mathbf{x})$-interpretable. The next theorem generalizes the previous argumentation to any continuously differentiable functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and to any interval argument $\mathbf{x} \in \mathbb{KR}^n$.

Theorem 3.1. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuously differentiable function, $\mathbf{x} \in \mathbb{KR}^n$, $c : \mathbb{IR}^n \longrightarrow \mathbb{R}^n$ such that $c(\mathbf{x}) \in \mathbf{x}$ and $\mathbf{g} : \mathbb{IR}^n \longrightarrow \mathbb{IR}^{1\times n}$ be an interval extension of f', i.e.*

$$\mathbf{g}_{1k}(\text{pro } \mathbf{x}) \supseteq \textit{range}\left(\frac{\partial f}{\partial x_k}, \text{pro } \mathbf{x}\right).$$

Then, the interval function $\mathbf{h}$ defined by

$$\mathbf{h}(\mathbf{x}) = f(c(\mathbf{x})) + \mathbf{g}(\text{pro } \mathbf{x}) \times (\mathbf{x} - c(\mathbf{x}))$$

is an AE-extension of f and is called a mean-value AE-extension of f.

Proof. First of all, for any $x \in \mathbb{R}^n$ we have $c(x) = x$ and therefore $\mathbf{h}(x) = f(x) + \mathbf{g}(\text{pro } \mathbf{x}) \times (x - x) = f(x)$. Then, for any $\mathbf{x}_{[1..n]} \in \mathbb{KR}^n$, define $\Delta_{[1..n]} = \mathbf{g}(\text{pro } \mathbf{x}_{[1..n]})^T \in \mathbb{IR}^n$ and $\tilde{x}_{[1..n]} = c(\mathbf{x}_{[1..n]})$ and $\mathbf{x}_0 := \mathbf{h}(\mathbf{x}_{[1..n]})$. We therefore have

$$\mathbf{x}_0 = f(\tilde{x}_{[1..n]}) + \Delta^T \times (\mathbf{x} - \tilde{x}) \tag{10}$$

Furthermore define the set of indices $\mathcal{A} = \mathcal{P}(\mathbf{x}_{[1..n]}) \cup \mathcal{I}(\mathbf{x}_0)$ and $\mathcal{E} = \mathcal{I}(\mathbf{x}_{[1..n]}) \cup \mathcal{P}(\mathbf{x}_0)$. So, we have to prove that the following quantified proposition is true:

$$\big(\forall x_{\mathcal{A}} \in \text{pro } \mathbf{x}_{\mathcal{A}}\big)\big(\exists x_{\mathcal{E}} \in \text{pro } \mathbf{x}_{\mathcal{E}}\big)\big(x_0 = f(x_{[1..n]})\big) \tag{11}$$

On one hand, by the expression (10), $\mathbf{x}_0$ is the generalized interval evaluation of the real function

$$m(x_{[1..n]}, \delta_{[1..n]}) := f(\tilde{x}_{[1..n]}) + \delta^T \times (x - \tilde{x})$$

evaluated at $\mathbf{x}_{[1..n]}$ and $\Delta_{[1..n]}$. As the expression of m involves only one occurrence of each variable we can apply Proposition 2.1 which proves that there exists a continuous function

$$s'_{\mathcal{E}} : (\text{pro } \mathbf{x}_{\mathcal{A}}, \Delta_{[1..n]})^T \longrightarrow \text{pro } \mathbf{x}_{\mathcal{E}}$$

which satisfies

$$x_{\mathcal{E}} = s'_{\mathcal{E}}(x_{\mathcal{A}}, \delta_{[1..n]}) \Longrightarrow x_0 = m(x_{[1..n]}, \delta_{[1..n]}) \tag{12}$$

On the other hand, as $\mathbf{g}$ is an interval extension of f' we have $\Delta^T \supseteq \text{range}\,(f', \text{pro } \mathbf{x})$ and we can apply Proposition 3.1 which proves that there exists a continuous function

$$s''_{[1..n]} : \text{pro } \mathbf{x}_{[1..n]} \longrightarrow \Delta_{[1..n]}$$

which satisfies

$$\delta_{[1..n]} = s''_{[1..n]}(x_{[1..n]}) \Longrightarrow f(x_{[1..n]}) = g(x_{[1..n]}, \delta_{[1..n]}) \tag{13}$$

Now, we construct the continuous function $s_{\mathcal{E}} : \text{pro } \mathbf{x}_{\mathcal{A}\cup\mathcal{E}} \longrightarrow \text{pro } \mathbf{x}_{\mathcal{E}}$ composing s' and s'' in the following way (notice that $\mathcal{A} \cup [1..n] \ \subseteq \ \mathcal{A} \cup \mathcal{E}$):

$$s_{\mathcal{E}}(x_{\mathcal{A}\cup\mathcal{E}}) = s'_{\mathcal{E}}\big(x_{\mathcal{A}}, s''_{[1..n]}(x_{[1..n]})\big) \tag{14}$$

Let us prove that

$$x_{\mathcal{E}} = s_{\mathcal{E}}(x_{\mathcal{A}\cup\mathcal{E}}) \Longrightarrow x_0 = f(x_{[1..n]}) \tag{15}$$

By the equation (14), $s_{\mathcal{E}}(x_{\mathcal{A}\cup\mathcal{E}}) = x_{\mathcal{E}}$ implies

$$\delta_{[1..n]} = s''_{[1..n]}(x_{[1..n]}) \wedge x_{\mathcal{E}} = s'_{\mathcal{E}}(x_{\mathcal{A}}, \delta_{[1..n]}])$$

Now, by (12) and (13), this latter implies

$$f(x_{[1..n]}) = g(x_{[1..n]}, \delta_{[1..n]}) \wedge x_0 = g(x_{[1..n]}, \delta_{[1..n]})$$

That is eventually $x_0 = f(x_{[1..n]})$. Finally, thanks to the Brouwer fixed point theorem, for each value of $x_\mathcal{A} \in \text{pro }\mathbf{x}_\mathcal{A}$, the function $s_\mathcal{E}(x_\mathcal{A}, .) : \text{pro }\mathbf{x}_\mathcal{E} \longrightarrow \text{pro }\mathbf{x}_\mathcal{E}$ has a fixed point. That is,

$$\big(\forall x_\mathcal{A} \in \text{pro }\mathbf{x}_\mathcal{A}\big)\big(\exists x_\mathcal{E} \in \text{pro }\mathbf{x}_\mathcal{E}\big)\big(s_\mathcal{E}(x_{\mathcal{A}\cup\mathcal{E}}) = x_\mathcal{E}\big)$$

which entails (11) thanks to the implication (15). □

Remark. Similarly to the classical mean-value extension, if the interval function $\mathbf{g}$ is not defined for all $\mathbf{x} \in \mathbb{IR}^n$ then the domain definition of the mean-value AE-extension has to be adapted.

The next example illustrates the way the mean-value AE-extension computes inner and outer approximations of the range of a continuously differentiable function.

Example 3.1. Consider the real function $f(x) = x^2$ and the interval $\mathbf{x} = [1, 1.2]$. As f is strictly increasing over $\mathbf{x}$, the exact range $\text{range}\,(f, \mathbf{x})$ can be computed in the following:

$$\text{range}\,(f, \mathbf{x}) = [f(\underline{\mathbf{x}}), f(\overline{\mathbf{x}})] = [1, 1.44]$$

Consider the interval $\Delta = [2, 2.4]$ which satisfies $\Delta \supseteq \text{range}\,(f', \mathbf{x})$. When evaluated at $\mathbf{x}$, the mean-value AE-extension leads to

$$\text{range}\,(f, \mathbf{x}) \subseteq f(\tilde{\mathbf{x}}) + \Delta(\mathbf{x} - \text{mid }\mathbf{x}) = [0.97, 1.45]$$

which is an outer approximation of the range of the original function f. When evaluated at dual $\mathbf{x}$, the mean-value AE-extension leads to

$$f(\tilde{\mathbf{x}}) + \Delta(\text{dual }\mathbf{x} - \text{mid }\mathbf{x}) = [1.41, 1.01]$$

The interval $\text{pro}\,[1.41, 1.01] = [1.01, 1.41]$ is indeed included inside $[1, 1.44]$. The figure 1 illustrates both the outer (left hand side graphic) and inner (right hand side graphic) approximations obtained thanks to the linearization of f.

Theorem 3.1 has proved that the interval

$$\mathbf{z} = f(\tilde{x}) + \mathbf{g}(\text{pro }\mathbf{x}) \times (\mathbf{x} - \tilde{x})$$

is $(f, \mathbf{x})$-interpretable. As f^* is the unique minimal AE-extension of f, this is equivalently stated by $f^*(\mathbf{x}) \subseteq \mathbf{z}$. It will be useful for several purposes to build an interval which is included inside $f^*(\mathbf{x})$. The next proposition provides such a construction.

Proposition 3.2. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuously differentiable function, $\mathbf{x} \in \mathbb{KR}^n$, $\tilde{x} \in \mathbf{x}$ and $\mathbf{g} : \mathbb{IR}^n \longrightarrow \mathbb{IR}^{1\times n}$ be an interval extension of f'. Define the interval*

$$\mathbf{z} = f(\tilde{x}) + \big(\text{dual }\mathbf{g}(\text{pro }\mathbf{x})\big) \times (\mathbf{x} - \tilde{x})$$

Then, $\mathbf{z} \subseteq f^*(\mathbf{x})$.

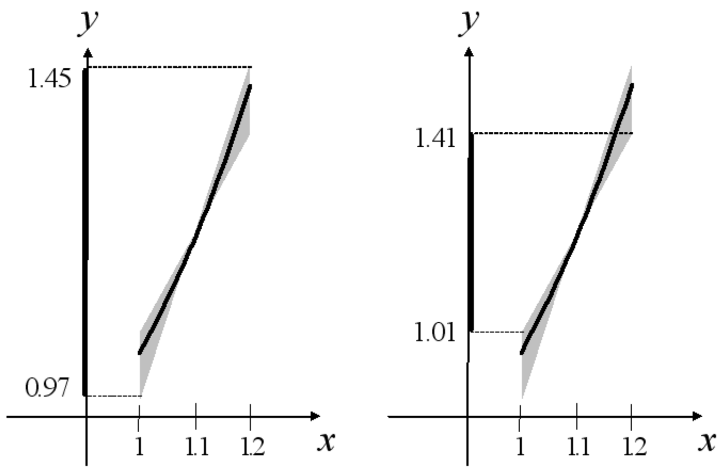

Figure 1: Graph of the function $f(x) = x^2$ together with the cone of the derivatives $f(\tilde{\mathbf{x}}) + f'(\mathbf{x})(x - \tilde{\mathbf{x}})$. The left hand side (respec. right hand side) graphic illustrates the outer (respec. inner) approximation of the range obtained thanks to the cone of the derivatives.

Proof. Define the continuous real function

$$m(x, \delta) = f(\tilde{x}) + \delta^T \times (x - \tilde{x})$$

The interval $\mathbf{z}$ is then the natural AE-extension of m evaluated at $\mathbf{x}$ and dual Δ, where $\Delta = \mathbf{g}(\text{pro }\mathbf{x})^T \in \mathbb{IR}^n$. As proved in [5], this natural AE-extension is minimal. Therefore, we have $\mathbf{z} = m^*\big(\mathbf{x}, \text{dual }\Delta\big)$. The interval $\mathbf{z'} := f^*(\mathbf{x})$ is $(f, \mathbf{x})$-interpretable so the following quantified proposition is true:

$$\big(\forall x_{\mathcal{P}} \in \mathbf{x}_{\mathcal{P}}\big)\big(\mathsf{Q}^{(\mathbf{z'})} z \in \text{pro }\mathbf{z'}\big)\big(\exists x_{\mathcal{I}} \in \text{pro }\mathbf{x}_{\mathcal{I}}\big)\big(z = f(x)\big)$$

Now, by Proposition 3.1 the following quantified proposition is also true:

$$\big(\forall x \in \text{pro }\mathbf{x}\big)\big(\exists \delta \in \Delta\big)\big(f(x) = m(x, \delta)\big)$$

Both previous quantified propositions obviously entail the following one:

$$\big(\forall x_{\mathcal{P}} \in \mathbf{x}_{\mathcal{P}}\big)\big(\mathsf{Q}^{(\mathbf{z'})} z \in \mathbf{z'}\big)\big(\exists x_{\mathcal{I}} \in \text{pro }\mathbf{x}_{\mathcal{I}}\big)\big(\exists \delta \in \mathbf{g}(\text{pro }\mathbf{x})\big)\big(z = m(x, \delta)\big)$$

Therefore, the interval $\mathbf{z'} = f^*(\mathbf{x})$ is $\big(m, \mathbf{x}, \text{dual }\Delta\big)$-interpretable, which implies

$$m^*\big(\mathbf{x}, \text{dual }\mathbf{g}(\mathbf{y})\big) \subseteq f^*(\mathbf{x})$$

That is, $\mathbf{z} \subseteq f^*(\mathbf{x})$. □

3.3 The mean-value AE-extensions of vector-valued real functions

Given a continuously differentiable vector-valued function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and an interval $\mathbf{x} \in \mathbb{KR}^n$, the mean-value AE-extensions of the real-valued functions f_i are now used in order to construct a $(f, \mathbf{x})$-interpretable interval $\mathbf{z} \in \mathbb{KR}^m$, i.e. $\mathbf{z}$ satisfies

$$\big(\forall x_{\mathcal{A}} \in \mathbf{x}_{\mathcal{A}}\big)\big(\forall z_{A'} \in \text{pro } \mathbf{z}_{A'}\big)\big(\exists z_{E'} \in \mathbf{z}_{E'}\big)\big(\exists x_{\mathcal{E}} \in \text{pro } \mathbf{x}_{\mathcal{E}}\big)\big(z = f(x)\big) \tag{16}$$

where $A = P(\mathbf{x})$, $E = I(\mathbf{x})$, $E' = P(\mathbf{z})$ and $A' = I(\mathbf{z})$. This construction is similar to the case of the natural AE-extensions for vector-valued functions presented in [5]. For $i \in [1..m]$ consider some mean-value AE-extensions $\mathbf{h}_i$ of the real-valued function f_i. The interval $\mathbf{z}$ naively defined by $\mathbf{z}_i = \mathbf{h}_i(\mathbf{x})$ is not $(f, \mathbf{x})$-interpretable in general because the conjunction

$$\bigwedge_{i \in [1..m]} \big(\forall x_{\mathcal{A}} \in \mathbf{x}_{\mathcal{A}}\big)\big(\mathsf{Q}^{(\mathbf{z}_i)} z_i \in \text{pro } \mathbf{z}_i\big)\big(\exists x_{\mathcal{E}} \in \text{pro } \mathbf{x}_{\mathcal{E}}\big)\big(z_i = f_i(x)\big) \tag{17}$$

where $\mathsf{Q}^{(\mathbf{z}_i)} = \exists$ if $i \in P'$ and $\mathsf{Q}^{(\mathbf{z}_i)} = \forall$ if $i \in I'$, does not implies (16) in general.

Example 3.2. Consider $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ and $\mathbf{x}_1 \in \mathbb{IR}$ and $\mathbf{x}_2, \mathbf{x}_3 \in \overline{\mathbb{IR}}$. Suppose that both $\mathbf{z}_1 = \mathbf{h}_1(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ and $\mathbf{z}_2 = \mathbf{h}_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ are improper. Then both following quantified propositions are true:

$$\big(\forall x_1 \in \mathbf{x}_1\big)\big(\forall z_1 \in \text{pro } \mathbf{z}_1\big)\big(\exists x_2 \in \text{pro } \mathbf{x}_2\big)\big(\exists x_3 \in \text{pro } \mathbf{x}_3\big)\big(z_1 = f_1(x)\big)$$

and

$$\big(\forall x_1 \in \mathbf{x}_1\big)\big(\forall z_2 \in \text{pro } \mathbf{z}_2\big)\big(\exists x_2 \in \text{pro } \mathbf{x}_2\big)\big(\exists x_3 \in \text{pro } \mathbf{x}_3\big)\big(z_1 = f_2(x)\big)$$

However, their conjunction does not imply

$$\big(\forall x_1 \in \mathbf{x}_1\big)\big(\forall z \in \text{pro } \mathbf{z}\big)\big(\exists x_2 \in \text{pro } \mathbf{x}_2\big)\big(\exists x_3 \in \text{pro } \mathbf{x}_3\big)\big(z = f(x)\big)$$

in general. So $\mathbf{z} \in \overline{\mathbb{IR}}^2$ is not $(f, \mathbf{x})$-interpretable in general.

In order to entail (16), the previous computations have to be modified in such a way that each variable which is existentially quantified inside (16) is existentially quantified in exactly one quantified proposition of the conjunction (17) and universally quantified in all the others. This is done in the same way as the natural AE-extension: an operation pro is inserted in before all but one occurrences of each variable. In the case of the mean-value AE-extension, each component has one occurrence of each variable. Therefore, the choice of the occurrence which is not preceded of an operation pro is done choosing one component of the vector-valued interval function.

Example 3.3. Like in the previous example, consider $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ and $\mathbf{x}_1 \in \mathbb{IR}$ and $\mathbf{x}_2, \mathbf{x}_3 \in \overline{\mathbb{IR}}$. The following intervals are computed:

$$\mathbf{z}_1 = \mathbf{h}_1(\mathbf{x}_1, \text{pro } \mathbf{x}_2, \mathbf{x}_3) \;\text{ and }\; \mathbf{z}_2 = \mathbf{h}_2(\text{pro } \mathbf{x}_1, \mathbf{x}_2, \text{pro } \mathbf{x}_3).$$

Suppose that both $\mathbf{z}_1$ and $\mathbf{z}_2$ are improper. Then both following quantified propositions are true (notice that $\mathbf{x}_1$ being proper we have $(\text{pro } \mathbf{x}_1) = \mathbf{x}_1$):

$$\big(\forall x_1 \in \mathbf{x}_1\big)\big(\forall x_2 \in \text{pro } \mathbf{x}_2\big)\big(\forall z_1 \in \text{pro } \mathbf{z}_1\big)\big(\exists x_3 \in \text{pro } \mathbf{x}_3\big)\big(z_1 = f_1(x)\big)$$

and

$$\big(\forall x_1 \in \mathbf{x}_1\big)\big(\forall x_3 \in \text{pro } \mathbf{x}_3\big)\big(\forall z_2 \in \text{pro } \mathbf{z}_2\big)\big(\exists x_2 \in \text{pro } \mathbf{x}_2\big)\big(z_2 = f_2(x)\big)$$

Under some additional hypothesis which are be fulfilled if the $\mathbf{h}_i$ are the mean-value AE-extensions of the functions f_i, their conjunction implies

$$\big(\forall x_1 \in \mathbf{x}_1\big)\big(\forall z \in \text{pro } \mathbf{z}\big)\big(\exists x_2 \in \text{pro } \mathbf{x}_2\big)\big(\exists x_3 \in \text{pro } \mathbf{x}_3\big)\big(z = f(x)\big)$$

Therefore, $\mathbf{z} \in \overline{\mathbb{IR}}^2$ is $(f, \mathbf{x})$-interpretable.

This can be formalized introducing a integral function $\pi : [1..n] \longrightarrow [1..m]$ which associates to the variable x_j for $j \in [1..n]$ the index of the AE-extension $\mathbf{h}_i$ for $i \in [1..m]$ in which it will be existentially quantified (i.e. it will not be preceded by an operation pro). Then, the AE-extensions $\mathbf{h}_i$ are used to construct $\mathbf{z}$ in the following way:

$$\mathbf{z}_i = \mathbf{h}_i(\mathbf{y})$$

where for $j \in [1..n]$ we have $\mathbf{y}_j = \mathbf{x}_j$ if $i = \pi(j)$ and $\mathbf{y}_j = \text{pro } \mathbf{x}_j$ otherwise. The proper components of $\mathbf{x}$ being not sensitive to the operation pro , the interval $\mathbf{z}$ satisfies

$$\bigwedge_{i \in [1..m]} \big(\forall x_{A \cup A_i} \in \mathbf{x}_{A \cup A_i}\big)\big(\mathsf{Q}^{(\mathbf{z}_i)} z \in \text{pro } \mathbf{z}_i\big)\big(\exists x_{E_i} \in \text{pro } \mathbf{x}_{E_i}\big)\big(z = f(x)\big) \quad (18)$$

where $E_i = E \cap \pi^{-1}(i)$ and $A_i = E \backslash E_i$. Now, as each existentially quantified variable appears in one and only one quantified proposition of the previous conjunction and this latter is likely to entail the quantified proposition (16).

Example 3.4. In previous example, the function π is defined by

$$\pi : (1 \rightarrow 1\ ;\ 2 \rightarrow 2\ ;\ 3 \rightarrow 1).$$

The implication "(18)$\Longrightarrow$(16)" is true because the $\mathbf{h}_i$ are the mean-value AE-extensions of the functions f_i (see [5] for a counter example for this implication when other AE-extensions are used). Similarly to the proof of the interpretation of the natural AE-extension of vector-valued functions, the proof of the implication "(18)$\Longrightarrow$(16)" involves the Brouwer fixed point theorem (see Appendix A). In order to formulate the mean-value AE-extensions of vector valued functions in a compact form, the following specific matrix/vector product is first defined.

Definition 3.1. Let $\mathbf{A} \in \mathbb{KR}^{m\times n}$, $\mathbf{x} \in \mathbb{KR}^n$ and $\pi : [1..n] \longrightarrow [1..m]$. The matrix product $\mathbf{A} *_\pi \mathbf{x} \in \mathbb{KR}^n$ is then defined in the following way: for all $i \in [1..m]$,

$$(\mathbf{A} *_\pi \mathbf{x})_i = \sum_{j\in\pi^{-1}(i)} \mathbf{A}_{ij}\mathbf{x}_j + \sum_{j\in([1..n]\backslash\pi^{-1}(i))} \mathbf{A}_{ij}(\text{pro } \mathbf{x})$$

A special case will be met several times: if $m = n$ and $\pi = \text{id}$ then the product $\mathbf{A}*_\pi\mathbf{x}$ is simply denoted by $\mathbf{A}*\mathbf{x}$. In this case, its definition is simplified to

$$(\mathbf{A} * \mathbf{x})_i = \mathbf{A}_{ii}\mathbf{x}_i + \sum_{j\in([1..n]\backslash\{i\})} \mathbf{A}_{ij}(\text{pro } \mathbf{x}_j)$$

Now, the mean-value AE-extension for vector valued function can be defined.

Theorem 3.2. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a continuously differentiable function, $\mathbf{x} \in \mathbb{KR}^n$, $c : \mathbb{IR}^n \longrightarrow \mathbb{R}^n$ such that $c(\mathbf{x}) \in \mathbf{x}$ and $\pi : [1..n] \longrightarrow [1..m]$. Consider an interval extension extension $\mathbf{g} : \mathbb{IR}^n \longrightarrow \mathbb{IR}^{m\times n}$ of f'. Then the interval function $\mathbf{h}$ defined by*

$$\mathbf{h}(\mathbf{x}) = f(\tilde{x}) + \mathbf{g}(\text{pro } \mathbf{x}) *_\pi (\mathbf{x} - c(\mathbf{x}))$$

is an AE-extension of f and is called a mean-value AE-extension of f.

Proof. First of all, whatever is $x \in \mathbb{R}^n$ we have $\mathbf{h}(x) = f(x) + \mathbf{g}(\text{pro } \mathbf{x}) \times (x - x) = f(x)$ because $c(x) = x$. Now consider any $\mathbf{x}_{[1..n]} \in \mathbb{KR}^n$ and define the interval matrix $\Delta = \mathbf{g}(\text{pro } \mathbf{x}_{[1..n]})$ and $\tilde{x}_{[1..n]} = c(\mathbf{x}_{[1..n]})$. So as to obtain homogeneous notations, the evaluation of $\mathbf{h}$ is done in the following way:

$$\mathbf{x}_{[n+1..n+m]} = f(\tilde{x}_{[1..n]}) + \Delta *_\pi (\mathbf{x}_{[1..n]} - \tilde{x}_{[1..n]}) \tag{19}$$

We have to prove that $\mathbf{x}_{[n+1..n+m]}$ is $(f, \mathbf{x}_{[1..n]})$-interpretable. Define the sets of indices $A = P(\mathbf{x}_{[1..n]}) \cup I(\mathbf{x}_{[n+1..n+m]})$ and $E = I(\mathbf{x}_{[1..n]}) \cup P(\mathbf{x}_{[n+1..n+m]})$. So, we have to prove that the following quantified proposition is true:

$$\big(\forall x_{\mathcal{A}} \in \mathbf{x}_{\mathcal{A}}\big)\big(\exists x_{\mathcal{E}} \in \mathbf{x}_{\mathcal{E}}\big)\big(f(x_{[1..n]}) = x_{[n+1..n+m]}\big) \tag{20}$$

Consider any $i \in [1..m]$. The i^{th} line of the equality (19) is

$$\mathbf{x}_{n+i} = f_i(\tilde{x}) + \sum_{j\in[1..n]} \Delta_{ij}(\mathbf{y}_j^{(i)} - \tilde{x})$$

where $\mathbf{y}_j^{(i)} = \mathbf{x}_j$ if $\pi(j) = i$ and $\mathbf{y}_j^{(i)} = \text{pro } \mathbf{x}_j$ otherwise. This corresponds to the mean-value AE-extension of the function f_i evaluated at $\mathbf{y}^{(i)}$. Define $E_i = E \cap \pi^{-1}(i)$ and $A_i = E\backslash E_i$. We know that $\mathbf{y}_j^{(i)}$ is proper if and only if either $\mathbf{x}_j$ is proper or $j \in A_i$ or equivalently, $\mathbf{y}_j^{(i)}$ is improper if and only if $j \in E_i$. Using the same reasoning as in the proof of Theorem 3.1, we obtain for each $i \in [1..m]$ a continuous function

$$s_{E_i} : \mathbf{x}_{A\cup A_i} \longrightarrow \mathbf{x}_{E_i}$$

which satisfies

$$x_{E_i} = s_{E_i}(x_{A \cup A_i}) \Longrightarrow x_{n+i} = f_i(x_{[1..n]}).$$

Thank to the definition of π, one can check the for any $i \in [1..m]$ and $i' \in [1..m]$ such that $i \neq i'$, we have $E_i \cap E_{i'} = \emptyset$, and that $\cup\{E_i | i \in [1..m]\} = E$. We also have $\cup\{A \cup A_i | i \in [1..m]\} = \mathcal{A} \cup \mathcal{E}$. So, the function $s_{\mathcal{E}} : \mathbf{x}_{\mathcal{A}\cup\mathcal{E}} \longrightarrow \mathbf{x}_E$ is well defined, continuous and furthermore satisfies

$$\begin{aligned} s_{\mathcal{E}}(x_{\mathcal{A}\cup\mathcal{E}}) = x_{\mathcal{E}} \quad &\Longrightarrow \quad \bigwedge_{i\in[1..m]} x_{E_i} = s_{E_i}(x_{A\cup A_i}) \\ &\Longrightarrow \quad \bigwedge_{i\in[1..m]} x_{n+i} = f_i(x_{[1..n]}) \\ &\Longrightarrow \quad x_{[n+1..n+m]} = f(x_{[1..n]}). \end{aligned}$$

Finally, thank to the Brouwer fixed point theorem, for any $x_{\mathcal{A}} \in \mathbf{x}_{\mathcal{A}}$ the function $s_{\mathcal{E}}(x_{\mathcal{A}}, .) : \mathbf{x}_{\mathcal{E}} \longrightarrow \mathbf{x}_{\mathcal{E}}$ has a fixed point. That is,

$$\big(\forall x_{\mathcal{A}} \in \mathbf{x}_{\mathcal{A}}\big)\big(\exists x_{\mathcal{E}} \in \mathbf{x}_{\mathcal{E}}\big)\big(s_{\mathcal{E}}(x_{\mathcal{A}\cup\mathcal{E}}) = x_{\mathcal{E}}\big)$$

which concludes the proof. □

In the case $m = 1$, Theorem 3.2 coincides with Theorem 3.1. In this case, there is only one possible integral function π which is $\pi(i) = 1$ for all $i \in [1..n]$ and

$$\mathbf{g}(\text{pro } \mathbf{x}) *_\pi (\mathbf{x} - c(\mathbf{x})) = \mathbf{g}(\text{pro } \mathbf{x}) \times (\mathbf{x} - c(\mathbf{x})).$$

Therefore, Theorem 3.2 can be used in general to define the mean-value AE-extensions. Also, if $\mathbf{x} \in \mathbb{IR}^n$ then

$$\mathbf{g}(\text{pro } \mathbf{x}) *_\pi (\mathbf{x} - c(\mathbf{x})) = \mathbf{g}(\mathbf{x}) \times (\mathbf{x} - c(\mathbf{x}))$$

and the mean-value AE-extension coincides with the classical interval mean-value extension.

Example 3.5. Consider the function

$$f(x) = \begin{pmatrix} 81x_1^2 + x_2^2 + 18x_1x_2 - 100 \\ x_1^2 + 81x_2^2 + 18x_1x_2 - 100 \end{pmatrix}$$

Consider the following interval extension of its derivative:

$$\mathbf{A} = \begin{pmatrix} 162(\text{pro } \mathbf{x}_1) + 18(\text{pro } \mathbf{x}_2) & 2(\text{pro } \mathbf{x}_2) + 18(\text{pro } \mathbf{x}_1) \\ 2(\text{pro } \mathbf{x}_1) + 18(\text{pro } \mathbf{x}_2) & 162(\text{pro } \mathbf{x}_2) + 18(\text{pro } \mathbf{x}_1) \end{pmatrix}$$

Then, we can build the following mean-value AE-extension of f:

$$\mathbf{h}(\mathbf{x}) = f(\text{mid } \mathbf{x}) + \mathbf{A} * (\mathbf{x} - \text{mid } \mathbf{x})$$

That is explicitly

$$\begin{pmatrix} f_1(\text{mid } \mathbf{x}) & + & \mathbf{A}_{11}(\mathbf{x}_1 - \text{mid } \mathbf{x}_1) & + & \mathbf{A}_{12}(\text{pro } \mathbf{x}_2 - \text{mid } \mathbf{x}_2) \\ f_2(\text{mid } \mathbf{x}) & + & \mathbf{A}_{21}(\text{pro } \mathbf{x}_1 - \text{mid } \mathbf{x}_1) & + & \mathbf{A}_{22}(\mathbf{x}_2 - \text{mid } \mathbf{x}_2) \end{pmatrix}$$

If $\mathbf{x} = ([1.1, 0.9], [1.1, 0.9])^T$ the mean-value AE-extension leads to

$$\mathbf{h}(\mathbf{x}) = (\ [14, -14]\ ,\ [14, -14]\ )^T$$

and so proves that pro $\mathbf{h}(\mathbf{x}, \mathbf{y}) \subseteq \text{range}\,(f, \text{pro}\ \mathbf{x}, \text{pro}\ \mathbf{y})$. As a consequence, there exists $x \in \text{pro}\ \mathbf{x}$ such that $f(x) = 0$. Indeed, $f(1, 1) = 0$.

4 On the quality of the mean-value AE-extension

In the particular case of real-valued functions, the mean-value extension has a quadratic order of convergence. However, in the general case, its order of convergence is linear.

Theorem 4.1. *With the notations of Theorem 3.2, suppose that furthermore* $\mathbf{g}$ *is locally Lipschitz continuous. Then,*

- *the mean-value AE-extension has a linear order of convergence.*
- *if $m = 1$, i.e. in the special case of Theorem 3.1, the mean-value AE-extension has a quadratic order of convergence.*

Proof. The mean value extension is a composition of: the minimal AE-extensions of $+$ and $\times$, the operation pro and the interval extension $\mathbf{g}$ of f'. All are locally Lipschitz continuous (see [5]). Therefore, their composition is also locally Lipschitz continuous. As a consequence, the mean-value AE-extension has a linear order of convergence (Proposition 7.2 [5]).

Now, we study the special case $m = 1$. Denote $\mathbf{g}(\text{pro}\ \mathbf{x})^T$ by $\Delta \in \mathbb{IR}^n$ and define both

$$\mathbf{z} = \mathbf{h}(\mathbf{x}) = f(\tilde{x}) + \Delta^T \times (\mathbf{x} - \tilde{x}) \quad \text{and} \quad \mathbf{z}' = f(\tilde{x}) + (\text{dual}\ \Delta)^T \times (\mathbf{x} - \tilde{x})$$

By Theorem 3.1 and Proposition 3.2, we have $\mathbf{z}' \subseteq f^*(\mathbf{x}) \subseteq \mathbf{z}$. Thanks to Lemma 2.5 of [12], this implies $\text{dist}(f^*(\mathbf{x}), \mathbf{z}) \leq \text{dist}(\mathbf{z}, \mathbf{z}')$. Therefore, it remains to bound the distance between $\mathbf{z}$ and $\mathbf{z}'$. Using the relations between the distance, the Kaucher addition and multiplication, we have

$$\begin{aligned}
\text{dist}(\mathbf{z}, \mathbf{z}') &= \text{dist}\big(\Delta^T \times (\mathbf{x} - \tilde{x})\ ,\ (\text{dual}\ \Delta)^T \times (\mathbf{x} - \tilde{x})\big) \\
&\leq \sum_{i \in [1..n]} \text{dist}\big(\Delta_i \times (\mathbf{x}_i - \tilde{x}_i), (\text{dual}\ \Delta_i) \times (\mathbf{x}_i - \tilde{x}_i)\big) \\
&\leq \sum_{i \in [1..n]} |\mathbf{x}_i - \tilde{x}_i|\,\text{dist}(\Delta_i, \text{dual}\ \Delta_i)
\end{aligned}$$

Now, as $\tilde{x} \in \mathbf{x}$, we have $|\mathbf{x}_i - \tilde{x}_i| \leq |\,\text{wid}\ \mathbf{x}_i| \leq ||\,\text{wid}\ \mathbf{x}||$. Furthermore, it is obvious that $\text{dist}(\Delta_i, \text{dual}\ \Delta_i) = \text{wid}\ \Delta_i \leq ||\,\text{wid}\ \Delta||$. Therefore we have

$$\text{dist}(\mathbf{z}, \mathbf{z}') \leq n\,||\,\text{wid}\ \mathbf{x}||\,||\,\text{wid}\ \Delta||$$

Now, as $\mathbf{g} : \mathbb{IR}^n \longrightarrow \mathbb{IR}^{1\times n}$ is locally Lipschitz continuous, so is $\mathbf{g}' : \mathbb{IR} \longrightarrow \mathbb{IR}^n$ defined by $\mathbf{g}'(\mathbf{x}) = \mathbf{g}(\mathbf{x})^T$. And because pro is locally Lipschitz continuous, the composition $\mathbf{g}' \circ \text{pro} : \mathbb{KR}^n \longrightarrow \mathbb{IR}^n$ is also locally Lipschitz continuous (with these notations we have $\Delta = \mathbf{g}' \circ \text{pro}\,(\mathbf{x})$). So, for all $\mathbf{x}^{\mathsf{ref}} \in \mathbb{IR}^n$, there exists $\gamma > 0$ such that whatever is $\mathbf{y} \in \mathbb{K}\mathbf{x}^{\mathsf{ref}}$, we have

$$\text{dist}(\Delta, \mathbf{g}' \circ \text{pro}\,(\mathbf{y})) \leq \gamma\,\text{dist}(\mathbf{x}, \mathbf{y})$$

So, choose $\mathbf{y} = \text{mid }\mathbf{x} \in \mathbf{x}^{\mathsf{ref}}$. On one hand, we have

$$||\,\text{rad }\Delta|| \leq \text{dist}(\Delta, \mathbf{g}' \circ \text{pro}\,(\text{mid }\mathbf{x}))$$

because $\mathbf{g}' \circ \text{pro}\,(\text{mid }\mathbf{x}) \in \mathbb{R}$. On the other hand, we have $\text{dist}(\mathbf{x}, \text{mid }\mathbf{x}) = ||\,\text{rad }\mathbf{x}||$. So we have $||\,\text{rad }\Delta|| \leq \gamma||\,\text{rad }\mathbf{x}||$ which is equivalent to $||\,\text{wid }\Delta|| \leq \gamma||\,\text{wid }\mathbf{x}||$. Therefore, $\text{dist}(\mathbf{z}, \mathbf{z}') \leq n\gamma||\,\text{wid }\mathbf{x}||^2$. Finally, we have therefore proved that, for all $\mathbf{x}^{\mathsf{ref}} \in \mathbb{IR}^n$, there exists $\gamma' = n\gamma$ such that

$$\text{dist}(\mathbf{h}(\mathbf{x}), f^*(\mathbf{x})) \leq \gamma'||\,\text{wid }\mathbf{x}||^2$$

where γ' does not depend on the choice of the box $\mathbf{x}^{\mathsf{ref}}$, which correspond to quadratic order of convergence. □

The next example illustrates that the mean-value AE-extension does not have a quadratic order of convergence for $m > 1$.

Example 4.1. Consider the function

$$f(x) = \begin{pmatrix} x_1 + 0.1x_2 \\ 0.1x_1 + x_2 \end{pmatrix}$$

which derivative is

$$f'(x) = \begin{pmatrix} 1 & 0.1 \\ 0.1 & 1 \end{pmatrix}$$

Denote this matrix by Δ. Consider the integral function $\pi : [1..2] \longrightarrow [1..2]$ defined by $\pi(1) = 2$ and $\pi(2) = 1$. The corresponding mean-value AE-extension of f is

$$\begin{aligned} \mathbf{g}(\mathbf{x}) &= f(\text{mid }\mathbf{x}) + \Delta *_\pi (\mathbf{x} - \text{mid }\mathbf{x}) \\ &= f(\text{mid }\mathbf{x}) + \begin{pmatrix} (\text{pro }\mathbf{x}_1 - \text{mid }\mathbf{x}_1) + 0.1(\mathbf{x}_2 - \text{mid }\mathbf{x}_2) \\ 0.1(\mathbf{x}_1 - \text{mid }\mathbf{x}_1) + (\text{pro }\mathbf{x}_2 - \text{mid }\mathbf{x}_2) \end{pmatrix} \end{aligned}$$

Define the intervals $\mathbf{x}_\epsilon = ([\epsilon, -\epsilon], [\epsilon, -\epsilon])^T$, so

$$\mathbf{g}(\mathbf{x}_\epsilon) = ([-0.9\epsilon, 0.9\epsilon], [-0.9\epsilon, 0.9\epsilon])^T$$

Denote this interval by $\mathbf{z}_\epsilon$. This latter interval is $(f, \mathbf{x}_\epsilon)$-interpretable, that is the following quantified proposition is true.

$$\big(\exists z \in \mathbf{z}_\epsilon\big)\big(\exists x \in \text{pro }\mathbf{x}_\epsilon\big)\big(f(x) = z\big)$$

Now, in order to investigate the order of convergence of this mean-value AE-extension, we consider an minimal $(f, \mathbf{x}_\epsilon)$-interpretable interval $\mathbf{z}^*_\epsilon$ which is more accurate than $\mathbf{z}_\epsilon$. As $\mathbf{x}$ is improper, at least one component of $\mathbf{z}^*_\epsilon$ is improper (because a proper interval cannot be minimal if $\mathbf{x}$ is improper). Therefore, we have

$$\text{dist}(\mathbf{z}_\epsilon, \mathbf{z}^*_\epsilon) \geq ||\,\text{rad }\,\mathbf{z}_\epsilon|| = 0.9||\,\text{wid }\,\mathbf{x}_\epsilon||$$

Finally, we have

$$\frac{\text{dist}(\mathbf{z}_\epsilon, \mathbf{z}^*_\epsilon)}{||\,\text{wid }\,\mathbf{x}_\epsilon||^2} \geq \frac{0.9\;||\,\text{wid }\,\mathbf{x}_\epsilon||}{||\,\text{wid }\,\mathbf{x}_\epsilon||^2} = \frac{0.9}{||\,\text{wid }\,\mathbf{x}_\epsilon||}$$

This ratio is not upper bounded if $\epsilon \to 0$, that is if wid $\mathbf{x}_\epsilon \to 0$. Therefore, this mean-value AE-extension does not have a quadratic convergence order.

This example also illustrates that the choice of the function π is important for the quality of the AE-extension: the choice $\pi(1) = 1$ and $\pi(2) = 2$ would give a much more accurate mean-value AE-extension. However, a efficient choice for π is not always possible, as illustrated in [5]. In such cases, a preconditioning step is necessary. The use of some preconditioning process together with the mean-value AE-extension is illustrated in the next section.

5 Inner approximation of the range continuous vector-valued real functions

The mean-value AE-extension is now used so as to build an inner approximation of a continuous function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ under the form of a skew box, i.e. under the form of the image of a box through a linear map. To this end, the mean-value AE-extension is associated to a preconditioning process. First of all, the following lemma is needed.

Lemma 5.1. *Let* $\mathbf{d} \in \mathbb{IR}$ *and* $\mathbf{x} \in \mathbb{IR}$ *such that* mid $\mathbf{x} = 0$.

1. *if* $0 \notin \mathbf{d}$ *then* wid $\big(\mathbf{d}(\text{dual }\,\mathbf{x})\big) = -\langle\mathbf{d}\rangle\,(\text{wid }\,\mathbf{x})$ *where* $\langle\mathbf{d}\rangle$ *is the mignitude of the interval* $\mathbf{x}$*, i.e.* $\langle\mathbf{d}\rangle = \min\{|\underline{\mathbf{d}}|, |\overline{\mathbf{d}}|\}$.
2. wid $\big(\mathbf{d}\,\mathbf{x}\big) = |\mathbf{d}|(\text{wid }\,\mathbf{x})$

Proof. Just apply the expressions of the Kaucher arithmetic in the following way.
1. On one hand, if $\mathbf{d} > 0$ then $\mathbf{d}(\text{dual }\,\mathbf{x}) = [\underline{\mathbf{d}}\overline{\mathbf{x}}, \underline{\mathbf{d}}\underline{\mathbf{x}}] = \langle\mathbf{d}\rangle\,(\text{dual }\,\mathbf{x})$. On the other hand, if $\mathbf{d} < 0$ then $\mathbf{d}(\text{dual }\,\mathbf{x}) = [\overline{\mathbf{d}}\underline{\mathbf{x}}, \overline{\mathbf{d}}\overline{\mathbf{x}}] = \langle\mathbf{d}\rangle\,(\text{dual }\,\mathbf{x})$ (the last equality being a consequence of $\underline{\mathbf{x}} = -\overline{\mathbf{x}}$ and $\overline{\mathbf{d}} = -\langle\mathbf{d}\rangle$). Finally, wid $(\langle\mathbf{d}\rangle\,(\text{dual }\,\mathbf{x})) = -\langle\mathbf{d}\rangle\,(\text{wid }\,\mathbf{x})$.
2. $\mathbf{d}\,\mathbf{x} = [-|\mathbf{d}|\overline{\mathbf{x}}, |\mathbf{d}|\overline{\mathbf{x}}]$. Therefore, wid $\big(\mathbf{d}\,\mathbf{x}\big) = |\mathbf{d}|(\text{wid }\,\mathbf{x})$. □

Theorem 5.1. *Let* $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *be a continuously differentiable function,* $\mathbf{x} \in \mathbb{IR}^n$*,* $\tilde{x} \in \mathbf{x}$ *and* $C \in \mathbb{R}^{n\times n}$ *be a non-singular matrix. Furthermore consider an interval matrix* $\mathbf{A} \in \mathbb{IR}^{n\times n}$ *which satisfies* $\forall x \in \mathbf{x}, f'(x) \in \mathbf{A}$*. Define the interval*

$$\mathbf{u} = Cf(\tilde{x}) + (C\mathbf{A}) * \text{dual}\,(\mathbf{x} - \tilde{x}) \tag{21}$$

If $\mathbf{u}$ *is improper, then*

$$\{C^{-1}u | u \in \text{pro}\ \mathbf{u}\} \subseteq \mathit{range}\,(f, \mathbf{x})$$

If furthermore $\mathbf{u}$ *is strictly improper then* $\mathbf{A}$ *is regular (and even strongly regular).*

Proof. As C is non-singular, $f(x) = z$ is equivalent to $Cf(x) = Cz$. Now the equation (21) corresponds to the mean-value AE-extension of the function Cf evaluated at dual $\mathbf{x}$ (the Jacoby matrix of the function Cf is $Cf'(x)$). Therefore, if $\mathbf{u}$ is improper, then the following proposition holds.

$$\big(\forall u \in \text{pro}\ \mathbf{u}\big)\big(\exists x \in \mathbf{x}\big)\big(u = Cf(x)\big)$$

Notice that $u = Cf(x)$ is equivalent to $C^{-1}u = f(x)$. Therefore, the previous quantified proposition is equivalent to the following one.

$$\big(\forall z \in \{C^{-1}u | u \in \text{pro}\ \mathbf{u}\}\big)\big(\exists x \in \mathbf{x}\big)\big(z = f(x)\big)$$

that is $\{C^{-1}u | u \in \mathbf{u}\} \subseteq \text{range}\,(f, \mathbf{x})$. It remains to study the regularity of the interval matrix $\mathbf{A}$. Suppose that $\mathbf{u}$ is strictly improper and denote $C\mathbf{A}$ by $\mathbf{A}'$. The i^{th} line of the equation (21) is

$$\mathbf{u}_i = (Cf(\tilde{x}))_i + \mathbf{A}'_{ii}(\text{dual}\ \mathbf{x}_i - \tilde{x}_i) + \sum_{j\neq i} \mathbf{A}'_{ij}(\mathbf{x}_j - \tilde{x}_j) \tag{22}$$

Suppose that for some $i \in [1..n]$ we have $0 \in \mathbf{A}'_{ii}$. By the definition of the Kaucher multiplication, a proper interval which contains 0 multiplied by an improper interval which proper projection contains 0 gives $[0, 0]$. So, we have $\mathbf{A}'_{ii}(\text{dual}\ \mathbf{x}_i - \tilde{x}_i) = 0$ and therefore $\mathbf{u}_i \in \mathbb{IR}$. This latter is in contradiction with the hypothesis that $\mathbf{u}$ is strictly improper, therefore we have $0 \notin \mathbf{A}'_{ii}$ for all $i \in [1..n]$. Similarly, we have wid $\mathbf{x} > 0$. Now, applying the lemma 5.1 to (22) we obtain

$$\text{wid}\ \mathbf{u}_i = 0 - \langle \mathbf{A}'_{ii} \rangle\,(\text{wid}\ \mathbf{x}_i) + \sum_{j\neq i} |\mathbf{A}'_{ij}|(\text{wid}\ \mathbf{x}_j)$$

Regrouping these componentwise equalities, we get

$$\text{wid}\ \mathbf{u} = -\,\langle \mathbf{A}' \rangle\,(\text{wid}\ \mathbf{x})$$

where $\langle \mathbf{A}' \rangle$ is the comparison matrix of $\mathbf{A}'$. Therefore, as $-\,\text{wid}\ \mathbf{u} > 0$, there exists a non null positive vector $v = (\text{wid}\ \mathbf{x})$ such that $\langle \mathbf{A}' \rangle\, v \geq 0$, which corresponds to the definition of a M-matrix. As $\langle \mathbf{A}' \rangle$ is a M-matrix, $\mathbf{A}' = C\mathbf{A}$ is a H-matrix and $\mathbf{A}$ is finally a strongly regular matrix ([16] theorem 4.1.2 and its corollary). □

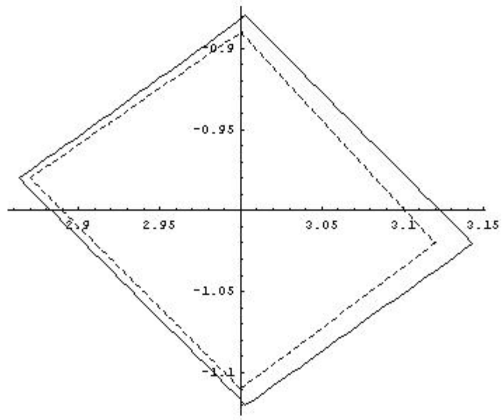

Figure 2: Inner approximation of the range of a vector-valued function.

Example 5.1. Consider the function

$$f(x) = \begin{pmatrix} x_1^6 + x_2^6 + x_1 x_2 \\ x_1^6 - x_2^6 - x_1 x_2 \end{pmatrix}$$

and the interval $\mathbf{x} = ([0.99, 1.01], [0.99, 1.01])^T$. Consider the following interval extension of f':

$$\mathbf{A} = \begin{pmatrix} 6\mathbf{x}_1^5 + \mathbf{x}_2 & 6\mathbf{x}_2^5 + \mathbf{x}_1 \\ 6\mathbf{x}_1^5 - \mathbf{x}_2 & -6\mathbf{x}_2^5 - \mathbf{x}_1 \end{pmatrix} \supseteq \{f'(x) \mid x \in \mathbf{x}\}\,.$$

Choosing $C = (\text{mid } \mathbf{A})^{-1}$ and $\tilde{x} = \text{mid } \mathbf{x}$, the application of Theorem 5.1 leads to

$$\mathbf{u} \approx ([0.18, 0.16], [0.27, 0.25])^T.$$

As $\mathbf{u}$ is improper, Theorem 5.1 proves that following skew box is an inner approximation of range $(f, \mathbf{x})$:

$$\{(\text{mid } \mathbf{A})u \mid u \in \text{pro } \mathbf{u}\}\,.$$

The exact range (continuous line) and the previous skew box (dotted line) are displayed on Figure 2.

Now, an existence test is derived from Theorem 5.1 in a simple way.

Corollary 5.1. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a continuously differentiable function, $\mathbf{x} \in \mathbb{IR}^n$ be non degenerated box, $\tilde{x} \in \mathbf{x}$ and $C \in \mathbb{R}^{n \times n}$ be a non-singular matrix. Furthermore consider an interval matrix $\mathbf{A} \in \mathbb{IR}^{n \times n}$ which satisfies $\forall x \in \mathbf{x}, f'(x) \subseteq \mathbf{A}$. Define the interval*

$$\mathbf{u} = Cf(\tilde{x}) + (C\mathbf{A}) * \text{dual}\,(\mathbf{x} - \tilde{x})$$

If $\mathbf{u} \subseteq 0$ *then there exists* $x \in \mathbf{x}$ *such that* $f(x) = 0$*. If furthermore* $\mathbf{u}$ *is strictly improper then there is only one solution inside* $\mathbf{x}$*.*

Proof. The inclusion $\mathbf{u} \subseteq 0$ is equivalent to $\mathbf{u}$ is improper and $0 \in \text{pro}\,\mathbf{u}$. Theorem 5.1 therefore entails

$$\{C^{-1}u | u \in \text{pro } \mathbf{u}\} \subseteq \text{range}\,(f, \mathbf{x}) \tag{23}$$

Now, $0 \in \text{pro}\,\mathbf{u}$ entails $0 \in \{C^{-1}u | u \in \text{pro } \mathbf{u}\}$ which finally entails $0 \in \text{range}\,(f, \mathbf{x})$. Therefore, there exists $x \in \mathbf{x}$ such that $f(x) = 0$. The uniqueness is proved noticing that $\mathbf{u}$ strictly improper entails $\mathbf{A}$ is strongly regular. Then, the regularity of the interval matrix $\mathbf{A}$ entails the uniqueness of the solution ([16] theorem 5.1.6). □

Surprisingly, this existence test is equivalent to existence test associated to the classical Hansen-Sengupta operator. Indeed, the componentwise expression of the test of the corollary 5.1 is

$$(Cf(\tilde{x}))_i + (C\mathbf{A})_{ii}(\text{dual } \mathbf{x}_i - \tilde{x}_i) + \sum_{j \neq i} (C\mathbf{A})_{ij}(\mathbf{x}_j - \tilde{x}_j) \subseteq 0$$

Now using the rules of the Kaucher arithmetic, this is equivalent to

$$-(Cf(\tilde{x}))_i - \sum_{j \neq i} (C\mathbf{A})_{ij}(\mathbf{x}_j - \tilde{x}_j) \subseteq \text{dual}\,\big((C\mathbf{A})_{ii}\big)(\mathbf{x}_i - \tilde{x}_i)$$

which is eventually equivalent to

$$\tilde{x}_i - \frac{1}{(C\mathbf{A})_{ii}}\Big((Cf(\tilde{x}))_i + \sum_{j \neq i} (C\mathbf{A})_{ij}(\mathbf{x}_j - \tilde{x}_j)\Big) \subseteq \mathbf{x}_i$$

because $0 \notin \text{pro}\,(C\mathbf{A})_{ii}$. This last expression corresponds to the existence test associated to the classical Hansen-Sengupta operator (see [16]). This link gives new lights on the existence test associated to the Hansen-Sengupta operator. Indeed, when this latter succeeds, it means that it has been proved that the following quantified propositions are true:

$$\begin{array}{c}
\big(\forall x_2 \in \mathbf{x}_2\big)\big(\forall x_3 \in \mathbf{x}_3\big) \cdots \big(\forall x_n \in \mathbf{x}_n\big)\big(\exists x_1 \in \mathbf{x}_1\big)\big(f_1(x) = 0\big) \\
\big(\forall x_1 \in \mathbf{x}_2\big)\big(\forall x_3 \in \mathbf{x}_3\big) \cdots \big(\forall x_n \in \mathbf{x}_n\big)\big(\exists x_2 \in \mathbf{x}_2\big)\big(f_2(x) = 0\big) \\
\vdots \qquad\qquad\qquad\qquad \vdots \\
\big(\forall x_1 \in \mathbf{x}_2\big)\big(\forall x_2 \in \mathbf{x}_2\big) \cdots \big(\forall x_{n-1} \in \mathbf{x}_{n-1}\big)\big(\exists x_n \in \mathbf{x}_n\big)\big(f_n(x) = 0\big)
\end{array}$$

So that all these quantified propositions are true, the function f has to be close to the identity. This explains why the Hansen-Sengupta existence test generally needs a midpoint inverse preconditioning to succeed.

6 Conclusion

In Modal Interval Revisited Part 1 ([5]), a new formulation of the modal intervals theory has been proposed: new extensions to generalized intervals, called AE-extensions, have been defined which enhance the interpretations of extensions to classical intervals. Thanks to this new framework, a new linearization process is proposed under the form of the mean-value AE-extension. This linearization process is compatible with both inner and outer approximation of functions ranges over boxes.

The advantages of the mean-value AE-extension in front of the natural AE-extension (and therefore on the modal rational extensions) are the same than the advantages of the classical mean-value extension in front of the classical natural extension: on one hand, it is more accurate for small intervals (its quadratic order of convergence has been established for real-valued functions). On the other hand, thanks to the linearization process it provides, it allows to apply to non-linear systems the algorithms dedicated to linear systems.

The usefulness of the mean-value AE-extension has been illustrated: some inner approximation of the range of continuously differentiable functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ over some small boxes has been constructed. Also, the well-known existence and uniqueness test associated to the classical Hansen-Sengupta operator has been easily derived from the mean-value AE-extension.

Future work

The newly introduced linearization process is compatible with inner approximation of non-linear AE-solution sets and this has to be investigated. Also, the mean-value AE-extension has been defined only for continuously differentiable functions. The introduction of slopes in place of derivatives should allow extending the scope of the mean-value AE-extension to non-differentiable functions like $\mathrm{abs}(x)$ or $\max(x, y)$ and obtaining more accurate computations.

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A Some useful theorems from real analysis

The mean value-theorem is usually stated in the following way.

Theorem (The mean-value theorem)**.** *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be differentiable, $a \in \mathbb{R}$ and $b \in \mathbb{R}$. Then, there exists $c \in a \vee b$ which satisfies*

$$f(b) = f(a) + f'(c)(b - a)$$

The Brouwer fixed point theorem is a famous classical existence theorem (see for example [10] or [16]).

Theorem (Brouwer fixed point theorem)**.** *Let $E \subseteq \mathbb{R}^n$ be nonempty, compact and convex, and $f : E \longrightarrow E$ be continuous. Then, there exists $x \in E$ such that $f(x) = x$.*

References

[1] G. Alefeld. On the Approximation of the Range of Values by Interval Expressions. *Computing*, 44:273–278, 1990.

[2] G. Alefeld and J. Herzberger. *Introduction to Interval Computations*. Computer Science and Applied Mathematics, 1974.

[3] J. Armengol, J. Vehí, L. Travé-Massuyès, and M.A. Sainz. Application of modal intervals to the generation of error-bounded envelopes. *Reliab. Comp.*, 7(2):171–185, 2001.

[4] O. Caprani and K. Mandsen. Mean Value Forms in Interval Analysis. *Computing*, 25:147–154, 1980.

[5] A. Goldsztejn. Modal Intervals Revisited Part 1: A Generalized Interval Natural Extension. *Submitted to Reliable Computing*.

[6] A. Goldsztejn, D. Daney, M. Rueher, and P. Taillibert. Modal Intervals Revisited: a Mean-Value Extension to Generalized Intervals. In *Proceedings of QCP 2005*.

[7] E. Hansen. On solving systems of equations using interval arithmetic. *Math. Compt.*, 22:374–384, 1968.

[8] P. Herrero, M.A. Sainz, J. Vehí, and L. Jaulin. Quantified Set Inversion with Applications to Control. In *IEEE International Symposium on Computer Aided Control Systems Design*, 2004.

[9] P. Herrero, M.A. Sainz, J. Vehí, and L. Jaulin. Quantified Set Inversion Algorithm with Applications to Control. In *Proc. of Interval Mathematics and Constrained Propagation Methods*, volume 11(5) of *Reliab. Comp.*, 2005.

[10] V.I. Istratescu. *Fixed point theory : an introduction*. Mathematics and its applications. D. Reidel, 1981.

[11] E. Kaucher. *Uber metrische und algebraische Eigenschaften einiger beim numerischen Rechnen auftretender Raume*. PhD thesis, Karlsruhe, 1973.

[12] E. Kaucher. Interval Analysis in the Extended Interval Space $\mathbb{IR}$. *Computing*, Suppl. 2:33–49, 1980.

[13] R. B. Kearfott, M. T. Nakao, A. Neumaier, S. M. Rump, S. P. Shary, and P. Van Hentenryck. Standardized Notation in Interval Analysis. 2002.

[14] V. Kreinovich, A. Lakeyev, J. Rohn, and P. Kahl. *Computational Complexity and Feasibility of Data Processing and Interval Computations.* Kluwer Academic Publishers, 1998.

[15] R. Moore. *Interval Analysis.* Prentice-Hall, Englewood Cliffs N. J., 1966.

[16] A. Neumaier. *Interval Methods for Systems of Equations.* Cambridge Univ. Press, 1990.

[17] A. Neumaier. Taylor forms - use and limits. *Reliab. Comp.*, 9:43–79, 2002.

[18] H.-J. Ortolf. Eine Verallgemeinerung der Intervallarithmetik. *Geselschaft fuer Mathematik und Datenverarbeitung, Bonn*, 11:1–71, 1969.

[19] H. Ratschek. Centered Forms. *SIAM J. Numer. Anal.*, 17(5):656–662, 1980.

[20] M.A. Sainz, E. Gardenyes, and L. Jorba. Formal Solution to Systems of Interval Linear or Non-Linear Equations. *Reliab. Comp.*, 8(3):189–211, 2002.

[21] M.A. Sainz, E. Gardenyes, and L. Jorba. Interval Estimations of Solutions Sets to Real-Valued Systems of Linear or Non-Linear Equations. *Reliab. Comp.*, 8(4):283–305, 2002.

[22] S.P. Shary. A new technique in systems analysis under interval uncertainty and ambiguity. *Reliab. Comp.*, 8:321–418, 2002.

[23] SIGLA/X group. Modal intervals (basic tutorial). *Applications of Interval Analysis to Systems and Control (Proceedings of MISC'99)*, pages 157–227, 1999.

[24] SIGLA/X group. Modal intervals. *Reliab. Comp.*, 7:77–111, 2001.

[25] Manna Z. and Waldinger R. *The Deductive Foundations of Computer Programming.* AddisonWesley, 1993.