

# NONUNIFORM BEHAVIORS FOR SKEW-EVOLUTION SEMIFLOWS IN BANACH SPACES

CODRUȚA STOICA MIHAIL MEGAN

**Abstract.** The paper emphasizes some asymptotic behaviors for skew-evolution semiflows in Banach spaces. These are defined by means of evolution semiflows and evolution cocycles. Some characterizations which generalize classical results are also provided. The approach is from nonuniform point of view.

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## 1 Preliminaries

The study of asymptotic properties, such as exponential dichotomy and exponential trichotomy, considered basic concepts that appear in the theory of dynamical systems, plays an important role in the study of stable, instable and central manifolds. Some of the original results concerning stability and instability were published in [7], [10] and [11] for a particular case of skew-evolution semiflows defined by means of semiflows and cocycles. In this very case was also defined and characterized the trichotomy on Banach spaces in [5].

Concerning previous results, C. Buşe presents in [1] the nonuniform exponential stability for evolutionary processes. Characterizations for the nonuniform exponential instability for evolution operators on Banach spaces were obtained by M. Megan, A.L. Sasu and B. Sasu in [2]. The study of the nonuniform exponential dichotomy for evolution families was emphasized by P. Preda and M. Megan in [8] and for evolution operators, also in Banach spaces, by M. Megan, A.L. Sasu and B. Sasu in [3]. Other asymptotic properties for evolution families were studied by the same authors in the nonuniform setting in [4].

In this paper we extend the asymptotic properties of exponential dichotomy and trichotomy for the newly introduced concept of skew-evolution semiflows defined on Banach spaces, which can be considered generalizations for evolution operators and skew-product semiflows. The results concerning

the nonuniform exponential trichotomy are generalizations of some Theorems proved for evolution operators in [6].

## 2 Notations. Definitions. Examples

Let us consider a metric space  $(X, d)$ , a Banach space  $V$  and  $\mathcal{B}(V)$  the space of all bounded linear operators from  $V$  into itself. Let  $V^*$  be the topological dual of  $V$ . We denote the sets  $T = \{(t, t_0) \in \mathbb{R}^2, t \geq t_0 \geq 0\}$  and  $Y = X \times V$ . Let  $P : Y \rightarrow Y$  be a projector given by  $P(x, v) = (x, P(x)v)$ , where  $P(x)$  is a projection on  $Y_x = \{x\} \times V$ ,  $x \in X$ .

**Definition 2.1** A mapping  $\varphi : T \times X \rightarrow X$  is called *evolution semiflow* on  $X$  if following relations hold:

- (s<sub>1</sub>)  $\varphi(t, t, x) = x, \forall (t, x) \in \mathbb{R}_+ \times X$
- (s<sub>2</sub>)  $\varphi(t, s, \varphi(s, t_0, x)) = \varphi(t, t_0, x), \forall (t, s), (s, t_0) \in T, x \in X$ .

**Definition 2.2** A mapping  $\Phi : T \times X \rightarrow \mathcal{B}(V)$  is called *evolution cocycle* over an evolution semiflow  $\varphi$  if:

- (c<sub>1</sub>)  $\Phi(t, t, x) = I$ , the identity operator on  $V, \forall (t, x) \in \mathbb{R}_+ \times X$
- (c<sub>2</sub>)  $\Phi(t, s, \varphi(s, t_0, x))\Phi(s, t_0, x) = \Phi(t, t_0, x), \forall (t, s), (s, t_0) \in T, x \in X$ .

**Definition 2.3** The mapping  $C : T \times Y \rightarrow Y$  defined by the relation  $C(t, s, x, v) = (\varphi(t, s, x), \Phi(t, s, x)v)$ , where  $\Phi$  is an evolution cocycle over an evolution semiflow  $\varphi$ , is called *skew-evolution semiflow* on  $Y$ .

**Example 2.1** We denote  $\mathcal{C} = \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$  the set of all continuous functions  $x : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , endowed with the topology of uniform convergence on compact subsets of  $\mathbb{R}_+$  and which is metrizable by means of the distance

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(x, y)}{1 + d_n(x, y)}, \text{ where } d_n(x, y) = \sup_{t \in [0, n]} |x(t) - y(t)|.$$

If  $x \in \mathcal{C}$  then for all  $t \in \mathbb{R}_+$  we denote  $x_t(s) = x(t + s)$ ,  $x_t \in \mathcal{C}$ . Let  $X$  be the closure in  $\mathcal{C}$  of the set  $\{f_t, t \in \mathbb{R}_+\}$ , where  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$  is a decreasing function with the property  $\lim_{t \rightarrow \infty} f(t) = l > 0$ . Then  $(X, d)$  is a metric space and the mapping

$$\varphi : T \times X \rightarrow X, \varphi(t, s, x) = x_{t-s}$$

is an evolution semiflow on  $X$ .

We consider the Banach space  $V = \mathbb{R}^n$ ,  $n \geq 1$ , with the norm

$$\|(v_1, \dots, v_n)\| = |v_1| + \dots + |v_n|.$$

The mapping  $\Phi : T \times X \rightarrow \mathcal{B}(V)$  given by

$$\Phi(t, s, x)(v_1, \dots, v_n) = \left( e^{\alpha_1 \int_s^t x(\tau-s)d\tau} v_1, \dots, e^{\alpha_n \int_s^t x(\tau-s)d\tau} v_n \right),$$

where  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ , is an evolution cocycle over  $\varphi$  and  $C = (\varphi, \Phi)$  is a skew-evolution semiflow on  $Y$ .

An interesting class of skew-evolution semiflows, useful to describe asymptotic properties, is given by

**Example 2.2** Let us consider a skew-evolution semiflow  $C = (\varphi, \Phi)$  and a parameter  $\lambda \in \mathbb{R}$ . We define the application

$$\Phi_\lambda : T \times X \rightarrow \mathcal{B}(V), \quad \Phi_\lambda(t, t_0, x) = e^{-\lambda(t-t_0)} \Phi(t, t_0, x). \quad (2.1)$$

It is to remark that  $C_\lambda = (\varphi, \Phi_\lambda)$ , where  $\Phi_\lambda$  verifies the conditions of Definition 2.2, is a skew-evolution semiflow and it will be defined as the  $\lambda$ -shift skew-evolution semiflow on  $Y$ .

**Definition 2.4** A skew-evolution semiflow  $C = (\varphi, \Phi)$  is said to be

(sm) *strongly measurable* if for all  $(t_0, x, v) \in T \times Y$  the mapping  $s \mapsto \|\Phi(s, t_0, x)v\|$  is measurable on  $[t_0, \infty)$ .

(ssm) *\*-strongly measurable* if for all  $(t, t_0, x, v^*) \in T \times X \times V^*$  the mapping  $s \mapsto \|\Phi(t, s, \varphi(s, t_0, x))^* v^*\|$  is measurable on  $[t_0, t]$ .

**Definition 2.5** The skew-evolution semiflow  $C$  is said to have *exponential growth* if there exist some applications  $M, \omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$  such that

$$\|\Phi(t, t_0, x)v\| \leq M(s)e^{\omega(s)(t-s)} \|\Phi(s, t_0, x)v\|, \quad (2.2)$$

for all  $(t, s), (s, t_0) \in T$  and all  $(x, v) \in Y$ .

**Remark 2.1** Let  $C = (\varphi, \Phi)$  be a skew-evolution semiflow with exponential growth and let  $C_{-\alpha} = (\varphi, \Phi_{-\alpha})$ ,  $\alpha > 0$ , be the  $-\alpha$ -shift skew-evolution semiflow, where the evolution cocycle  $\Phi_{-\alpha}$  is given by relation (2.1). We have

$$\|\Phi_{-\alpha}(t, t_0, x)v\| = e^{\alpha(t-t_0)} \|\Phi(t, t_0, x)v\| \leq M(t_0)e^{[\alpha+\omega(t_0)](t-t_0)} \|v\|$$

for all  $(t_0, x, v) \in \mathbb{R}_+ \times Y$ , where the functions  $M$  and  $\omega$  are given by Definition 2.5. Hence,  $C_{-\alpha}$  has also exponential growth.

**Definition 2.6** The skew-evolution semiflow  $C$  is said to have *exponential decay* if there exist some applications  $M, \omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$  such that

$$\|\Phi(s, t_0, x)v\| \leq M(t)e^{\omega(t)(t-s)} \|\Phi(t, t_0, x)v\|, \quad (2.3)$$

for all  $(t, s), (s, t_0) \in T$  and all  $(x, v) \in Y$ .

**Remark 2.2** Let  $C = (\varphi, \Phi)$  be a skew-evolution semiflow with exponential decay and let  $C_\alpha = (\varphi, \Phi_\alpha)$ ,  $\alpha > 0$ , be the  $\alpha$ -shifted skew-evolution semiflow, where the evolution cocycle  $\Phi_\alpha$  is given by relation (2.1). Following relations

$$\|\Phi_\alpha(s, t_0, x)v\| = e^{-\alpha(s-t_0)} \|\Phi(s, t_0, x)v\| \leq M(t)e^{[\omega(t)+\alpha](t-s)} \|\Phi_\alpha(t, t_0, x)v\|$$

hold for all  $(t, s), (s, t_0) \in T$  and all  $(x, v) \in Y$ , where the functions  $M$  and  $\omega$  are given by Definition 2.6. Hence,  $C_\alpha$  has exponential decay.

**Remark 2.3** Sometimes it is useful to consider in Definition 2.5 or in Definition 2.6 the particular case  $\omega(s) \equiv \omega$ ,  $\forall s \geq 0$ .

### 3 Exponential stability and instability

Let  $C : T \times Y \rightarrow Y$ ,  $C(t, s, x, v) = (\varphi(t, s, x), \Phi(t, s, x)v)$  be a skew-evolution semiflow on  $Y$ .

**Definition 3.1** The skew-evolution semiflow  $C$  is called

(s) *stable* if there exists a mapping  $N : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$  such that

$$\|\Phi(t, t_0, x)v\| \leq N(s) \|\Phi(s, t_0, x)v\| \quad (3.1)$$

for all  $(t, s), (s, t_0) \in T$  and all  $(x, v) \in Y$

(es) *exponentially stable* if there exist a mapping  $N : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$  and a constant  $\nu > 0$  such that

$$\|\Phi(t, t_0, x)v\| \leq N(s)e^{-\nu(t-s)} \|\Phi(s, t_0, x)v\|, \quad (3.2)$$

for all  $(t, s), (s, t_0) \in T$  and all  $(x, v) \in Y$ .

**Remark 3.1** The exponential stability of a skew-evolution semiflow implies the stability and, further, the exponential growth.

In what follows we will present an example of a skew-evolution semiflow that is exponentially stable but not uniformly exponentially stable.

**Example 3.1** Let  $X = \mathbb{R}_+$  and  $V = \mathbb{R}$ . We consider the continuous function

$$f : \mathbb{R}_+ \rightarrow [1, \infty), \quad f(n) = e^{2n} \quad \text{and} \quad f\left(n + \frac{1}{e^{n^2}}\right) = 1$$

and the mapping

$$\Phi_f : T \times \mathbb{R}_+ \rightarrow \mathcal{B}(\mathbb{R}), \quad \Phi_f(t, s, x)v = \frac{f(s)}{f(t)} e^{-(t-s)}v.$$

Then  $C_f = (\varphi, \Phi_f)$  is a skew-evolution semiflow on  $Y = \mathbb{R}_+ \times \mathbb{R}$  over all evolution semiflows  $\varphi$  on  $\mathbb{R}_+$ . As

$$\|\Phi_f(t, s, x)v\| \leq f(s)e^{-(t-s)}|v|, \quad \forall (t, s, x, v) \in T \times Y,$$

it follows that  $C_f$  is exponentially stable and, according to Remark 3.1, stable. On the other hand, as

$$\Phi_f\left(n + \frac{1}{e^{n^2}}, n, x\right) = e^{2n - e^{-n^2}} \rightarrow \infty \text{ when } n \rightarrow \infty,$$

it follows that  $C_f$  is not uniformly exponentially stable.

**Definition 3.2** The skew-evolution semiflow  $C$  is called

(is) *instable* if there exists a mapping  $N : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$  such that

$$N(t) \|\Phi(t, t_0, x)v\| \geq \|\Phi(s, t_0, x)v\| \quad (3.3)$$

for all  $(t, s), (s, t_0) \in T$  and all  $(x, v) \in Y$

(eis) *exponentially instable* if there exist a mapping  $N : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$  and a constant  $\nu > 0$  such that

$$N(t) \|\Phi(t, t_0, x)v\| \geq e^{\nu(t-s)} \|\Phi(s, t_0, x)v\| \quad (3.4)$$

for all  $(t, s), (s, t_0) \in T$  and all  $(x, v) \in Y$ .

**Remark 3.2** The exponential instability of a skew-evolution semiflow implies the instability and, also, the exponential decay.

There exist skew-evolution semiflows that are exponentially instable but not uniformly exponentially instable, as following example shows.

**Example 3.2** Let  $X = \mathbb{R}_+$  and  $V = \mathbb{R}$ . We consider the function

$$f : \mathbb{R}_+ \rightarrow [1, \infty), \quad f(n) = 1 \text{ and } f\left(n + \frac{1}{e^{n^2}}\right) = e^{2n}$$

and the mapping

$$\Phi_f : T \times \mathbb{R}_+ \rightarrow \mathcal{B}(\mathbb{R}), \quad \Phi_f(t, s, x)v = \frac{f(s)}{f(t)} e^{(t-s)}v.$$

Then  $C_f = (\varphi, \Phi_f)$  is a skew-evolution semiflow on  $Y = \mathbb{R}_+ \times \mathbb{R}$  for all evolution semiflows  $\varphi$  on  $\mathbb{R}_+$ . We have

$$\|\Phi_f(t, s, x)v\| \geq \frac{1}{f(t)} e^{(t-s)}|v|, \quad \forall (t, s, x, v) \in T \times Y,$$

which proves that  $C_f$  is exponentially instable and, as in Remark 3.2, instable. But, as

$$\Phi_f\left(n + \frac{1}{e^{n^2}}, n, x\right) = e^{-2n + e^{-n^2}} \rightarrow 0 \text{ for } n \rightarrow \infty,$$

we obtain that  $C_f$  is not uniformly exponentially instable.

## 4 Exponential dichotomy

Let  $C : T \times Y \rightarrow Y$ ,  $C(t, s, x, v) = (\varphi(t, s, x), \Phi(t, s, x)v)$  be a skew-evolution semiflow on  $Y$ .

**Definition 4.1** Two projector families  $\{P_k\}_{k \in \{1,2\}}$  are said to be *compatible* with a skew-evolution semiflow  $C = (\varphi, \Phi)$  if

$$(dc_1) \quad P_1(x) + P_2(x) = I, \quad P_1(x)P_2(x) = P_2(x)P_1(x) = 0$$

$$(dc_2) \quad P_k(\varphi(t, s, x))\Phi(t, s, x)v = \Phi(t, s, x)P_k(x)v, \quad k \in \{1, 2\}$$

for all  $t \geq s \geq t_0 \geq 0$  and all  $(x, v) \in Y$ .

**Definition 4.2** The skew-evolution semiflow  $C = (\varphi, \Phi)$  is called

(d) *dichotomic* if there exist two projectors  $P_1$  and  $P_2$  compatible with  $C$  and some mappings  $N_1, N_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$  such that

$$\|\Phi(t, t_0, x)P_1(x)v\| \leq N_1(s) \|\Phi(s, t_0, x)P_1(x)v\| \quad (4.1)$$

$$\|\Phi(s, t_0, x)P_2(x)v\| \leq N_2(t) \|\Phi(t, t_0, x)P_2(x)v\| \quad (4.2)$$

for all  $(t, s), (s, t_0) \in T$  and all  $(x, v) \in Y$ .

(ed) *exponentially dichotomic* if there exist two projectors  $P_1$  and  $P_2$  compatible with  $C$ , some mappings  $N_1, N_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$  and some constants  $\nu_1, \nu_2 > 0$  such that

$$e^{\nu_1(t-s)} \|\Phi(t, t_0, x)P_1(x)v\| \leq N_1(s) \|\Phi(s, t_0, x)P_1(x)v\| \quad (4.3)$$

$$e^{\nu_2(t-s)} \|\Phi(s, t_0, x)P_2(x)v\| \leq N_2(t) \|\Phi(t, t_0, x)P_2(x)v\| \quad (4.4)$$

for all  $(t, s), (s, t_0) \in T$  and all  $(x, v) \in Y$ .

**Remark 4.1** (i) An exponentially dichotomic skew-evolution semiflow is dichotomic;

(ii) For  $P_2 = 0$  in Definition 4.2 are obtained the stability, respectively the exponential stability properties for skew-evolution semiflows;

(iii) For  $P_1 = 0$  we obtain in Definition 4.2 the instability and the exponential instability properties for skew-evolution semiflows.

**Remark 4.2** Without any loss of generality we can consider

$$N(t) = \max\{N_1(t), N_2(t)\}, \quad t \geq 0 \text{ and } \nu = \min\{\nu_1, \nu_2\}.$$

We will call  $N_1, N_2, \nu_1, \nu_2$ , respectively  $N, \nu$  the *dichotomic characteristics* associated to the skew-evolution semiflow  $C$ .

**Remark 4.3** Let us consider that the shifted skew-evolution semiflows  $C_\lambda = (\varphi, \Phi_\lambda)$  and  $C_\mu = (\varphi, \Phi_\mu)$ , where  $\Phi_\lambda$  and  $\Phi_\mu$  are evolution cocycles defined by relation (2.1) with  $\lambda < \mu$ , are exponentially dichotomic with characteristics

$$N_\lambda : \mathbb{R}_+ \rightarrow \mathbb{R}_+^* \text{ and } \nu_\lambda > 0, \text{ respectively } N_\mu : \mathbb{R}_+ \rightarrow \mathbb{R}_+^* \text{ and } \nu_\mu > 0.$$

If we denote

$$N(t) = \max\{N_\lambda(t), N_\mu(t)\} \text{ and } \nu = \min\{\nu_\lambda, \nu_\mu\}$$

then these are appropriate for both  $C_\lambda$  and  $C_\mu$ .

There exist exponentially dichotomic skew-evolution semiflows that are not uniformly exponentially dichotomic, as in the next

**Example 4.1** Let  $X = \mathbb{R}_+$  and  $V = \mathbb{R}^2$  endowed with the norm

$$\|(v_1, v_2)\| = |v_1| + |v_2|, \quad v = (v_1, v_2) \in V.$$

The mapping  $\Phi : T \times X \rightarrow \mathcal{B}(V)$ , defined by

$$\Phi(t, t_0, x)(v_1, v_2) = (e^{t \sin t - s \sin s - 2t + 2s} v_1, e^{2t - 2s - 3t \cos t + 3s \cos s} v_2)$$

is an evolution cocycle over all evolution semiflows  $\varphi$ . We consider the projectors compatible with  $C$

$$P_1(x)(v_1, v_2) = (v_1, 0) \text{ and } P_2(x)(v_1, v_2) = (0, v_2).$$

As

$$t \sin t - s \sin s - 2t + 2s \leq -t + 3s, \quad \forall (t, s) \in T,$$

we obtain that

$$\|\Phi(t, s, x)P_1(x)v\| \leq e^{2s} e^{-(t-s)} |v_1|, \quad \forall (t, s, x, v) \in T \times Y.$$

Similarly, as

$$2t - 2s - 3t \cos t + 3s \cos s \geq -t - 5s, \quad \forall (t, s) \in T,$$

it follows that

$$e^{6t} \|\Phi(t, s, x)P_2(x)v\| \geq e^{5(t-s)} |v_2|, \quad \forall (t, s, x, v) \in T \times Y.$$

The skew-evolution semiflow  $C = (\varphi, \Phi)$  is exponentially dichotomic with the dichotomic characteristics

$$N(t_0) = e^{6t_0} \text{ and } \nu = 2$$

and, according to Remark 4.1, dichotomic. But, as

$$\Phi\left(2n\pi, 2n\pi - \frac{\pi}{2}, x\right) = e^{2n\pi - \frac{3\pi}{2}} \rightarrow \infty \text{ as } n \rightarrow \infty,$$

and

$$\Phi\left(2n\pi, 2n\pi + \frac{\pi}{2}, x\right) = e^{-6n\pi - \pi} \rightarrow 0 \text{ as } n \rightarrow \infty,$$

we obtain that  $C$  is not uniformly exponentially dichotomic.

In what follows we will denote

$$C_k(t, s, x, v) = (\varphi(t, s, x), \Phi_k(t, s, x)v), \quad \forall (t, t_0, x, v) \in T \times Y, \quad \forall k \in \{1, 2\},$$

where

$$\Phi_k(t, t_0, x) = \Phi(t, t_0, x)P_k(x), \quad \forall (t, t_0) \in T, \quad \forall x \in X, \quad \forall k \in \{1, 2\}.$$

We give an integral characterization for the dichotomy property of skew-evolution semiflows.

**Theorem 4.1** *A strongly measurable skew-evolution semiflow  $C = (\varphi, \Phi)$  is exponentially dichotomic if and only if there exist two projectors  $P_1$  and  $P_2$  compatible with  $C$  such that  $C_1$  has exponential growth and  $C_2$  has exponential decay, some functions  $M_1, M_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$  and some constants  $\alpha, \beta > 0$  such that*

(ed<sub>1</sub>)

$$\int_{t_0}^t e^{\alpha(\tau-t_0)} \|\Phi_1(\tau, t_0, x)v\| d\tau \leq M_1(t_0) \|P_1(x)v\| \quad (4.5)$$

(ed<sub>2</sub>)

$$\int_{t_0}^t e^{\beta(t-\tau)} \|\Phi_2(\tau, t_0, x)v\| d\tau \leq M_2(t) \|\Phi_2(t, t_0, x)v\| \quad (4.6)$$

for all  $(t, t_0) \in T$  and all  $(x, v) \in Y$ .

**Proof.** *Necessity.* As the skew-evolution semiflow  $C$  is exponentially dichotomic, there exist two projectors  $P_1$  and  $P_2$  compatible with  $C$ , a function  $N_1 : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$  and a constant  $\nu_1 > 0$  such that

$$\|\Phi(t, s, x)P_1(x)v\| \leq N_1(s)e^{-\nu_1(t-s)} \|P_1(x)v\|$$

for all  $t \geq s \geq 0$  and all  $(x, v) \in Y$ . We consider  $\alpha$  such that  $\nu_1 = 2\alpha$ . Following relations hold

$$\int_{t_0}^t e^{\alpha(\tau-t_0)} \|\Phi(\tau, t_0, x)v\| d\tau \leq N_1(t_0) \|v\| \int_{t_0}^t e^{\alpha(\tau-t_0)} e^{-\nu_1(\tau-t_0)} d\tau \leq M_1(t_0) \|v\|,$$

where we have denoted

$$M_1(t_0) = \frac{N_1(t_0)}{\alpha}.$$

Hence, relation (4.5) is obtained.

Also, there exist a function  $N_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$  and a constant  $\nu_2 > 0$  such that

$$e^{\nu_2(t-s)} \|\Phi(s, t_0, x)P_2(x)v\| \leq N_2(t) \|\Phi(t, t_0, x)P_2(x)v\|$$

for all  $t \geq t_0 \geq 0$  and all  $(x, v) \in Y$ . We will consider  $\beta$  such that  $\nu_2 = 2\beta$ . We have

$$\begin{aligned} & \int_{t_0}^t e^{\beta(t-\tau)} \|\Phi(\tau, t_0, x)P_2(x)v\| d\tau \leq \\ & \leq N_2(t) \int_{t_0}^t e^{\beta(t-\tau)} e^{-\nu_2(t-\tau)} \|\Phi(t, t_0, x)P_2(x)v\| d\tau \leq M_2(t) \|\Phi(t, t_0, x)P_2(x)v\|, \end{aligned}$$

where we have denoted

$$M_2(t) = \frac{N_2(t)}{\beta}.$$

Relation (4.6) is then obtained.

*Sufficiency.* As  $C_1$  has exponential growth, similarly as in Theorem 2.3 of [9], proved for evolution operators, there exists a nondecreasing function  $f : [0, \infty) \rightarrow [1, \infty)$  with the property  $\lim_{t \rightarrow \infty} f(t) = \infty$  such that

$$\|\Phi(t, t_0, x)v\| \leq f(t-s) \|\Phi(s, t_0, x)v\|,$$

for all  $(t, s), (s, t_0) \in T$  and all  $(x, v) \in Y$ .

For  $t \geq t_0 + 1$  we have

$$\begin{aligned} & \|\Phi(t, t_0, x)P_1(x)v\| e^{\alpha(t-t_0)} \int_0^1 \frac{e^{-\alpha u}}{f(u)} du = \\ & = e^{\alpha(t-t_0)} \|\Phi(t, t_0, x)P_1(x)v\| \int_{t-1}^t \frac{e^{-\alpha(t-\tau)}}{f(t-\tau)} d\tau = \\ & = \int_{t-1}^t \frac{\|\Phi(t, t_0, x)P_1(x)v\|}{f(t-\tau)} e^{\alpha(\tau-t_0)} d\tau \leq \int_{t-1}^t \|\Phi(\tau, t_0, x)P_1(x)v\| e^{\alpha(\tau-t_0)} d\tau \leq \\ & \leq M_1(t_0) \|P_1(x)v\|. \end{aligned}$$

For  $t \in [t_0, t_0 + 1)$  we have

$$\|\Phi_{-\alpha}(t, t_0, x)P_1(x)v\| \leq f(1)e^\alpha \|P_1(x)v\|,$$

where the evolution cocycle  $\Phi_{-\alpha}$  is given by relation (2.1). We obtain

$$\|\Phi_{-\alpha}(t, t_0, x)P_1(x)v\| \leq N_1(t_0) \|P_1(x)v\|,$$

for all  $(t, t_0) \in T$  and all  $(x, v) \in Y$ , where we have denoted

$$N_1(t_0) = f(1)e^\alpha + M_1(t_0) \left[ \int_0^1 \frac{e^{-\alpha u}}{f(u)} du \right]^{-1}.$$

It follows that

$$\|\Phi(t, t_0, x)P_1(x)v\| \leq N_1(t_0)e^{-\alpha(t-t_0)} \|P_1(x)v\|,$$

for all  $t \geq t_0 \geq 0$  and all  $(x, v) \in Y$ . Hence, relation (4.3) was proved.

As  $C_2$  has exponential decay, by a similar deduction used to prove Theorem 3.3 of [9] for evolution operators, there exists a nondecreasing function  $g : [0, \infty) \rightarrow [1, \infty)$  with the property  $\lim_{t \rightarrow \infty} g(t) = \infty$  such that

$$\|\Phi(s, t_0, x)v\| \leq g(t-s) \|\Phi(t, t_0, x)v\|,$$

for all  $(t, s), (s, t_0) \in T$  and all  $(x, v) \in Y$ .

For  $t \geq s \geq t_0 \geq 0$ ,  $x \in X$ ,  $v \in V$  we have

$$\begin{aligned} \|\Phi(s, t_0, x)P_2(x)v\| e^{\beta(t-s)} \int_0^1 \frac{e^{\beta u}}{g(u)} du &= \\ &= \int_{s-1}^s \|\Phi(s, t_0, x)P_2(x)v\| \frac{e^{\beta(t-s)}}{g(s-\tau)} e^{\beta(s-\tau)} d\tau \leq \\ &\leq \int_{s-1}^s \|\Phi(\tau, t_0, x)P_2(x)v\| e^{\beta(t-\tau)} d\tau \leq M_2(t) \|\Phi(t, t_0, x)P_2(x)v\|. \end{aligned}$$

We obtain

$$\|\Phi(s, t_0, x)P_2(x)v\| \leq N_2(t) e^{-\beta(t-s)} \|\Phi(t, t_0, x)P_2(x)v\|,$$

for all  $t \geq t_0 \geq 0$  and all  $(x, v) \in Y$ , where we have denoted

$$N_2(t) = M_2(t) \left[ \int_0^1 \frac{e^{\beta u}}{g(u)} \right]^{-1}.$$

Relation (4.4) is then obtained.

Hence, the skew-evolution semiflow  $C$  is exponentially dichotomic.  $\square$

## 5 Exponential trichotomy

Let  $C : T \times Y \rightarrow Y$ ,  $C(t, s, x, v) = (\varphi(t, s, x), \Phi(t, s, x)v)$  be a skew-evolution semiflow on  $Y$ .

**Definition 5.1** Three projector families  $\{P_k\}_{k \in \{1,2,3\}}$  are said to be *compatible* with a skew-evolution semiflow  $C = (\varphi, \Phi)$  if

$$(tc_1) \quad P_1(x) + P_2(x) + P_3(x) = I, \quad P_i(x)P_j(x) = P_j(x)P_i(x) = 0, \quad \forall i, j \in \{1, 2, 3\}, \quad i \neq j$$

$$(tc_2) \quad P_k(\varphi(t, s, x))\Phi(t, s, x)v = \Phi(t, s, x)P_k(x)v, \quad \forall k \in \{1, 2, 3\},$$

for all  $(t, s), (s, t_0) \in T$  and all  $(x, v) \in Y$ .

**Definition 5.2** A skew-evolution semiflow  $C = (\varphi, \Phi)$  is said to be

(t) *trichotomic* if there exist three projectors  $P_1, P_2$  and  $P_3$  compatible with  $C$  and some functions  $N_1, N_2, N_3 : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$  such that

$$\|\Phi(t, t_0, x)P_1(x)v\| \leq N_1(t_0) \|\Phi(s, t_0, x)P_1(x)v\| \quad (5.1)$$

$$\|\Phi(s, t_0, x)P_2(x)v\| \leq N_2(t_0) \|\Phi(t, t_0, x)P_2(x)v\| \quad (5.2)$$

$$\begin{aligned} \|\Phi(s, t_0, x)P_3(x)v\| &\leq N_3(t_0) \|\Phi(t, t_0, x)P_3(x)v\| \leq \\ &\leq N_3^2(t_0) \|\Phi(s, t_0, x)P_3(x)v\| \end{aligned} \quad (5.3)$$

for all  $(t, s), (s, t_0) \in T$  and all  $(x, v) \in Y$ .

(*et*) *exponentially trichotomic* if there exist three projectors  $P_1, P_2$  and  $P_3$  compatible with  $C$ , some functions  $N_1, N_2, N_3, N_4 : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$  and some constants  $\nu_1, \nu_2, \nu_3, \nu_4$  with the properties  $\nu_1 < \nu_2 \leq 0 \leq \nu_3 < \nu_4$  such that

$$\|\Phi(t, t_0, x)P_1(x)v\| \leq N_1(s) \|\Phi(s, t_0, x)P_1(x)v\| e^{\nu_1(t-s)} \quad (5.4)$$

$$N_4(t) \|\Phi(t, t_0, x)P_2(x)v\| \geq \|\Phi(s, t_0, x)P_2(x)v\| e^{\nu_4(t-s)} \quad (5.5)$$

$$\|\Phi(t, t_0, x)P_3(x)v\| \leq N_3(s) \|\Phi(s, t_0, x)P_3(x)v\| e^{\nu_3(t-s)} \quad (5.6)$$

$$N_2(t) \|\Phi(t, t_0, x)P_3(x)v\| \geq \|\Phi(s, t_0, x)P_3(x)v\| e^{\nu_2(t-s)} \quad (5.7)$$

for all  $(t, s), (s, t_0) \in T$  and all  $(x, v) \in Y$ .

**Remark 5.1** (*i*) An exponentially trichotomic skew-evolution semiflow is trichotomic;

(*ii*) For  $P_1 = 0$  in Definition 5.2 are obtained the properties of dichotomy, respectively exponential dichotomy;

(*iii*) For  $P_2 = P_3 = 0$  the properties of stability, respectively exponential stability follow from Definition 5.2;

(*iv*) If  $P_1 = P_3 = 0$  the properties of instability and exponential instability are obtained from Definition 5.2.

**Remark 5.2** Without any loss of generality we can chose

$$N(t) = \max\{N_1(t), N_2(t), N_3(t), N_4(t)\}, t \geq 0$$

respectively

$$\alpha = -\nu_1 = \nu_4 > 0 \text{ and } \beta = -\nu_2 = \nu_3 > 0.$$

We call  $N_1, N_2, N_3, N_4, \nu_1, \nu_2, \nu_3, \nu_4$ , respectively  $N, \alpha, \beta$  the trichotomic characteristics of the skew-evolution semiflow  $C$ .

**Example 5.1** Let us consider  $(X, d)$  a metric space given as in Example 2.1. The mapping

$$\varphi : T \times X \rightarrow X, \varphi(t, s, x)(\tau) = x(t - s + \tau)$$

is an evolution semiflow on  $X$ .

Let  $V = \mathbb{R}^3$  with the norm

$$\|(v_1, v_2, v_3)\| = |v_1| + |v_2| + |v_3|.$$

The mapping  $\Phi : T \times X \rightarrow \mathcal{B}(V)$ , given by

$$\Phi(t, s, x)(v) = \left( e^{-2(t-s)f(0) + \int_0^t x(\tau) d\tau} v_1, e^{t-s + \int_0^t x(\tau) d\tau} v_2, e^{-(t-s)f(0) + 2 \int_0^t x(\tau) d\tau} v_3 \right)$$

is an evolution cocycle.

We consider the projections

$$P_1(x)(v) = (v_1, 0, 0), \quad P_2(x)(v) = (0, v_2, 0), \quad P_3(x)(v) = (0, 0, v_3).$$

The skew-evolution semiflow  $C = (\varphi, \Phi)$  is exponentially trichotomic with associated trichotomic characteristics

$$\nu_1 = \nu_2 = -f(0), \quad \nu_3 = f(0) \quad \text{and} \quad \nu_4 = 1$$

$$N_1(t_0) = e^{t_0 f(0)}, \quad N_2(t) = e^{-2t}, \quad N_3(t_0) = e^{2t_0 f(0)} \quad \text{and} \quad N_4(t) = e^{-t}.$$

A characterization for the property of exponential trichotomy can be given by the next

**Theorem 5.1** *Let  $C = (\varphi, \Phi)$  be a skew-evolution semiflow and three projectors  $P_1, P_2$  and  $P_3$  compatible with  $C$  such that  $C_1$  has exponential growth and is  $*$ -strongly measurable and  $C_2$  has exponential decay and is strongly measurable. Then  $C$  is exponentially trichotomic if and only if there exist a mapping  $\tilde{N} : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$  and a constant  $\alpha > 0$ , a mapping  $\overline{N} : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$  and a constant  $\beta > 0$ , some functions  $\tilde{M}, \overline{M} : [0, \infty) \rightarrow (0, \infty)$ , some non-decreasing functions  $\tilde{g}, \overline{g} : [0, \infty) \rightarrow (0, \infty)$  with the property  $\lim_{t \rightarrow \infty} \tilde{g}(t) = \lim_{t \rightarrow \infty} \overline{g}(t) = \infty$  such that*

(et<sub>1</sub>)

$$\int_{t_0}^t e^{\alpha(t-s)} \|\Phi(t, s, \varphi(s, t_0, x))^* P_1(x) v^*\| ds \leq \tilde{N}(t_0) \|P_1(x) v^*\|, \quad (5.8)$$

for all  $(t, t_0) \in T$  and all  $(x, v^*) \in X \times V^*$  with  $\|v^*\| \leq 1$

(et<sub>2</sub>)

$$\int_0^t e^{-\beta(s-t_0)} \|\Phi(s, t_0, x) P_2(x) v\| ds \leq \overline{N}(t) e^{-\beta(t-t_0)} \|\Phi(t, t_0, x) P_2(x) v\|, \quad (5.9)$$

for all  $(t, t_0) \in T$  and all  $(x, v) \in Y$

(et<sub>3</sub>)

$$\|\Phi(t, t_0, x) P_3(x) v\| \leq \tilde{M}(s) \tilde{g}(t-s) \|\Phi(s, t_0, x) P_3(x) v\| \quad (5.10)$$

for all  $(t, s), (s, t_0) \in T$  and all  $(x, v) \in Y$   
(et<sub>4</sub>)

$$\|\Phi(s, t_0, x)P_3(x)v\| \leq \overline{M}(t)\overline{g}(t-s)\|\Phi(t, t_0, x)P_3(x)v\| \quad (5.11)$$

for all  $(t, s), (s, t_0) \in T$  and all  $(x, v) \in Y$ .

**Proof.** *Necessity.* (et<sub>1</sub>) We consider

$$\alpha = -\frac{\nu_1}{2}$$

and we obtain

$$\begin{aligned} \int_{t_0}^t e^{\alpha(t-s)} \|\Phi(t, s, x)^* P_1(x)v^*\| ds &\leq \int_{t_0}^t N_1(t_0) e^{\frac{\nu_1}{2}(t-s)} \|P_1(x)v^*\| ds \leq \\ &\leq \tilde{N}(t_0) \|P_1(x)v^*\|, \end{aligned}$$

for all  $(t, t_0, x, v^*) \in T \times X \times V^*$ , where we have denoted

$$\tilde{N}(t_0) = -\frac{2}{\nu} N_1(t_0).$$

(et<sub>2</sub>) Let us define

$$\beta = \frac{\nu}{2} > 0,$$

where the existence of constant  $\nu$  is assured by hypothesis and by Definition 3.2. Hence, we obtain

$$\begin{aligned} &\int_0^t e^{-\beta(s-t_0)} \|\Phi(s, t_0, x)v\| ds \leq \\ &\leq N(t) \int_{t_0}^t e^{-\beta(s-t_0)} \|\Phi(t, t_0, x)v\| e^{-\nu(t-s)} ds \leq \beta^{-1} N(t) e^{-\beta(t-t_0)} \|\Phi(t, t_0, x)v\|. \end{aligned}$$

(et<sub>3</sub>) It is obtained immediately if we consider

$$\tilde{M}(u) = N_3(u), \quad u \geq 0 \text{ and } \tilde{g}(v) = e^{\nu_3 v}, \quad v \geq 0.$$

(et<sub>4</sub>) It follows for

$$\overline{M}(u) = N_2(u), \quad u \geq 0 \text{ and } \overline{g}(v) = e^{-\nu_2 v}, \quad v \geq 0.$$

*Sufficiency.* (et<sub>1</sub>) Let  $t \geq t_0 + 1$  and  $s \in [t_0, t_0 + 1)$ . Then

$$\begin{aligned} &e^{-[\alpha+\omega(t_0)]} \left| \left\langle v^*, e^{\alpha(t-t_0)} \Phi(t, t_0, x)v \right\rangle \right| = \\ &= e^{-[\alpha+\omega(t_0)]} \int_{t_0}^{t_0+1} \left| \left\langle \Phi(t, s, x)^* v^*, e^{\alpha(t-t_0)} \Phi(s, t_0, x)v \right\rangle \right| ds \leq \end{aligned}$$

$$\begin{aligned}
&\leq \int_{t_0}^{t_0+1} e^{\alpha(t-s)} \|\Phi(t, s, \varphi(s, t_0, x))^* v^*\| e^{-\omega(t_0)(s-t_0)} \|\Phi(s, t_0, x)v\| ds \leq \\
&\leq M(t_0) \|v\| \int_{t_0}^t e^{\alpha(t-s)} \|\Phi(t, s, \varphi(s, t_0, x))^* v^*\| ds \leq M(t_0)N(t_0) \|v\| \|v^*\|,
\end{aligned}$$

where the functions  $M, \omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$  are given by Definition 2.5. By taking supremum over  $\|v^*\| \leq 1$  we obtain

$$e^{-[\alpha+\omega(t_0)]e^{\alpha(t-t_0)}} \|\Phi(t, t_0, x)v\| \leq M(t_0)N(t_0) \|v\|, \quad \forall t \geq t_0 + 1,$$

and, further,

$$\|\Phi(t, t_0, x)v\| \leq M_1(t_0)e^{-\alpha(t-t_0)} \|v\|, \quad \forall t \geq t_0 + 1,$$

where we have denoted

$$M_1(t_0) = M(t_0)N(t_0)e^{[\alpha+\omega(t_0)]}, \quad t_0 \geq 0.$$

For  $t \in [t_0, t_0 + 1)$  we have

$$\|\Phi(t, t_0, x)v\| \leq M(t_0)e^{\omega(t_0)(t-t_0)} \|v\| \leq M_2(t_0)e^{-\alpha(t-t_0)} \|v\|,$$

where we have denoted

$$M_2(t_0) = M(t_0)e^{[\alpha+\omega(t_0)]}, \quad t_0 \geq 0.$$

Hence

$$\|\Phi(t, t_0, x)v\| \leq [M_1(t_0) + M_2(t_0)]e^{-\alpha(t-t_0)} \|v\|, \quad \forall (t, t_0, x, v) \in T \times X \times V,$$

which proves relation (5.4) of Definition 5.1.

(et<sub>2</sub>) We denote  $K = \int_0^1 e^{-\beta u} f(u) du$ , where function  $f$  is given as in Theorem 3.3 of [9]. We obtain successively

$$\begin{aligned}
K \|v\| &= \int_{t_0}^{t_0+1} e^{-\beta(\tau-t_0)} f(\tau-t_0) \|\Phi(t_0, t_0, x)v\| d\tau \leq \\
&\leq \int_{t_0}^{t_0+1} e^{-\beta(\tau-t_0)} \|\Phi(\tau, t_0, x)v\| d\tau \leq M(t) \|\Phi_\beta(t, t_0, x)v\| = \\
&= M(t)e^{-\beta(t-t_0)} \|\Phi(t, t_0, x)v\|
\end{aligned}$$

for all  $(t, t_0) \in T$  and all  $(x, v) \in Y$ . Hence, relation (5.5) of Definition 5.1 was proved.

(et<sub>3</sub>) As  $\lim_{t \rightarrow \infty} \tilde{g}(t) = \infty$ , there exists  $\delta > 0$  such that  $\tilde{g}(\delta) > 1$ . Let  $(t, s) \in T$ . Then there exists  $n \in \mathbb{N}$  and  $r \in [0, \delta)$  such that  $t = s + n\delta + r$ . We obtain successively

$$\|\Phi(t, t_0, x)P_3(x)v\| = \|\Phi(s + n\delta + r, t_0, x)P_3(x)v\| \leq$$

$$\begin{aligned}
&\leq \tilde{g}(r)\widetilde{M}(s+n\delta)\|\Phi(s+n\delta,t_0,x)P_3(x)v\| \leq \\
&\leq \tilde{g}(\delta)\widetilde{M}(s+n\delta)\|\Phi(s+n\delta,t_0,x)P_3(x)v\| \leq \\
&\leq [\tilde{g}(\delta)]^2\widetilde{M}(s+(n-1)\delta+r)\|\Phi(s+(n-1)\delta,t_0,x)P_3(x)v\| \leq \dots \leq \\
&\leq [\tilde{g}(\delta)]^{n+1}\widetilde{M}(s+(n-1)\delta)\dots\widetilde{M}(s)\|\Phi(s+r,t_0,x)P_3(x)v\|,
\end{aligned}$$

for all  $(t, s)$ ,  $(s, t_0) \in T$  and all  $(x, v) \in Y$ . If we define

$$N_3(u) = \tilde{g}(\delta)\widetilde{M}(u+(n-1)\delta+r)\dots\widetilde{M}(u+r)\widetilde{M}(u) \text{ and } \nu_3 = \frac{\ln \tilde{g}(\delta)}{\delta}$$

we obtain relation (5.6) of Definition 5.1.

(*et*<sub>4</sub>) Without loss of generality we can consider  $\bar{g}(1) > 1$ . Let  $(t, t_0) \in T$  and  $n = [t - t_0]$ . We obtain  $t_0 + n \leq t < t_0 + n + 1$ . Following relations hold for all  $(x, v) \in Y$

$$\begin{aligned}
\|\Phi(t, t_0, x)P_3(x)v\| &\geq \frac{1}{\overline{M}(t)}\frac{1}{\bar{g}(1)}\|\Phi(t-1, t_0, x)P_3(x)v\| \geq \\
&\geq \frac{1}{\overline{M}(t)}\frac{1}{\overline{M}(t-1)}\frac{1}{[\bar{g}(1)]^2}\|\Phi(t-2, t_0, x)P_3(x)v\| \geq \dots \geq \\
&\geq \frac{1}{\overline{M}(t)}\frac{1}{\overline{M}(t-1)}\dots\frac{1}{\overline{M}(t-(n-1))}\frac{1}{[\bar{g}(1)]^n}\|\Phi(t-n, t_0, x)P_3(x)v\| \geq \\
&\geq \frac{1}{\overline{M}(t)}\frac{1}{\overline{M}(t-1)}\dots\frac{1}{\overline{M}(t-(n-1))}\frac{1}{\overline{M}(t-t_0-n)}\frac{1}{[\bar{g}(1)]^n}\frac{1}{\bar{g}(t-t_0-n)}\|P_3(x)v\|.
\end{aligned}$$

If we denote

$$N_2(t) = \bar{g}(1)\overline{M}(t)\overline{M}(t-1)\dots\overline{M}(1) \text{ and } \nu_2 = \bar{g}(1),$$

relation (5.7) of Definition 5.1 is obtained.

Hence, the skew-evolution semiflow  $C$  is exponentially trichotomic.  $\square$

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CODRUȚA STOICA  
Institut de Mathématiques  
Université Bordeaux 1  
351 Cours de la Libération  
F-33405 Talence Cedex France  
e-mail: [codruta.stoica@math.u-bordeaux1.fr](mailto:codruta.stoica@math.u-bordeaux1.fr)

MIHAIL MEGAN  
Faculty of Mathematics  
West University of Timișoara  
Bd. Pârvan, No.4  
300223 Timișoara  
Romania  
E-mail: [megan@math.uvt.ro](mailto:megan@math.uvt.ro)