Equivalent Definitions for Uniform Exponential Trichotomy of Evolution Operators in Banach Spaces

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Abstract

The aim of this paper is to give necessary and sufficient conditions for the uniform exponential trichotomy property of nonlinear evolution operators in Banach spaces. The obtained results are generalizations for infinite-dimensional case of some well-known results of Elaydi and Hajek on exponential trichotomy of differential systems.

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1 Introduction

It is well known that in the last decades the theory of asymptotic behaviors of evolution operators has witnessed an explosive development. A number of long standing open problems have recently been solved and the theory seems to have obtained a certain degree of maturity. There are various conditions characterizing exponentially stable or dichotomic evolution operators on Banach or Hilbert spaces.

In recent years, the techniques used in the investigation of the exponential stability have been generalized for the case of exponential dichotomy.

The concept of uniform exponential trichotomy is a natural generalization of the classical concept of uniform exponential dichotomy. In the study of the trichotomy, the main idea is to obtain a decomposition of the space at every moment into three closed subspaces: the stable subspace, the unstable subspace and the center manifold. For the finite dimensional case some concepts of trichotomy have been considered by Sacker and Sell in [8] and by Elaydi and Hajek in [1], [2] and [3]. The exponential trichotomy property in the infinite dimensional case has been studied in [4], [5], [6] and [7].

The aim of the present paper is to give two characterizations of the uniform exponential trichotomy of nonlinear evolution operators on $\mathbb{R}_+$. We consider a concept of exponential trichotomy which is a direct generalization of the concept of uniform exponential dichotomy. The obtained results are extensions for nonlinear infinite dimensional case of some well known results of Elaydi and Hajek ([1], [2] and [3]) on exponential trichotomy of linear differential systems.

It is important to observe that in our paper we consider a very general concept of nonlinear evolution operators.

2 Definitions and notations

Let X be a real or complex Banach space. The norm on X will be denoted by $\|\cdot\|$. The set of all mappings from X into itself is denoted by $\mathfrak{F}(X)$. Let T be the set of all pairs (t, t_0) of real numbers with $t \geq t_0 \geq 0$.

Definition 2.1 A mapping $E : T \to \mathfrak{F}(X)$ with the property

$$E(t,s)E(s,t_0) = E(t,t_0), \ \forall (t,s), (s,t_0) \in T \tag{2.1}$$

is called *evolution operator* on X.

Example 2.1 If $f : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ then the mapping $E_f : T \to \mathfrak{F}(\mathbb{R})$ defined by

$$E_f(t,t_0)x = \frac{f(t)}{f(t_0)}x$$

is an evolution operator on $\mathbb{R}$.

Example 2.2 If $(S(t))_{t \geq 0}$ is a nonlinear semigroup on X, then the mapping $E_f : T \to \mathfrak{F}(X)$ given by $E(t,s) = S(t-s)$, where $t \geq s \geq 0$, defines an evolution operator on X.

Definition 2.2 An application $P : \mathbb{R}_+ \to \mathfrak{F}(X)$ is said to be a *projection family* on X if

$$P(t)^2 = P(t), \ \forall t \in \mathbb{R}_+. \tag{2.2}$$

Definition 2.3 Three projection families P_0, P_1, $P_2 : \mathbb{R}_+ \to \mathfrak{F}(X)$ are said to be *compatible* with the evolution operator $E : T \to \mathfrak{F}(X)$ if

(c_1) $P_0(t) + P_1(t) + P_2(t) = I$ (the identity operator) for all $t \geq 0$;

(c_2) $P_i(t)P_j(t) = 0$ for all $t \geq 0$ and all $i, j \in \{0, 1, 2\}, \ i \neq j$;

(c_3) $\|P_i(t)x + P_j(t)x\|^2 = \|P_i(t)x\|^2 + \|P_j(t)x\|^2$ for all $t \geq 0$, all $x \in X$ and all $i, j \in \{0, 1, 2\}, \ i \neq j$;

(c_4) $E(t,t_0)P_k(t_0) = P_k(t)E(t,t_0)$ for all $(t,t_0) \in T$ and all $k \in \{0, 1, 2\}$.

In what follows we will denote

$$E_k(t,t_0) = E(t,t_0)P_k(t_0) = P_k(t)E(t,t_0) \tag{2.3}$$

for all $(t,t_0) \in T$ and all $k \in \{0,1,2\}$.

Remark 2.1 E_0, E_1 and E_2 are evolution operators on X.

Definition 2.4 An evolution operator $E : T \to \mathfrak{F}(X)$ is said to be *uniformly exponentially trichotomic* if there exist some constants N_0, N_1, $N_2 > 1$, ν_0, ν_1, $\nu_2 > 0$ and three projection families P_0, P_1 and P_2 compatible with E such that

(t_1) $e^{\nu_1(t-s)} \|E_1(t,t_0)x\| \le N_1 \|E_1(s,t_0)x\|$

(t_2) $e^{\nu_2(t-s)} \|E_2(s,t_0)x\| \le N_2 \|E_2(t,t_0)\|$

(t_3) $\|E_0(s,t_0)x\| \le N_0 e^{\nu_0(t-s)} \|E_0(t,t_0)x\|$

(t_4) $\|E_0(t,t_0)x\| \le N_0 e^{\nu_0(t-s)} \|E_0(s,t_0)x\|$

for all $t \ge s \ge t_0 \ge 0$ and all $x \in X$.

Remark 2.2 In Definition 2.4 one can consider

$$N_0 = N_1 = N_2 = N \text{ and } \nu_1 = \nu_2 = \nu.$$

Otherwise we can denote

$$N = \max\{N_0, N_1, N_2\} \text{ and } \nu = \min\{\nu_1, \nu_2\}.$$

Remark 2.3 For the particular case $P_0(t) = 0$, $t \ge 0$, we obtain the uniform exponential dichotomy property. Thus, the uniform exponential trichotomy is a natural generalization of the uniform exponential dichotomy property.

Example 2.3 Let f_0, f_1, $f_2 : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ be three functions defined by

$$f_0(t) = 1, \ f_1(t) = e^{-t}, \ f_2(t) = e^t.$$

It is easy to observe that the evolution operators E_{f_0}, E_{f_1} and E_{f_2}, defined as in Example 2.1, are uniformly exponentially trichotomic.

Example 2.4 Let us consider $X = \mathbb{R}^3$ with the norm

$$\|(x_1,x_2,x_3)\| = |x_1| + |x_2| + |x_3|, \ x = (x_1,x_2,x_3) \in X.$$

Let $\varphi : \mathbb{R}_+ \to (0,\infty)$ be a decreasing continuous function with the property that there exists $\lim_{t\to\infty} \varphi(t) = l > 0$.

Then the mapping $E : T \to \mathfrak{F}(X)$ defined by

$$E(t,t_0)x = (e^{-\int_{t_0}^{t} \varphi(\tau)d\tau} x_1,\ e^{\int_{t_0}^{t} \varphi(\tau)d\tau} x_2,\ e^{-(t-t_0)\varphi(0)} x_3)$$

is an evolution operator on X.

Let us consider the projections defined by

$$P_1(t)(x_1,x_2,x_3) = (x_1,0,0)$$

$$P_2(t)(x_1,x_2,x_3) = (0,x_2,0)$$

$$P_3(t)(x_1,x_2,x_3) = (0,0,x_3).$$

for all $t \geq 0$ and all $x = (x_1,x_2,x_3) \in X$.

Following relations hold

$$\|E(t,t_0)P_1(t_0)x)\| \leq e^{-l(t-s)} \|E(s,t_0)P_1(t_0)x)\|$$

$$\|E(t,t_0)P_2(t_0)x)\| \geq e^{l(t-s)} \|E(s,t_0)P_2(t_0)x)\|$$

$$\|E(t,t_0)P_3(t_0)x)\| \leq e^{\varphi(0)(t-s)} \|E(s,t_0)P_3(t_0)x)\|$$

$$\|E(t,t_0)P_3(t_0)x)\| \geq e^{-\varphi(0)(t-s)} \|E(s,t_0)P_3(t_0)x)\|$$

for all $t \geq s \geq t_0 \geq 0$ and all $x \in X$.

It follows that E is uniformly exponentially trichotomic.

3 The main results

It is well known that the uniform exponential dichotomy involves two commuting families of projections. In order to obtain a characterization of the uniform exponential trichotomy property using two commuting projection families we introduce the following

Definition 3.1 Two projection families $Q_1,\ Q_2 : \mathbb{R}_+ \to \mathfrak{F}(X)$ are said to be *compatible* with the evolution operator $E : T \to \mathfrak{F}(X)$ if

(c_1') $Q_1(t)Q_2(t) = Q_2(t)Q_1(t) = 0$

(c_2') $\|[Q_1(t) + Q_2(t)]\, x\|^2 = \|Q_1(t)x\|^2 + \|Q_2(t)x\|^2$

(c_3') $\|[I - Q_1(t)]\, x\|^2 = \|[I - Q_1(t) - Q_2(t)]\, x\|^2 + \|Q_2(t)x\|^2$

(c_4') $\|[I - Q_2(t)]\, x\|^2 = \|[I - Q_1(t) - Q_2(t)]\, x\|^2 + \|Q_1(t)x\|^2$

(c_5') $E(t,t_0)Q_k(t_0) = Q_k(t)E(t,t_0),\ k \in \{1,2\}$

for all $t \geq 0$, $(t,t_0) \in T$ and all $x \in X$.

Remark 3.1 If X is a real Hilbert space then statement (c_1') of Definition 3.1 implies (c_2'), (c_3') and (c_4').

The first main result of this paper is

Theorem 3.1 *The evolution operator $E : T \to \mathfrak{F}(X)$ is uniformly exponentially trichotomic if and only if there exist some constants $N > 1$, ν, $\nu_0 > 0$ and two projection families Q_1, $Q_2 : \mathbb{R}_+ \to \mathfrak{F}(X)$ compatible with E such that*

(t_1') $e^{\nu(t-s)} \|E(t,t_0)Q_1(t_0)x\| \leq N \|E(s,t_0)Q_1(t_0)x\|$

(t_2') $e^{\nu(t-s)} \|E(s,t_0)Q_2(t_0)x\| \leq N \|E(t,t_0)Q_2(t_0)x\|$

(t_3') $\|E(s,t_0)[I - Q_1(t_0)]x\| \leq Ne^{\nu_0(t-s)} \|E(t,t_0)[I - Q_1(t_0)]x\|$

(t_4') $\|E(t,t_0)[I - Q_2(t_0)]x\| \leq Ne^{\nu_0(t-s)} \|E(s,t_0)[I - Q_2(t_0)]x\|$

for all $t \geq s \geq t_0 \geq 0$ and all $x \in X$.

Proof. *Necessity.* If we denote $Q_1 = P_1, Q_2 = P_2$ then the conditions (c_1'), (c_2') of Definition 3.1 respectively (c_5') result from (c_2), (c_3) respectively (c_4) of Definition 2.3.

For (c_3') we observe that by (c_3) we have

$$\|[I - Q_1(t)]x\|^2 = \|[I - P_1(t)]x\|^2 = \|[P_0(t) + P_2(t)]x\|^2 =$$
$$= \|P_0(t)x\|^2 + \|P_2(t)x\|^2 = \|[I - P_1(t) - P_2(t)]x\|^2 + \|Q_2(t)x\|^2 =$$
$$= \|[I - Q_1(t) - Q_2(t)]x\|^2 + \|Q_2(t)x\|^2$$

for all $t \geq 0$ and all $x \in X$.

Similarly for (c_4') we have

$$\|[I - Q_2(t)]x\|^2 = \|[I - P_2(t)]x\|^2 = \|[P_0(t) + P_1(t)]x\|^2 =$$
$$= \|P_0(t)x\|^2 + \|P_1(t)x\|^2 = \|[I - Q_1(t) - Q_2(t)]x\|^2 + \|Q_1(t)x\|^2$$

for all $t \geq 0$ and all $x \in X$.

Thus, the projection families Q_1 and Q_2 are compatible with E.

The relations (t_1') respectively (t_2') result from (t_1) respectively (t_2) of Definition 2.4.

Using conditions (c_3), (c_4), (t_2) and (t_3) we obtain

$$\|E(s,t_0)[I - Q_1(t_0)]x\|^2 = \|E(s,t_0)[P_0(t_0) + P_2(t_0)]x\|^2 =$$
$$= \|P_0(s)E(s,t_0)x\|^2 + \|P_2(s)E(s,t_0)x\|^2 = \|E_0(s,t_0)x\|^2 + \|E_2(s,t_0)x\|^2 \leq$$
$$\leq N^2 e^{2\nu_0(t-s)} \|E_0(t,t_0)x\|^2 + N^2 e^{-2\nu(t-s)} \|E_2(t,t_0)x\|^2 \leq$$

$$\leq N^2 e^{2\nu_0(t-s)} \|E(t,t_0)[P_0(t_0)+P_2(t_0)]x\|^2 =$$
$$= N^2 e^{2\nu_0(t-s)} \|E(t,t_0)[I-Q_1(t_0)]x\|^2$$

for all $t \geq s \geq t_0 \geq 0$ and all $x \in X$, which proves (t_3').

The proof of (t_4') is similar.

Sufficiency. If we denote $P_0 = I - Q_1 - Q_2, P_1 = Q_1$ and $P_2 = Q_2$ then from (c_1')-(c_5') the statements (c_1)-(c_4) are obtained immediately.

Moreover, $(t_1') \Leftrightarrow (t_1)$ and $(t_2') \Leftrightarrow (t_2)$.

We observe that $P_0 = (I - Q_1)(I - Q_2)$ and by (t_3') we obtain

$$\|E_0(s,t_0)x\| = \|E(s,t_0)P_0(t_0)x\| = \|E(s,t_0)[I-Q_1(t_0)][I-Q_2(t_0)]x\| \leq$$
$$\leq Ne^{\nu_0(t-s)} \|E(t,t_0)[I-Q_1(t_0)][I-Q_2(t_0)]x\| = Ne^{\nu_0(t-s)} \|E_0(t,t_0)x\|$$

for all $t \geq s \geq t_0 \geq 0$ and all $x \in X$. Thus (t_3) is proved.

Similarly, from (t_4') and the remark that $P_0 = (I-Q_2)(I-P_1)$ inequality (t_4) is obtained.

It follows that E is uniformly exponentially trichotomic. □

In order to obtain a characterization of the uniform exponential trichotomy property using four commuting projection families, we introduce the following

Definition 3.2 Four projection families $R_1,\ R_2,\ R_3,\ R_4 : \mathbb{R}_+ \to \mathfrak{F}(X)$ are said to be *compatible* with the evolution operator $E : T \to \mathfrak{F}(X)$ if

(c_1'') $R_1(t) + R_3(t) = R_2(t) + R_4(t) = I$

(c_2'') $R_1(t)R_2(t) = R_2(t)R_1(t) = 0$ and $R_3(t)R_4(t) = R_4(t)R_3(t)$

(c_3'') $\|[R_1(t) + R_2(t)]x\|^2 = \|R_1(t)x\|^2 + \|R_2(t)x\|^2$

(c_4'') $\|[R_1(t) + R_3(t)R_4(t)]x\|^2 = \|R_1(t)x\|^2 + \|R_3(t)R_4(t)x\|^2$

(c_5'') $\|[R_2(t) + R_3(t)R_4(t)]x\|^2 = \|R_2(t)x\|^2 + \|R_3(t)R_4(t)x\|^2$

(c_6'') $E(t,t_0)R_k(t_0) = R_k(t)E(t,t_0),\ k \in \{1,2,3,4\}$

for all $t \geq 0, (t,t_0) \in T$ and for all $x \in X$.

Remark 3.2 In the particular case when X is a real Hilbert space the conditions (c_1'') and (c_2'') imply (c_3''), (c_4'') and (c_5'') in Definition 3.2.

The second main result of this paper is the following

Theorem 3.2 *The evolution operator $E : T \to \mathfrak{F}(X)$ is uniformly exponentially trichotomic if and only if there exist $N > 1$, ν, $\nu_0 > 0$ and four projection families $R_1,\ R_2,\ R_3,\ R_4 : \mathbb{R}_+ \to \mathfrak{F}(X)$ compatible with E such that*

(t_1'') $e^{\nu(t-s)} \|E(t,t_0)R_1(t_0)x\| \leq N \|E(s,t_0)R_1(t_0)x\|$
(t_2'') $e^{\nu(t-s)} \|E(s,t_0)R_2(t_0)x\| \leq N \|E(t,t_0)R_2(t_0)x\|$
(t_3'') $\|E(s,t_0)R_3(t_0)x\| \leq Ne^{\nu_0(t-s)} \|E(t,t_0)R_3(t_0)x\|$
(t_4'') $\|E(t,t_0)R_4(t_0)x\| \leq Ne^{\nu_0(t-s)} \|E(s,t_0)R_4(t_0)x\|$
for all $t \geq s \geq t_0 \geq 0$ *and all* $x \in X$.

Proof. *Necessity.* We suppose that E is uniformly exponentially trichotomic and we denote $R_1 = P_1, R_2 = P_2$, $R_3 = I - P_1$, $R_4 = I - P_2$ where P_0, P_1, P_2 are given by Definition 2.4. Then $R_3R_4 = R_4R_3 = P_0$ and the conditions (c_1'')-(c_6'') of Definition 3.2 result immediately from (c_1)-(c_4). Thus we obtain that the projection families R_1, R_2, R_3 and R_4 are compatible with E.

It is obvious that $(t_1'') \Leftrightarrow (t_1)$ and $(t_2'') \Leftrightarrow (t_2)$.

In order to prove (t_3'') we observe that $R_3 = I - P_1 = P_0 + P_2$ and similarly as in the proof of (t_3') from Theorem 3.1 we obtain the desired result. Similarly is also proved (t_4'').

Sufficiency. If we denote $P_1 = R_1$, $P_2 = R_2$ and $P_0 = R_3R_4$ then, from (c_1'') and (c_2''), we obtain

$$P_0 + P_1 + P_2 = (I - R_1)(I - R_2) + R_1 + R_2$$

and hence (c_1) holds.

By (c_1'') and (c_2'') it follows that

$$P_0P_1 = (I - R_1)(I - R_2)R_1 = (I - R_1 - R_2)R_1 = 0$$

$$P_0P_2 = (I - R_1)(I - R_2)R_2 = (I - R_1 - R_2)R_2 = 0$$

$$P_1P_2 = R_1R_2 = 0$$

and hence the condition (c_2) is verified.

From (c_1'')-(c_6'') it results immediately (c_3) and thus we obtain that the projection families P_0, P_1 and P_2 are compatible with E.

It is obvious that $(t_1'') \Leftrightarrow (t_1)$ and $(t_2'') \Leftrightarrow (t_2)$.

To prove (t_3) we observe that (t_3'') implies

$$\|E_0(s,t_0)x\| = \|E(s,t_0)P_0(t_0)x\| = \|E(s,t_0)R_3(t_0)R_4(t_0)x\| \leq$$

$$\leq Ne^{\nu_0(t-s)} \|E(t,t_0)R_3(t_0)R_4(t_0)x\| = Ne^{\nu_0(t-s)} \|E_0(t,t_0)x\|$$

for all $t \geq s \geq t_0 \geq 0$ and all $x \in X$.

Similarly, we prove that (t_4'') implies (t_4).

Finally, we conclude that E is uniformly exponentially trichotomic. □

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