# On subexponentiality of the Lévy measure of the diffusion inverse local time; with applications to penalizations

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**Abstract**

For a recurrent linear diffusion on $\mathbf{R}_+$ we study the asymptotics of the distribution of its local time at 0 as the time parameter tends to infinity. Under the assumption that the Lévy measure of the inverse local time is subexponential this distribution behaves asymtotically as a multiple of the Lévy measure. Using spectral representations we find the exact value of the multiple. For this we also need a result on the asymptotic behavior of the convolution of a subexponential distribution and an arbitrary distribution on $\mathbf{R}_+$. The exact knowledge of the asymptotic behavior of the distribution of the local time allows us to analyze the process derived via a penalization procedure with the local time. This result generalizes the penalizations obtained in Roynette, Vallois and Yor [22] for Bessel processes.

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# 1 Introduction

**1.** Let $X$ be a linear regular recurrent diffusion taking values in $\mathbf{R}_+$ with 0 an instantaneously reflecting boundary and $+\infty$ a natural boundary. Let $\mathbf{P}_x$ and $\mathbf{E}_x$ denote, respectively, the probability measure and the expectation associated with $X$ when started from $x \geq 0$. We assume that $X$ is defined in the canonical space $C$ of continuous functions $\omega : \mathbf{R}_+ \mapsto \mathbf{R}_+$. Let

$$\mathcal{C}_t := \sigma\{\omega(s) : s \leq t\}$$

denote the smallest $\sigma$-algebra making the co-ordinate mappings up to time $t$ measurable and take $\mathcal{C}$ to be the smallest $\sigma$-algebra including all $\sigma$-algebras $\mathcal{C}_t,\ t \geq 0$.

We let $m$ and $S$ denote the speed measure and the scale function of $X$, respectively. We normalize $S$ by $S(0) = 0$ and remark that $S(+\infty) = +\infty$ since we assume $X$ to be recurrent. It is also assumed that $m$ does not have atoms. Recall that $X$ has a jointly continuous transition density $p(t; x, y)$ with respect to $m$, i.e.,

$$\mathbf{P}_x(X_t \in A) = \int_A p(t; x, y)\, m(dy),$$

where $A$ is a Borel subset of $\mathbf{R}_+$. Moreover, $p$ is symmetric in $x$ and $y$, that is, $p(t; x, y) = p(t; y, x)$. The Green or the resolvent kernel of $X$ is defined for $\lambda > 0$ via

$$R_\lambda(x, y) := \int_0^\infty \mathrm{e}^{-\lambda t}\, p(t; x, y)\, dt, \tag{1.1}$$

Let $\{L_t^{(y)} : t \geq 0\}$ denote the local time of $X$ at $y$ normalized via

$$L_t^{(y)} = \lim_{\delta \downarrow 0} \frac{1}{m((y, y+\delta))} \int_0^t \mathbf{1}_{[y,y+\delta)}(X_s)\, ds. \tag{1.2}$$

For $y = 0$ we write simply $L_t$, and define for $\ell \geq 0$

$$\tau_\ell := \inf\{s : L_s > \ell\}, \tag{1.3}$$

i.e., $\tau := \{\tau_\ell : \ell \geq 0\}$ is the right continuous inverse of $\{L_t\}$. As is well known $\tau$ is an increasing Lévy process, in other words, a subordinator and its Lévy exponent is given by

$$\begin{aligned} \mathbf{E}_0\left(\exp(-\lambda \tau_\ell)\right) &= \exp\left(-\ell / R_\lambda(0, 0)\right) \\ &= \exp(-\ell \int_0^\infty \nu(dv)(1 - \mathrm{e}^{-\lambda v})), \end{aligned} \tag{1.4}$$

where $\nu$ is the Lévy measure of $\tau$. The assumption that the speed measure does not have an atom at 0 implies that $\tau$ does not have a drift.

**2.** We are interested in the asymptotic behavior of the distribution of $L_t$ as $t$ tends to infinity. The basic assumption under which this study is done is the subexponentiality of the Lévy measure of $\tau$ (see Section 4). The subexponentiality assumption is equivalent with the relation (cf. Proposition 4.1)

$$\mathbf{P}(\tau_\ell \geq t) \underset{t\to+\infty}{\sim} \ell\,\nu((t,+\infty)) \quad \forall\, \ell > 0.$$

Here and throughout the paper the notation

$$f(x) \underset{x\to a}{\sim} g(x),$$

where $f$ and $g$ are real valued functions and $a$ is allowed to take also "values" $+\infty$ or $-\infty$, means that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = 1.$$

Since $\tau$ is the inverse of $L$, it also holds (see Proposition 4.1)

$$\mathbf{P}_0(L_t \leq \ell) \underset{t\to+\infty}{\sim} \ell\,\nu((t,+\infty)).$$

To extend this for an arbitrary starting state $x > 0$, we first show that (see Proposition 4.2)

$$\mathbf{P}_x(H_0 > t) \underset{t\to+\infty}{\sim} S(x)\,\nu((t,+\infty)),$$

where $H_0 := \inf\{t : \, X_t = y\}$, and then (see Proposition 4.3)

$$\mathbf{P}_x(L_t \leq \ell) \underset{t\to+\infty}{\sim} (S(x) + \ell)\,\nu((t,+\infty)). \tag{1.5}$$

Our motivation for relation (1.5) arose from the desire to generalize the penalization result obtained for Bessel processes in Roynette, Vallois and Yor [22] (see also [19] and [21]). From our point of view, since many of the penalization results are derived for Brownian motion and Bessel processes, it is important to increase understanding of the assumptions needed to guarantee the validity of such results for more general diffusions. In particular, we prove that (see Theorem 5.2 and Example 5.3)

$$\lim_{t\to\infty} \frac{\mathbf{E}_0(h(L_t)\,|\,\mathcal{C}_u)}{\mathbf{E}_0(h(L_t))} = S(X_u)h(L_u) + 1 - H(L_u) =: M_u^h \qquad \text{a.s.}, \tag{1.6}$$

where $h$ is a probability density function on $\mathbf{R}_+$ (with some nice properties) and $H$ is the corresponding distribution function.

**3.** The paper is organised as follows. In the next section basic properties on subexponentiality are presented and a new result (Lemma 2.4) on the limiting behavior of the convolution of an subexponential and a more general distribution is derived. In Section 3 we study the spectral representations of the hitting time distributions and the Lévy measure. In Section 4 results on subexponentiality and the spectral representations are combined to yield relation (1.5). Hereby we also need a weak form of a Tauberian theorem given as Lemma 6.1 in Appendix. The application in penalizations is discussed in Section 5. To make the paper more readable we state and prove first the general theorem on penalizations. After this the penalization with local time is treated and (1.6) is proved. The paper is concluded by characterizing the law of the canonical process under the penalized measure induced by the martingale $M^h$. Using absolute continuity and the compensation formula for excursions we are able to shorten the proof when compared with the one in [22].

## 2 Subexponentiality

In this section we present some basic results on subexponential probability distributions. Later, in Section 4, it is assumed that the probability distribution induced by the tail of the Lévy measure of $\tau$ is subexponential. This assumption allows us to deduce the crucial limiting behavior of the first hitting time distribution (see Proposition 4.2).

**Definition 2.1.** *The probability distribution function $F$ on $(0, +\infty)$ such that*

$$F(0+) = 0, \quad F(x) < 1 \quad \forall x > 0, \quad \lim_{x\to\infty} F(x) = 1 \tag{2.1}$$

*is called subexponential if*

$$\lim_{x\to+\infty} \overline{F * F}(x) \,/\, \overline{F}(x) = 2 \tag{2.2}$$

*where $*$ denotes the convolution and $\overline{F}(x) := 1 - F(x)$ the complementary distribution function.*

For the following two lemmas and their proofs we refer Chistyakov [3] and Embrechts et al. [5].

**Lemma 2.2.** *If $F$ is a probability distribution function satisfying (2.1) and*

$$\overline{F}(x) \underset{x\to\infty}{\sim} x^{-\alpha}\, H(x)$$

*with $\alpha \geq 0$ and $H$ a slowly varying function then $F$ is subexponential.*

**Lemma 2.3.** *If $F$ is subexponential then*

*(i) uniformly on compact $y$-sets*

$$\lim_{x\to\infty} \overline{F}(x+y)/\overline{F}(x) = 1, \tag{2.3}$$

*(ii) for all $\varepsilon > 0$,*

$$\lim_{x\to+\infty} \mathrm{e}^{\varepsilon\, x}\overline{F}(x) = +\infty \tag{2.4}$$

The proof of the next lemma uses some ideas from Teugels [25] p. 1006.

**Lemma 2.4.** *Let $F$ and $G$ be two probability distributions on $\mathbf{R}_+$. Assume that*

*(1)* $F$ *is subexponential,*

*(2)* $\lim_{x\to\infty} \overline{G}(x)/\overline{F}(x) = c > 0$.

*Then*

$$\lim_{x\to\infty} \overline{F*G}(x)/\left(\overline{G}(x) + \overline{F}(x)\right) = 1. \tag{2.5}$$

*Proof.* Let $\varepsilon \in (0,1)$. By assumption *(2)* there exists $\delta = \delta(\varepsilon)$ such that for $x > \delta$

$$c\,(1-\varepsilon)\overline{F}(x) \leq \overline{G}(x) \leq c\,(1+\varepsilon)\overline{F}(x). \tag{2.6}$$

Observe that

$$\begin{aligned}\overline{F*G}(x) = 1 - F*G(x) &= 1 - F(x) + F(x) - \int_0^x G(x-y)\, dF(y) \\ &= \overline{F}(x) + \int_0^x \overline{G}(x-y)\, dF(y).\end{aligned}$$

We assume now, throughout the proof, that $x > \delta$ and write

$$\frac{\overline{F*G}(x)}{\overline{G}(x)+\overline{F}(x)} = \frac{\overline{G}(x)}{\overline{G}(x)+\overline{F}(x)}\left(I_1(x)+I_2(x)\right) + \frac{\overline{F}(x)}{\overline{G}(x)+\overline{F}(x)}, \tag{2.7}$$

where

$$I_1(x) := \int_0^{x-\delta} \frac{\overline{G}(x-y)}{\overline{G}(x)}\, dF(y)$$

and

$$I_2(x) := \int_{x-\delta}^{x} \frac{\overline{G}(x-y)}{\overline{G}(x)}\, dF(y).$$

Obviously, by assumption *(2)*, the claim (2.5) follows if we show that

$$\lim_{x\to\infty} I_1(x) = 1 \tag{2.8}$$

and

$$\lim_{x\to\infty} I_2(x) = 0. \tag{2.9}$$

*Proof of (2.9).* Since $\overline{G}(x-y) \le 1$ we have

$$\begin{aligned} I_2(x) \le \int_{x-\delta}^{x} \frac{dF(y)}{\overline{G}(x)} &= \frac{F(x)-F(x-\delta)}{\overline{G}(x)} \\ &= \frac{\overline{F}(x-\delta)-\overline{F}(x)}{\overline{G}(x)} \\ &= \frac{\overline{F}(x)}{\overline{G}(x)}\left(\frac{\overline{F}(x-\delta)}{\overline{F}(x)} - 1\right). \end{aligned}$$

Using now (2.3) and assumption *(2)* yields (2.9).
*Proof of (2.8).* Since

$$\overline{G}(x-y) \ge \overline{G}(x)$$

we have

$$I_1(x) = \int_0^{x-\delta} \frac{\overline{G}(x-y)}{\overline{G}(x)}\, dF(y) \ge F(x-\delta).$$

Consequently,

$$\liminf_{x\to+\infty} I_1(x) \ge 1. \tag{2.10}$$

To derive an upper estimate, notice first that

$$I_1(x) \le \frac{1+\varepsilon}{1-\varepsilon} \int_0^{x-\delta} \frac{\overline{F}(x-y)}{\overline{F}(x)} \, dF(y), \tag{2.11}$$

because, from (2.6),

$$x > \delta \quad \Rightarrow \quad \overline{G}(x) \ge c(1-\varepsilon)\overline{F}(x)$$

and

$$y \le x - \delta \quad \Rightarrow \quad x - y \ge \delta \quad \Rightarrow \quad \overline{G}(x-y) \le c(1+\varepsilon)\overline{F}(x-y).$$

Next we develop the integral term in (2.11) as follows

$$\begin{aligned}
&\int_0^{x-\delta} \overline{F}(x-y)\, dF(y) \\
&\qquad = \int_0^{x-\delta} (1 - F(x-y))\; dF(y) \\
&\qquad = F(x-\delta) - \int_0^{x-\delta} F(x-y)\, dF(y) \\
&\qquad = F(x) - \int_0^{x-\delta} F(x-y)\, dF(y) + F(x-\delta) - F(x) \\
&\qquad = F(x) - \int_0^{x-\delta} F(x-y)\, dF(y) - \int_{x-\delta}^{x} dF(y) \\
&\qquad = F(x) - \int_0^{x-\delta} F(x-y)\, dF(y) \\
&\qquad\qquad - \int_{x-\delta}^{x} F(x-y)\, dF(y) - \int_{x-\delta}^{x} \overline{F}(x-y)\, dF(y) \\
&\qquad \le F(x) - \int_0^{x} F(x-y)\, dF(y).
\end{aligned}$$

Hence,

$$\int_0^{x-\delta} \overline{F}(x-y)\, dF(y) \le F(x) - F * F(x) = \overline{F * F}(x) - \overline{F}(x).$$

Consequently, from (2.11),

$$I_1(x) \le \left(\frac{1+\varepsilon}{1-\varepsilon}\right) \left(\frac{\overline{F * F}(x) - \overline{F}(x)}{\overline{F}(x)}\right),$$

and using (2.2) and letting $\varepsilon \to 0$ we obtain

$$\limsup_{x\to+\infty} I_1(x) \le 1$$

which together with (2.10) proves (2.8) completing the proof of Lemma 2.4 □

# 3 Spectral representations

Spectral representations play a crucial role in our study of asympotic properties of the hitting time distributions. In this section we recall basic properties of these representations and derive some useful estimates. For references on spectral theory of strings, we list [7], [9], [4], [10], [14], [12] , [13], [11], and [15].

Besides the diffusion $X$ itself, it is important to study $X$ when killed at the first hitting time of 0, denoted $\widehat{X} = \{\widehat{X}_t : t \ge 0\}$, i.e., the diffusion with the sample paths

$$\widehat{X}_t := \begin{cases} X_t, & t < H_0, \\ \partial, & t \ge H_0, \end{cases} \tag{3.1}$$

where $H_0 := \inf\{t :\ X_t = y\}$, and $\partial$ is a point isolated from $\mathbf{R}_+$ (a "cemetary" point). Then $\{\widehat{X}_t : t \ge 0\}$ is a diffusion with the same scale and speed as $X$. Let $\hat{p}$ denote the transition density of $\widehat{X}$ with respect to $m$ :

$$\mathbf{P}_x(\widehat{X}_t \in dy) = \mathbf{P}_x(X_t \in dy; t < H_0) = \hat{p}(t; x, y)\, m(dy). \tag{3.2}$$

Recall that the density of the $\mathbf{P}_x$-distribution of $H_0$ exists and is given by

$$f_{x0}(t) := \mathbf{P}_x(H_0 \in dt)/dt = \lim_{y\downarrow 0} \frac{\hat{p}(t; x, y)}{S(y)}. \tag{3.3}$$

Moreover, the Lévy measure $\nu$ of the inverse local time $\tau$, see (1.2) and (1.3), is absolutely continuous with respect to the Lebesgue measure, and the density of $\nu$ satisfies

$$\dot{\nu}(v) := \nu(dv)/dv = \lim_{x\downarrow 0} \frac{f_{x0}(v)}{S(x)} \tag{3.4}$$

We define now the basic eigenfunctions $A(x;\gamma)$ and $C(x;\gamma)$ associated with $X$ and $\widehat{X}$, respectively, via the integral equations (recall that $S$ is continuous and $m$ has no atoms)

$$A(x;\gamma) = 1 - \gamma \int_0^x dS(y) \int_0^y m(dz)\, A(z;\gamma),$$

$$C(x;\gamma) = S(x) - \gamma \int_0^x dS(y) \int_0^y m(dz)\, C(z;\gamma), \tag{3.5}$$

and the initial values

$$A(0;\gamma) = 1, \quad A'(0;\gamma) := \lim_{x\downarrow 0} \frac{A(x;\gamma) - 1}{S(x)} = 0, \tag{3.6}$$

$$C(0;\gamma) = 0, \quad C'(0;\gamma) := \lim_{x\downarrow 0} \frac{C(x;\gamma)}{S(x)} = 1. \tag{3.7}$$

Let $\{A_n\}$and $\{C_n\}$ be two families of functions defined by

$$A_0(x) = 1, \qquad A_{n+1}(x) = \int_0^x dS(y) \int_0^y m(dz)\, A_n(z) \tag{3.8}$$

and

$$C_0(x) = S(x), \qquad C_{n+1}(x) = \int_0^x dS(y) \int_0^y m(dz)\, C_n(z), \tag{3.9}$$

respectively. Then the functions $A(x;\gamma)$ and $C(x;\gamma)$ are explicitly given by

$$A(x;\gamma) = \sum_{n=0}^{\infty} (-\gamma)^n\, A_n(x). \tag{3.10}$$

and

$$C(x;\gamma) = \sum_{n=0}^{\infty} (-\gamma)^n\, C_n(x), \tag{3.11}$$

respectively (see Kac and Krein [7] p. 29). In the next lemma we give an estimate which shows that the series for $C$ converges rapidly for all values on $\gamma$ and $x \geq 0$. A similar estimate for $A$ can be found in Dym and McKean [4] p. 162.

**Lemma 3.1.** *The functions* $x \mapsto C_n(x)$, $x \geq 0$, $n = 0, 1, 2, \dots,$ *are positive, increasing and satisfy*

$$C_n(x) \leq \frac{1}{n!}\, S(x) \left( \int_0^x M(y)\, dS(y) \right)^n \tag{3.12}$$

*where* $M(z) = m(0, z)$.

*Proof.* The fact that $C_n$ are positive and increasing is immediate from (3.9). Clearly (3.12) holds for $n = 0$. Hence, consider

$$\begin{aligned} C_{n+1}(x) &= \int_0^x dS(y) \int_0^y m(du)\, C_n(u) \\ &\leq \int_0^x dS(y) \int_0^y m(du)\, \frac{1}{n!}\, S(u) \left( \int_0^u M(z)\, dS(z) \right)^n \\ &\leq \frac{1}{n!}\, S(x) \int_0^x dS(y) \int_0^y m(du) \left( \int_0^u M(z)\, dS(z) \right)^n \\ &\leq \frac{1}{n!}\, S(x) \int_0^x dS(y) \left( \int_0^y M(z)\, dS(z) \right)^n M(y) \\ &= \frac{1}{(n+1)!}\, S(x) \left( \int_0^x M(y) dS(y) \right)^{n+1}, \end{aligned}$$

where we have used the facts that $x \mapsto S(x)$ is increasing and $x \mapsto M(x)$ is positive. □

**Lemma 3.2.** *The function* $x \mapsto C(x; \gamma)$ *satisfies the inequality*

$$|C(x; \gamma)| \leq S(x)\, \exp \left( |\gamma| \int_0^x M(z) dS(z) \right). \tag{3.13}$$

*Proof.* This follows readily from (3.11) and (3.12). □

$>$From Krein's theory of strings it is known (see [4] p.176, and [7, 14, 13]) that there exists a $\sigma$-finite measure denoted $\Delta$, called the principal spectral measure of $X$, with the property

$$\int_0^\infty \frac{\Delta(dz)}{z+1} < \infty \tag{3.14}$$

such that the transition density of $X$ can be represented as

$$p(t;x,y) = \int_0^\infty \mathrm{e}^{-\gamma t}\, A(x;\gamma)\, A(y;\gamma)\, \Delta(d\gamma). \tag{3.15}$$

We remark that from the assumption that $m$ does not have an atom at 0 it follows (see [4] p.192) that $\Delta([0,\infty)) = \infty$.

Analogously, for the killed process $\widehat{X}$ there exists (see [12], [15]) a $\sigma$-finite measure, denoted $\widehat{\Delta}$ and called the principal spectral measure of $\widehat{X}$, such that

$$\int_0^\infty \frac{\widehat{\Delta}(dz)}{z(z+1)} < \infty, \tag{3.16}$$

and

$$\int_0^\infty \frac{\widehat{\Delta}(dz)}{z} = \infty. \tag{3.17}$$

The transition density of $\widehat{X}$ can be represented as

$$\hat{p}(t;x,y) = \int_0^\infty \mathrm{e}^{-\gamma t}\, C(x;\gamma)\, C(y;\gamma)\, \widehat{\Delta}(d\gamma). \tag{3.18}$$

The result of the next proposition can be found also in [15]. Since the proof in [15] is not complete in all details we found it worthwhile to give here a new proof.

**Proposition 3.3.** *(i) The density of the* $\mathbf{P}_x$*-distribution of the first hitting time* $H_0$ *has the spectral representation*

$$f_{x0}(t) = \int_0^\infty \mathrm{e}^{-\gamma t}\, C(x;\gamma)\, \widehat{\Delta}(d\gamma). \tag{3.19}$$

*(ii) The density of the Lévy measure of the inverse local time at 0 has the spectral representation*

$$\dot{\nu}(t) = \int_0^\infty \mathrm{e}^{-\gamma t}\, \widehat{\Delta}(d\gamma). \tag{3.20}$$

*Proof.* (i) Combining (3.3) and (3.18) yields

$$\begin{aligned} f_{x0}(t) &= \lim_{y\downarrow 0} \frac{\hat{p}(t;x,y)}{S(y)}. \\ &= \lim_{y\downarrow 0} \int_0^\infty \mathrm{e}^{-\gamma t}\, C(x;\gamma)\, \frac{C(y;\gamma)}{S(y)}\, \widehat{\Delta}(d\gamma). \end{aligned}$$

We show that the limit can be taken inside the integral by the Lebesgue dominated convergence theorem. Let $t > 0$ be fixed an choose $\varepsilon$ such that

$$t - \int_0^\varepsilon M(z) dS(z) \geq t/2.$$

Then, from Lemma 3.2, for $\gamma > 0$ and $0 < y < \varepsilon$ we have

$$\mathrm{e}^{-\gamma t} \frac{|C(y;\gamma)|}{S(y)} \leq \exp\left(-\gamma\left(t - \int_0^y M(z) dS(z)\right)\right) \leq \mathrm{e}^{-\gamma t/2}$$

Consequently, it remains to show that

$$\int_0^\infty \mathrm{e}^{-\gamma t/2} \, |C(x;\gamma)| \, \widehat{\Delta}(d\gamma) < \infty. \tag{3.21}$$

By the Cauchy-Schwartz inequality

$$\begin{aligned}
&\left(\int_0^\infty \mathrm{e}^{-\gamma t/2} \, |C(x;\gamma)| \, \widehat{\Delta}(d\gamma)\right)^2 \\
&\qquad \leq \int_0^\infty \mathrm{e}^{-\gamma t/2} \, (C(x;\gamma))^2 \, \widehat{\Delta}(d\gamma) \int_0^\infty \mathrm{e}^{-\gamma t/2} \widehat{\Delta}(d\gamma) \\
&\qquad = \hat{p}(t/2; x, x) \int_0^\infty \mathrm{e}^{-\gamma t/2} \widehat{\Delta}(d\gamma).
\end{aligned}$$

Clearly, $\hat{p}(t/2; x, x) < \infty$ and, by (3.16), $\int_0^\infty \mathrm{e}^{-\gamma t/2} \widehat{\Delta}(d\gamma) < \infty$. These estimates allow us to use the Lebesgue dominated convergence theorem and since (cf. (3.7))

$$\lim_{y \to 0} C(y;\gamma)/S(y) = C'(0;\gamma) = 1$$

the proof of (i) is complete. Representation (3.20) can be proved similarly using formula (3.4), (3.19), (3.7) and the estimates derived above. We leave the details to the reader. □

**Remark 3.4.** *Consider*

$$\begin{aligned}
\int_0^\infty (1 \wedge t) \, \dot{\nu}(t) \, dt &= \int_0^\infty dt \, (1 \wedge t) \int_0^\infty \widehat{\Delta}(d\gamma) \, \mathrm{e}^{-\gamma t} \\
&= \int_0^\infty \widehat{\Delta}(d\gamma) \int_0^\infty dt \, (1 \wedge t) \, \mathrm{e}^{-\gamma t}.
\end{aligned}$$

*A straightforward integration yields*

$$\int_0^\infty (1\wedge t)\,\mathrm{e}^{-\gamma t}\,dt = \frac{1}{\gamma^2}\left(1-\mathrm{e}^{-\gamma}\right),$$

*and, consequently,* (3.16) *is equivalent with (cf. [12])*

$$\int_0^\infty (1\wedge t)\,\dot{\nu}(t)\,dt < \infty,$$

*which is the crucial property of the Lévy measure of a subordinator. For (3.17), see [7] p. 82. and [15].*

**Example 3.5.** *Let $R = \{R_t : t \geq 0\}$ and $\widehat{R} = \{\widehat{R}_t : t \geq 0\}$ be Bessel processes of dimension $0 < \delta < 2$ reflected at 0 and killed at 0, respectively. We compute explicit spectral representations associated with $R$ and $\widehat{R}$.*

*From, e.g., [2] p. 133 the following information concerning $R$ and $\widehat{R}$ can be found:*

*Speed measure*

$$m(dx) = 2\,x^{1-2\alpha}\,dx \qquad \alpha := (2-\delta)/2. \tag{3.22}$$

*Scale function*

$$S(x) = \frac{1}{2\alpha}\,x^{2\alpha} \tag{3.23}$$

*Transition density of $R$ (w.r.t. $m$)*

$$p(t;x,y) = \frac{1}{2t}(xy)^\alpha \exp\left(-\frac{x^2+y^2}{2t}\right) I_{-\alpha}\left(\frac{xy}{t}\right), \quad x,y>0. \tag{3.24}$$

*Transition density of $\widehat{R}$ (w.r.t. $m$)*

$$\hat{p}(t;x,y) = \frac{1}{2t}(xy)^\alpha \exp\left(-\frac{x^2+y^2}{2t}\right) I_{\alpha}\left(\frac{xy}{t}\right), \quad x,y>0. \tag{3.25}$$

*To find the Krein measure $\Delta$ associated with $R$ we exploit formulas (3.15) and (3.24) with $x = y = 0$ and use*

$$I_\nu(z) \sim \frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^\nu, \quad z\to 0$$

*to obtain*

$$p(t;0,0) = \lim_{x,y\to 0} p(t;x,y) = \frac{t^{-(1-\alpha)}}{2^{1-\alpha}\,\Gamma(1-\alpha)} = \int_0^\infty \mathrm{e}^{-\gamma t}\,\Delta(d\gamma).$$

*Inverting the Laplace transform yields*

$$\Delta(d\gamma) = \frac{\gamma^{-\alpha}\,d\gamma}{2^{1-\alpha}\,(\Gamma(1-\alpha))^2}. \tag{3.26}$$

*We apply formula (3.10), (3.22), and (3.23) to find the function* $A(x;\gamma)$*, and, hence, compute first directly via (3.8)*

$$A_n(x) = \frac{\Gamma(1-\alpha)\,x^{2n}}{2^n\,\Gamma(n+1)\,\Gamma(n+1-\alpha)}, \qquad n = 0,1,2,\ldots.$$

*Consequently, after some manipulations, we have*

$$A(x;\gamma) = \Gamma(1-\alpha)\,2^{-\alpha}\,\left(x\sqrt{2\gamma}\right)^{\alpha}\,J_{-\alpha}\left(x\sqrt{2\gamma}\right),$$

*where* $J$ *denotes the usual Bessel function of the first kind, i.e.,*

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{\nu+2n}}{\Gamma(n+1)\,\Gamma(\nu+n+1)},$$

*and, finally, putting pieces together into (3.15) yields*

$$p(t;x,y) = \frac{1}{2}\int_0^\infty \mathrm{e}^{-\gamma t}\,(xy)^{\alpha}\,J_{-\alpha}\left(x\sqrt{2\gamma}\right)\,J_{-\alpha}\left(y\sqrt{2\gamma}\right)\,d\gamma. \tag{3.27}$$

*Next we compute the Krein measure* $\widehat{\Delta}$ *associated with* $\widehat{R}$*. For this, we deduce from (3.3), (3.4), (3.23), and (3.25)*

$$\dot{\nu}(t) = \lim_{x,y\to 0} \frac{\hat{p}(t;x,y)}{S(x)S(y)} = \frac{2^{1-\alpha}\,\alpha\,t^{-(1+\alpha)}}{\Gamma(\alpha)} = \int_0^\infty \mathrm{e}^{-\gamma t}\,\widehat{\Delta}(d\gamma), \tag{3.28}$$

*and, consequently, inverting the Laplace transform gives*

$$\widehat{\Delta}(d\gamma) = \frac{2^{1-\alpha}\,\gamma^{\alpha}}{(\Gamma(\alpha))^2}\,d\gamma \tag{3.29}$$

*Similarly as above, we apply formula (3.11) to find the function $C(x;\gamma)$, and, hence, compute first directly via (3.9)*

$$C_n(x) = \frac{\Gamma(\alpha)\, x^{2\alpha+2n}}{2^{n+1}\,\Gamma(n+1)\,\Gamma(n+1+\alpha)}, \quad n = 0,1,2,\dots.$$

*Consequently, after some manipulations,*

$$C(x;\gamma) = \Gamma(\alpha)\, 2^{(\alpha-2)/2}\, \gamma^{-\alpha/2}\, x^{\alpha}\, J_\alpha\left(x\sqrt{2\gamma}\right).$$

*and*

$$\hat{p}(t;x,y) = \frac{1}{2}\int_0^\infty \mathrm{e}^{-\gamma t}\,(xy)^\alpha\, J_\alpha\left(x\sqrt{2\gamma}\right)\, J_\alpha\left(y\sqrt{2\gamma}\right)\, d\gamma. \qquad (3.30)$$

*See also Karlin and Taylor [8] p. 338.*

**Example 3.6.** *Taking above $\alpha = 1/2$ yields formulas for Brownian motion. Recall*

$$J_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\,\sin z, \quad \text{and} \quad J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}}\,\cos z.$$

*Consequently, from (3.27)*

$$\begin{aligned} p(t;x,y) &= \frac{1}{\pi}\int_0^\infty \mathrm{e}^{-\gamma t}\cos(x\sqrt{2\gamma})\cos(y\sqrt{2\gamma})\,\frac{d\gamma}{\sqrt{2\gamma}} \qquad (3.31)\\ &= \frac{1}{2\sqrt{2\pi t}}\left(\mathrm{e}^{-(x-y)^2/(2t)} + \mathrm{e}^{-(x+y)^2/(2t)}\right), \end{aligned}$$

*and from (3.30)*

$$\begin{aligned} \hat{p}(t;x,y) &= \frac{1}{\pi}\int_0^\infty \mathrm{e}^{-\gamma t}\,\frac{\sin(x\sqrt{2\gamma})}{\sqrt{2\gamma}}\,\frac{\sin(y\sqrt{2\gamma})}{\sqrt{2\gamma}}\,\sqrt{2\gamma}\,d\gamma\\ &= \frac{1}{2\sqrt{2\pi t}}\left(\mathrm{e}^{-(x-y)^2/(2t)} - \mathrm{e}^{-(x+y)^2/(2t)}\right). \end{aligned}$$

*Moreover,*

$$f_{x0}(t) = \frac{1}{\pi}\int_0^\infty \mathrm{e}^{-\gamma t}\,\sin(x\sqrt{2\gamma})\,d\gamma = \frac{x}{t^{3/2}\sqrt{2\pi}}\,\mathrm{e}^{-x^2/(2t)},$$

*and*

$$\dot{\nu}(t) = \frac{1}{\pi}\int_0^\infty \mathrm{e}^{-\gamma t}\,\sqrt{2\gamma}\,d\gamma = \frac{1}{t^{3/2}\sqrt{2\pi}}. \qquad (3.32)$$

*>From (3.31) we obtain $\Delta(d\gamma) = d\gamma/(\pi\sqrt{2\gamma})$, and from (3.32) $\widehat{\Delta}(d\gamma) = \sqrt{2\gamma}\,d\gamma/\pi$. See also Karlin and Taylor [8] p. 337 and 393, and [2] p. 120.*

**Proposition 3.7.** *(i) The complementary $\mathbf{P}_x$-distribution function of $H_0$ has the spectral representation*

$$\mathbf{P}_x(H_0 > t) = \int_0^\infty \frac{1}{\gamma}\, \mathrm{e}^{-\gamma t}\, C(x;\gamma)\, \widehat{\Delta}(d\gamma). \tag{3.33}$$

*(ii) The Lévy measure has the spectral representation*

$$\nu((t,\infty)) = \int_t^\infty \dot{\nu}(s)\, ds = \int_0^\infty \frac{1}{\gamma}\, \mathrm{e}^{-\gamma t}\, \widehat{\Delta}(d\gamma). \tag{3.34}$$

*Proof.* Formulas (3.33) and (3.34) follow from (3.19) and (3.20), respectively, using Fubini's theorem. To obtain (3.34) is straightforward but for (3.33) the applicability of Fubini's theorem needs to be justified. Indeed, from (3.19) we have informally

$$\begin{aligned}\mathbf{P}_x(H_0 > t) &= \int_t^\infty f_{x0}(s)\, ds = \int_t^\infty ds \int_0^\infty \widehat{\Delta}(d\gamma)\, \mathrm{e}^{-\gamma s}\, C(x;\gamma)\\ &= \int_0^\infty \widehat{\Delta}(d\gamma) \int_t^\infty ds\, \mathrm{e}^{-\gamma s}\, C(x;\gamma)\end{aligned}$$

leading to (3.33). To make this rigorous, we verify that for all $x > 0$

$$\int_0^\infty \frac{1}{\gamma}\, \mathrm{e}^{-\gamma t}\, |C(x;\gamma)|\, \widehat{\Delta}(d\gamma) < \infty.$$

Consider first for $\varepsilon > 0$

$$K_1 := \int_0^\varepsilon \frac{1}{\gamma}\, \mathrm{e}^{-\gamma t}\, |C(x;\gamma)|\, \widehat{\Delta}(d\gamma).$$

By the basic estimate (3.13) for $0 < \gamma < \varepsilon$

$$|C(x;\gamma)| \leq S(x)\, \exp\left(\varepsilon \int_0^x M(z) dS(z)\right).$$

and, consequently,

$$K_1 \leq S(x)\, \exp\left(\varepsilon \int_0^x M(z) dS(z)\right) \int_0^\infty \frac{1}{\gamma}\, \mathrm{e}^{-\gamma t}\, \widehat{\Delta}(d\gamma) < \infty$$

by (3.16). Next, let

$$K_2 := \int_\varepsilon^\infty \frac{1}{\gamma}\, \mathrm{e}^{-\gamma t}\, |C(x;\gamma)|\, \widehat{\Delta}(d\gamma).$$

By the Cauchy-Schwartz inequality

$$K_2^2 \leq \int_\varepsilon^\infty \gamma^{-2}\, \mathrm{e}^{-\gamma t}\, \widehat{\Delta}(d\gamma) \int_\varepsilon^\infty \mathrm{e}^{-\gamma t}\, (C(x;\gamma))^2\, \widehat{\Delta}(d\gamma).$$

The first term on the right hand side is finite by (3.16). For the second term we have

$$\int_\varepsilon^\infty \mathrm{e}^{-\gamma t}\, (C(x;\gamma))^2\, \widehat{\Delta}(d\gamma) \leq \int_0^\infty \mathrm{e}^{-\gamma t}\, (C(x;\gamma))^2\, \widehat{\Delta}(d\gamma).$$
$$\leq \hat{p}(t;x,x) < \infty.$$

The proof of (3.33) is now complete. □

# 4 Asymtotic behavior of the distribution of $L_t$ as $t \to +\infty$

We make the following assumption concerning the Lévy measure of the inverse local time process $\{\tau_\ell : \ell \geq 0\}$ valid throughout the rest of the paper (if nothing else is stated)

(A) *The probability distribution function*

$$x \mapsto \frac{\nu(1,x]}{\nu(1,+\infty)}, \quad x > 1,$$

*is assumed to be subexponential.*

It is known, see Sato [24] p. 164, that Assumption (A) is equivalent with

$$\mathbf{P}(\tau_\ell \geq t) \underset{t\to+\infty}{\sim} \ell\, \nu((t,+\infty)) \quad \forall\, \ell > 0, \tag{4.1}$$

and also with

*The law of $\tau_\ell$ is subexponential for every $\ell > 0$.* (4.2)

**Proposition 4.1.** *For any fixed $\ell > 0$, it holds*

$$\mathbf{P}_0(L_t \leq \ell) \underset{t\to+\infty}{\sim} \ell\, \nu((t,+\infty)). \tag{4.3}$$

*Proof.* The claim follows immediately from (4.1) since

$$\mathbf{P}_0(L_t \le \ell) = \mathbf{P}(\tau_\ell \ge t).$$

□

Our goal is to study the asymptotic behavior of $L_t$ under $\mathbf{P}_x$. For this, we analyze first the distribution of the hitting time $H_0$. The proof of the next proposition is based on Lemma 6.1 stated and proved in Section 6 below.

**Proposition 4.2.** *For any $x > 0$, it holds*

$$\mathbf{P}_x(H_0 > t) \underset{t\to+\infty}{\sim} S(x)\,\nu((t, +\infty)). \tag{4.4}$$

*Proof.* Recall from (3.33) and (3.34) in Proposition 3.7 the spectral representations

$$\mathbf{P}_x(H_0 > t) = \int_0^\infty \frac{1}{\gamma}\,\mathrm{e}^{-\gamma t}\; C(x;\gamma)\,\widehat{\Delta}(d\gamma) \tag{4.5}$$

and

$$\nu((t, +\infty)) = \int_0^\infty \frac{1}{\gamma}\,\mathrm{e}^{-\gamma t}\,\widehat{\Delta}(d\gamma). \tag{4.6}$$

We apply Lemma 6.1 with $\mu(d\gamma) = \widehat{\Delta}(d\gamma)/\gamma$, $g_1(\gamma) = C(x;\gamma)$ and $g_2(\gamma) = S(x)$. Then, the mapping $t \mapsto \mathbf{P}_x(H_0 > t)$ has the rôle of $f_1$ and $t \mapsto S(x)\,\nu((t, +\infty))$ the rôle of $f_2$. Condition (6.1) takes the form

$$\lim_{t\to\infty} S(x)\,\nu((t, +\infty))\,\mathrm{e}^{bt} = 0$$

and this holds by Assumption (A) and (2.4). Moreover, condition (6.2) means now

$$\lim_{\gamma\to 0} C(x;\gamma)/S(x) = 1$$

and this is true since using estimate (3.12) in (3.11) we obtain

$$\left|\frac{C(x;\gamma)}{S(x)} - 1\right| \le \alpha\,|\gamma|\mathrm{e}^{\beta\,|\gamma|}$$

with some $\alpha$ and $\beta$ depending only on $x$. Consequently, (6.3) in Lemma 6.1 holds and, hence, the proof of the proposition is complete. □

The main result of this section is as follows.

**Proposition 4.3.** *For any $x > 0$ and $\ell > 0$, it holds*

$$\mathbf{P}_x(L_t \leq \ell) \underset{t\to+\infty}{\sim} (S(x) + \ell)\, \nu((t, +\infty)). \tag{4.7}$$

*Proof.* Since $L_t$ increases only when $X$ is at 0 we may write

$$\begin{aligned}\mathbf{P}_x(L_t \leq \ell) &= \mathbf{P}_x(H_0 > t) + \mathbf{P}_x(H_0 < t\,,\, L_t \leq \ell)\\ &= \mathbf{P}_x(H_0 > t) + \mathbf{P}_x(H_0 < t\,,\, L_{t-H_0} \circ \theta_{H_0} \leq \ell)\\ &= \mathbf{P}_x(H_0 > t) + \mathbf{P}_x(H_0 < t\,,\, t - H_0 \leq \hat{\tau}_\ell),\end{aligned}$$

where $\theta.$ denotes the usual shift operator and $\hat{\tau}_\ell$ is a subordinator starting from 0, independent of $H_0$ and identical in law with $\tau_\ell$ (under $\mathbf{P}_0$), by the strong Markov property. Consequently,

$$\begin{aligned}\mathbf{P}_x(L_t \leq \ell) &= \mathbf{P}_x(H_0 > t) + \mathbf{P}_x(\hat{\tau}_\ell + H_0 \geq t) - \mathbf{P}_x(\hat{\tau}_\ell + H_0 \geq t\,,\, H_0 > t)\\ &= \mathbf{P}_x(\hat{\tau}_\ell + H_0 \geq t).\end{aligned}$$

We use Lemma 2.4 and take therein $F$ to be the $P_x$-distribution $\hat{\tau}_\ell$ (which is the same as the $P_0$-distribution $\tau_\ell$) and $G$ the $P_x$-distribution of $H_0$. Then, by (4.2), $F$ is subexponential and from (4.3) and (4.4) we have

$$\lim_{t\to\infty} \frac{\mathbf{P}_x(H_0 > t)}{\mathbf{P}_x(\hat{\tau}_\ell > t)} = \frac{S(x)}{\ell} > 0.$$

Consequently, by Lemma 2.4,

$$\lim_{t\to\infty} \frac{\mathbf{P}_x(\hat{\tau}_\ell + H_0 > t)}{\mathbf{P}_x(\hat{\tau}_\ell > t) + \mathbf{P}_x(H_0 > t)} = 1,$$

in other words,

$$\begin{aligned}\mathbf{P}_x(L_t \leq \ell) &\underset{t\to\infty}{\sim} \mathbf{P}_x(H_0 > t) + \mathbf{P}_x(\hat{\tau}_\ell > t)\\ &\underset{t\to\infty}{\sim} S(x)\, \nu((t, \infty)) + \ell\, \nu((t, \infty)),\end{aligned}$$

as claimed. □

**Example 4.4.** For a Bessel process of dimension $d \in (0, 2)$ reflected at 0 we have from (3.28) in Example 3.5

$$\nu((t, +\infty)) = \frac{2^{1-\alpha}}{\Gamma(\alpha)}\, t^{-\alpha},$$

and Assumption (A) holds by Lemma 2.2. Consequently,

$$\mathbf{P}_x(L_t < \ell) \underset{t\to\infty}{\sim} (S(x) + \ell)\,\nu((t, +\infty)).$$

where the scale function is as in Example 3.5. Taking here $\alpha = 1/2$ gives formulae for reflecting Brownian motion. We remark that our normalization of the local time (see (1.2)) is different from the one used in Roynette et al. [22] Section 2. In our case, from (1.4) and (3.28) it follows (cf. also [2] p. 133 where the resolvent kernel is explicitly given) that

$$\mathbf{E}_0\left(\exp(-\lambda\tau_\ell)\right) = \exp\left(-\ell\,\frac{\Gamma(1-\alpha)}{\Gamma(\alpha)}\,2^{1-\alpha}\,\lambda^\alpha\right). \tag{4.8}$$

Comparing now formula (2.11) in [22] with (4.8) it is seen that

$$\widehat{L}_t = 2\alpha\, L_t$$

where $\widehat{L}$ denotes the local time used in [22].

# 5 Penalization of the diffusion with its local time

## 5.1 General theorem of penalization

Recall that $(C, \mathcal{C}, \{\mathcal{C}_t\})$ denotes the canonical space of continuous functions, and let $\mathbf{P}$ be a probability measure defined therein. In the next theorem we present the general penalization result which we then specialize to the penalization with local time.

**Theorem 5.1.** *Let $\{F_t : t \geq 0\}$ be a stochastic process (so called weight process) satisfying*

$$0 < \mathbf{E}(F_t) < \infty \quad \forall\, t > 0.$$

*Suppose that for any $u \geq 0$*

$$\lim_{t\to\infty} \frac{\mathbf{E}(F_t\,|\,\mathcal{C}_u)}{\mathbf{E}(F_t)} =: M_u \tag{5.1}$$

*exists a.s. and*

$$\mathbf{E}(M_u) = 1. \tag{5.2}$$

*Then*

*1)* $M = \{M_u : u \geq 0\}$ *is a non-negative martingale with* $M_0 = 1$,

*2) for any* $u \geq 0$ *and* $\Lambda \in \mathcal{C}_u$

$$\lim_{t\to\infty} \frac{\mathbf{E}(\mathbf{1}_\Lambda\, F_t)}{\mathbf{E}(F_t)} = \mathbf{E}(\mathbf{1}_\Lambda\, M_u) =: \mathbf{Q}^{(u)}(\Lambda), \tag{5.3}$$

*3) there exits a probability measure* $\mathbf{Q}$ *on* $(C, \mathcal{C})$ *such that for any* $u > 0$

$$\mathbf{Q}(\Lambda) = \mathbf{Q}^{(u)}(\Lambda) \qquad \forall \Lambda \in \mathcal{C}_u.$$

*Proof.* We have (cf. Roynette et al. [20])

$$\frac{\mathbf{E}(\mathbf{1}_{\Lambda_u}\, F_t)}{\mathbf{E}(F_t)} = \mathbf{E}\left(\mathbf{1}_{\Lambda_u}\, \frac{\mathbf{E}(F_t\,|\,\mathcal{C}_u)}{\mathbf{E}(F_t)}\right),$$

and by (5.1) and (5.2) the family of random variables

$$\left\{\frac{\mathbf{E}(F_t\,|\,\mathcal{C}_u)}{\mathbf{E}(F_t)} \,:\, t \geq 0\right\}$$

is uniformly integrable by Sheffe's lemma (see, e.g., Meyer [17]), and, hence, (5.3) holds in $\mathbf{L}^1(\Omega)$. To verify the martingale property of $M$ notice that if $u < v$ then $\Lambda_u \in \mathcal{C}_v$ and by (5.3) we have also

$$\lim_{t\to\infty} \frac{\mathbf{E}(\mathbf{1}_{\Lambda_u}\, F_t)}{\mathbf{E}(F_t)} = \mathbf{E}(\mathbf{1}_{\Lambda_u}\, M_v).$$

Consequently,

$$\mathbf{E}(\mathbf{1}_{\Lambda_u}\, M_v) = \mathbf{E}(\mathbf{1}_{\Lambda_u}\, M_u),$$

i.e., $M$ is a martingale. Since the family $\{\mathbf{Q}^{(u)} : u \geq 0\}$ of probability measures is consistent, claim 3) follows from Kolmogorov's existence theorem (see, e.g., Billingsley [1] p. 228-230). □

## 5.2 Penalization with local time

We are interested in analyzing the penalizations of diffusion $X$ with the weight process given by

$$F_t := h(L_t), \quad t \geq 0 \tag{5.4}$$

with a suitable function $h$. In particular, if $h = \mathbf{1}_{[0,\ell)}$ for some fixed $\ell > 0$ then $F_t = \mathbf{1}_{\{L_t<\ell\}}$. In the next theorem we prove under some assumtions on $h$ the validity of the basic penalization hypotheses (5.1) and (5.2) for the weight process $\{F_t : t \geq 0\}$. The explicit form of the corresponding martingale $M^h$ is given. In Section 6.3 it is seen that $M^h$ remains to be a martingale for more general functions $h$, and properties of $X$ under the probability measure induced by $M^h$ are discussed.

In Roynette et al. [22] this kind of penalizations via local times of Bessel processes with dimension parameter $d \in (0, 2)$ are studied. Our work generalizes Theorem 1.5 in [22] for diffusions with subexponential Lévy measure.

**Theorem 5.2.** *Let $h : [0,\infty) \mapsto [0,\infty)$ be a Borel measurable, right-continuous and non-increasing function with compact support in $[0, K]$ for some given $K > 0$. Assume also that*

$$\int_0^K h(y)\, dy = 1,$$

*and define for $x \geq 0$*

$$H(x) := \int_0^x h(y)\, dy.$$

*Then for any $u \geq 0$*

$$\lim_{t\to\infty} \frac{\mathbf{E}_0(h(L_t)\,|\,\mathcal{C}_u)}{\mathbf{E}_0(h(L_t))} = S(X_u)h(L_u) + 1 - H(L_u) =: M_u^h \qquad a.s. \tag{5.5}$$

*and*

$$\mathbf{E}_0\left(M_u^h\right) = 1. \tag{5.6}$$

*Consequently, statements 1), 2) and 3) in Theorem 5.1 hold.*

*Proof.* *I)* We prove first (5.5).
a*)* To begin with, the following result on the behavior of the denominator in (5.5) is needed: for any $a \geq 0$

$$\mathbf{E}_a(h(L_t)) \underset{t\to+\infty}{\sim} (S(a)h(0) + 1)\ \nu((t,\infty)). \tag{5.7}$$

To show this, let $\mu$ denote the measure induced by $h$, i.e., $\mu(dy) = -dh(y)$. Then

$$h(x) = \int_{(x,K]} \mu(dy) = \int_{(0,K]} \mathbf{1}_{\{y>x\}}\mu(dy), \tag{5.8}$$

and, consequently,

$$\mathbf{E}_a(h(L_t)) = \mathbf{E}_a\left(\int_{(0,K]} \mathbf{1}_{\{\ell > L_t\}} \mu(d\ell)\right) = \int_{(0,K]} \mathbf{P}_a(L_t < \ell)\mu(d\ell).$$

By Proposition 4.3

$$\lim_{t\to\infty} \frac{\mathbf{P}_a(L_t < \ell)}{\nu((t,\infty))} = S(a) + \ell.$$

Moreover, for $\ell \leq K$

$$\frac{\mathbf{P}_a(L_t < \ell)}{\nu((t,\infty))} \leq \frac{\mathbf{P}_a(L_t < K)}{\nu((t,\infty))} \to S(a) + K \quad \text{as } t \to \infty,$$

and, by the dominated convergence theorem,

$$\lim_{t\to\infty} \int_{(0,K]} \frac{\mathbf{P}_a(L_t < \ell)}{\nu((t,\infty))} \mu(d\ell) = \int_{(0,K]} (S(a) + \ell)\, \mu(d\ell).$$

Hence,

$$\mathbf{E}_a(h(L_t)) \underset{t\to+\infty}{\sim} \left(\int_{(0,K]} (S(a) + \ell)\, \mu(d\ell)\right) \nu((t,\infty)), \tag{5.9}$$

and the integral in (5.9) can be evaluated as follows:

$$\begin{aligned}
\int_{(0,K]} (S(a) + \ell)\, \mu(d\ell) &= S(a) \int_{(0,K]} \mu(d\ell) + \int_{(0,K]} \ell\, \mu(d\ell) \\
&= S(a)h(0) + \int_{(0,K]} \mu(d\ell) \int_0^\ell du \\
&= S(a)h(0) + \int_0^K du \int_{(u,K)} \mu(d\ell) \\
&= S(a)h(0) + \int_0^K h(u) du \\
&= S(a)h(0) + 1.
\end{aligned}$$

This concludes the proof of (5.7).
*b)* To proceed with the proof of (5.5), recall that $\{L_s : s \geq 0\}$ is an additive functional, that is, $L_t = L_u + L_{t-u} \circ \theta_u$ for $t > u$ where $\theta_u$ is the usual shift operator. Hence, by the Markov property, for $t > u$

$$\mathbf{E}_0(h(L_t) \,|\, \mathcal{C}_u) = \mathbf{E}_0(h(L_u + L_{t-u} \circ \theta_u) \,|\, \mathcal{C}_u) = H(X_u, L_u, t-u) \tag{5.10}$$

with

$$H(a,\ell,r) := \mathbf{E}_a(h(\ell + L_r)).$$

By (5.7), since $x \mapsto h(\ell + x)$ is non-increasing with compact support,

$$H(a,\ell,r) \underset{t\to+\infty}{\sim} \left( S(a)h(\ell) + \int_0^\infty h(\ell+u)du \right) \nu((t,\infty)).$$

Bringing together (5.10) and (5.7) with $a = 0$ yields

$$\lim_{t\to\infty} \frac{\mathbf{E}_0(h(L_t)\,|\,\mathcal{C}_u)}{\mathbf{E}_0(h(L_t))} = \frac{S(X_u)h(L_u) + \int_{L_u}^\infty h(x)dx}{\int_0^\infty h(x)dx}$$

completing the proof of (5.5).

*II)* To verify (5.6), we show that $\{M_t^h \,:\, t \geq 0\}$ defined in (5.5) is a non-negative martingale with $M_0^h = 1$ (cf. Theorem 5.1 statement *1)*).

*a)* First, consider the process $S(X) = \{S(X_t) \,:\, t \geq 0\}$. Since the scale function is increasing $S(X)$ is a non-negative linear diffusion. Moreover, e.g., from Meleard [16], it is, in fact, a sub-martingale with the Doob-Meyer decomposition

$$S(X_t) = \tilde{Y}_t + \tilde{L}_t, \tag{5.11}$$

where $\tilde{Y}$ is a martingale and $\tilde{L}$ is a non-decreasing adapted process. Because $\tilde{L}$ increases only when $S(X)$ is at 0 or, equivalently, $X$ is at 0 $\tilde{L}$ is a local time of $X$ at 0. Consequently, $\tilde{L}$ is a multiple of $L$ a.s. (for the normalization of $L$, see (1.2)), i.e., there is a non-random constant $c$ such that for all $t \geq 0$

$$\tilde{L}_t = c\, L_t. \tag{5.12}$$

We claim that $\tilde{L}$ coincides with $L$, that is $c = 1$. To show this, recall that

$$\mathbf{E}_x(L_t^{(y)}) = \int_0^t p(s;x,y)\, ds,$$

which yields (cf. (1.1))

$$R_\lambda(0,0) = \int_0^\infty \lambda\, \mathrm{e}^{-\lambda t}\, \mathbf{E}_0(L_t)\, dt.$$

>From (5.11) and (5.12) we have $\mathbf{E}_0(L_t) = \dfrac{1}{c}\, \mathbf{E}_0(S(X_t))$, and, hence,

$$\begin{aligned} R_\lambda(0,0) &= \frac{1}{c} \int_0^\infty \lambda\, \mathrm{e}^{-\lambda t}\, \mathbf{E}_0(S(X_t))\, dt \\ &= \frac{1}{c} \int_0^\infty S(y)\, \lambda\, R_\lambda(0,y)\, m(dy). \end{aligned} \tag{5.13}$$

Next recall that the resolvent kernel can be expressed as

$$R_\lambda(x,y) = w_\lambda^{-1}\,\psi_\lambda(x)\,\varphi_\lambda(y), \quad 0 \le x \le y, \tag{5.14}$$

where $w_\lambda$ is a constant (Wronskian) and $\varphi_\lambda$ and $\psi_\lambda$ are the fundamental decreasing and increasing, respectively, solutions of the generalized differential equation

$$\frac{d}{dm}\frac{d}{dS}\,u = \lambda u \tag{5.15}$$

characterized (probabilistically) by

$$\mathbf{E}_x\left(\mathrm{e}^{-\lambda H_y}\right) = \frac{R_\lambda(x,y)}{R_\lambda(y,y)}. \tag{5.16}$$

Consequently, (5.13) is equivalent with

$$\begin{aligned}
\varphi_\lambda(0) &= \frac{1}{c}\int_0^\infty S(y)\,\lambda\,\varphi_\lambda(y)\,m(dy) \\
&= \frac{1}{c}\int_0^\infty S(y)\,\frac{d}{dm}\frac{d}{dS}\,\varphi_\lambda(y)\,m(dy). \\
&= \frac{1}{c}\int_0^\infty dS(y)\,\int_y^\infty m(dz)\,\frac{d}{dm}\frac{d}{dS}\,\varphi_\lambda(z). \\
&= \frac{1}{c}\int_0^\infty dS(y)\left(\frac{d}{dS}\varphi_\lambda(+\infty) - \frac{d}{dS}\,\varphi_\lambda(y)\right),
\end{aligned} \tag{5.17}$$

where for the third equality we have used Fubini's theorem. Next we claim that

$$\frac{d}{dS}\,\varphi_\lambda(+\infty) := \lim_{x\to\infty}\frac{d}{dS}\,\varphi_\lambda(x) = 0. \tag{5.18}$$

To prove this, recall that the Wronskian (a constant) is given for all $z \ge 0$ by

$$w_\lambda = \varphi_\lambda(z)\,\frac{d}{dS}\,\psi_\lambda(z) + \psi_\lambda(z)\left(-\frac{d}{dS}\,\varphi_\lambda(z)\right). \tag{5.19}$$

Notice that both terms on the right hand side are non-negative. Since the boundary point $+\infty$ is assumed to be natural it holds that $\lim_{z\to\infty} H_z = +\infty$ a.s. and, therefore, (cf. (5.16))

$$\lim_{z\to\infty}\psi_\lambda(z) = +\infty.$$

Consequently, letting $z \to +\infty$ in (5.19) we obtain (5.18). Now (5.17) takes the form

$$\varphi_\lambda(0) = -\frac{1}{c}\left(\varphi_\lambda(+\infty) - \varphi_\lambda(0)\right).$$

This implies that $c = 1$ since $\varphi_\lambda(+\infty) = 0$ by the assumption that $+\infty$ is natural (cf. (5.16)).
*b)* To proceed with the proof that $M^h$ is a martingale, consider first the case with continuously differentiable $h$. Then, applying (5.11),

$$dM_t^h = h(L_t)(d\tilde{Y}_t + dL_t) + S(X_t)h'(L_t)dL_t - h(L_t)dL_t = h(L_t)dY_t, \quad (5.20)$$

where we have used that

$$S(X_t)h'(L_t)dL_t = S(0)h'(L_t)dL_t = 0.$$

Consequently, $M^h$ is a continuous local martingale, and it is a continuous martingale if for any $T > 0$ the process $\{M_t^h : 0 \le t \le T\}$ is uniformly integrable (u.i.). To prove this, we use again (5.11) and write

$$M_t^h = h(L_t)\tilde{Y}_t + h(L_t)L_t + 1 - H(L_t). \quad (5.21)$$

Since $h$ is non-increasing and and has a compact support in $[0, K]$ we have

$$|h(L_t)L_t + 1 - H(L_t)| \le K \sup_{x\in[0,K]} h(x) + \int_0^\infty h(u)du$$

showing that $\{h(L_t)L_t + 1 - H(L_t) : t \ge 0\}$ is u.i. Moreover, since $\{h(L_t) : t \ge 0\}$ is bounded and $\{\tilde{Y}_t : 0 \le t \le T\}$ is u.i. it follows that $\{h(L_t)\tilde{Y}_t : 0 \le t \le T\}$ is u.i.. Consequently, $\{M_t^h : 0 \le t \le T\}$ is u.i., as claimed, and, hence, $\{M_t^h : t \ge 0\}$ is a true martingale implying (5.6). By the monotone class theorem (see, e.g., Meyer [17] T20 p. 28) we can deduce that $\{M_t^h : t \ge 0\}$ remains a martingale if the assumtion "$h$ is continuously differentiable" is relaxed to be "$h$ is bounded and Borel-measurable". The proof of Theorem 5.2 is now complete. □

**Example 5.3.** *Let* $h(x) := \mathbf{1}_{[0,\ell)}(x)$ *with* $\ell > 0$. *Then*

$$h(0) = 1, \quad \int_x^\infty h(y)dy = (\ell - x)^+, \quad \int_0^\infty h(y)dy = \ell,$$

*and the martingale $M^h$ takes the form*

$$\begin{aligned} M_u^h &= \frac{1}{\ell}\left(S(X_u)\mathbf{1}_{\{L_u<\ell\}} + (\ell - L_u)_+\right) \\ &= \frac{1}{\ell}\left(S(X_u) + \ell - L_u\right)\mathbf{1}_{\{L_u<\ell\}} \\ &= \frac{1}{\ell}\left(S(X_{u\wedge\tau_\ell}) + \ell - L_{u\wedge\tau_\ell}\right) \\ &= 1 + \frac{1}{\ell}\left(S(X_{u\wedge\tau_\ell}) - L_{u\wedge\tau_\ell}\right). \\ &= 1 + \frac{1}{\ell}\tilde{Y}_{u\wedge\tau_\ell}. \end{aligned}$$

## 5.3 The law of $X$ under the penalized measure

In this section we study the process $X$ under the penalized measure $\mathbf{Q}$ introduced in Theorem 5.2. In fact, we consider a more general situation, and assume that $h$ is "only" a Borel measurable and non-negative function defined on $\mathbf{R}_+$ such that

$$\int_0^\infty h(x)dx = 1. \tag{5.22}$$

For such a function $h$ we define

$$M_t^h := S(X_t)h(L_t) + 1 - H(L_t), \tag{5.23}$$

where

$$H(x) := \int_0^x h(y)dy.$$

It can be proved (see Roynette et al. [20] Section 3.2 and [22] Section 3) that $\{M_t^h : t \geq 0\}$ is also in this more general case a martingale such that $\mathbf{E}_0(M_t^h) = 1$ and $\lim_{t\to\infty} M_t^h = 0$. Therefore, we may define, for each $u \geq 0$, a probability measure $\mathbf{Q}^h$ on $(\mathcal{C}, \mathcal{C}_u)$ by setting

$$\mathbf{Q}^h(\Lambda_u) := \mathbf{E}_0\left(\mathbf{1}_{\Lambda_u} M_u^h\right) \qquad \Lambda_u \in \mathcal{C}_u. \tag{5.24}$$

The notation $\mathbf{E}^h$ is used for the expectation with respect to $\mathbf{Q}^h$. Next two propositions constitute a generalization of Theorem 1.5 in [22].

**Proposition 5.4.** *Under $\mathbf{Q}^h$, the random variable $L_\infty := \lim_{t\to\infty} L_t$ is finite a.s. and*

$$\mathbf{Q}^h(L_\infty \in d\ell) = h(\ell)\, d\ell.$$

*Proof.* For $u \geq 0$ and $\ell \geq 0$ it holds $\{L_u \geq \ell\} \in \mathcal{C}_u$, and, consequently,

$$\mathbf{Q}^h(L_u \geq \ell) = \mathbf{E}_0\left(\mathbf{1}_{\{L_u \geq \ell\}}\, M_u^h\right) = \mathbf{E}_0\left(\mathbf{1}_{\{\tau_\ell \leq u\}}\, M_u^h\right).$$

By optional stopping,

$$\mathbf{E}_0\left(\mathbf{1}_{\{\tau_\ell \leq u\}}\, M_u^h\right) = \mathbf{E}_0\left(\mathbf{1}_{\{\tau_\ell \leq u\}}\, M_{\tau_\ell}^h\right),$$

but

$$\begin{aligned} M_{\tau_\ell}^h &= S(X_{\tau_\ell})h(L_{\tau_\ell}) + 1 - H(L_{\tau_\ell}) = S(0)h(\ell) + 1 - H(\ell) \\ &= \int_\ell^\infty h(y)\,dy. \end{aligned} \tag{5.25}$$

As a result,

$$\mathbf{Q}^h(L_u \geq \ell) = \left(\int_\ell^\infty h(y)\,dy\right)\, \mathbf{P}_0(\tau_\ell \leq u).$$

Letting here $u \to \infty$ and using the fact that $\tau_\ell$ is finite $\mathbf{P}_0$−a.s. shows that

$$\mathbf{Q}^h(L_\infty \geq \ell) = \int_\ell^\infty h(y)\,dy.$$

Moreover, from assumption (5.22) it now follows that $L_\infty$ is $\mathbf{Q}^h$-a.s. finite, and the proof is complete. □

In the proof of the next proposition we use the process $X^\uparrow = \{X_t^\uparrow : t \geq 0\}$ which is obtained from $\widehat{X}$ (cf. (3.1)) by conditioning $\widehat{X}$ not to hit 0. The process $X^\uparrow$ can be described as Doob's $h$-transform of $\widehat{X}$, see, e.g., Salminen, Vallois and Yor [23] p.105. The probability measure and the expectation operator associated with $X^\uparrow$ are denoted by $\mathbf{P}^\uparrow$ and $\mathbf{E}^\uparrow$, respectively. The transition density and the speed measure associated with $X^\uparrow$ are given by

$$p^\uparrow(t;x,y) := \frac{\hat{p}(t;x,y)}{S(y)S(x)}, \qquad m^\uparrow(dy) := S(y)^2\, m(dy). \tag{5.26}$$

Notice (cf. (3.3)) that

$$p^\uparrow(t;0,y) := \lim_{x\downarrow 0} p^\uparrow(t;x,y) = \frac{f_{y0}(t)}{S(y)}. \tag{5.27}$$

Consequently, we have the formula

$$1 = \mathbf{P}_0^\uparrow\left(X_t^\uparrow > 0\right) = \int_0^\infty p^\uparrow(t;0,y)\, m^\uparrow(dy) = \int_0^\infty f_{y0}(t)\, S(y)\, m(dy). \tag{5.28}$$

**Proposition 5.5.** *Let* $\lambda$ *denote the last exit time from 0, i.e.,*

$$\lambda := \sup\{t\,:\, X_t = 0\}$$

*with* $\lambda = 0$ *if* $\{\cdot\} = \emptyset$*. Then*

**1)** $\mathbf{Q}^h(0 < \lambda < \infty) = 1$,

**2)** *under* $\mathbf{Q}^h$

- **a)** $\{X_t\,:\, t \le \lambda\}$ *and* $\{X_{\lambda+t}\,:\, t \ge 0\}$ *are independent,*
- **b)** *conditionally on* $L_\infty = \ell$*, the process* $\{X_t\,:\, t \le \lambda\}$ *is distributed as* $\{X_t\,:\, t \le \tau_\ell\}$ *under* $\mathbf{P}_0$*, in other words,*

$$\mathbf{E}^h\left(F(X_t\,:\, t \le \lambda)\, f(L_\infty)\right) = \int_0^\infty f(\ell) h(\ell) \mathbf{E}_0\left(F(X_t\,:\, t \le \tau_\ell\right) d\ell. \tag{5.29}$$

  *where* $F$ *is a bounded and measurable functional defined in the canonical space* $(C, \mathcal{C}, (\mathcal{C}_t))$ *and* $f : [0,\infty) \mapsto [0,\infty)$ *is a bounded and measurable function.*
- **c)** *the process* $\{X_{\lambda+t}\,:\, t \ge 0\}$ *is distributed as* $\{X_t^\uparrow\,:\, t \ge 0\}$ *started from 0.*

*Proof.* Consider for a given $T > 0$

$$\Delta := \mathbf{E}^h\left(F_1(X_u\,:\, u \le \lambda)\, F_2(X_{\lambda+v}\,:\, v \le T)\, f(L_\lambda)\, \mathbf{1}_{\{0<\lambda<\infty\}}\right),$$

where $F_1$ and $F_2$ are bounded and measurable functionals defined in the canonical space $(C, \mathcal{C}, (\mathcal{C}_t))$ and $f : [0,\infty) \mapsto [0,\infty)$ is a bounded and measurable function. For $N > 0$ define

$$\lambda_N := \sup\{u \le N\,:\, X_u = 0\}$$

and

$$\Delta_N^{(1)} := \mathbf{E}^h\left(F_1(X_u\,:\, u \le \lambda_N)\, F_2(X_{\lambda_N+v}\,:\, v \le T)\, f(L_{\lambda_N})\, \mathbf{1}_{\{\lambda_N+T<N\}}\right).$$

Then

$$\Delta = \lim_{N\to\infty} \Delta_N^{(1)}.$$

By absolute continuity, cf. (5.24),

$$\begin{aligned}\Delta_N^{(1)} &= \mathbf{E}_0\left(F_1(X_u\,:\,u\le\lambda_N)\,F_2(X_{\lambda_N+v}\,:\,v\le T)\,f(L_{\lambda_N})\,\mathbf{1}_{\{\lambda_N+T<N\}}\,M_N^h\right)\\ &= \mathbf{E}_0\Big(F_1(X_u\,:\,u\le\lambda_N)\,F_2(X_{\lambda_N+v}\,:\,v\le T)\,f(L_{\lambda_N})\,\mathbf{1}_{\{\lambda_N+T<N\}}\\ &\qquad\qquad\times\left(S(X_N)h(L_N)+1-H(L_N)\right)\Big).\end{aligned}$$

Since $F_1, F_2$, and $f$ are bounded and

$$\lim_{N\to\infty}(1-H(L_N))=0\quad \mathbf{P}_0\text{-}a.s.$$

we have

$$\begin{aligned}\Delta = \lim_{N\to\infty}\mathbf{E}_0\Big(&F_1(X_u\,:\,u\le\lambda_N)\,F_2(X_{\lambda_N+v}\,:\,v\le T)\,f(L_{\lambda_N})\\ &\times\mathbf{1}_{\{\lambda_N+T<N\}}\,S(X_N)h(L_N)\Big).\end{aligned}$$

Let $\Delta_N^{(2)}$ denote the expression after the limit sign. Then we write

$$\begin{aligned}\Delta_N^{(2)} = \mathbf{E}_0\Big(\sum_\ell &F_1(X_u\,:\,u\le\tau_{\ell-})\,F_2(X_{\tau_{\ell-}+v}\,:\,v\le T)\\ &\times f(\ell)\;\mathbf{1}_{\{\tau_{\ell-}+T<N<\tau_\ell\}}\,S(X_N)h(\ell)\Big),\end{aligned}$$

where $\{\tau_\ell\}$ is the right continuous inverse of $\{L_t\}$ (see (1.3)). By the Master formula (see Revuz and Yor [18] p. 475 and 483)

$$\begin{aligned}\Delta_N^{(2)} = \int_0^\infty &d\ell\,h(\ell)f(\ell)\mathbf{E}_0\Big(F_1(X_u\,:\,u\le\tau_\ell)\\ &\times\int_{\mathcal{E}}\mathbf{n}(de)\;F_2(e_v\,:\,v\le T)\,\mathbf{1}_{\{T\le N-\tau_\ell\le\zeta(e)\}}\,S(e_{N-\tau_\ell})\Big),\end{aligned}$$

where $\mathcal{E}$ denotes the excursion space, $e$ is a generic excursion, $\zeta(e)$ is the life time of the excursion $e$, and $\mathbf{n}$ is the Itô measure in the excursion space (see, e.g., [18] p. 480 and [23]). We claim that

$$\begin{aligned}I := \int_{\mathcal{E}} F_2(e_v\,:\,v\le T)\,\mathbf{1}_{\{T\le T'\le\zeta(e)\}}\,S(e_{T'})\,\mathbf{n}(de)&\\ = \mathbf{E}_0^{\uparrow}\left(F_2(X_v\,:\,v\le T)\right).&\end{aligned}\tag{5.30}$$

Notice that the right hand side of (5.30) does not depend on $T'$. We prove (5.30) for $F_2$ of the form

$$F_2(e_v : v \le T) = G(e_{t_1}, \dots, e_{t_k}), \quad t_1 < t_2 < \cdots < t_k = T,$$

where $G$ is a bounded and measurable function. For simplicity, take $k = 2$ and use Theorem 2 in [23] to obtain (for notation and results needed, see (3.2), (3.3), (5.26) and (5.27))

$$\begin{aligned} I &= \int_{[0,\infty)^3} f_{x_1,0}(t_1)\, \hat{p}(t_2 - t_1; x_1, x_2)\, \hat{p}(T' - t_2; x_2, x_3) \\ &\qquad\qquad \times G(x_1, x_2)\, S(x_3)\, m(dx_1)\, m(dx_2)\, m(dx_3) \\ &= \int_{[0,\infty)^3} S(x_1)\, f_{x_1,0}(t_1)\, p^{\uparrow}(t_2 - t_1; x_1, x_2)\, \hat{p}^{\uparrow}(T' - t_2; x_2, x_3) \\ &\qquad\qquad \times G(x_1, x_2)\, S(x_2)^2\, S(x_3)^2\, m(dx_1)\, m(dx_2)\, m(dx_3) \\ &= \mathbf{E}_0^{\uparrow}\left(G(X_{t_1}, X_{t_2})\right) \end{aligned}$$

proving (5.30). Consequently, we have (for all $N$)

$$\Delta_N^{(2)} = \mathbf{E}_0^{\uparrow}\left(F_2(X_v : v \le T)\right) \int_0^\infty d\ell\, h(\ell) f(\ell) \mathbf{E}_0\left(F_1(X_u : u \le \tau_\ell)\right),$$

and choosing here $F_1$, $F_2$, and $f$ appropriately implies all the claims of Proposition. In particular, $F_1 = F_2 = 1$ and $f = 1$ yields $\mathbf{Q}_0^h(0 < \lambda < \infty) = 1$, and, hence, $L_\infty = L_\lambda$ $\mathbf{Q}_0^h$-a.s. □

# 6 Appendix: a technical lemma

The following lemma could be viewed as a "weak" form of the Tauberian theorem (cf. Feller [6] Theorem 1 p. 443) stating, roughly speaking, that if two functions behave similarly at zero then their Laplace transforms behave similarly at infinity.

**Lemma 6.1.** *Let $\mu$ be a $\sigma$-finite measure on $[0, +\infty)$ and $g_1$ and $g_2$ two real valued functions such that for some $\lambda_0 > 0$*

$$C_i := \int_{[0,+\infty)} \mathrm{e}^{-\lambda_0 \gamma}\, |g_i(\gamma)|\, \mu(d\gamma) < \infty, \quad i = 1, 2.$$

*Assume also that* $g_2(\gamma) > 0$ *for all* $\gamma$*. Introduce for* $\lambda \geq \lambda_0$

$$f_i(\lambda) := \int_{[0,+\infty)} \mathrm{e}^{-\lambda\gamma}\, g_i(\gamma)\, \mu(d\gamma), \quad i = 1, 2.$$

*and suppose*

$$\lim_{\lambda\to+\infty} f_2(\lambda)\, \mathrm{e}^{b\lambda} = +\infty \qquad \text{for all } b > 0. \tag{6.1}$$

*Then*

$$g_1(\gamma) \underset{\gamma\to 0}{\sim} g_2(\gamma) \tag{6.2}$$

*implies*

$$f_1(\lambda) \underset{\lambda\to+\infty}{\sim} f_2(\lambda) \tag{6.3}$$

*Proof.* By property (6.2) there exist two functions $\theta_*$ and $\theta^*$ such that for some $\varepsilon > 0$ and for all $\gamma \in (0, \varepsilon)$

$$\theta_*(\varepsilon)\, g_2(\gamma) \leq g_1(\gamma) \leq \theta^*(\varepsilon)\, g_2(\gamma). \tag{6.4}$$

and

$$\lim_{\varepsilon\to 0} \theta_*(\varepsilon) = \lim_{\varepsilon\to 0} \theta^*(\varepsilon) = 1. \tag{6.5}$$

We assume also that $\theta_*(\varepsilon) > 0$ and $\theta^*(\varepsilon) > 0$. Letting $\lambda \geq 2\lambda_0$ we have for $\gamma \geq \varepsilon$

$$\lambda\gamma \geq \lambda_0\gamma + \frac{\lambda\gamma}{2} \geq \lambda_0\gamma + \frac{\lambda\varepsilon}{2}$$

and

$$\int_\varepsilon^\infty \mathrm{e}^{-\lambda\gamma}\, |g_i(\gamma)|\, \mu(d\gamma) \leq \mathrm{e}^{-\lambda\varepsilon/2} \int_\varepsilon^\infty \mathrm{e}^{-\lambda_0\gamma}\, |g_i(\gamma)|\, \mu(d\gamma) \leq \mathrm{e}^{-\lambda\varepsilon/2}\, C_i. \tag{6.6}$$

Furthermore, from (6.4)

$$\begin{aligned} \int_0^\varepsilon \mathrm{e}^{-\lambda\gamma}\, g_1(\gamma)\, \mu(d\gamma) &\leq \theta^*(\varepsilon) \int_0^\varepsilon \mathrm{e}^{-\lambda\gamma}\, g_2(\gamma)\, \mu(d\gamma) \\ &\leq \theta^*(\varepsilon) \int_0^\infty \mathrm{e}^{-\lambda\gamma}\, g_2(\gamma)\, \mu(d\gamma) \\ &\leq \theta^*(\varepsilon)\, f_2(\lambda) \end{aligned} \tag{6.7}$$

since $g_2$ is assumed to be positive. Writing

$$f_1(\lambda) = \int_0^\varepsilon \mathrm{e}^{-\lambda\gamma}\, g_1(\gamma)\, \mu(d\gamma) + \int_\varepsilon^\infty \mathrm{e}^{-\lambda\gamma}\, g_1(\gamma)\, \mu(d\gamma)$$

the estimates in (6.6) and (6.7) yield

$$f_1(\lambda) \leq \theta^*(\varepsilon)\, f_2(\lambda) + \mathrm{e}^{-\lambda\varepsilon/2}\, C_1,$$

which after dividing with $f_2(\lambda) > 0$ implies using (6.1) and (6.5)

$$\limsup_{\lambda\to+\infty} \frac{f_1(\lambda)}{f_2(\lambda)} = 1. \tag{6.8}$$

For a lower bound, consider

$$\begin{aligned}
f_1(\lambda) &= \int_{[0,\infty)} \mathrm{e}^{-\lambda\gamma}\, g_1(\gamma)\, \mu(d\gamma) \\
&\geq \int_{[0,\varepsilon)} \mathrm{e}^{-\lambda\gamma}\, g_1(\gamma)\, \mu(d\gamma) - \int_\varepsilon^\infty \mathrm{e}^{-\lambda\gamma}\, |g_1(\gamma)|\, \mu(d\gamma) \\
&\geq \theta_*(\varepsilon) \int_{[0,\varepsilon)} \mathrm{e}^{-\lambda\gamma}\, g_2(\gamma)\, \mu(d\gamma) - \mathrm{e}^{-\lambda\varepsilon/2}\, C_1 \\
&\geq \theta_*(\varepsilon) \left( f_2(\theta) - \int_\varepsilon^\infty \mathrm{e}^{-\lambda\gamma}\, g_2(\gamma)\, \mu(d\gamma) \right) - \mathrm{e}^{-\lambda\varepsilon/2}\, C_1. \\
&\geq \theta_*(\varepsilon)\, f_2(\theta) - \theta_*(\varepsilon)\, \mathrm{e}^{-\lambda\varepsilon/2}\, C_2 - \mathrm{e}^{-\lambda\varepsilon/2}\, C_1.
\end{aligned}$$

Hence,

$$\frac{f_1(\lambda)}{f_2(\lambda)} \geq \theta_*(\varepsilon) - (\theta_*(\varepsilon) C_2 - C_1)\, \frac{1}{\mathrm{e}^{\lambda\varepsilon/2}\, f_2(\lambda)}$$

showing that

$$\liminf_{\lambda\to+\infty} \frac{f_1(\lambda)}{f_2(\lambda)} \geq 1,$$

and completing the proof. □

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