Asymptotics of the maximal radius of an L^r-optimal sequence of quantizers

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Abstract

Let P be a probability distribution on $\mathbb{R}^d$ (equipped with an Euclidean norm). Let $r, s > 0$ and assume $(\alpha_n)_{n\geq 1}$ is an (asymptotically) $L^r(P)$-optimal sequence of n-quantizers. In this paper we investigate the asymptotic behavior of the maximal radius sequence induced by the sequence $(\alpha_n)_{n\geq 1}$ and defined to be for every $n \geq 1$, $\rho(\alpha_n) = \max\{|a|,\ a \in \alpha_n\}$. We show that if $\mathrm{card}(\mathrm{supp}(P))$ is infinite, the maximal radius sequence goes to $\sup\{|x|, x \in \mathrm{supp}(P)\}$ as n goes to infinity. We then give the rate of convergence for two classes of distributions with unbounded support : distributions with exponential tails and distributions with polynomial tails.

1 Introduction

Quantization has become an important field of information theory since the early 1940's. Nowadays, it plays an important rule in Digital Signal Processing (DSP), the basis of many areas of technology, from mobile phones to modems and multimedia PCs. In DSP, quantization is the process of approximating a continuous range of values or a very large set of discrete values by a relatively small set of discrete values. A common use of quantization is the conversion of a continuous signal into a digital signal. This is performed in analog-to-digital converters with a given quantization level. Beside these fields, quantization has recently become a domain of interest in Numerical Probability specially in numerical pricing of financial derivatives when their prices read as an expectation (or involve conditional expectations) of some random processes (see e.g. [1]).

From a mathematical point of view, the L^r-optimal quantization problem at level n for a $\mathbb{R}^d$-valued random vector X lying in $L^r(\Omega, \mathcal{A}, \mathbb{P})$ consists in finding the best approximation of X by $q(X)$, where q is a Borel function taking at most n values. This reads as the following minimization problem:

$$e_{n,r}(X) = \inf\left\{\|X - q(X)\|_r, q : \mathbb{R}^d \xrightarrow{\text{Borel}} \mathbb{R}^d,\ \text{card}(q(\mathbb{R}^d)) \leq n\right\}.$$

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Note that in fact $e_{n,r}(X)$ only depends on the distribution $P = \mathbb{P}_X$ of X so that we will also use the notation $e_{n,r}(P)$. However, for any Borel function $q : \mathbb{R}^d \to \alpha$ we have

$$|X - q(X)| \geq \min_{a\in\alpha} d(X, a) = d(X, \alpha) = |X - \widehat{X}^\alpha| \quad \mathbb{P} \text{ a.s}$$

so that the quantization problem reduces to

$$\begin{aligned} e_{n,r}(X) &= \inf\{\|X - \widehat{X}^\alpha\|_r, \alpha \subset \mathbb{R}^d, \ \text{card}(\alpha) \leq n\} \\ &= \inf_{\substack{\alpha\subset\mathbb{R}^d \\ \text{card}(\alpha)\leq n}} \left(\int_{\mathbb{R}^d} d(x, \alpha)^r dP(x) \right)^{1/r}. \end{aligned} \tag{1.1}$$

where $\widehat{X}^\alpha = \sum_{a\in\alpha} a\mathbf{1}_{\{X\in C_a(\alpha)\}}$ and $(C_a(\alpha))_{a\in\alpha}$ corresponds to a Voronoi partition of $\mathbb{R}^d$ (with respect to a norm $|\cdot|$ on $\mathbb{R}^d$), that is, a Borel partition of $\mathbb{R}^d$ satisfying for every $a \in \alpha$,

$$C_a(\alpha) \subset \{x \in \mathbb{R}^d : |x - a| = \min_{b\in\alpha} |x - b|\}.$$

For every $n \geq 1$, the infimum in (1.1) holds as a finite minimum reached (at least) at one grid $\alpha^\star$. In this case $\alpha^\star$ is called an $L^r(P)$**-optimal** (or L^r-optimal for X) and a sequence of n-quantizers $(\alpha_n)_{n\geq1}$ is $L^r(P)$-optimal if for every $n \geq 1$, α_n is $L^r(P)$-optimal. A sequence $(\alpha_n)_{n\geq1}$ is said **asymptotically** $L^r(P)$**-optimal** if

$$\int_{\mathbb{R}^d} d(x, \alpha_n)^r P(dx) \ = \ e^r_{n,r}(X) + o(e^r_{n,r}(X)) \qquad \text{as } n \to \infty.$$

Moreover the L^r-quantization error $e_{n,r}(X)$ decreases to 0 as n goes to infinity and if there is an $(r + \eta)$-moment of X, for $\eta > 0$, the so-called Zador's theorem recalled below rules its rate of convergence to 0.

Zador's Theorem (see [4]). Let $P = P_a + P_s$ be the Lebesgue decomposition of P with respect to the Lebesgue measure λ_d, where P_a denotes the absolutely continuous part and P_s the singular part of P. Suppose $\mathbb{E}|X|^{r+\eta} < +\infty$ for some $\eta > 0$. Then

$$\lim_{n\to+\infty} n^{r/d}(e_{n,r}(P))^r = Q_r(P).$$

with

$$Q_r(P) = J_{r,d} \left(\int_{\mathbb{R}^d} f^{\frac{d}{d+r}} d\lambda_d \right)^{\frac{d+r}{d}} = J_{r,d} \, \|f\|_{\frac{d}{d+r}} \ \in [0, +\infty),$$

$$J_{r,d} = \inf_{n\geq1} n^{r/d} e^r_{n,r}(U([0, 1]^d)) \in (0, +\infty),$$

where $U([0, 1]^d)$ denotes the uniform distribution on the set $[0, 1]^d$ and $f = \frac{dP_a}{d\lambda_d}$. Note that the moment assumption : $\mathbb{E}|X|^{r+\eta} < +\infty$ ensure that $\|f\|_{\frac{d}{d+r}}$ is finite.

Very little is known about the geometric properties of optimal quantizers. In this paper we address a first problem in this direction: we study the asymptotic behavior of the radii of a sequence $(\alpha_n)_{n\geq1}$ of L^r-optimal quantizers. The maximal radius (or simply radius) $\rho(\alpha)$ of a quantizer $\alpha \subset \mathbb{R}^d$ is defined by

$$\rho(\alpha) = \max\{|a|, \ a \in \alpha\}.$$

In our framework, $|\cdot|$ will be an Euclidean norm on $\mathbb{R}^d$. For the sake of simplicity, we will denote from now on by $(\rho_n)_{n\geq 1}$ the sequence $(\rho(\alpha_n))_{n\geq 1}$ of radii of a sequence $(\alpha_n)_{n\geq 1}$ of optimal quantizers (although it may be not unique).

We will show that, as soon as $\mathrm{supp}(P)$ is unbounded, $\lim_{n\to+\infty} \rho_n = +\infty$. Besides, our key inequalities to get the upper and lower estimates of the maximal radius sequence are provided in Theorem 3.1 and Theorem 3.2. The first theorem yields amount others the maximal rate of convergence of $\bar{F}_r(\frac{\rho_n}{c_{r,d}+\varepsilon})$ (when $n \to +\infty$) to 0, for every $\varepsilon > 0$, with $c_{r,d} = 1$ if $d = 1; r \geq 1$ and $c_{r,d} = 2$ otherwise. It claims that this rate is at most equals to $n^{-(1+r/d)}$.

Theorem 3.2 maintains in particular that for every $u > 1$, $\bar{F}_r(\rho_n u)$ goes to 0, as n goes to infinity, at a rate at less equals to $n^{-\frac{r+\nu}{d}}$, where ν is such that the random vector X has an $(r, r+\nu)$-distribution (see Definition 3.1). We will see later on that the index $\nu_X^\star$ ensuring that X has an $(r, r+\nu)$-distribution for every $\nu \in (0, \nu^\star)$ will play a crucial role in the lower limit estimates of the maximal radius sequence.

Then we will emphasize how knowing the asymptotic behavior of the function $-\log \bar{F}_r$ allow to derive the asymptotic estimates of ρ_n (or $\log \rho_n$). As an important example we can already mention distributions with density function f satisfying

$$f(x) \propto \frac{(\log|x|)^\beta}{|x|^c} \mathbf{1}_{\{|x|>1\}} \qquad x \in \mathbb{R}^d,\ \beta \in \mathbb{R},\ c > r+d$$

for which the optimal rate of convergence of $\log \rho_n$ is computed and given by

$$\lim_{n\to+\infty} \frac{\log \rho_n}{\log(n)} = \frac{1}{c-r-d}\frac{r+d}{d}.$$

Of course, this result is less accurate as giving the rate of convergence of the sequence (ρ_n) itself for which the exact limit can not be computed with our approach because the upper and lower limits make appear no identified constants. Another example concerns distributions with exponential tail for which the upper and lower rates of convergence of the sequence (ρ_n) are provided. This is the case for the normally distributed random vector on $\mathbb{R}^d$ for which we have

$$\sqrt{\frac{2(r+d)}{d}} \leq \liminf_{n\to+\infty} \frac{\rho_n}{\sqrt{\log(n)}} \leq \limsup_{n\to+\infty} \frac{\rho_n}{\sqrt{\log(n)}} \leq 2\sqrt{\frac{2(r+d)}{d}}.$$

Our general conjecture for such distributions, which is proved when $d = 1$ and $r \geq 1$, is that the liminf bound is sharp, that is,

$$\lim_{n\to+\infty} \frac{\rho_n}{\sqrt{\log(n)}} = \sqrt{\frac{2(r+d)}{d}}.$$

Moreover, an alternative approach is given for the lower limit estimates. This approach is based on random quantization and relies ρ_n to the expectation of an $i.i.d$ sequence of random variables distributed as X.

The paper is organized as follows. In Section 2, upper and lower estimates of the maximal radius sequence are given and the exact limit is provided when the cardinal of the support of P is infinite. This limit corresponds to $\sup\{|x|,\ x \in \mathrm{supp}(P)\}$ and Section 3 is entirely devoted to the convergence rate of the sequence of radii to this limit value.

Notations : Throughout the paper X will denote an $\mathbb{R}^d$-valued random vector defined in the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with distribution P having a moment of order $r > 0$ $i.e.$ $\mathbb{E}|X|^r < +\infty$. We define

$$L^{r+}(\mathbb{P}) = \bigcup_{\varepsilon>0} L^{r+\varepsilon}(\mathbb{P}).$$

We will denote by λ_d the Lebesgue measure on $\big(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)\big)$. We will also denote by $\bar{F}$ the survival function of X, that is, the $(0,1]$-valued function defined on $\mathbb{R}_+$ by

$$\bar{F} : x \mapsto \bar{F}(x) = \mathbb{P}(\{|X| > x\})$$

and for every $r > 0$, we define the generalized survival function of X by

$$\bar{F}_r : x \mapsto \bar{F}_r(x) = \mathbb{E}\big(|X|^r \mathbf{1}_{\{|X|>x\}}\big).$$

Note that this last function is defined on $\mathbb{R}_+$ and takes values on the set $(0, \mathbb{E}|X|^r]$.

For a given set A, $\overline{A}$ will stand for its closure, ∂A its boundary, $\mathrm{Conv}(A)$ its convex hull and $\mathring{A}$ or $\mathrm{Int}(A)$ its interior. The cardinal of A is denoted by $\mathrm{card}(A)$. For every $x \geq 0$, $[x]$ will denote the integral part of x.

2 Asymptotics of the the maximal radius sequence

In this section we give an asymptotic upper bound and a lower bound of the sequence of radii. For distributions supported by a infinite set, the exact limit is provided.

Proposition 2.1. *Let $X \in L^r_{\mathbb{R}^d}(\mathbb{P})$. Let $(\alpha_n)_{n\geq 1}$ be a sequence of n-quantizers such that $e_{n,r}(X) \to 0$ as $n \to +\infty$. Then,*

$$\liminf_{n\to+\infty} \rho_n \geq \sup\{|x|,\ x \in \mathrm{supp}(P)\}. \tag{2.1}$$

Remark that this result also holds for any norm on $\mathbb{R}^d$.

***Proof*.** Let $x \in$ supp(P). Suppose that there exists $\varepsilon_0 > 0$ and a subsequence $(\rho_{n_k})_{k\geq 1}$ such that

$$\forall k \geq 1, \qquad \rho_{n_k} < |x| - 2\varepsilon_0. \tag{2.2}$$

Thus

$$\exists\, \eta > 0 \text{ such that } \forall k,\ d(B(0,\rho_{n_k}), B(x,\varepsilon_0)) > \eta > 0.$$

Then one has for every $k \geq 1$,

$$\begin{aligned} e_{n_k,r}(X) &= \|d(X, \alpha_{n_k})\|_r \\ &\geq \|d(X, B(0,\rho_{n_k}))\|_r \qquad\qquad (\text{since } \ \alpha_{n_k} \subset B(0,\rho_{n_k})) \\ &\geq \|d(X, B(0,\rho_{n_k}))\mathbf{1}_{\{X\in B(x,\varepsilon_0)\}}\|_r \\ &\geq \|d(B(x,\varepsilon_0), B(0,\rho_{n_k}))\mathbf{1}_{\{X\in B(x,\varepsilon_0)\}}\|_r \\ &= d(B(x,\varepsilon_0), B(0,\rho_{n_k}))\mathbb{P}(X \in B(x,\varepsilon_0))^{1/r} \\ &> \eta\, \mathbb{P}(X \in B(x,\varepsilon_0))^{1/r} > 0. \end{aligned}$$

This is not possible since $e_{n,r}(X) \to 0$. Then, we have shown that

$$\forall x \in \ \mathrm{supp}(P), \quad \liminf_n \rho_n \geq |x|.$$

Hence $\liminf_n \rho_n \geq \sup\{|x|,\ x \in \mathrm{supp}(P)\}$. ☐

Among other results, the next proposition provides the limit of the sequence $(\rho_n)_{n\geq 1}$ when the support of P is infinite.

Proposition 2.2. ***(a)*** *Let α be an L^r-optimal quantizer at level n. If $\mathrm{card}(\mathrm{supp}(P)) \geq n$ then*

$$\alpha \subset \overline{\mathrm{Conv}(\mathrm{supp}(P))} \quad \textit{and} \quad \rho_n \leq \sup\{|x|,\ x \in \mathrm{supp}(P)\}. \tag{2.3}$$

(b) *If $\mathrm{card}(\mathrm{supp}(P)) = +\infty$ then*

$$\lim_{n\to+\infty} \rho_n = \sup_{n\geq 1} \rho_n = \sup\{|x|,\ x \in \mathrm{supp}(P)\}. \tag{2.4}$$

for any $L^r(P)$-optimal sequence of quantizers $(\alpha_n)_{n\geq 1}$.

Proof. **(a)** If α is L^r-optimal at level n then $\mathrm{card}(\alpha) = n$ since $\mathrm{card}(\mathrm{supp}(P)) \geq n$ (see [9]).

Now, suppose that $\alpha \not\subset \overline{\mathrm{Conv}(\mathrm{supp}(P))}$. Then let $a \in \alpha \cap \left(\overline{\mathrm{Conv}(\mathrm{supp}(P))}\right)^c$ and set

$$\alpha' = (\alpha\backslash\{a\}) \cup \{\Pi(a)\}$$

where Π denotes the projection on the non empty closed convex set $\overline{\mathrm{Conv}(\mathrm{supp}(P))}$. The projection is 1-Lipschitz and X is $\mathbb{P}$-$a.s$ $\mathrm{supp}(P)$-valued, hence

$$d(X,a) \geq d(\Pi(X),\Pi(a)) \overset{\mathbb{P}\text{-}a.s}{=} d(X,\Pi(a)).$$

It follows that

$$d(X,\alpha) \geq d(X,\alpha') \qquad \mathbb{P}\text{-}a.s.$$

Since α is $L^r(P)$-optimal at level n and $\mathrm{card}(\alpha') \leq \mathrm{card}(\alpha) = n$,

$$\mathbb{E}(d(X,\alpha')^r) = \mathbb{E}(d(X,\alpha)^r)$$

so that the three statements hold:

- $d(X,\alpha') = d(X,\alpha) \quad \mathbb{P}-a.s$
- $\Pi(a) \notin \alpha\backslash\{a\}$ since α' is $L^r(P)$-optimal (which implies that $\mathrm{card}(\alpha') = n$),
- $\mathbb{P}(X \in C_{\Pi(a)}(\alpha')) > 0$ (otherwise $\alpha'\backslash\{\Pi(a)\}$ would be optimal).

On the other hand, $X \in \mathrm{Conv}(\mathrm{supp}(P))$ $\mathbb{P}$-$a.s$ so that

$$\big(a - \Pi(a) | X - \Pi(a)\big) \leq 0 \quad \mathbb{P}\text{-}a.s.$$

Consequently

$$\begin{aligned} |X-a|^2 - |X-\Pi(a)|^2 &= 2(\Pi(a) - a|X - \Pi(a)) + |a - \Pi(a)|^2 \\ &\geq |a-\Pi(a)|^2 > 0 \qquad \text{since } a \notin \overline{\mathrm{Conv}(\mathrm{supp}(P))}. \end{aligned}$$

As a consequence

$$d(X,\alpha') < d(X,\alpha) \qquad \mathbb{P}\text{-a.s on } \{X \in \mathring{C}_{\Pi(\mathbf{a})}(\alpha')\}$$

where $\mathring{C}_{\Pi(a)}(\alpha') = \{\xi \in \mathbb{R}^d,\ d(\xi,\Pi(a)) < d(\xi,\alpha\backslash\{a\})\}$ since the norm is Euclidean.

This implies that $\mathbb{P}(X \in \mathring{C}_{\Pi(\mathbf{a})}(\alpha')) = 0$ and then $\mathbb{P}(X \in \partial C_{\Pi(\mathbf{a})}(\alpha')) > 0$; this is impossible since α' is L^r-optimal (see [4]). Hence $\alpha \subset \overline{\mathrm{Conv}(\mathrm{supp}(P))}$.

Now, let us prove that $\rho_n \leq \sup\{|x|,\ x \in \mathrm{supp}(P)\}$. Note first that this assertion is obvious if $\mathrm{supp}(P)$ is unbounded.

On the other hand if $\mathrm{supp}(P)$ is bounded then it is compact and so is $\mathrm{Conv}(\mathrm{supp}(P))$. Let $x_0 \in \mathrm{Conv}(\mathrm{supp}(P))$ be such that $|x_0| = \sup\{|x|, x \in \mathrm{Conv}(\mathrm{supp}(P))\}$. Thus

$$x_0 = \lambda_0 \xi_1 + (1-\lambda_0)\xi_2, \qquad \xi_1, \xi_2 \in \mathrm{supp}(P)$$

and $\lambda \longmapsto |\lambda \xi_1 + (1-\lambda)\xi_2|$ is convex so that it reaches its maximum at $\lambda = 0$ or $\lambda = 1$. Consequently $x_0 \in \mathrm{supp}(P)$.

(b) This follows from the assertion about $\rho(\alpha_n)$ in the item (a) and from Proposition 2.1. $\square$

Remark 2.1. *If the norm on $\mathbb{R}^d$ is an arbitrary norm, the assertion (a) of the proposition may fail. An example is given with the l_∞-norm in [4], p. 25.*

3 Convergence rate of the maximal radius sequence

We first start by giving two examples of distributions for which the sharp convergence rate of the maximal radius sequence can be computed rather easily. In fact the semi-closed forms established in [7] for the L^r-optimal quantizers of the exponential and the Pareto distributions and summed up in the following proposition allow to derive some sharp asymptotics for the maximal radius sequence $(\rho_n)_{n\geq 1}$ induced by the unique sequence of L^r-optimal quantizers at level n. These rates will be very useful to validate the asymptotic rates obtained by others approaches.

Proposition 3.1. *(see [7])* ***(a)*** *Let $r > 0$ and let X be an exponentially distributed random variable with scale parameter $\lambda > 0$. Then, for every $n \geq 1$, the L^r-optimal quantizer $\alpha_n = (\alpha_{n,1}, \cdots, \alpha_{n,n})$ is unique and given by*

$$\alpha_{n,k} = \frac{1}{\lambda}\left(\frac{a_n}{2} + \sum_{i=n+1-k}^{n-1} a_i\right), \quad 1 \leq k \leq n, \tag{3.1}$$

where $(a_k)_{k\geq 1}$ is an $\mathbb{R}_+$-valued sequence recursively defined by the following implicit equation:

$$a_0 := +\infty, \quad \phi_r(-a_{k+1}) := \phi_r(a_k),\ k \geq 0$$

with $\phi_r(x) := \int_0^{x/2} |u|^{r-1}\mathrm{sign}(u)e^{-u}du$ (convention : $0^0 = 1$).

Furthermore, the sequence $(a_k)_{k\geq 1}$ decreases to zero and for every $k \geq 1$,

$$a_k = \frac{r+1}{k}\left(1 + \frac{c_r}{k} + O(\frac{1}{k^2})\right)$$

for some positive real constant c_r.
(b) *Let $r > 0$ and let X be a random variable having a Pareto distribution with index $\gamma > r$. Let f be the density function : $f(x) = \gamma x^{-(\gamma+1)}\mathbf{1}_{\{x>1\}}$. Then for every $n \geq 1$, the L^r-optimal quantizer $\alpha_n = (\alpha_{n,1}, \cdots, \alpha_{n,n})$ is unique and given by*

$$\alpha_{n,k} = \frac{1}{1+a_n}\prod_{i=n-k+1}^{n-1}(1+a_i), \quad 1 \leq k \leq n, \tag{3.2}$$

where $(a_k)_{k\geq 1}$ is an $\mathbb{R}_+$-valued sequence recursively defined by the following implicit equation:

$$a_0 = +\infty, \quad \phi_\gamma\left(-\frac{a_{k+1}}{1+a_{k+1}}\right) := \phi_\gamma(a_k),\ k \geq 1,$$

with $\phi_\gamma(x) := \int_0^{x/2} \gamma |u|^{r-1}\text{sign}(u)(1+u)^{-(\gamma+1)}du$. The sequence $(a_k)_{k\geq 1}$ decreases to zero and there is some positive real constant c such that for every $k \geq 1$,

$$a_k = \frac{r+1}{(\gamma-r)k}\left(1 + \frac{c}{k} + O(\frac{1}{k^2})\right).$$

Let us give now the sharp asymptotic derived from these semi-closed forms.

Proposition 3.2. ***(a)*** *Let $r > 0$ and let X be an exponentially distributed random variable with parameter $\lambda > 0$. Then*

$$\rho_n \quad = \quad \frac{r+1}{\lambda}\log(n) + \frac{C_r}{\lambda} + \mathrm{O}(\frac{1}{n}), \tag{3.3}$$

where C_r is a real constant depending only on r.

(b) *Let $r > 0$ and let X be a random variable with Pareto distribution of index γ such that $\gamma > r$. Then,*

$$\log(\rho_n) = \frac{r+1}{\gamma - r}\log(n) + C_r + O(\frac{1}{n}), \tag{3.4}$$

where C_r is a real constant depending only on r.

Proof. **(a)** It follows from (3.1) that

$$\lambda\,\rho_n = \frac{a_n}{2} + \sum_{i=1}^{n-1} a_i$$

where the sequence $(a_n)_{n\geq 1}$ decreases to zero and satisfies for every $n \geq 1,\ a_n = \big((r+1)/n\big)\big(1 + c_r/n + \mathrm{O}(1/n^2)\big)$, for some real constant c_r. Thus,

$$\begin{aligned}\lambda\,\rho_n &= \frac{a_n}{2} + (r+1)\sum_{i=1}^{n-1}\frac{1}{i} + c_r\sum_{i=1}^{n-1}\frac{1}{i^2} + \sum_{i=1}^{n-1}\mathrm{O}(1/i^3)\\ &= (r+1)\log(n) + C_r + \mathrm{O}\big(\frac{1}{n}\big).\end{aligned}$$

(b) It follows from (3.2) that

$$\rho_n = \frac{1}{1+a_n}\prod_{i=1}^{n-1}(1+a_i)$$

where $(a_n)_{n\geq 1}$ is an $\mathbb{R}_+$-valued sequence, decreasing to zero and satisfying : $\forall n \geq 1$, $a_n = \frac{r+1}{(\gamma-r)n}\big(1 + c_r/n + \mathrm{O}(1/n^2)\big)$, for some real constant c_r.

Then,

$$\begin{aligned}\log(\rho_n) &= -\log(1+a_n) + \sum_{i=1}^{n-1}\big(a_i - \frac{a_i^2}{2} + \mathrm{O}(a_i^3)\big)\\ &= \frac{r+1}{\gamma-r}\log(n) + C_r + \mathrm{O}\big(\frac{1}{n}\big)\end{aligned}$$

where we used that $\sum_{i=1}^{\infty} a_i^2 < \infty$ and $\sum_{i=1}^{\infty}\mathrm{O}(a_i^3) < \infty$. □

3.1 Upper estimate

We investigate in this section the rate of convergence of (ρ_n) to infinity. Let us give first some definitions and some hypotheses which will be useful later on.

Let $(\alpha_n)_{n\geq 1}$ be an $L^r(P)$-optimal sequence of quantizers at level n. For $n \geq 1$, we define $M(\alpha_n)$ to be

$$M(\alpha_n) = \{a \in \alpha_n \text{ such that } |a| = \max_{b\in\alpha_n} |b|\}.$$

We will need the following assumption on P

$$\textbf{(H)} \;\equiv\qquad P(dx) \geq \varepsilon_0\, \mathbf{1}_{\{x\in\bar{B}(x_0,r_0)\}}\lambda_d(dx), \quad \varepsilon_0,\ r_0 > 0,\ x_0 \in \mathbb{R}^d.$$

In the one dimensional setting, we will need the following specific assumption depending on $r \in [1, +\infty)$:

$(\mathbf{G_r})$ $P = f \cdot \lambda_d$ where f is non-increasing to 0 on $[A, +\infty)$ for some real constant A and

$$\lim_{y\to+\infty} \int_1^{+\infty} (u-1)^{r-1} \frac{f(uy)}{f(y)} du = 0. \tag{3.5}$$

Let us make some brief comments on theses assumptions as well as some simple criterions.

• Note that Assumption **(H)** holds as soon as X has a density f which is bounded away from 0 on a closed ball $\bar{B}(x_0, r_0), r_0 > 0, x_0 \in \mathbb{R}^d$, i.e. $\varepsilon_0 := \min_{x\in\bar{B}(x_0,r_0)} f(x) > 0$. This is a very light assumption satisfied by all usual distributions (Gaussian distribution, the exponential distribution, the Pareto distribution, etc).

• Assumption $(\mathbf{G_r})$ holds for distributions with density functions of the form

$$f(x) \;\propto\; |x|^c\, e^{-\vartheta|x|^\kappa} \qquad x \in \mathbb{R};\ \vartheta, \kappa > 0;\ c > -1.$$

Indeed, we have for large enough y, f is non-increasing and

$$\begin{aligned}\int_1^{+\infty} (u-1)^{r-1} \frac{f(uy)}{f(y)} du &= \int_1^{+\infty} (u-1)^{r-1} u^c e^{-\vartheta y^\kappa(u^\kappa-1)} du \\ &\leq \int_1^{+\infty} (u-1)^{r-1} u^c e^{-\vartheta A^\kappa(u^\kappa-1)} du < +\infty.\end{aligned}$$

The existence of the last integral follows from the existence of the moment of every order.

It follows from the Lebesgue convergence theorem that (3.5) holds. Then Assumption $(\mathbf{G_r})$ holds in particular for the Gaussian distribution, for the Weibull distribution and for the Gamma distribution. However it fails for example for the Pareto distribution. But, we will see later that we do not need this assumption for distributions with polynomial tails to estimate the sequence $(\log \rho_n)_{n\geq 1}$.

Let us recall the L^r-stationary property which will be also useful. Assume $P = f \cdot \lambda_d$. The so-called L^r-distorsion function $D^X_{n,r} : (\mathbb{R}^d)^n \longrightarrow \mathbb{R}_+$ is defined by :

$$\alpha = (\alpha_1, \cdots, \alpha_n) \longmapsto \mathbb{E}\left(\min_{i=1,\cdots,n} |X - \alpha_i|^r\right).$$

Then, for every $r \geq 1$, $D^X_{n,r}$ is differentiable at any codebook having pairwise distinct components and (see [6] for details)

$$\nabla D^X_{n,r}(\alpha) = r\left(\int_{C_i(\alpha)} (\alpha_i - u)|u - \alpha_i|^{r-2} f(u) du\right)_{1\leq i\leq n}. \tag{3.6}$$

An optimal L^r-quantizer at level n $\alpha = \{\alpha_1, \cdots, \alpha_n\}$ for P has full size n, so that,

$$\nabla D^X_{n,r}(\alpha) = 0.$$

α is said to satisfy an L^r-stationary property.

When $d = 1$ then for any (ordered) quantizer $\alpha_n = \{x_1^{(n)}, \cdots, x_n^{(n)}\}$, $x_1^{(n)} < \cdots < x_n^{(n)}$) at level n, its Voronoi partition is given by

$$C_1(\alpha_n) = (-\infty, x^{(n)}_{\frac{1}{2}}],\ C_n(\alpha_n) = (x^{(n)}_{n-\frac{1}{2}}, +\infty),\ C_i(\alpha_n) = (x^{(n)}_{i-\frac{1}{2}}, x^{(n)}_{i+\frac{1}{2}}],\ i = 2, \cdots, n-1,$$

with $x^{(n)}_{i-\frac{1}{2}} = \frac{x_i^{(n)} + x_{i-1}^{(n)}}{2}$ and $x^{(n)}_{i+\frac{1}{2}} = \frac{x_i^{(n)} + x_{i+1}^{(n)}}{2}$.

The main result of this section is the following.

Theorem 3.1. *Suppose that X has an unbounded support and that* ***(H)*** *holds. Let $(\alpha_n)_{n \geq 1}$ be an $L^r(P)$-optimal sequence of quantizers. Then,*

(a)

$$\lim_{\varepsilon \downarrow 0} \liminf_{n \to +\infty} \left(n^{1+\frac{r}{d}} \bar{F}_r\big(\frac{\rho_n}{2+\varepsilon}\big) \right) \geq C_{r,d,U}. \tag{3.7}$$

(b) *If $d = 1,\ r \geq 1$ and if furthermore $(\mathbf{G_r})$ holds then,*

$$\lim_{\varepsilon \downarrow 0} \liminf_{n \to +\infty} \left(n^{r+1} \bar{F}_r\big(\frac{\rho_n}{1+\varepsilon}\big) \right) \geq C_{r,1,U}. \tag{3.8}$$

$C_{r,d,U}$ is a positive real constant depending on r, d and the uniform distribution U on $[0,1]^d$.

The Lemmas below are used to prove this result.

Lemma 3.1. *Let X be an $\mathbb{R}^d$ valued random variable with unbounded support and probability distribution P and let $(\alpha_n)_{n \geq 1}$ be an $L^r(P)$-optimal sequence of n-quantizers, $r > 0$. Let $(\rho_n)_{n \geq 1}$ be the maximal radius sequence induced by $(\alpha_n)_{n \geq 1}$. Then,*

(a) *$\forall\, \varepsilon > 0,\ \exists\, n_\varepsilon$ such that $\forall n \geq n_\varepsilon$,*

$$\forall\, a \in M(\alpha_n),\ \forall\, \xi \in C_a(\alpha_n), \qquad |\xi| \geq \frac{\rho_n}{2+\varepsilon}. \tag{3.9}$$

(b) *If $d = 1,\ r \geq 1$ and if furthermore $(\mathbf{G_r})$ holds then, for large enough n,*

$$\forall\, a \in M(\alpha_n),\ \forall\, \xi \in C_a(\alpha_n), \qquad |\xi| \geq \frac{\rho_n}{1+\varepsilon}. \tag{3.10}$$

Proof. **(a)** Since (α_n) is $L^r(P)$-optimal, $e_{n,r}(X) \to 0$ as $n \to +\infty$. Then, the following asymptotic density property of (α_n) in the support of P holds:

$$\forall \varepsilon > 0,\ \forall x \in \operatorname{supp}(P) \qquad B(x, \varepsilon) \cap \alpha_n \neq \emptyset. \tag{3.11}$$

Otherwise, if there exists $x \in \operatorname{supp}(P), \varepsilon > 0$ and a subsequence $(\alpha_{n_k})_{k \geq 1}$ so that $\forall k \geq 1,\ B(x,\varepsilon) \cap \alpha_{n_k} = \emptyset$, then, for every $k \geq 1$,

$$e_{n_k,r}(X) \geq \|d(X, \alpha_{n_k}) \mathbf{1}_{X \in B(x,\varepsilon/2)}\|_r \geq \frac{\varepsilon}{2} P(B(x, \varepsilon/2))^{1/r} > 0.$$

Which contradicts the fact that $e_{n,r}(X) \to 0$ as $n \to +\infty$.

Now, to prove the result assume first $0 \in \operatorname{supp}(P)$. Let $\varepsilon > 0,\ a \in M(\alpha_n)$. Then, $\exists N_1 \in \mathbb{N}$ such that $B(0,\varepsilon) \cap \alpha_n \neq \emptyset,\ \forall n \geq N_1$. Now $\rho_n \to +\infty$ implies that $B(0,\varepsilon) \cap (\alpha_n \backslash M(\alpha_n)) \neq \emptyset$ for $n \geq N_1'$.

Let $b \in B(x,\varepsilon) \cap (\alpha_n \backslash M(\alpha_n))$. We have for every $\xi \in C_a(\alpha_n)$,

$$|\xi - b|^2 \geq |\xi - a|^2,$$

namely

$$\begin{aligned} 2(\xi | a - b) &\geq |a|^2 - |b|^2 \qquad (\geq 0) \\ &\geq \rho_n^2 - |b|^2. \end{aligned}$$

Now, $|\xi||a-b| \geq (\xi | a - b)$, then,

$$|\xi||a-b| \geq \frac{(\rho_n + |b|)(\rho_n - |b|)}{2}.$$

Moreover, $|a - b| \leq |a| + |b| \leq \rho_n + |b|$. One finally gets

$$|\xi| \geq \frac{\rho_n - |b|}{2} \geq \frac{\rho_n - \varepsilon}{2}.$$

Since $\rho_n \to +\infty$ as $n \to +\infty$, $|\xi| \geq \frac{\rho_n}{2+\varepsilon}$, for large enough n.

If $0 \notin \operatorname{supp}(P)$ we show likewise that $|\xi| \geq \frac{\rho_n - |x_0| - \varepsilon}{2},\ \forall x_0 \in \operatorname{supp}(P)$ which implies the announced result since $\rho_n \to +\infty$.

(b) We will make an abuse of notation by considering that

$$\rho_n = \rho_n^+ := \max\{x,\ x \in \alpha\}.$$

In what follows all results on $\rho_-(\alpha) := \max\{-x,\ x \in \alpha\}$ can be derived by using $-X$ instead of X. Let $\alpha_n = \{x_1^{(n)}, \cdots, x_n^{(n)}\}$ and suppose that (up to a subsequence)

$$\frac{x_{n-1}^{(n)}}{x_n^{(n)}} \to \rho < 1.$$

Let $\varepsilon > 0$ such that $\rho + \varepsilon < 1$. We have for large enough n,

$$\frac{x_{n-1}^{(n)}}{x_n^{(n)}} < \rho + \varepsilon < 1$$

or equivalently,

$$\frac{x_{n-1}^{(n)} + x_n^{(n)}}{2} < x_n^{(n)} \left(\frac{1 + \rho + \varepsilon}{2} \right). \tag{3.12}$$

Let ρ' be such that $0 < \rho' < \frac{1-(\rho+\varepsilon)}{2}$, that is,

$$\left(\frac{1 + \rho + \varepsilon}{2} \right) < 1 - \rho' < 1. \tag{3.13}$$

It follows from (3.12) and (3.12) that

$$\begin{aligned}\int_{\frac{x_{n-1}^{(n)}+x_n^{(n)}}{2}}^{x_n^{(n)}}\left(1-\frac{u}{x_n^{(n)}}\right)^{r-1}f(u)du &\geq \int_{\frac{x_n^{(n)}(1+\rho+\varepsilon)}{2}}^{x_n^{(n)}(1-\rho')}\left(1-\frac{u}{x_n^{(n)}}\right)^{r-1}f(u)du\\ &\geq (\rho')^{r-1}\int_{\frac{x_n^{(n)}(1+\rho+\varepsilon)}{2}}^{x_n^{(n)}(1-\rho')}f(u)du\\ &\geq \rho'' x_n^{(n)}f(c_n)\end{aligned}$$

with $\rho''=(\rho')^{r-1}(\frac{1}{2}-\rho'-\frac{\rho+\varepsilon}{2})>0$ and $c_n\in(x_n^{(n)}(1+\rho+\varepsilon)/2, x_n^{(n)}(1-\rho'))$.

On the other hand, we have

$$\frac{1}{x_n^{(n)}f(x_n^{(n)})}\int_{x_n^{(n)}}^{+\infty}\left(\frac{u}{x_n^{(n)}}-1\right)^{r-1}f(u)du=\int_1^{+\infty}(u-1)^{r-1}\frac{f(ux_n^{(n)})}{f(x_n^{(n)})}du.$$

It follows from Assumption $(\mathbf{G_r})$ that

$$\limsup_{n\to+\infty}\frac{1}{x_n^{(n)}f(x_n^{(n)})}\int_{x_n^{(n)}}^{+\infty}\left(\frac{u}{x_n^{(n)}}-1\right)^{r-1}f(u)du=0.$$

Consequently we have for large enough n,

$$\frac{1}{x_n^{(n)}f(x_n^{(n)})}\int_{x_n^{(n)}}^{+\infty}\left(\frac{u}{x_n^{(n)}}-1\right)^{r-1}f(u)du<\rho''$$

so that ($A<c_n<x_n^{(n)}$ for large enough n and f is non-increasing in $[A,+\infty)$)

$$\int_{x_n^{(n)}}^{+\infty}\left(\frac{u}{x_n^{(n)}}-1\right)^{r-1}f(u)du<\rho'' x_n^{(n)}f(x_n^{(n)})\leq\rho'' x_n^{(n)}f(c_n)\leq\int_{\frac{x_{n-1}^{(n)}+x_n^{(n)}}{2}}^{x_n^{(n)}}\left(1-\frac{u}{x_n^{(n)}}\right)^{r-1}f(u)du$$

which is not possible since the L^r-stationary equation implies

$$\int_{\frac{x_{n-1}^{(n)}+x_n^{(n)}}{2}}^{x_n^{(n)}}\left(1-\frac{u}{x_n^{(n)}}\right)^{r-1}f(u)du=\int_{x_n^{(n)}}^{+\infty}\left(\frac{u}{x_n^{(n)}}-1\right)^{r-1}f(u)du.$$

We have then shown that

$$\lim_{n\to+\infty}\frac{x_n^{(n)}}{x_{n-1}^{(n)}}=1.$$

It follows that $\forall\varepsilon>,\exists\, n_\varepsilon$ such that $\forall n\geq n_\varepsilon,\ x_n^{(n)}<(1+\varepsilon)x_{n-1}^{(n)}$. Thus,

$$\rho_n=x_n^{(n)}<(1+\varepsilon)x_{n-1}^{(n)}<(1+\varepsilon)\xi\quad\forall\,\xi\in C_a(\alpha_n), a\in M(\alpha_n).$$

This completes the proof. $\square$

Lemma 3.2. *Let* $(\alpha_n)_{n\geq 1}$ *be a sequence of* L^r*-optimal* n*-quantizers of the distribution* P*. Suppose that* ***(H)*** *holds. Then for large enough* n*,*

$$e^r_{n,r}(X) - e^r_{n+1,r}(X) \geq C_{r,d,U}\, n^{-\frac{r+d}{d}}, \tag{3.14}$$

where

$$C_{r,d,U} = 2^{-(r+d)} \left(\frac{r}{d+r}\right)^{1/r} \left(\frac{d}{d+r}\right)^{d/r} \frac{\varepsilon_0}{1+\varepsilon_0}\, Q_{d+r}(U(\bar{B}(x_0, \frac{r_0}{2}))). \tag{3.15}$$

Proof. **Step** 1. Let $y \in \mathbb{R}^d$. Without lost of generality we temporarily set $\delta_n = d(y, \alpha_n)$. Following the lines of the proof of Theorem 2 in [5] we have for every $x \in B(y, \delta_n/2)$ and $a \in \alpha_n$,

$$|x-a| \geq |y-a| - |x-a| \geq \delta_n/2$$

and hence

$$d(x, \alpha_n) \geq \delta_n/2 \geq |x-y|, \quad x \in B(y, \delta_n/2).$$

It follows, by setting $\beta_n = \alpha_n \cup \{y\}$, that

$$d(x, \beta_n) = |x-y|, \quad x \in B(y, \delta_n/2).$$

Consequently for every $b \in (0, 1/2)$,

$$\begin{aligned} e^r_{n,r}(X) - e^r_{n+1,r}(X) &\geq \int d(x,\alpha_n)^r dP(x) - \int d(x,\beta_n)^r dP(x) \\ &\geq \int_{B(y,\delta_n b)} (d(x,\alpha_n)^r - d(x,\beta_n)^r) dP(x) \\ &= \int_{B(y,\delta_n b)} (d(x,\alpha_n)^r - |x-y|^r) dP(x) \\ &\geq \int_{B(y,\delta_n b)} ((\delta_n/2)^r - (\delta_n b)^r) dP(x) \\ &= (2^{-r} - b^r)\, \delta_n^r\, P(B(y, \delta_n b)). \end{aligned}$$

Step 2. Now, coming back to the core of our proof let x_0 and r_0 be as in **(H)**. For every $y \in \bar{B}(x_0, \frac{r_0}{2})$ we have

$$\begin{aligned} e^r_{n,r}(X) - e^r_{n+1,r}(X) &\geq (2^{-r} - b^r)\, \delta_n^r\, P(B(y, (b\,\delta_n) \wedge \frac{r_0}{2})) \\ &\geq (2^{-r} - b^r)\, \delta_n^r \varepsilon_0 \left((b\,\delta_n)^d \wedge (\frac{r_0}{2})^d\right) \mathbf{1}_{\{y \in \bar{B}(x_0, \frac{r_0}{2})\}}. \end{aligned}$$

One checks that

$$\sup_{y \in \bar{B}(x_0, \frac{r_0}{2})} d(y, \alpha_n) \to 0.$$

Otherwise $\exists y_\infty \in \bar{B}(x_0, \frac{r_0}{2}),\ \eta > 0$ and a subsequence $(\alpha_{\varphi(n)})_{n\geq 1}$ of $(\alpha_n)_{n\geq 1}$ such that for every $n \geq 1,\ d(y_\infty, \alpha_{\varphi(n)}) > \frac{\eta}{2}$. Then

$$\int d(\alpha_{\varphi(n)}, \xi)^r P(d\xi) \geq \int_{\bar{B}(y_\infty, \frac{\eta}{4})} d(\alpha_{\varphi(n)}, \xi)^r P(d\xi).$$

Moreover for every $\xi \in B(y_{\infty,\frac{\eta}{4}})$

$$d(\alpha_{\varphi(n)}, \xi) \geq d(y_\infty, \alpha_{\varphi(n)}) - d(y_\infty, \xi) \geq \frac{\eta}{2} - \frac{\eta}{4}$$

so that

$$\int d(\alpha_{\varphi(n)}, \xi)^r P(d\xi) \geq (\frac{\eta}{2})^r P(B(y_{\infty,\frac{\eta}{4}})).$$

This contradicts the fact that $e_{n,r}(X) \to 0$ as n goes to infinity. Consequently, for large enough n,

$$\sup_{y \in \bar{B}(x_0, \frac{r_0}{2})} d(y, \alpha_n) \leq \frac{r_0}{2}$$

so that

$$e^r_{n,r}(X) - e^r_{n+1,r}(X) \geq (2^{-r} - b^r)\, b^d d(y, \alpha_n)^{d+r} \varepsilon_0 \, \mathbf{1}_{\{y \in \bar{B}(x_0, \frac{r_0}{2})\}}.$$

It follows that

$$\begin{aligned} \int_{\bar{B}(x_0, \frac{r_0}{2})} (e^r_{n,r}(X) - e^r_{n+1,r}(X)) \lambda_d(dy) &\geq (2^{-r} - b^r)\, \varepsilon_0 \, b^d \int_{\bar{B}(x_0, \frac{r_0}{2})} d(y, \alpha_n)^{d+r} \lambda_d(dy) \\ &\geq (2^{-r} - b^r)\, \varepsilon_0 \;\, b^d \, \lambda_d(\bar{B}(x_0, \frac{r_0}{2}))\, e^{r+d}_{n,r+d}(U(\bar{B}(x_0, \frac{r_0}{2}))). \end{aligned}$$

Then,

$$e^r_{n,r}(X) - e^r_{n+1,r}(X) \geq (2^{-r} - b^r)\, \varepsilon_0 \, b^d \, e^{r+d}_{n,r+d}(U(\bar{B}(x_0, \frac{r_0}{2}))).$$

Consequently, for large enough n,

$$e^r_{n,r}(X) - e^r_{n+1,r}(X) \geq (2^{-r} - b^r)\, \frac{\varepsilon_0}{1 + \varepsilon_0}\, b^d \, Q_{d+r}(U(\bar{B}(x_0, \frac{r_0}{2})))\, n^{-\frac{d+r}{d}}.$$

As a function of b, the right hand side of this last inequality reaches its maximum at $b^\star = \frac{1}{2}\left(\frac{d}{d+r}\right)^{1/r}$. Which completes the proof. ☐

Proof of Theorem 3.1. Let $a \in M(\alpha_n)$ and $\varepsilon > 0$. We have,

$$\mathbb{E}|X - \widehat{X}^{\alpha_{n-1}}|^r \leq \mathbb{E}|X - \widehat{X}^{\alpha_n \setminus \{a\}}|^r$$

and

$$\begin{aligned} \mathbb{E}|X - \widehat{X}^{\alpha_n \setminus a}|^r &= \mathbb{E}\big(|X - \widehat{X}^{\alpha_n}|^r \mathbf{1}_{\{X \in C^c_a(\alpha_n)\}}\big) + \mathbb{E}\big(\min_{b \in \alpha_n \setminus \{a\}} |X - b|^r \mathbf{1}_{\{X \in C_a(\alpha_n)\}}\big) \\ &\leq \mathbb{E}|X - \widehat{X}^{\alpha_n}|^r + \mathbb{E}\big(\min_{b \in \alpha_n \setminus \{a\}} (|X| + |b|)^r \mathbf{1}_{\{X \in C_a(\alpha_n)\}}\big). \end{aligned}$$

It follows from Lemma 3.1 (a) that $\exists\, n_\varepsilon \in \mathbb{N}$ such that $\forall n \geq n_\varepsilon,\ |X| > \frac{\rho_n}{2+\varepsilon}$ on $\{X \in C_a(\alpha_n)\}$. Consequently, for all $b \in \alpha_n \backslash \{a\},\ |b| \leq |a| = \rho_n < (2 + \varepsilon)\, |X|$.

Hence,

$$\mathbb{E}|X - \widehat{X}^{\alpha_{n-1}}|^r - \mathbb{E}|X - \widehat{X}^{\alpha_n}|^r \leq (3 + \varepsilon)^r \mathbb{E}\big(|X|^r \mathbf{1}_{\{|X| > \frac{\rho_n}{2+\varepsilon}\}}\big).$$

Lemma 3.2 yields (since $(n-1)^{-\frac{r+d}{d}} \sim n^{-\frac{r+d}{d}}$ as $n \to +\infty$)

$$(1 + \varepsilon)^{-1} C_{r,d,U}\, n^{-\frac{r+d}{d}} \leq (3 + \varepsilon)^r \mathbb{E}\big(|X|^r \mathbf{1}_{\{|X| > \frac{\rho_n}{2+\varepsilon}\}}\big)$$

for large enough n, so that for every $\varepsilon > 0$,

$$\liminf_n \left(n^{\frac{r+d}{d}} \bar{F}_r\big(\frac{\rho_n}{2+\varepsilon}\big)\right) \geq \frac{C_{r,d,U}}{(3+\varepsilon)^r(1+\varepsilon)}.$$

Taking the limit as $\varepsilon \to 0$ gives the statement (3.7). Assertion (3.8) is proved as above by using Lemma 3.1 (b) instead of Lemma 3.1 (a). $\square$

Recall that $\bar{F}_r(x) = \mathbb{E}\big(|X|^r \mathbf{1}_{\{|X|>x\}}\big)$. It is clear that this function is non-increasing and goes to 0 as $x \to +\infty$ (provided $\mathbb{E}|X|^r < +\infty$). Consequently, $-\log \bar{F}_r(x)$ is monotone nondecreasing and goes to $+\infty$ as x goes to $+\infty$. Moreover, we know that if a function f defined on $(0, +\infty)$ is increasing to $+\infty$ (at $+\infty$), its generalized inverse function $f^{\leftarrow}$ defined by $\forall x > 0$,

$$f^{\leftarrow}(x) = \inf\{t > 0,\ f(t) \geq x\} \tag{3.16}$$

is monotone increasing to $+\infty$. On the other hand, the following result holds (see [2]): If furthermore f is regularly varying (at $+\infty$) with index $1/\delta, \delta > 0$, then there exists a function ψ, regularly varying with index δ and satisfying

$$\psi(f(x)) \sim f(\psi(x)) \sim x \quad \text{as } x \to +\infty. \tag{3.17}$$

The function ψ is an asymptotic inverse of f and it is not necessarily increasing neither continuous. Moreover, ψ is unique up to asymptotic equivalence and $f^{\leftarrow}$ is one version of ψ.

We next show that for distributions with exponential tails, specifying either the asymptotic inverse ϕ_r (if any) of the function $-\log \bar{F}_r$ or finding some asymptotic upper bound ψ_r of ϕ_r (having some " nice" properties) leads to an upper estimate of the maximal radius sequence. This estimate is connected to the chosen function ψ_r.

When the distribution has a polynomial tail, we will look for the asymptotic inverse function of $-\log \bar{F}_r(e^x)$ or some asymptotic upper bound ψ_r of it to provide an upper estimate of $(\log \rho_n)_{n\geq 1}$.

Proposition 3.3. *Assume that the distribution P of X has an unbounded support and satisfies (**H**). Let $(\alpha_n)_{n\geq 1}$ be an $L^r(P)$-optimal sequence of n-quantizers.*

(a) *If ψ_r is a measurable nondecreasing function, regularly varying with index δ and*

$$\psi_r(-\log \bar{F}_r(x)) \geq x + o(x) \quad \textit{as } x \to +\infty, \tag{3.18}$$

then

$$\limsup_{n\to+\infty} \frac{\rho_n}{\psi_r(\log(n))} \leq 2\left(1 + \frac{r}{d}\right)^{\delta}. \tag{3.19}$$

If $d = 1$ and $r \geq 1$ and if $(\mathbf{G_r})$ holds then, one has

$$\limsup_{n\to+\infty} \frac{\rho_n}{\psi_r(\log(n))} \leq (r+1)^{\delta}. \tag{3.20}$$

In particular if $-\log \bar{F}_r$ has regular variation with index $1/\delta$ then (3.19) holds with $\psi_r = (-\log \bar{F}_r)^{\leftarrow}$.

(b) *If ψ_r is a measurable nondecreasing function, regularly varying with index δ and*

$$\psi_r(-\log \bar{F}_r(e^x)) \geq x + o(x) \quad \textit{as } x \to +\infty, \tag{3.21}$$

then

$$\limsup_{n\to+\infty} \frac{\log \rho_n}{\psi_r(\log(n))} \leq \left(1 + \frac{r}{d}\right)^{\delta}. \tag{3.22}$$

If $-\log \bar{F}_r(e^{\cdot})$ has regular variation of index $1/\delta$ then (3.22) holds with $\psi_r = (-\log \bar{F}_r(e^{\cdot}))^{\leftarrow}$.

Prior to the proof, let us make some comment on the proposition. First note that the measurability of ψ_r is necessary to define the regular varying property. On the other hand we have for every $r > 0$ and for every $x > 0$,

$$\bar{F}_r(x) \geq x^r \bar{F}(x).$$

Then

$$-\log \bar{F}_r(x) \leq -\log \bar{F}(x) - r\log(x).$$

According to the nondecreasing hypothesis on ψ we have for every $x > 1$,

$$\psi_r(-\log \bar{F}_r(x)) \leq \psi_r(-\log \bar{F}(x) - r\log(x)) \leq \psi_r(-\log \bar{F}(x)) \tag{3.23}$$

since $log(x) > 0$. Hence if Assumption (3.18) holds then

$$\psi_r(-\log \bar{F}(x)) \geq x + o(x).$$

We will see further on that for distributions with exponential tails, the function ψ_r in the statement (a) of the proposition does not depend on r. However in the situation of the item (b) of the proposition, Assumption (3.21) implies that

$$\psi_r(-\log \bar{F}(e^x)) \geq (r+1)x + o(x).$$

Consequently, taking $\bar{F}$ instead of $\bar{F}_r$ in Assumption (3.21) will induce a loss of precision in the upper estimate of $\log \rho_n$.

Also remark that if $-\log \bar{F}_r$ (resp. $-\log \bar{F}_r(e^{\bullet})$) is measurable, locally bounded and regularly varying with index $1/\delta, \delta > 0$ then its generalized inverse function ϕ_r (resp. Φ_r) is measurable increasing to $+\infty$, regularly varying with index δ and, $\phi_r(-\log \bar{F}_r(x)) = x + o(x)$ (resp. $\Phi_r(-\log \bar{F}_r(e^x)) = x + o(x)$). Consequently, both inequalities (3.19) and (3.20) (resp. claim (3.22)) hold with ϕ_r (resp. Φ_r) in place of ψ_r. However, ϕ_r (resp. Φ_r) is in general not easy to compute and the examples below show that it is often easier to exhibit directly a function ψ_r satisfying the announced hypotheses without inducing any asymptotic loss of accuracy.

We prove now the proposition.

Proof. **(a)** It follows from (3.7) and (3.8) that for every $\varepsilon > 0$, there is a positive real constant $C_{r,d,U,\varepsilon}$ depending on the indexing parameters such that

$$n^{-\frac{d+r}{d}} C_{r,d,U,\varepsilon} \leq \bar{F}_r\big(\frac{\rho_n}{c_{r,d}+\varepsilon}\big)$$

where (from now on) $c_{r,1} = 1$ if $r > 1$; $c_{r,d} = 2$ otherwise. Therefore, one has

$$\frac{r+d}{d}\log(n) - \log(C_{r,d,U,\varepsilon}) \geq -\log \bar{F}_r\big(\frac{\rho_n}{c_{r,d}+\varepsilon}\big).$$

Combining the fact that ψ_r is nondecreasing and Assumption (3.18) yield

$$\begin{aligned}\psi_r\big(\frac{r+d}{d}\log(n) - \log(C_{r,d,U,\varepsilon})\big) &\geq \psi_r\big(-\log \bar{F}_r\big(\frac{\rho_n}{c_{r,d}+\varepsilon}\big)\big) \\ &\geq \frac{\rho_n}{c_{r,d}+\varepsilon} + o(\rho_n).\end{aligned}$$

Moreover, dividing by $\psi_r(\log(n))$ (which is positive for large enough n) yields

$$\frac{\rho_n}{\psi_r(\log(n))} \leq (c_{r,d}+\varepsilon)\big(1+\frac{o(\rho_n)}{\rho_n}\big)^{-1}\frac{\psi_r\big(\frac{r+d}{d}\log(n)-\log(C_{r,d,U,\varepsilon})\big)}{\psi_r(\log(n))}.$$

It follows from the regular varying hypothesis on ψ_r and $\lim_n \rho_n = +\infty$ that

$$\limsup_{n\to+\infty}\frac{\rho_n}{\psi_r(\log(n))} \leq (c_{r,d}+\varepsilon)\left(\frac{r+d}{d}\right)^{\delta}, \quad \forall \varepsilon > 0.$$

The result follows by letting $\varepsilon \to 0$.
(b) As previously, one derives from (3.7) and from Assumption (3.21) and the nondecreasing hypothesis on ψ_r that

$$\begin{aligned}\psi_r\big(\frac{r+d}{d}\log(n)-\log(C_{r,d,U,\varepsilon})\big) &\geq \psi_r\big(-\log \bar F_r\big(\frac{\rho_n}{c_{r,d}+\varepsilon}\big)\big)\\ &\geq \log\rho_n - \log(c_{r,d}+\varepsilon) + o(\log\rho_n).\end{aligned}$$

It follows that

$$\frac{\log\rho_n}{\psi_r(\log(n))} \leq \left(1-\frac{\log(c_{r,d}+\varepsilon)}{\log\rho_n}+\frac{o(\log\rho_n)}{\log\rho_n}\right)^{-1}\frac{\psi_r\big(\frac{r+d}{d}\log(n)-\log(C_{r,d,U,\varepsilon})\big)}{\psi_r(\log(n))}.$$

Owing to the regular varying hypothesis on ψ_r and the fact that $\lim_n \rho_n = +\infty$, we have

$$\limsup_{n\to+\infty}\frac{\log\rho_n}{\psi_r(\log(n))} \leq \left(\frac{r+d}{d}\right)^{\delta}.$$

□

We next give an explicit asymptotic upper bound for the convergence rate of the maximal radius sequence in the sense that the function ψ_r is made explicit. These bounds are derived on the rate of decay of the generalized survival function $\bar F_r$.

Criterion 3.1. ***(a)*** *Let X be a random variable with unbounded support. Let $r>0$ and let $(\alpha_n)_{n\geq 1}$ be an L^r-optimal sequence of n-quantizers for X. Let $\kappa>0$ such that $e^{|X|^\kappa}\in L^{0+}(\mathbb{P})$. Set*

$$\theta^\star = \sup\big\{\theta>0,\ \limsup_{x\to+\infty} e^{\theta x^\kappa}\bar F_r(x)<+\infty\big\} = \sup\big\{\theta>0,\ \mathbb{E}\,e^{\theta|X|^\kappa}<+\infty\big\}. \tag{3.24}$$

Then $\theta^\star\in(0,+\infty]$ and

$$\limsup_{n\to+\infty}\frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} \leq 2\left(\frac{r+d}{d\,\theta^\star}\right)^{1/\kappa}. \tag{3.25}$$

When $d=1$ and $r\geq 1$, if $(\mathbf{G_r})$ holds then

$$\limsup_{n\to+\infty}\frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} \leq \left(\frac{r+1}{\theta^\star}\right)^{1/\kappa}.$$

(b) *Let $X\in L^{r+}(\mathbb{P})$ be a random variable with unbounded support. Set*

$$\zeta^\star = \sup\big\{\zeta>0,\ \limsup_{x\to+\infty} x^{\zeta-r}\bar F_r(x)<+\infty\big\} = \sup\big\{\zeta>r,\ \mathbb{E}|X|^\zeta<+\infty\big\}. \tag{3.26}$$

Then $\zeta^\star \in (r, +\infty]$ *and*

$$\limsup_{n\to+\infty} \frac{\log \rho_n}{\log(n)} \le \frac{1}{\zeta^\star - r}\frac{r+d}{d}. \tag{3.27}$$

Prior to the proof we can make the following remark.

Remark 3.1. If $X \in \bigcap\limits_{r>0} L^r(\mathbb{P})$ then $\zeta^\star = +\infty$ and consequently $\lim\limits_{n\to+\infty} \frac{\log \rho_n}{\log(n)} = 0$.

Proof . **(a)** The equalities in (3.24) and (3.26) are obvious.

Let $\theta \in (0, \theta^\star)$. We have

$$\mathbb{E}\big(|X|^r \mathbf{1}_{\{|X|>x\}}\big) = \mathbb{E}\big(|X|^r \mathbf{1}_{\left\{e^{\theta|X|^\kappa} > e^{\theta x^\kappa}\right\}}\big) \le e^{-\theta x^\kappa}\mathbb{E}\big(|X|^r e^{\theta|X|^\kappa}\big).$$

Now, the right hand side of this last inequality is finite because if $\theta' \in (\theta, \theta^\star)$,

$$|x|^r e^{\theta|x|^\kappa} \le 1 + C_{\theta,\theta'} e^{\theta'|x|^\kappa}.$$

As a consequence,

$$-\log \bar{F}_r(x) \ge \theta x^\kappa + C_X, \qquad C_X \in \mathbb{R}.$$

Let $\psi_\theta(y) = \big(\frac{y}{\theta}\big)^{1/\kappa}$. As a function of y, ψ_θ is continuous (then measurable) increasing to $+\infty$, regularly varying with index $\delta = \frac{1}{\kappa}$ and we have

$$\psi_\theta(-\log \bar{F}_r(x)) \ge \left(x^\kappa + \frac{C_X}{\theta}\right)^{1/\kappa} = x + o(x), \quad \text{as } x \to +\infty.$$

It follows from Proposition 3.3 (a) that

$$\limsup_{n\to+\infty} \frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} \le c_{r,d}\left(\frac{d+r}{d\,\theta}\right)^{1/\kappa} \qquad \forall \theta \in (0, \theta^\star).$$

Letting $\theta \to \theta^\star$ completes the proof.

(b) Let $\zeta \in (r, \zeta^\star)$. We have

$$\begin{aligned} \mathbb{E}\big(|X|^r \mathbf{1}_{\{|X|>x\}}\big) &= \mathbb{E}\big(|X|^r \mathbf{1}_{\{1 < x^{-\zeta+r}|X|^{\zeta - r}\}}\big) \\ &\le x^{-\zeta+r}\mathbb{E}|X|^\zeta. \end{aligned}$$

Then

$$-\log \bar{F}_r(x) \ge (\zeta - r)\log(x) + C$$

so that by setting $\psi_r(x) = \frac{x}{\zeta - r}$, it follows from Proposition 3.3 (b) that

$$\limsup_{n\to+\infty} \frac{\log \rho_n}{\log(n)} \le \frac{1}{\zeta - r}\frac{r+d}{d}.$$

Letting ζ go to $\zeta^\star$ yields the assertion (3.27) . ☐

Remark 3.2. Note that the choice of the function ψ_r in the statement (a) of Proposition 3.3 does not depend on r as approved in the proof of the item (a) of the previous criterion. But for distributions with polynomial tails the choice of ψ_r clearly depends on r.

We now give more explicit results for specified density functions.

Corollary 3.1. ***(a)*** *Suppose that the density f of X satisfies*

$$f(x) \;\propto\; |x|^c\, e^{-\vartheta|x|^\kappa} \qquad x\in\mathbb{R}^d;\ \vartheta,\kappa>0;\ c>-d. \tag{3.28}$$

Then $\theta^\star=\vartheta$ and

$$\limsup_{n\to+\infty}\frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} \le \frac{c_{r,d}}{\vartheta^{1/\kappa}}\left(1+\frac{r}{d}\right)^{1/\kappa}. \tag{3.29}$$

(b) *If the density of X reads*

$$f(x) \;\propto\; \frac{(\log|x|)^\beta}{|x|^c}\,\mathbf{1}_{\{|x|>1\}} \qquad x\in\mathbb{R}^d,\ \beta\in\mathbb{R},\ c>r+d \tag{3.30}$$

then $\zeta^\star=c-d$ and

$$\limsup_{n\to+\infty}\frac{\log\rho_n}{\log(n)} \le \frac{1}{c-d-r}\,\frac{r+d}{d}. \tag{3.31}$$

(c) *The statement* (3.29) *(resp* (3.31)*) holds if the density of X is simply equivalent to the specified density in* (3.28) *(resp in* (3.30)*).*

Notice that the restriction $c>r+d$ in (3.30) ensures that $\mathbb{E}|X|^r<+\infty$.

Proof. **(a)** We have

$$\begin{aligned}\bar F_r(x)=\mathbb{E}\big(|X|^r\mathbf{1}_{\{|X|>x\}}\big) &= K\int_{\{|u|>x\}}|u|^{r+c}\,e^{-\vartheta|u|^\kappa}d\lambda_d(u)\\ &= K\,V_d\int_x^{+\infty}\rho^{r+c+d-1}\,e^{-\vartheta\rho^\kappa}d\rho\end{aligned}$$

where $K=\int|x|^c e^{-\vartheta|x|^\kappa}d\lambda_d(x)$ is the normalizing positive real constant in (3.28) and V_d denotes the volume of the Euclidean unit ball. Integrating by parts and using usual integral comparison rules yields

$$\bar F_r(x)=C_{d,\vartheta,\kappa}\,x^{r+c+d-\kappa}\,e^{-\vartheta x^\kappa}\big(1+o(1)\big)$$

for a positive real constant $C_{d,\vartheta,\kappa}$. It follows that if $\theta<\vartheta$ then

$$\limsup_{x\to+\infty} e^{\theta x^\kappa}\bar F_r(x)<+\infty$$

and if $\theta>\vartheta$ then

$$\limsup_{x\to+\infty} e^{\theta x^\kappa}\bar F_r(x)=+\infty.$$

Which means that $\theta^\star=\vartheta$ and the statement (3.29) follows from Criterion 3.1 (a).

(b) We have for every $r>0$, for every $x>1$,

$$\bar F_r(x)=\int_{\{|u|>x\}}\frac{(\log|u|)^\beta}{|u|^{c-r}}d\lambda_d(u)=V_d\int_x^{+\infty}\frac{(\log\rho)^\beta}{\rho^{c'}}d\rho.$$

with $c' = c - r - d + 1$ ($c' > 1$) and V_d defined as previously. Integrating by parts and multiplying by $x^{\zeta - r}$ yields

$$x^{\zeta-r}\bar{F}_r(x) = \frac{K\ V_d}{c-(r+d)}\log(x)^{\beta}x^{\zeta-c+d}(1+o(1)).$$

Consequently, $\zeta^\star = c - d$. The statement (3.31) follows from Criterion 3.1 (b).

(c) Obvious from the forgoing. □

We now give some examples for usual distributions.

Example 3.1. **(a)** • If $X \sim \mathcal{N}(0; I_d)$, we have

$$f(x) = (2\pi)^{-d/2}e^{-\frac{1}{2}|x|^2}.$$

It follows from the item (a) of the previous corollary (with $\vartheta = 1/2, \kappa = 2, c = 0$) that for every $r > 0$, for every $d \geq 1$,

$$\limsup_{n\to+\infty} \frac{\rho_n}{\sqrt{\log(n)}} \leq 2\sqrt{2\left(1+\frac{r}{d}\right)}.$$

In particular, when $r \geq 1, d = 1$ we have $\limsup\limits_{n\to+\infty} \frac{\rho_n}{\sqrt{\log(n)}} \leq \sqrt{2(r+1)}$.

• For a double Gamma distribution in the real line where

$$f(x) = \frac{\lambda^a}{2\Gamma(a)}|x|^{a-1}e^{-\lambda|x|}, \quad x \in \mathbb{R};\ \lambda, a > 0$$

or a Gamma distribution for which

$$f(x) = \frac{\lambda^a}{\Gamma(a)}x^{a-1}e^{-\lambda x}\mathbf{1}_{\{x\geq 0\}}, \quad \lambda, a > 0$$

we have (from Corollary 3.1 (a) with $c = a-1, \vartheta = \lambda, \kappa = 1$) for every $r \in (0,1)$,

$$\limsup_{n\to+\infty} \frac{\rho_n}{\log(n)} \leq \frac{2(r+1)}{\lambda}$$

and in case $r \geq 1$ we have

$$\limsup_{n\to+\infty} \frac{\rho_n}{\log(n)} \leq \frac{r+1}{\lambda};$$

which coincides with the sharp rate given in (3.3).

• When X has a logistic distribution with density $f(x) = \frac{e^{-x}}{(1+e^{-x})^2}$ we have

$$f(x)\ \sim\ e^{-x} \qquad \text{as } x \to +\infty.$$

Then, it follows from Corollary 3.1 (c) that $(\rho_n)_{n\geq 1}$ has the same upper asymptotic as the exponential distribution with parameter $\lambda = 1$.

• As concern the Weibull distribution with shape parameter $\kappa > 0$ with density function

$$f(x) = \kappa x^{\kappa-1}e^{-x^\kappa}\mathbf{1}_{\{x\geq 0\}}$$

it follows from Corollary 3.1, (a) (with $\vartheta = 1$) that for $r \in (0,1)$,

$$\limsup_{n\to+\infty} \frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} \leq 2(r+1)^{1/\kappa}$$

and if $r \geq 1$

$$\limsup_{n\to+\infty} \frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} \leq (r+1)^{1/\kappa}.$$

(b) Suppose X is a random variable having a Pareto distribution with index $\gamma > r$. The density function reads

$$f(x) = \gamma x^{-(\gamma+1)} \mathbf{1}_{\{x>1\}}.$$

Then we deduce from Corollary 3.1 (b) (with $d = 1, c = \gamma + 1, \beta = 0$) that

$$\limsup_{n\to+\infty} \frac{\log \rho_n}{\log(n)} \leq \frac{r+1}{\gamma - r}.$$

3.2 Lower estimate

In this section we investigate the asymptotic lower estimate of the maximal radius sequence $(\rho_n)_{n\geq1}$ induced by an L^r-optimal sequence of n-quantizers. First we introduce the family of the (r,s)-distributions which will play a crucial role to obtain the optimal lower estimate for the rate of the maximal radius sequence.

Let $r > 0, s > r$. Since the L^r-norm is increasing, it is clear that for every $s \leq r$ any L^r-optimal sequence of quantizers $(\alpha_n)_{n\geq1}$ is L^s**-rate optimal** $i.e.$

$$\limsup_{n\to+\infty} n^{1/d} \|X - \widehat{X}^{\alpha_n}\|_s < +\infty. \tag{3.32}$$

But if $s > r$ (and $X \in L^s(\mathbb{P})$) this asymptotic rate optimality usually fails . So is always the case when $s > r + d$ and X has a probability distribution f satisfying $\lambda_d(f > 0) = +\infty$, see [5]. But it is established in [11] that some linear transformation of the L^r-optimal quantizers (α_n) makes possible to overcome the critical exponent $r + d$, that is, one can always construct an L^s-rate-optimal sequence of quantizers by a linear transformation of the L^r-optimal sequence of quantizers (α_n). However there is some distributions for which (3.32) holds for every $s \in [r, r+d)$. This leads to the following definition:

Definition 3.1. *Let $s, r > 0, s > r$. A random vector $X \in L^s(\mathbb{P})$ has an (r,s)-**distribution** if any L^r-optimal sequence $(\alpha_n)_{n\geq1}$ is L^s-rate optimal.*

For $\nu \in (0,d)$, sufficient conditions such that X has an $(r, r+\nu)$-distribution are provided in [5]. Let us mention two criterions ensuring that a random vector X has an $(r, r+\nu)$-distribution. The first one deals with distributions with radial tails.

Criterion 3.2. ***(a)*** *Let $d \geq 1$. If $f = h(|\cdot|)$ on $B_{|\cdot|}(0,N)^c$ with $h : (R, +\infty) \to \mathbb{R}_+,\ R \in \mathbb{R}_+$, a decreasing function and $|\cdot|$ any norm on $\mathbb{R}^d$. If*

$$\int f(cx)^{-\frac{r+\nu}{r+d}} dP(x) < +\infty \tag{3.33}$$

for some $c > 1$. Then X has an $(r, r+\nu)$-distribution.
(b) *If $d = 1$ and if $\mathrm{supp}(P) \subset [R_0, +\infty)$ for some $R_0 \in \mathbb{R}$ and $f_{|(R_0',+\infty)}$ is decreasing for $R_0' \geq R_0$. If further Assumption (3.33) holds for some $c > 1$. Then X has an $(r, r+\nu)$-distribution.*

The following criterion works for distributions with non radial tails.

Criterion 3.3. *Let $r > 0,\ \nu \in (0,d),\ P = f \cdot \lambda_d$ and $\int |x|^{r+\eta} P(dx) < +\infty$ for some $\eta > 0$. Assume that $\mathrm{supp}(P)$ is convex and that f satisfies the local growth control assumption : there exists real numbers $\varepsilon \geq 0,\ \eta \in (0,1/2),\ M, C > 0$ such that*

$$\forall x, y \in \mathit{supp}(P),\ |x| \geq M,\ |y-x| \leq 2\eta|x| \implies f(y) \geq C f(x)^{1+\varepsilon}.$$

If

$$\int f(x)^{-\frac{(r+\nu)(1+\varepsilon)}{r+d}} dP(x) < +\infty, \tag{3.34}$$

then X has an $(r, r+\nu)$-distribution.

Furthermore a necessary condition for X (with density f) to have an $(r, r+\nu)$-distribution is (see [5]) :

$$X \text{ has an } (r, r+\nu)\text{-distribution} \implies \int f(x)^{-\frac{(r+\nu)}{d+r}} dP(x) < +\infty. \tag{3.35}$$

It follows from (3.33) and (3.35) that the Gaussian distribution, the Weibull and the Gamma distributions have an $(r, r+\nu)$-distribution if and only if $\nu \in (0,d)$. The Pareto distribution with index $\gamma > r$ has an $(r, r+\nu)$-distribution if and only if $\nu \in (0, \frac{\gamma - r}{\gamma+1})$.

Now, suppose that X has an $(r, r+\nu)$-distribution for some $\nu \in (0,d)$ and set

$$\nu_X^\star := \sup\{\nu > 0 \text{ s.t. } X \text{ has an } (r, r+\nu)\text{-distribution}\}.$$

Note that

$$\{\nu > 0 \text{ s.t. } X \text{ has an } (r, r+\nu)\text{-distribution}\} = (0, \nu_X^\star) \text{ or } (0, \nu_X^\star]$$

and that $X \in L^{r+\nu}(\mathbb{P}),\ \forall \nu \in (0, \nu_X^\star)$. When

$$\{\nu > 0 \text{ s.t. } X \text{ has an } (r, r+\nu)\text{-distribution}\} = \emptyset$$

we set $\nu_X^\star = 0$.

This index $\nu_X^\star$ will play a crucial role to determine the lower bound of the maximal radius sequence. Recall that if X has a density f satisfying $\lambda_d(f > 0) = +\infty$ then a necessary condition for X to have $(r, r+\nu)$-distribution is that $\nu < d$. Which means that $\nu_X^\star \leq d$. However, this inequality may stand strictly as approved by the Pareto distribution with index γ for which $\nu_X^\star = \frac{\gamma - r}{\gamma + 1} < 1$.

We present below two different approaches to get the lower bound for the maximal radius sequence. The first one involves the generalized survival functions $\bar{F}_r$ like for upper bounds and is based on tail estimates. The second one is probably more original. It is based on random quantization: its specificity is to provide a close connection between the sequence $(\rho_n)_{n\geq 1}$ and the maximum of the norm of an $i.i.d$ sequence of random variables with distributions P.

3.2.1 Distribution tail approach

The main result of this section is the following theorem.

Theorem 3.2. *Let $r > 0$ and let X be a $\mathbb{R}^d$-valued random variable with probability distribution P. Let $(\alpha_n)_{n\geq 1}$ be an $L^r(P)$-optimal sequence of n-quantizers. For every $\nu \in (0, \nu_X^\star)$, the following statements hold:*

(a)

$$\limsup_{n\to+\infty}\ \sup_{c>0}\Big(c^{r+\nu}\ n^{\frac{r+\nu}{d}}\bar{F}\big(\rho_n+c\big)\Big)<+\infty. \tag{3.36}$$

(b)

$$\limsup_{n\to+\infty}\ \sup_{u>1}\Big(\big(1-1/u\big)^{r+\nu}\ n^{\frac{r+\nu}{d}}\bar{F}_r\big(u\rho_n\big)\Big)<+\infty. \tag{3.37}$$

Proof. **(a)** Let $c>0$ and let $\nu\in(0,\nu_X^\star)$. Then

$$\mathbb{E}|X-\widehat{X}^{\alpha_n}|^{r+\nu}\geq\mathbb{E}\left(\min_{a\in\alpha_n}|X-a|^{r+\nu}\mathbf{1}_{\{|X|>\rho_n+c\}}\right).$$

On the events $\{|X|>\rho_n+c\}$, we have: $|X|>\rho_n+c>\rho_n\geq|a|,\ \forall a\in\alpha_n$. Then

$$\begin{aligned}\mathbb{E}|X-\widehat{X}^{\alpha_n}|^{r+\nu} &\geq \mathbb{E}\left(\min_{a\in\alpha_n}|X-a|^{r+\nu}\mathbf{1}_{\{|X|>\rho_n+c\}}\right)\\ &\geq \mathbb{E}\left(\min_{a\in\alpha_n}\big(|X|-|a|\big)^{r+\nu}\mathbf{1}_{\{|X|>\rho_n+c\}}\right)\\ &\geq \mathbb{E}\Big(\big(|X|-\rho_n\big)^{r+\nu}\mathbf{1}_{\{|X|>\rho_n+c\}}\Big)\\ &\geq c^{r+\nu}\,\mathbb{P}\left(\{|X|>\rho_n+c\}\right).\end{aligned}\tag{3.38}$$

It follows that

$$\mathbb{E}|X-\widehat{X}^{\alpha_n}|^{r+\nu}\geq\sup_{c>0}\left(c^{r+\nu}\,\mathbb{P}\big(\{|X|>\rho_n+c\}\big)\right).$$

Since X has an $(r,r+\nu)$-distribution we have

$$\limsup_n\ n^{\frac{r+\nu}{d}}\|X-\widehat{X}^{\alpha_n}\|_{r+\nu}^{r+\nu}<+\infty.$$

Which completes the proof.

(b) is proved like (a). Inequality (3.38) becomes: for every $u>1$,

$$\mathbb{E}|X-\widehat{X}^{\alpha_n}|^{r+\nu}\geq\mathbb{E}\Big(\big(|X|-\rho_n\big)^{r+\nu}\mathbf{1}_{\{|X|>u\rho_n\}}\Big)\geq\mathbb{E}\left(|X|^{r+\nu}\left(1-1/u\right)^{r+\nu}\mathbf{1}_{\{|X|>u\rho_n\}}\right).$$

Then,

$$\mathbb{E}|X-\widehat{X}^{\alpha_n}|^{r+\nu}\geq\sup_{u>1}\left[(1-1/u)^{r+\nu}\,\mathbb{E}\left(|X|^{r+\nu}\mathbf{1}_{\{|X|>u\rho_n\}}\right)\right].$$

Inequality (3.37) follows by noticing that $\limsup_n n^{\frac{r+\nu}{d}}\|X-\widehat{X}^{\alpha_n}\|_{r+\nu}^{r+\nu}<+\infty$. □

Like for the upper estimate, given the asymptotic inverse function ϕ of $-\log\bar{F}$ or given an asymptotic lower bound ψ of ϕ satisfying some standard hypotheses specified below, we provide the asymptotic lower estimate for the maximal radius sequence for distributions with exponential tails.

For distributions with polynomial tails, we will rather look for the asymptotic inverse function $\Phi_{r,\nu}$, $\nu\in(0,\nu_X^\star)$ (if any) of $-\log\bar{F}_{r+\nu}(e^x)$ or some asymptotic lower bound of it to provide a lower estimate of $\log\rho_n$.

Proposition 3.4. *Let $r > 0$ and let X be an $\mathbb{R}^d$-valued random variable with distribution P. Suppose that X has an unbounded support. Let $(\alpha_n)_{n\geq 1}$ be an $L^r(P)$-optimal sequence of n-quantizers.*

(a) *If ψ is a measurable nondecreasing function going to $+\infty$ as $x \to +\infty$, regularly varying with index δ and satisfying*

$$\psi(-\log \bar{F}(x)) \leq x + o(x), \tag{3.39}$$

then

$$\liminf_{n\to+\infty} \frac{\rho_n}{\psi(\log(n))} \geq \left(\frac{r+\nu_X^\star}{d}\right)^\delta. \tag{3.40}$$

If $-\log \bar{F}$ is regularly varying of index $1/\delta$ then (3.40) holds with $\psi = (-\log \bar{F})^{\leftarrow}$.

(b) *Let $\nu \in (0, \nu_X^\star)$. If there is a measurable nondecreasing function $\psi_{r,\nu}(x)$ going to $+\infty$ as $x \to +\infty$, regularly varying with index δ and satisfying*

$$\psi_{r,\nu}(-\log \bar{F}_{r+\nu}(e^x)) \leq x + o(x), \tag{3.41}$$

then

$$\liminf_{n\to+\infty} \frac{\log \rho_n}{\psi_{r,\nu}(\log(n))} \geq \left(\frac{r+\nu}{d}\right)^\delta. \tag{3.42}$$

In particular if $-\log \bar{F}_{r+\nu}(e^x)$ has regular variation with index $1/\delta$ then (3.42) holds with $\psi_{r,\nu}(x) = (-\log \bar{F}_{r+\nu}(e^x))^{\leftarrow}$.

Let us provide a few comments on this proposition. We have for every $r > 0$ and for every $x > 0$, $\bar{F}_r(x) > x^r \bar{F}(x)$. Then

$$-\log \bar{F}_r(x) \leq -\log \bar{F}(x) - r\log(x).$$

According to the nondecreasing hypothesis on ψ we have for every $x > 1$

$$\psi(-\log \bar{F}_r(x)) \leq \psi(-\log \bar{F}(x) - r\log(x)) \leq \psi(-\log \bar{F}(x))$$

so that if (3.39) holds then for every $r > 0$, for every $\nu \in (0, \nu_X^\star)$,

$$\psi(-\log \bar{F}_{r+\nu}(x)) \leq x + o(x).$$

Reproducing the proof (given below) of Proposition 3.4 (a) by using (3.37) instead of (3.36) shows that (3.40) still holds true. This means that for distribution with exponential tails, the function ψ do not depend on r and ν even if in Assumption (3.39) we take the generalized survival function $\bar{F}_{r+\nu}$ in place of the regular survival function $\bar{F}$. However for distributions with polynomial tails like Pareto distribution the function $\psi_{r,\nu}$ in (3.41) may depend on r and taking the regular survival function $\bar{F}$ in place of the generalized survival function $\bar{F}_{r+\nu}$ would make lose the dependance upon r and consequently lead to a less accurate result.

We next prove the proposition.

Proof. **(a)** Assume $\nu_X^\star > 0$ and let $\nu \in (0, \nu_X^\star)$. It follows from (3.37) that for large enough n,

$$-\log \bar{F}(\rho_n + c) \geq -\log(C_{\nu,c}) + \frac{r+\nu}{d}\log(n)$$

where $C_{\nu,c}$ is a positive real constant depending on the indexing parameters. It follows from the fact that ψ is nondecreasing and goes to $+\infty$ and from Assumption (3.39) that

$$\frac{\rho_n}{\psi(\log(n))} \geq \left(1+\frac{c}{\rho_n}+\frac{o(\rho_n)}{\rho_n}\right)^{-1} \frac{\psi(\frac{r+\nu}{d}\log(n)-\log(C_{\nu,c}))}{\psi(\log(n))}.$$

Since ψ is regularly varying with index δ we have

$$\liminf_{n\to+\infty} \frac{\rho_n}{\psi(\log(n))} \geq \left(\frac{r+\nu}{d}\right)^{\delta}, \qquad \forall \nu \in (0,\nu_X^\star).$$

Letting $\nu \to \nu_X^\star$ give the announced result. If $\nu_X^\star = 0$, one follows the same proof with $\nu = 0$.

(b) This is proved like the statement **(b)** in Proposition 3.3 by considering $\bar{F}_{r+\nu}$ instead of $\bar{F}_r$, for $\nu \in (0,\nu_X^\star)$. □

The next criterion is the lower limit counterpart of Criterion 3.1.

Criterion 3.4. ***(a)*** *Let X be an $\mathbb{R}^d$-valued random variable with unbounded support and suppose that*

$$\theta_\star = \inf\left\{\theta>0,\ \liminf_{x\to+\infty}\ e^{\theta x^\kappa}\mathbb{P}(|X|>x)>0\right\} \in (0,+\infty]. \tag{3.43}$$

Then

$$\liminf_{n\to+\infty} \frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} \geq \left(\frac{r+\nu_X^\star}{d\,\theta_\star}\right)^{1/\kappa}. \tag{3.44}$$

(b) *Let X be a random variable with unbounded support such that $\nu_X^\star > 0$. Set*

$$\zeta_\star = \inf\left\{\zeta>0,\ \forall\nu\in(0,\nu_X^\star),\ \liminf_{x\to+\infty}\ x^{\zeta-r-\nu}\bar{F}_{r+\nu}(x)>0\right\} \in [r+\nu_X^\star,+\infty]. \tag{3.45}$$

Then

$$\liminf_{n\to+\infty} \frac{\log\rho_n}{\log(n)} \geq \frac{1}{\zeta_\star-r-\nu_X^\star}\,\frac{r+\nu_X^\star}{d}. \tag{3.46}$$

Proof . **(a)** Let $\theta \in (\theta_\star,+\infty)$. Then

$$\bar{F}(x) \geq C\,e^{-\theta x^\kappa}$$

for large enough x and for a positive real constant C. Therefore

$$-\log\bar{F}(x) \leq \theta x^\kappa(1-\frac{\log(C)}{x^\kappa})$$

so that by setting $\psi_\theta(x) = (x/\theta)^{1/\kappa}$ we have

$$\psi_\theta(-\log\bar{F}(x)) \leq x+o(x).$$

It follows from Proposition 3.4 (a) that

$$\liminf_{n\to+\infty} \frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} \geq \frac{1}{\theta^{1/\kappa}}\left(\frac{r+\nu_X^\star}{d}\right)^{1/\kappa}.$$

We let θ go to $\theta_\star$ to get the announced result.

(b) Let $\zeta \in (0, \zeta_\star)$. We have, for every $\nu \in (0, \nu_X^\star)$,

$$\bar{F}_{r+\nu}(x) \geq C_\nu \, x^{-\zeta+r+\nu}$$

for large enough x and for a positive real constant C_ν. Then, by setting $\psi_{r,\nu}(x) = \frac{x}{\zeta-r-\nu}$ we get $\psi_{r,\nu}(-\log \bar{F}_{r+\nu}(x)) \leq \log(x) + o(\log(x))$. It follows from Proposition 3.4 (b) that

$$\liminf_{n\to+\infty} \frac{\log \rho_n}{\log(n)} \geq \frac{1}{\zeta - r - \nu} \frac{r+\nu}{d}.$$

The right hand side of this last inequality is increasing on $(0, \nu_X^\star)$ (as a function of ν) and on $(0, \zeta_\star)$ (as a function of ζ) so that by letting ν and ζ go respectively to $\nu_X^\star$ and $\zeta_\star$ we get

$$\liminf_{n\to+\infty} \frac{\log \rho_n}{\log(n)} \geq \frac{1}{\zeta_\star - r - \nu_X^\star} \frac{r+\nu_X^\star}{d}.$$

□

Corollary 3.2. ***(a)*** *If the density function of X reads*

$$f(x) \;\propto\; |x|^c \, e^{-\vartheta |x|^\kappa} \; x \in \mathbb{R}^d; \; \vartheta, \kappa > 0; \; c > -d \tag{3.47}$$

then $\nu_X^\star = d$ and $\theta^\star = \theta_\star = \vartheta$. In this case we have for every $r > 0$, for every $d \geq 1$,

$$\frac{1}{\vartheta^{1/\kappa}} \left(1 + \frac{r}{d}\right)^{1/\kappa} \leq \liminf_{n\to+\infty} \frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} \leq \limsup_{n\to+\infty} \frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} \leq \frac{2}{\vartheta^{1/\kappa}} \left(1 + \frac{r}{d}\right)^{1/\kappa}. \tag{3.48}$$

When $d = 1$ and $r \geq 1$ we have

$$\lim_{n\to+\infty} \frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} = \left(\frac{r+1}{\vartheta}\right)^{1/\kappa}. \tag{3.49}$$

(b) *If X has a density f satisfying*

$$f(x) \;\propto\; \frac{(\log|x|)^\beta}{|x|^c} \, \mathbf{1}_{\{|x|>1\}} \qquad x \in \mathbb{R}^d, \; \beta \in \mathbb{R}, \; c > r + d \tag{3.50}$$

then $\nu_X^\star = d\left(1 - \frac{r+d}{c}\right) \in (0, d)$ and $\zeta_\star = \zeta^\star = c - d$. Furthermore we have for every $r > 0$ and every $d \geq 1$,

$$\lim_{n\to+\infty} \frac{\log \rho_n}{\log(n)} = \frac{1}{c - r - d} \frac{r+d}{d}. \tag{3.51}$$

(c) *The claim (3.48) (resp (3.51)) holds if the density of X is simply equivalent to the specified density in (3.47) (resp in (3.50)).*

Proof . **(a)** It is obvious from Criterion 3.2 and (3.35) that $\nu^\star_X = d$. Let K be the normalizing positive real constant in (3.47). We have for every $x > 0$,

$$\begin{aligned}\mathbb{P}(|X| > x) &= K \int_{\{|u|>x\}} |u|^c \, e^{-\vartheta |u|^\kappa} d\lambda_d(u)\\ &= K_d \int_x^{+\infty} \rho^{c+d-1} \, e^{-\vartheta \rho^\kappa} d\rho\\ &= K_d \, \frac{x^{c+d-\kappa}}{\vartheta} e^{-\vartheta x^\kappa}(1+o(1))\end{aligned}$$

where we used an integration by parts and usual integral comparison criterions. Consequently, if $\theta > \vartheta$ then

$$\liminf_{x\to+\infty} \, e^{\theta x^\kappa} \mathbb{P}(|X| > x) = +\infty$$

and if $\theta < \vartheta$ then

$$\liminf_{x\to+\infty} \, e^{\theta x^\kappa} \mathbb{P}(|X| > x) = 0.$$

Which means that $\theta_\star = \vartheta$ and the statement (3.48) follows from Criterion 3.1 (a) and Criterion 3.4 (a).

(b) Let $\bar{c} > 1$. We have

$$\int_{\{f>0\}} f(\bar{c}x)^{-\frac{r+\nu}{r+d}} f(x) d\lambda_d(x) = (\bar{c})^{-\frac{r+\nu}{r+d}} \int_{\{|x|>1\}} \frac{(\log|u|)^{\beta'}}{|u|^{c'}} d\lambda_d(u) = K_{d,\bar{c}} \int_1^{+\infty} \frac{(\log\rho)^{\beta'}}{\rho^{c'-d+1}} d\rho$$

with $c' = c(1 - \frac{r+\nu}{r+d})$. $K_{d,\bar{c}}$ is some positive real constant and $\beta' \in \mathbb{R}$. We deduce that if

$$c' > d \iff \nu < d\left(1 - \frac{r+d}{c}\right) \text{ then } \int_{\{f>0\}} f(\bar{c}x)^{-\frac{r+\nu}{r+d}} f(x) d\lambda_d(x) < +\infty$$

and if

$$c' < d \iff \nu > d\left(1 - \frac{r+d}{c}\right) \text{ then } \int_{\{f>0\}} f(x)^{-\frac{r+\nu}{r+d}} f(x) d\lambda_d(x) = +\infty$$

so that (from Criterion 3.2 and Statement (3.35)) $\nu^\star_X = d\left(1 - \frac{r+d}{c}\right)$.

Let us show that $\zeta_\star = c - d$. For every $r > 0$, for every $\nu \in (0, \nu^\star_X)$, for every $x > 1$, integrating by parts and using integral comparison criterions yield

$$\begin{aligned}\bar{F}_{r+\nu}(x) = \mathbb{E}\left(|X|^{r+\nu} \mathbf{1}_{\{|X|>x\}}\right) &= \int_{\{|u|>x\}} \frac{(\log|u|)^\beta}{|u|^{c-r-\nu}} d\lambda_d(u)\\ &= V_d \int_x^{+\infty} \frac{(\log\rho)^\beta}{\rho^{c'}} d\rho\\ &= V_d \, \frac{x^{-c'+1}}{c'-1} \log(x)^\beta (1+o(1))\end{aligned}$$

with $c' := c - r - \nu - d + 1 > 1$. It follows that

$$x^{\zeta-r-\nu} \bar{F}_{r+\nu}(x) = \frac{V_d}{c'-1} \log(x)^\beta x^{\zeta-c+d}(1+o(1))$$

so that for every $r > 0$ and for every $\nu \in (0, \nu_X^\star)$, if $\zeta > c - d$ then

$$\liminf_{x \to +\infty} x^{\zeta - r - \nu} \bar{F}_{r+\nu}(x) = +\infty$$

and if $\zeta < c - d$,

$$\liminf_{x \to +\infty} x^{\zeta - r - \nu} \bar{F}_{r+\nu}(x) = 0.$$

Hence $\zeta_\star = c - d$. It follows from Criterion 3.1 (b) and Criterion 3.4 (b) that

$$\frac{1}{\zeta_\star - r - \nu_X^\star} \frac{r + \nu_X^\star}{d} \leq \liminf_{n \to +\infty} \frac{\log \rho_n}{\log(n)} \leq \limsup_{n \to +\infty} \frac{\log \rho_n}{\log(n)} \leq \frac{1}{c - r - d} \frac{r + d}{d}.$$

Now $\big($recall that $\zeta_\star = c - d$ and $\nu_X^\star = d\left(1 - \frac{r+d}{c}\right)\big)$,

$$\frac{1}{\zeta_\star - r - \nu_X^\star} \frac{r + \nu_X^\star}{d} = \frac{1}{c - r - d} \frac{r + d}{d}$$

so that

$$\lim_{n \to +\infty} \frac{\log \rho_n}{\log(n)} = \frac{1}{c - r - d} \frac{r + d}{d}.$$

(c) Obvious from what forgoes. ☐

We deal now with examples.

Example 3.2. **(1)** It follows from Corollary 3.2 (a) that

- When $X \sim \mathcal{N}(0, I_d)$, for every $r > 0$, for every $d \geq 1$,

$$\sqrt{\frac{2(r+d)}{d}} \leq \liminf_{n \to +\infty} \frac{\rho_n}{\sqrt{\log(n)}} \leq \limsup_{n \to +\infty} \frac{\rho_n}{\sqrt{\log(n)}} \leq 2\sqrt{\frac{2(r+d)}{d}}.$$

In case $d = 1$ and $r \geq 1$ we have

$$\lim_{n \to +\infty} \frac{\rho_n}{\sqrt{\log(n)}} = \sqrt{2(r+1)}.$$

- If $X \sim \Gamma(a, \lambda), a > 0, \lambda > 0$ or if X has a double gamma distribution we have for every $r \geq 1$,

$$\lim_{n \to +\infty} \frac{\rho_n}{\log(n)} = \frac{r+1}{\lambda}$$

(which coincides to the exact rate given in (3.3) for the exponential distribution) and for every $r \in (0, 1)$,

$$\frac{r+1}{\lambda} \leq \liminf_{n \to +\infty} \frac{\rho_n}{\log(n)} \leq \limsup_{n \to +\infty} \frac{\rho_n}{\log(n)} \leq \frac{2(r+1)}{\lambda}.$$

- As concern the logistic distribution, the maximal radius sequence has the same asymptotic as the exponential distribution with parameter $\lambda = 1$ following Corollary 3.2 (c).

- For a Weibull distribution with shape parameter $\kappa > 0$ we have for every $r \geq 1$,

$$\lim_{n \to +\infty} \frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} = (r+1)^{1/\kappa}.$$

For $r \in (0,1)$, one has

$$(r+1)^{1/\kappa} \leq \liminf_{n\to+\infty} \frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} \leq \limsup_{n\to+\infty} \frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} \leq 2(r+1)^{1/\kappa}.$$

(2) Suppose X is a random variable having a Pareto distribution with index $\gamma > r$ where the density reads $f(x) = \gamma x^{-(\gamma+1)}\mathbf{1}_{\{x>1\}}$. It follows from Corollary 3.2 (b) (with $c = \gamma+1$, $d = 1$)

$$\lim_{n\to+\infty} \frac{\log \rho_n}{\log(n)} = \frac{r+1}{\gamma - r}. \tag{3.52}$$

We retrieve of course the sharp rate given in (3.4).

3.2.2 An alternative approach by random quantization approach

Let $X \sim P$. Random quantization is another tool to compute the lower estimate of the maximal radius sequence. It makes a connection between ρ_n and the maximum of an $i.i.d$ sequence of random variables with distributions P.

Theorem 3.3. *Let $r > 0$ and let X be a random variable taking values in $\mathbb{R}^d$ with probability distribution P with $P_a \neq 0$. Assume $(\alpha_n)_{n\geq 1}$ is an $L^r(P)$-optimal sequence of n-quantizers. Let $(X_k)_{k\geq 1}$ be an $i.i.d$ sequence of $\mathbb{R}^d$-valued random variables with probability distribution P. For every $\nu \in (0, \nu_X^\star)$,*

$$\liminf_{n\to+\infty} \big(\rho_n - \mathbb{E}\big(\max_{k\leq [n^{(r+\nu)/d}]} |X_k|\big)\big) \geq -\, C_\nu \tag{3.53}$$

where C_ν is a positive real constant.

Proof. Let $\nu \in (0, \nu_X^\star)$ and let $\widehat{X}_k^{\alpha_n} = \sum_{a\in\alpha_n} a\mathbf{1}_{\{X_k \in C_a(\alpha_n)\}}$. We have,

$$\begin{aligned}
\rho_n &\geq \max_{k\leq m} |\widehat{X}_k^{\alpha_n}| \\
&\geq \sum_{k=1}^m \max_{l\leq m} |\widehat{X}_l^{\alpha_n}|\mathbf{1}_{\{|X_k| > \max_{i\neq k}|X_i|\}} \\
&\geq \sum_{k=1}^m |\widehat{X}_k^{\alpha_n}|\mathbf{1}_{\{|X_k| > \max_{i\neq k}|X_i|\}} \\
&\geq \sum_{k=1}^m \big(|X_k| - |X_k - \widehat{X}_k^{\alpha_n}|\big)\mathbf{1}_{\{|X_k| > \max_{i\neq k}|X_i|\}} \\
&\geq \mathbb{E}\max_{k\leq m}|X_k| - \sum_{k=1}^m \mathbb{E}\left(|X_k - \widehat{X}_k^{\alpha_n}|\mathbf{1}_{\{|X_k| > \max_{i\neq k}|X_i|\}}\right).
\end{aligned}$$

Furthermore,

$$\forall k \geq 1, \qquad |X_k - \widehat{X}_k^{\alpha_n}|\mathbf{1}_{\{|X_k| > \max_{i\neq k}|X_i|\}} \overset{\mathscr{L}}{=} |X_1 - \widehat{X}_1^{\alpha_n}|\mathbf{1}_{\{|X_1| > \max_{i\neq 1}|X_i|\}}.$$

Hence,

$$\begin{aligned}
\rho_n &\geq \mathbb{E}\max_{k\leq m}|X_k| - m\,\mathbb{E}\left(|X_1 - \widehat{X}_1^{\alpha_n}|\mathbf{1}_{\{|X_1| > \max_{i\neq 1}|X_i|\}}\right) \\
&\geq \mathbb{E}\max_{k\leq m}|X_k| - m\,\|X_1 - \widehat{X}_1^{\alpha_n}\|_{r+\nu}\left(\mathbb{P}\big(|X_1| > \max_{i\neq 1}|X_i|\big)\right)^{1-1/(r+\nu)}.
\end{aligned}$$

Since the events

$$\{|X_k| > \max_{i\neq k}|X_i|\},\ k = 1,\cdots,m$$

are pairwise disjoint with the same probability we have

$$\mathbb{P}\big(|X_1| > \max_{i\neq 1}|X_i|\big) \leq \frac{1}{m}.$$

Finally,

$$\rho_n \geq \mathbb{E}\max_{k\leq m}|X_k| - m^{\frac{1}{r+\nu}}\ \|X_1 - \widehat{X}_1^{\alpha_n}\|_{r+\nu}.$$

It follows, by setting $m = [n^{(r+\nu)/d}]$, that

$$\liminf_{n\to+\infty}\big(\rho_n - \mathbb{E}\big(\max_{k\leq[n^{(r+\nu)/d}]}|X_k|\big)\big) \geq -\limsup_{n\to+\infty} n^{\frac{1}{d}}\|X_1 - \widehat{X}_1^{\alpha_n}\|_{r+\nu}.$$

However, since X has an $(r, r+\nu)$-distribution, the upper limit on the right hand side of the equation is finite. □

Example 3.3. (Exponential distribution) Let $r > 0$ and let X be an exponentially distributed random variable with parameter $\lambda > 0$. If $(\alpha_n)_{n\geq 1}$ is an L^r-optimal sequence of n-quantizers for X then Theorem 3.3 implies

$$\liminf_{n\to+\infty}\ \frac{\rho_n}{\log(n)} \geq \frac{r+1}{\lambda}. \tag{3.54}$$

which is the sharp rate given by (3.3).

Indeed, let $\nu \in (0, \nu_X^\star)$ and let $(X_i)_{\{i=1,\cdots,[n^{r+\nu}]\}}$, be an $i.i.d$ exponentially distributed sequence of random variables with parameter λ. We have for every $u \geq 0$,

$$\mathbb{P}(\max_{i\leq[n^{r+\nu}]}X_i \geq u) = 1 - \mathbb{P}(X_1 \leq u)^{[n^{r+\nu}]} = 1 - F(u)^{[n^{r+\nu}]},$$

where F is the distribution function of X (we will denote by f its density function). Then

$$\begin{aligned}
\mathbb{E}\big(\max_{i\leq[n^{r+\nu}]}X_i\big) &= \int_0^{+\infty}\mathbb{P}(\max_{i\leq[n^{r+\nu}]}X_i \geq u)du = \int_0^{+\infty}(1 - (1-e^{-\lambda u})^{[n^{r+\nu}]})du \\
&= \int_0^{+\infty}\Big(\sum_{i=0}^{[n^{r+\nu}]-1}F(u)^i\Big)e^{-\lambda u}du \\
&= \int_0^{+\infty}\Big(1 + F(u) + \cdots + F(u)^{[n^{r+\nu}]-1}\Big)\frac{f(u)}{\lambda}du \\
&= \frac{1}{\lambda}(1 + \frac{1}{2} + \cdots + \frac{1}{[n^{r+\nu}]}) \\
&\geq \frac{1}{\lambda}\log(1 + [n^{r+\nu}]) \ \geq\ \frac{r+\nu}{\lambda}\log(n).
\end{aligned}$$

Consequently, it follows from the super-additivity of the liminf that for every $\nu \in (0,1)$,

$$\begin{aligned}
\liminf_{n\to+\infty}\frac{\rho_n}{\log(n)} &\geq \liminf_{n\to+\infty}\frac{\rho_n - \mathbb{E}\big(\max_{i\leq[n^{r+\nu}]}\big)}{\log(n)} + \liminf_{n\to+\infty}\frac{\mathbb{E}\big(\max_{i\leq[n^{r+\nu}]}\big)}{\log(n)} \\
&\geq \frac{r+\nu}{\lambda}.
\end{aligned}$$

The result follows by letting ν go to $\nu_X^\star = 1$.

Example 3.4. (Pareto distribution) Let X be a random variable having a Pareto distribution with index $\gamma > 0$. If $(\alpha_n)_{n\geq 1}$ is an asymptotically L^r-optimal sequence of n-quantizers for X, r is such that $\gamma > r$, then Theorem 3.3 yields

$$\liminf_{n\to+\infty} \frac{\log \rho_n}{\log(n)} \geq \frac{r+1}{\gamma+1}.$$

Which is not the sharp rate given by (3.4).

Notice that if $\gamma > r$ then $X \in L^{r+\eta}(\mathbb{P})$ for $\eta \in \big(0, \gamma - r\big)$. Now, to prove this result, let $\nu \in (0, \nu_X^\star)$ and let $(X_i)_{\{i=1,\cdots,[n^{r+\nu}]\}}$ be an $i.i.d$ sequence of random variables with Pareto distribution with index γ. We have

$$\forall m \geq 1, \forall u \geq 1, \quad \mathbb{P}(\max_{i\leq m} X_i \leq u) = (1-u^{-\gamma})^m.$$

Then, the density function of $\max_{1\leq i\leq m} X_i$ is $m\gamma u^{-(\gamma+1)}(1-u^{-\gamma})^{m-1}$.

Hence

$$\begin{aligned}
\mathbb{E}\big(\max_{1\leq i\leq m} X_i\big) &= m\gamma \int_1^{+\infty} x^{-\gamma}(1-x^{-\gamma})^{m-1}dx \\
&= m\int_0^1 u^{-1/\gamma}(1-u)^{m-1}du \qquad (u = x^{-\gamma}) \\
&= m\, B(1-\frac{1}{\gamma}, m) \\
&= \frac{\Gamma(1-\frac{1}{\gamma})\Gamma(m+1)}{\Gamma(m+1-\frac{1}{\gamma})} \\
&\sim \Gamma(1-\frac{1}{\gamma})\, m^{\frac{1}{\gamma}} \qquad \text{as } m \to +\infty
\end{aligned}$$

where we used Stirling's formula for the last statement. We finally set $m = [n^{r+\nu}]$ to get

$$\mathbb{E}\big(\max_{1\leq i\leq [n^{r+\nu}]} X_i\big) \sim \Gamma(1-\frac{1}{\gamma})\, n^{\frac{r+\nu}{\gamma}}.$$

It follows from (3.53) that for every $\varepsilon \in (0,1)$,

$$\rho_n - (1-\varepsilon)\Gamma(1-\frac{1}{\gamma})\, n^{\frac{r+\nu}{\gamma}} \geq -\varepsilon - \mathrm{C}_\nu.$$

Dividing both side of the inequality by $n^{\frac{r+\nu}{\gamma}}$ and taking the logarithm yields

$$\log \rho_n - \frac{r+\nu}{\gamma}\log(n) \geq \log\left((1-\varepsilon)\Gamma(1-\frac{1}{\gamma}) - (\varepsilon + \mathrm{C}_\nu)n^{-\frac{r+\nu}{\gamma}}\right).$$

Consequently

$$\liminf_{n\to+\infty} \frac{\log \rho_n}{\log(n)} \geq \frac{r+\nu}{\gamma}$$

for every $\nu \in (0, \nu_X^\star)$. The announced result follows by letting ν go to $\nu_X^\star = \frac{\gamma - r}{\gamma+1}$.

Comment. Let ϕ be the inverse (if any) function of $-\log \bar{F}$. It can be noticed that in both previous examples we have

$$\mathbb{E}\big(\max_{k\le[n^{r+\nu_X^\star}]}|X_k|\big) \sim \phi((r+\nu_X^\star)\log(n)) \qquad \text{as } n\to+\infty \tag{3.55}$$

which, for distributions with exponential tail leads to the same asymptotic lower bound for the sequence $(\rho_n)_{n\ge1}$ as in (3.40). For Pareto distribution, using the approximation (3.55) to compute the asymptotic lower estimate of the maximal radius sequence make us loose the "$-r$" term in the exact asymptotic. To recover this reminding term we have simply to consider the inverse function of $-\log \bar{F}_{r+\nu_X^\star}$ (as done in the previous section) instead of $-\log \bar{F}$, and, the random quantization approach clearly does not allow us to do so.

3.2.3 A conjecture about the sharp rate

The previous results related to distributions with exponential tails strongly suggest the following conjecture: suppose X is a distribution with exponential tail in the sense of claim (3.43). Then for every $r>0$, for every $d\ge1$,

$$\lim_{n\to+\infty}\frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} = \Big(\frac{r+d}{d\,\theta^\star}\Big)^{1/\kappa}.$$

This conjecture is proved for $d=1$ and $r\ge1$. To be satisfied for hight dimension we need to proof that the geometric statement (3.9) of Lemma 3.1 (a) holds true with "$1+\varepsilon$" instead of "$2+\varepsilon$" like in 1-dimension. Although this inequality looks quite intuitive in any dimension its proof seems out of reach when $d\ge2$.

3.2.4 Numerical experiments

We now attempt to focus on numerical experiment of the maximal radius sequence $(\rho_n)_{n\ge1}$ for the quadratic optimal quantizers of the Gaussian, the Weibull and the exponential distributions. A whole package of quadratic optimal n-quantizers of the $\mathcal{N}(0,I_d)$ distributions are available in the website

`www.quantize.maths-fi.com`

for $d\in\{1,\cdots,10\}$ and $n\in\{1,\cdots,5000\}$. When $d=1$, these L^2-optimal grids are obtained by the Newton method, see e.g. [10] for details. For the exponential distribution the quadratic optimal quantizers are computed by using the semi-closed formulae given in Proposition 3.1.

As concern the Weibull distribution with shape parameter $\kappa=2$, we compute the quadratic optimal quantizers up to 3000 using the Lloyd's I algorithm described in [10] (see [8] for a more itemized description of the algorithm).

In these three cases we depicted the ratio between ρ_n and the expected asymptotic optimal rate. For the exponential distribution we represent the graph of $\frac{\rho_n}{3\log(n)}$ as a function of the grid sizes (see Figure 1). One remarks that the convergence of $\frac{\rho_n}{3\log(n)}$ to 1 as n goes to infinity is almost instantaneous.

However, the cases of the Gaussian and the Weibull distributions are more delicate. Indeed, for the Gaussian distribution the ratio $\frac{\rho_n}{\sqrt{6\log(n)}}$ seems increasing but has not reached yet the value 9 even for a grid size equals 100000, as emphasized by Figure 1 (right hand side graph). For the Weibull distribution, $\frac{\rho_n}{\sqrt{3\log(n)}}$ also seems increasing but takes values around 0.927 for a grid size equal to 3000 (see Figure 2). Then for both cases, the convergence to 1 of the ratio between the maximal radius and the expected asymptotic optimal rate seems increasing but very low.

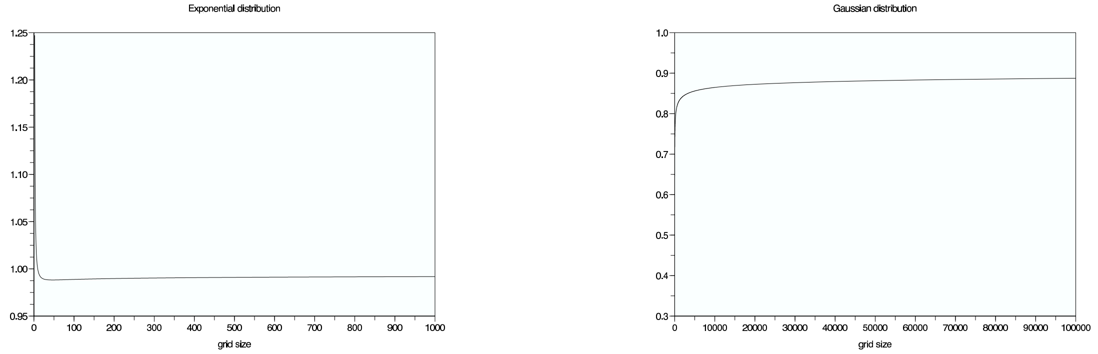

Figure 1: Left: $\frac{\rho_n}{3\log(n)}$ as a function of the grid size n for the exponential distribution. Right: $\frac{\rho_n}{\sqrt{6\log(n)}}$ as a function of the grid size n for the normal distribution.

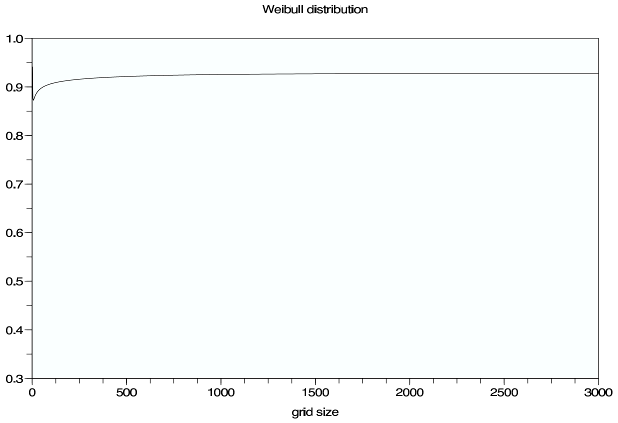

Figure 2: $\frac{\rho_n}{\sqrt{3\log(n)}}$ as a function of the grid size for the Weibull distribution with shape parameter $\kappa = 2$.

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