Exponential Instability of Skew-Evolution Semiflows in Banach Spaces

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Abstract. This paper emphasizes a couple of characterizations for the exponential instability property of skew-evolution semiflows in Banach spaces, defined by means of evolution semiflows and evolution cocycles. Some Datko type results for this asymptotic behavior are proved. There is provided a unified treatment for the uniform case.

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1 Introduction

The study of the asymptotic behaviors of skew-product semiflows, that has witnessed lately an impressive development, has been used in the theory of evolution equations in infinite dimensional spaces. It was essential that the theory was approached from point of view of asymptotic properties for the evolution semigroup associated to the skew-product semiflows. Some results on the instability of skew-product flows can be found in [2].

A particular concept of skew-evolution semiflow introduced by us in [3] is considered to be more interesting for the study of evolution equations connected to the theory of evolution operators. Some asymptotic behaviors for skew-evolution semiflows have been presented in [4].

In this paper we emphasize the property of exponential instability for skew-evolution semiflows defined by means of evolution semiflows and evolution cocycles. We remark that Theorem 3.1 and Theorem 3.2 of this paper are approaches for the uniform exponential instability property, extending Theorem 11 from [1] concerning the case of the property of uniform exponential stability. The skew-evolution semiflows considered in this paper are not necessary strongly continuous.

2 Notations and definitions. Preliminary results

Let us consider X a metric space, V a Banach space, $\mathcal{B}(V)$ the space of all bounded operators from V into itself. We denote $\mathcal{T} = \left\{(t, t_0) \in \mathbb{R}_+^2 : t \geq t_0\right\}$, respectively $Y = X \times V$ and the norm of vectors on V and operators on $\mathcal{B}(V)$ is denoted by $\|\cdot\|$. Let I be the identity operator on V.

Definition 2.1 A mapping $\varphi : \mathcal{T} \times X \to X$ is called *evolution semiflow* on X if it satisfies the following properties

(s_1) $\varphi(t, t, x) = x, \ \forall (t, x) \in \mathbb{R}_+ \times X$

(s_2) $\varphi(t, s, \varphi(s, t_0, x)) = \varphi(t, t_0, x), \ \forall (t, s), (s, t_0) \in \mathcal{T}, \ \forall x \in X.$

Definition 2.2 A mapping $\Phi : \mathcal{T} \times X \to \mathcal{B}(V)$ that satisfies the following properties

(c_1) $\Phi(t, t, x) = I, \ \forall t \geq 0, \ \forall x \in X$

(c_2) $\Phi(t, s, \varphi(s, t_0, x))\Phi(s, t_0, x) = \Phi(t, t_0, x), \ \forall (t, s), (s, t_0) \in \mathcal{T}, \ \forall x \in X$

(c_3) there exists a nondecreasing function $f : \mathbb{R}_+ \to \mathbb{R}_+^*$ such that

$$\|\Phi(t, t_0, x)\| \leq f(t - t_0), \ \forall t \geq t_0 \geq 0, \ \forall x \in X \tag{2.1}$$

is called *evolution cocycle* over the evolution semiflow φ.

Definition 2.3 A function $\xi : \mathcal{T} \times Y \to Y$ defined by

$$\xi(t, s, x, v) = (\varphi(t, s, x), \Phi(t, s, x)v), \ \forall (t, s, x, v) \in \mathcal{T} \times Y \tag{2.2}$$

where Φ is an evolution cocycle over the evolution semiflow φ, is called *skew-evolution semiflow* on Y.

Example 2.1 Let $f : \mathbb{R} \to \mathbb{R}_+$ be a function which is nondecreasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$ such that there exist

$$\lim_{t \to \pm\infty} f(t) = l > 0.$$

We consider the metric space

$$\mathcal{C}(\mathbb{R}, \mathbb{R}) = \{h : \mathbb{R} \to \mathbb{R} \mid h \text{ continuous}\},$$

with the topology of uniform convergence on compact subsets of $\mathbb{R}$.

Let X be the closure in $\mathcal{C}(\mathbb{R}, \mathbb{R})$ of the set of all functions $f_t, \ t \in \mathbb{R}$, where $f_t(\tau) = f(t + \tau), \ \forall \tau \in \mathbb{R}$. Then X is a metric space and the mapping

$$\varphi : \mathcal{T} \times X \to X, \ \varphi(t, s, x) = x_{t-s}$$

is an evolution semiflow on X.

Let $V = \mathbb{R}^2$ be a Banach space with the norm $\|(v_1, v_2)\| = |v_1| + |v_2|$. The mapping $\Phi : \mathcal{T} \times X \to \mathcal{B}(V)$ given by

$$\Phi(t, s, x)(v_1, v_2) = \left(e^{\alpha_1 \int_s^t x(\tau - s)d\tau} v_1, e^{\alpha_2 \int_s^t x(\tau - s)d\tau} v_2\right)$$

where $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ is fixed, is an evolution cocycle and $\xi = (\varphi, \Phi)$ is a skew-evolution semiflow on Y.

We introduce a particular class of skew-evolution semiflows in the next

Definition 2.4 A skew-evolution semiflow $\xi = (\varphi, \Phi)$ has *uniform exponential decay* if there exist $N > 1$ and $\omega > 0$ such that

$$\|\Phi(s, t_0, x)v\| \leq Ne^{\omega(t-s)} \|\Phi(t, t_0, x)v\|, \ \forall t \geq s \geq t_0 \geq 0, \ \forall (x, v) \in Y. \tag{2.3}$$

A characterization of the uniform exponential decay is given by

Proposition 2.1 *The skew-evolution semiflow $\xi = (\varphi, \Phi)$ has uniform exponential decay if and only if there exists a decreasing function $g : [0, \infty) \to (0, \infty)$ with the properties $\lim_{t\to\infty} g(t) = 0$ and*

$$\|\Phi(t, t_0, x)v\| \geq g(t - t_0) \|v\|, \ \forall t \geq t_0 \geq 0, \ \forall (x, v) \in Y.$$

Proof. *Necessity.* It is a simple verification.

Sufficiency. According to the property of function g, there exists a constant $\lambda > 0$ such that $g(\lambda) < 1$. For all $t \geq t_0 \geq 0$, there exist $n \in \mathbb{N}$ and $r \in [0, \lambda)$ such that

$$t - t_0 = n\lambda + r.$$

Following inequalities

$$\|\Phi(t, t_0, x)v\| \geq g(r) \|\Phi(t_0 + n\lambda, t_0, x)v\| \geq ... \geq g(\lambda)^{n+1} \|v\| \geq N_1 e^{-\omega(t-t_0)} \|v\|$$

hold for all $(x, v) \in Y$, where we have denoted

$$N_1 = g(\lambda) \text{ and } \omega = -\lambda^{-1} \ln g(\lambda).$$

The property of uniform exponential decay for ξ is thus obvious. □

Definition 2.5 A skew-evolution semiflow $\xi = (\varphi, \Phi)$ is said to be *uniformly instable* if there exists $N > 1$ such that

$$\|\Phi(s, t_0, x)v\| \leq N \|\Phi(t, t_0, x)v\|, \ \forall t \geq s \geq t_0 \geq 0, \ \forall (x, v) \in Y.$$

We can obtain a characterization of the former property as in the next

Proposition 2.2 *A skew-evolution semiflow $\xi = (\varphi, \Phi)$ with uniform exponential decay is uniformly instable if there exists $M > 1$ such that*

$$M \|\Phi(t, t_0, x)v\| \geq \|\Phi(s, t_0, x)v\|, \ \forall t \geq s + 1 > s \geq t_0 \geq 0, \ \forall (x, v) \in Y.$$

Proof. Let us consider a function g is given as in Proposition 2.1. Then there exists $\lambda > 1$ such that $g(\lambda) < 1$.

Let $s \geq 0$. For all $t \in [s, s+1)$, by the same result, we obtain following inequalities

$$\|\Phi(t, t_0, x)v\| \geq g(t-s) \|\Phi(s, t_0, x)v\| \geq g(\lambda) \|\Phi(s, t_0, x)v\| .$$

Hence, if we denote

$$N = \max \left\{M, g(\lambda)^{-1}\right\} > 1,$$

the property of uniform instability for ξ has been proved. □

Definition 2.6 A skew-evolution semiflow $\xi = (\varphi, \Phi)$ is called *uniformly exponentially instable* if there exist $N > 1$ and $\nu > 0$ with the property

$$\|\Phi(s, t_0, x)v\| \leq Ne^{-\nu(t-s)} \|\Phi(t, t_0, x)v\| , \ \forall t \geq s \geq t_0 \geq 0, \ \forall (x, v) \in Y. \tag{2.4}$$

Example 2.2 We consider the metric space X and an evolution semiflow on X defined as in Example 2.1.

Let $V = \mathbb{R}$. We consider $\Phi : \mathcal{T} \times X \to \mathcal{B}(V)$ given by

$$\Phi(t, t_0, x)v = e^{\int_{t_0}^{t} x(\tau - t_0)d\tau} v$$

which is an evolution cocycle. Then the skew-evolution semiflow $\xi = (\varphi, \Phi)$ is uniformly exponentially instable with $N = 1$ and $\nu = l > 0$.

The following result are characterizations for the property of uniform exponential instability, by means of other asymptotic properties and, also, of a special class of skew-evolution semiflow.

Proposition 2.3 *A skew-evolution semiflow $\xi = (\varphi, \Phi)$ with uniform exponential decay is uniformly exponentially instable if and only if there exists a decreasing function $h : [0, \infty) \to (0, \infty)$ with property $\lim_{t\to\infty} h(t) = 0$ such that*

$$\|v\| \leq h(t - t_0) \|\Phi(t, t_0, x)v\| , \ \forall t \geq t_0 \geq 0, \ \forall (x, v) \in Y.$$

Proof. It is similar with the proof of Proposition 2.1. □

Proposition 2.4 *A skew-evolution semiflow $\xi = (\varphi, \Phi)$ has uniform exponential decay if and only if there exists a constant $\alpha > 0$ such that the a skew-evolution semiflow $\xi_{-\alpha} = (\varphi, \Phi_{-\alpha})$, where $\Phi_{-\alpha}(t, t_0, x) = e^{\alpha(t-t_0)}\Phi(t, t_0, x)$, $(t, t_0) \in \mathcal{T}, \ x \in X$, is uniformly exponentially instable.*

Proof. *Necessity.* If ξ has uniform exponential decay then there exist $M \geq 1$ and $\omega > 0$ such that

$$e^{-\omega(t-s)} \|\Phi(s,t_0,x)v\| \leq M \|\Phi(t,t_0,x)v\|, \ \forall t \geq s \geq t_0 \geq 0, \ \forall (x,v) \in \mathcal{Y}.$$

We consider $\alpha = 2\omega > 0$ and we obtain

$$e^{\omega(t-s)} \|\Phi_{-\alpha}(s,t_0,x)v\| \leq M \|\Phi_{-\alpha}(t,t_0,x)v\|$$

for all $t \geq s \geq t_0 \geq 0$ and all $(x,v) \in \mathcal{Y}$, which shows that $\xi_{-\alpha}$ is uniformly exponentially instable.

Suficiency. If there exists $\alpha > 0$ such that $\xi_{-\alpha}$ is uniformly exponentially instable, then there exist $N > 1$ $\beta > 0$ such that

$$N \|\Phi_{-\alpha}(t,t_0,x)v\| = Ne^{\alpha(t-t_0)} \|\Phi(t,t_0,x)v\| \geq$$

$$\geq e^{\beta(t-s)} e^{\alpha(s-t_0)} \|\Phi(s,t_0,x)v\| = e^{\beta(t-s)} \|\Phi_{-\alpha}(s,t_0,x)v\|.$$

It follows that

$$N \|\Phi(t,t_0,x)v\| \geq e^{(\beta-\alpha)(t-s)} \|\Phi(s,t_0,x)v\| \geq e^{-\nu(t-s)} \|\Phi(s,t_0,x)v\|$$

for all $t \geq s \geq t_0 \geq 0$ and all $(x,v) \in \mathcal{Y}$, where we have denoted

$$\nu = \begin{cases} \alpha - \beta, & \text{if } \alpha > \beta \\ 1, & \text{if } \alpha \leq \beta \end{cases}$$

Hence, the uniform exponential decay for ξ is proven. □

A connection between the asymptotic behaviors of skew-evolution semiflows presented in Definition 2.4, Definition 2.6 and Definition 2.5 is given by

Remark 2.1 *The property of uniform exponential instability of a skew-evolution semiflow implies the uniform instability and, further, the uniform exponential decay.*

3 The main results

In this section we will give two characterizations for the property of uniform exponential instability in the case of a particular class of skew-evolution semiflows introduced by the following

Definition 3.1 A skew-evolution semiflow $\xi = (\varphi, \Phi)$ is called *strongly measurable* if the mapping $t \mapsto \|\Phi(t,t_0,x)v\|$ is measurable on $[t_0,\infty)$, for all $(t_0,x,v) \in \mathbb{R}_+ \times Y$.

Theorem 3.1 *A strongly measurable skew-evolution semiflow $\xi = (\varphi, \Phi)$ is uniformly exponentially instable if and only if it is uniformly instable and there exists $M \geq 1$ such that*

$$\int_{t_0}^{t} \|\Phi(s,t_0,x)v\|\, ds \leq M \|\Phi(t,t_0,x)v\|\,,\ \forall t \geq t_0 \geq 0,\ \forall (x,v) \in Y. \quad (3.1)$$

Proof. *Necessity.* Let ξ be uniformly exponentially instable. According to Remark 2.1, as ξ is also uniformly instable, there exist $N > 1$ and $\nu > 0$ such that

$$\int_{t_0}^{t} \|\Phi(s,t_0,x)v\|\, ds \leq N \|\Phi(t,t_0,x)v\| \int_{t_0}^{t} e^{-\nu(t-s)} ds \leq M \|\Phi(t,t_0,x)v\|$$

for all $t \geq t_0 \geq 0$ and all $(x,v) \in Y$, where we have denoted $M = N\nu^{-1}$.

Sufficiency. As ξ is uniformly instable, there exists $N > 1$ such that following inequality holds

$$\|v\| \leq N \|\Phi(\tau,t_0,x)v\|\,, \forall \tau \geq t_0 \geq 0$$

and, further, by hypothesis, there exists $M \geq 1$ such that

$$(t-s)\|\Phi(s,t_0,x)v\| \leq N \int_{s}^{t} \|\Phi(\tau,t_0,x)v\|\, d\tau \leq MN \|\Phi(t,t_0,x)v\|$$

for all $t \geq s \geq t_0 \geq 0$ and all $(x,v) \in Y$.

It follows that

$$\|v\| \leq \frac{MN}{(t-t_0+1)} \|\Phi(t,t_0,x)v\|\,,\ \forall t \geq t_0 \geq 0,\ \forall (x,v) \in Y.$$

As by Remark 2.1 ξ has also exponential decay, then, according to Proposition 2.3, the property of uniformly exponentially instability for ξ is obtained. $\square$

Theorem 3.2 *A strongly measurable skew-evolution semiflow $\xi = (\varphi, \Phi)$ is uniformly exponentially instable if and only if it has uniform exponential decay and there exists $M \geq 1$ such that relation* (3.1) *hold.*

Proof. Let function g be given as in Proposition 2.1.

Following relations hold for all $t \geq s+1 > s \geq t_0 \geq 0$ and all $(x,v) \in Y$

$$\|\Phi(s,t_0,x)v\| \int_{0}^{1} g(\tau) d\tau = \int_{s}^{s+1} g(u-s) \|\Phi(s,t_0,x)v\|\, du \leq$$

$$\leq \int_{s}^{s+1} \|\Phi(u,t_0,x)v\|\, du \leq \int_{t_0}^{t} \|\Phi(u,t_0,x)v\|\, du \leq M \|\Phi(t,t_0,x)v\|\,.$$

The property of uniform instability for ξ is obtained by Proposition 2.2, where we have considered

$$N = \int_0^1 g(\tau)d\tau,$$

Further, by Theorem 3.1 the proof is concluded. □

As a conclusion we obtain the next

Corollary 3.1 *Let ξ be a skew-evolution semiflow for which there exists $M \geq 1$ such that relation* (3.1) *hold. Following properties are equivalent:*
(*i*) *ξ has uniform exponential decay;*
(*ii*) *ξ is uniformly instable;*
(*iii*) *ξ is uniformly exponentially instable.*

Proof. It is obtained according to Theorem 3.1, Theorem 3.2 and Remark 2.1. □

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