ON UNIFORM EXPONENTIAL TRICHOTOMY OF EVOLUTION OPERATORS IN BANACH SPACES

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Abstract. This paper presents necessary and sufficient conditions for uniform exponential trichotomy of nonlinear evolution operators in Banach spaces. Thus are obtained results which extend well-known results for uniform exponential stability in the linear case.

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1 Introduction

One of the most notable results in the theory of stability of linear evolution operators has been proved by Datko in [2]. Generalizations of this result were obtained in [1], [5], [7] and [14] for exponential stability, in [9] and [11] for exponential instability and in [8], [10] and [15] for the case of exponential dichotomy.

In this paper we shall extend these results in two directions. First, we shall consider the case of uniform exponential trichotomy property ([3], [4], [6], [13]) and second, we shall not assume the linearity of evolution operators.

A unified treatment for uniform asymptotic behaviors (exponential decay, exponential growth, exponential stability, exponential instability, exponential dichotomy, exponential trichotomy) of nonlinear evolution operators is given. Examples that motivate the extension of the asymptotic behaviors for the nonlinear case are given in [5]. In our paper we obtain some theorems which extend well-known results for uniform exponential stability established in the linear case.

Let X be a real or complex Banach space. The norm on X will be denoted by $\|\cdot\|$. The set of all mappings from X into itself is denoted by $\mathfrak{F}(X)$. Let T be the set of all pairs (t, t_0) of real numbers with the property $t \geq t_0 \geq 0$.

2 Evolution operators

Definition 2.1 *A mapping* $E : T \to \mathfrak{F}(X)$ *is called evolution operator on* X *if it has the property*

$$E(t,s)E(s,t_0) = E(t,t_0),\ \forall (t,s),(s,t_0) \in T. \tag{2.1}$$

In order to emphasize the necessity of extending the study of evolution operators in the nonlinear setting, we will consider the next

Example 2.1 *Let us consider the Cauchy problem*

$$\begin{cases} \dot{v}(t) = Av(t),\ t > 0 \\ v(0) = v_0 \end{cases}$$

on a Banach space X *with nonlinear operator* A*. If* A *generates a nonlinear strongly continuous semigroup* $(S(t))_{t\geq 0}$*, then* $E(t,s) = S(t-s)$*, where* $t \geq s \geq 0$*, defines an evolution operator on* X*.*

Definition 2.2 *The evolution operator* $E : T \to \mathfrak{F}(X)$ *is said to be with*
(i) uniform exponential decay if there exist $M > 1$ *and* $\omega > 0$ *such that*

$$\|E(s,t_0)x\| \leq Me^{\omega(t-s)} \|E(t,t_0)x\| \tag{2.2}$$

for all $t \geq s \geq t_0 \geq 0$ *and all* $x \in X$;
(ii) uniform exponential growth if there are $M > 1$ *and* $\omega > 0$ *such that*

$$\|E(t,t_0)x\| \leq Me^{\omega(t-s)} \|E(s,t_0)x\| \tag{2.3}$$

for all $t \geq s \geq t_0 \geq 0$ *and all* $x \in X$.

Lemma 2.1 *The evolution operator* $E : T \to \mathfrak{F}(X)$ *has uniform exponential decay if and only if there exists a nondecreasing function* $f : [0,\infty) \to (1,\infty)$ *with the property*

$$\lim_{t\to\infty} f(t) = \infty$$

such that

$$\|E(s,t_0)x\| \leq f(t-s) \|E(t,t_0)x\|$$

for all $t \geq s \geq t_0 \geq 0$ *and all* $x \in X$.

Proof.

Necessity. It follows from Definition 2.2 (i) for $f(t) = Me^{\omega t}$.

Sufficiency. If $t \geq s \geq t_0 \geq 0$ then there exists a natural number n such that $n \leq t - s < n+1$. If we denote $M = f(1)$ and $\omega = \ln M$, then by hypothesis we have

$$\|E(s,t_0)x\| \leq M \|E(s+1,t_0)x\| \leq M^2 \|E(s+2,t_0)x\| \leq$$

$$\leq M^n \|E(s+n,t_0)x\| \leq M^{n+1} \|E(t,t_0)x\| =$$
$$= Me^{n\omega} \|E(t,t_0)x\| \leq Me^{\omega(t-s)} \|E(t,t_0)x\|$$

for all $t \geq s \geq t_0 \geq 0$ and all $x \in X$.

Finally, we conclude that E has exponential decay. □

Lemma 2.2 *The evolution operator* $E : T \to \mathfrak{F}(X)$ *has uniform exponential growth if and only if there exists a nondecreasing function* $g : [0,\infty) \to (1,\infty)$ *with the property*

$$\lim_{t\to\infty} g(t) = \infty$$

such that

$$\|E(t,t_0)x\| \leq g(t-s) \|E(s,t_0)x\|$$

for all $t \geq s \geq t_0 \geq 0$ *and all* $x \in X$.

Proof. It is similar with the proof of Lemma 2.1. □

3 Uniform exponential trichotomy of evolution operators

Let E be an evolution operator on the Banach space X.

Definition 3.1 *An application* $P : \mathbb{R}_+ \to \mathfrak{F}(X)$ *is said to be a projection family on* X *if*

$$P(t)^2 = P(t), \ \forall t \in \mathbb{R}_+. \tag{3.1}$$

Definition 3.2 *Three projection families* $P_0, P_1, P_2 : \mathbb{R}_+ \to \mathfrak{F}(X)$ *are said to be compatible with the evolution operator* $E : T \to \mathfrak{F}(X)$ *if*

(ct_1) $P_0(t) + P_1(t) + P_2(t) = I$, $\forall t \geq 0$

(ct_2) $P_i(t)P_j(t) = 0$, $i,j \in \{0,1,2\}$, $i \neq j$, $\forall t \geq 0$

(ct_3) $E(t,t_0)P_k(t_0) = P_k(t)E(t,t_0)$, $\forall (t,t_0) \in T$ *and* $k \in \{0,1,2\}$.

In what follows we will denote

$$E_k(t,t_0) = E(t,t_0)P_k(t_0) = P_k(t)E(t,t_0)$$

for all $(t,t_0) \in T$ and $k \in \{0,1,2\}$.

Remark 3.1 *If* E *is an evolution operator on* X*, then* E_0*,* E_1 *and* E_2 *are also evolution operators on* X*, fact proved by the following relations*

$$E_k(t,s)E_k(s,t_0) = E(t,s)P_k(s)E(s,t_0)P_k(t_0) =$$

$$= P_k(t)E(t,t_0)P_k(t_0) = E_k(t,t_0),$$

for all $t \geq s \geq t_0 \geq 0$ *and* $k \in \{0,1,2\}$.

Definition 3.3 *An evolution operator* $E : T \to \mathfrak{F}(X)$ *on a Banach space* X *is said to be uniformly exponentially trichotomic if there exist* $N_0, N_1, N_2 > 1$, $\nu_0, \nu_1, \nu_2 > 0$ *and three projection families* P_0, P_1 *and* P_2 *compatible with* E *such that*

(uet_0) $\|E_0(s,t_0)x\| \leq N_0 e^{\nu_0(t-s)} \|E_0(t,t_0)x\| \leq N_0^2 e^{2\nu_0(t-s)} \|E_0(s,t_0)x\|$

(uet_1) $e^{\nu_1(t-s)} \|E_1(t,t_0)x\| \leq N_1 \|E_1(s,t_0)x\|$

(uet_2) $e^{\nu_2(t-s)} \|E_2(s,t_0)x\| \leq N_2 \|E_2(t,t_0)x\|$

for all $t \geq s \geq t_0 \geq 0$ *and all* $x \in X$.

Remark 3.2 *For* $P_0 = 0$ *we obtain the property of uniform exponential dichotomy for evolution operators studied in [8], [10] and [15]. It is obvious that if the evolution operator* E *is uniformly exponentially dichotomic then it is uniformly exponentially trichotomic.*

Remark 3.3 *If* $P_0 = P_2 = 0$ *the property of uniform exponential stability is obtained, as in [1], [2], [7] and [12]. It follows that a uniformly exponentially stable evolution operator is uniformly exponentially dichotomic and, further, uniformly exponentially trichotomic.*

Remark 3.4 *Without any loss of generality, in Definition 3.4 we can suppose that*

$$N_0 = N_1 = N_2 = N \text{ and } \nu_1 = \nu_2 = \nu$$

because otherwise we can consider

$$N = \max\{N_0, N_1, N_2\} \text{ and } \nu = \min\{\nu_1, \nu_2\}.$$

Example 3.1 *Let us consider* $X = \mathbb{R}^3$ *with the norm*

$$\|(x_1, x_2, x_3)\| = |x_1| + |x_2| + |x_3|, \ x = (x_1, x_2, x_3) \in X.$$

Let $\varphi : \mathbb{R}_+ \to (0, \infty)$ *be a decreasing continuous function with the property that there exists* $\lim\limits_{t\to\infty} \varphi(t) = l > 0$.

Then the mapping $E : T \to \mathfrak{F}(X)$ *defined by*

$$E(t,t_0)x = (e^{-\int_{t_0}^t \varphi(\tau)d\tau} x_1, \ e^{\int_{t_0}^t \varphi(\tau)d\tau} x_2, \ e^{-(t-t_0)\varphi(0) + \int_{t_0}^t \varphi(\tau)d\tau} x_3)$$

is an evolution operator on X.

Let us consider the projections defined by

$$P_1(t)(x_1, x_2, x_3) = (x_1, 0, 0)$$

$$P_2(t)(x_1, x_2, x_3) = (0, x_2, 0)$$

$$P_3(t)(x_1, x_2, x_3) = (0, 0, x_3).$$

for all $t \geq 0$ *and all* $x = (x_1, x_2, x_3) \in X$.

Following relations hold

$$\|E(t,t_0)P_1(t_0)x)\| \leq e^{-l(t-s)}\, \|E(s,t_0)P_1(t_0)x)\|$$

$$\|E(t,t_0)P_2(t_0)x)\| \geq e^{l(t-s)}\, \|E(s,t_0)P_2(t_0)x)\|$$

$$\|E(t,t_0)P_3(t_0)x)\| \leq e^{\varphi(0)(t-s)}\, \|E(s,t_0)P_3(t_0)x)\|$$

$$\|E(t,t_0)P_3(t_0)x)\| \geq e^{-\varphi(0)(t-s)}\, \|E(s,t_0)P_3(t_0)x)\|$$

for all $t \geq s \geq t_0 \geq 0$ *and all* $x \in X$.

We conclude that E *is uniformly exponentially trichotomic.*

Theorem 3.1 *Let* $E : T \to \mathfrak{F}(X)$ *be an evolution operator on the Banach space* X *with the property that there exist three projection families* P_0, P_1 *and* P_2 *compatible with* E. *Then* E *is uniformly exponentially trichotomic if and only if there exist two nondecreasing functions* $f, g : [0,\infty) \to (1,\infty)$ *with the property*

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) = \infty$$

such that

(uet_0') $\|E_0(s,t_0)x\| \leq f(t-s)\, \|E_0(t,t_0)x\| \leq f^2(t-s)\, \|E_0(s,t_0)x\|$
(uet_1') $g(t-s)\, \|E_1(t,t_0)x\| \leq \ \|E_1(s,t_0)x\|$
(uet_2') $g(t-s)\, \|E_2(s,t_0)x\| \leq \ \|E_2(t,t_0)x\|$
for all $t \geq s \geq t_0 \geq 0$ *and all* $x \in X$.

Proof. *Necessity.* As the evolution operator $E : T \to \mathfrak{F}(X)$ is uniformly exponentially trichotomic it follows from Definition 3.3 that there exist three projection families P_0, P_1 and P_2 compatible with E such that E_0 has uniform exponential growth and uniform exponential decay, E_1 is uniformly exponentially stable and E_2 is uniformly exponentially instable.

According to Lemma 2.1 and Lemma 2.2 there exists a nondecreasing function $f : [0,\infty) \to (1,\infty)$ with the property

$$\lim_{t\to\infty} f(t) = \infty$$

such that

$$\|E_0(s,t_0)x\| \leq f(t-s)\, \|E_0(t,t_0)x\|$$

and

$$\|E_0(t,t_0)x\| \leq f(t-s)\, \|E_0(s,t_0)x\|$$

for all $t \geq s \geq t_0 \geq 0$ and all $x \in X$.

Hence (uet_0') is proved.

By a similar proof as in Lemma 2.1 one can characterize the properties of uniform exponential stability for E_1 and uniform exponential instability for E_2 (see [12]) by means of a nondecreasing function $g : [0, \infty) \to (1, \infty)$ with the property

$$\lim_{t \to \infty} g(t) = \infty$$

such that

$$g(t-s) \, \|E_1(t, t_0)x\| \leq \|E_1(s, t_0)x\|$$

respectively

$$g(t-s) \, \|E_2(s, t_0)x\| \leq \|E_2(t, t_0)x\|$$

for all $t \geq s \geq t_0 \geq 0$ and all $x \in X$, which completes the proof of (uet'_1) and (uet'_2).

Sufficiency. According to Lemma 2.1 and Lemma 2.2, the two inequalities of statement (uet'_0) imply that E_0 has exponential decay and exponential growth.

The inequality (uet'_1) characterize the property of uniform exponential stability for E_1 and (uet'_2) shows that E_2 is uniformly exponentially unstable, as in [12]. Thus, according to Definition 3.3, E is uniformly exponentially trichotomic. □

Definition 3.4 *The evolution operator $E : T \to \mathfrak{F}(X)$ is said to be strongly measurable if for every $(t_0, x) \in \mathbb{R}_+ \times X$ the mapping $t \to \|E(t, t_0)x\|$ is measurable.*

Theorem 3.2 *Let $E : T \to \mathfrak{F}(X)$ be an evolution operator on the Banach space X with the property that there exist three projection families P_0, P_1 and P_2 compatible with E such that the evolution operators E_1 and E_2 are strongly measurable.*

Then E is uniformly exponentially trichotomic if and only if

(i) E_0 and E_1 have uniform exponential growth;

(ii) E_0 and E_2 have uniform exponential decay;

(iii) there exists $M \geq 1$ such that following inequalities hold

$$\int_s^t \|E_1(\tau, t_0)x\| \, d\tau \leq M \, \|E_1(s, t_0)x\| \tag{3.2}$$

and

$$\int_s^t \|E_2(\tau, t_0)x\| \, d\tau \leq M \, \|E_2(t, t_0)x\| \tag{3.3}$$

for all $t \geq s \geq t_0 \geq 0$ and all $x \in X$.

Proof. The property of uniform exponential trichotomy is equivalent with the existence of three projection families P_0, P_1 and P_2 compatible with E such that E_0 is with uniform exponential growth and uniform exponential decay, E_1 is uniformly exponentially stable and E_2 is uniformly exponentially instable.

It is sufficient to prove that if the evolution operator E_1 has uniform exponential growth and satisfies (3.2) than it is uniformly stable. Indeed, if we denote by

$$\frac{1}{N} = \int_0^1 \frac{du}{g(u)}$$

where function g is given by Lemma 2.2, then

$$\frac{\|E_1(t,t_0)x\|}{N} = \int_{t-1}^{t} \frac{\|E_1(t,t_0)x\|}{g(t-\tau)} d\tau \leq$$

$$\leq \int_s^t \|E_1(\tau,t_0)x\| \, d\tau \leq M \, \|E_1(s,t_0)x\|$$

and hence

$$\|E_1(t,t_0)x\| \leq MN \, \|E(s,t_0)x\|$$

for all $t \geq s+1$, $s \geq t_0 \geq 0$ and all $x \in X$.

If $t \in [s, s+1]$ then

$$\|E_1(t,t_0)x\| \leq g(t-s) \, \|E_1(s,t_0)x\| \leq g(1) \, \|E_1(s,t_0)x\|$$

for all $s \geq t_0 \geq 0$ and all $x \in X$.

Finally, we deduce that E_1 is uniformly exponentially stable.

Similarly, it is sufficient to prove that if the evolution operator E_2 has uniform exponential decay and satisfies relation (3.3), then it is uniformly instable.

Indeed, if we denote by

$$\frac{1}{N} = \int_0^1 \frac{du}{f(u)}$$

where function f is given by Lemma 2.1, then

$$\frac{\|E_2(s,t_0)x\|}{N} = \int_s^{s+1} \frac{\|E_2(s,t_0)x\|}{f(v-s)} dv \leq \int_s^{s+1} \|E_2(v,t_0)x\| \, dv \leq$$

$$\leq \int_{t_0}^{t} \|E_2(v,t_0)x\| \, dv \leq M \, \|E_2(t,t_0)x\|$$

and hence

$$\|E_2(s,t_0)x\| \leq MN \, \|E_2(t,t_0)x\|$$

for all $t \geq s \geq t_0 \geq 0$ and all $x \in X$ and so E_2 is uniformly exponentially instable. □

Remark 3.5 *Theorem 3.2 can be considered a generalization of a well-known result due to Datko (Theorem* 11 *from [2]). We remark that our proofs are not generalizations of Datko's proof for the characterization of the uniform exponential stability property.*

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